Limit sets of free groups, Hausdorff dimension and subshifts of finite type

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§0. Introduction

Let X be a locally compact separable metric space which has an infinite diameter. Assume that X has a natural compact boundary ∂X with a corresponding induced (cordal) metric. Suppose that G is a discrete group of isometries of X which acts properly discontinuously on X. Set

$$orb(O,G) = \{U : U = gO, g \in G\}, O \in X,$$
$$\Lambda(G,O) = closure(orb(O,G)) \cap \partial X.$$

The simplest invariants of G are the Hausdorff dimension $\delta(G,O)$ of $\Lambda(G,O)$ and the critical exponent $\kappa(G)$ of the related Poincaré series. (We will show that $\kappa(G)$ is independent of the point $O \in X$.) A more sophisticated invariant is the family of Patterson-Sullivan measures on $\Lambda(G,O)$. See for example [**Pat**], [**Sul**] and [**Tuk**] for the case where $X = \mathbf{H}^n$ and G is a discrete group of

hyperbolic isometries. In this case $\Lambda(G,O)$ and $\delta(G,O)$ do not depend on the choice of O. We will let $\Lambda(G) := \Lambda(G,O), \delta(G) := \delta(G,O)$ when these quantities do not depend on O.

Assume in addition that F is a finitely generated free group on a set S of r generators. It is well known that $\Lambda(F,O)$ can be described as a standard subsfift of finite type (SFT) on $2r$ symbols - Γ^∞ such that certain points of $\Lambda(F,O)$ have a multiple representation as points of Γ^∞.

(If F is a Fuchsian group then a countable number of points of the limit set have a double representation.) One can view Γ^∞ as the boundary $\partial\mathcal{K}(F,S)$ of the infinite $2r$ regular tree $\mathcal{K}(F,S)$. (It is the tree induced by $orb(O,F)$.

In the Fuchsian case $\mathcal{K}(F,S)$ corresponds to the geometrical tiling induced by S.) Let ϕ be a metric on $\mathcal{K}(F,S)$. ϕ induces a metric d_ϕ on Γ^∞ which makes Γ^∞ a compact space (under some mild assumptions on ϕ). Let $\delta(\phi)$ be the Hausdorff dimension of Γ^∞ in d_ϕ. There are two natural choices for ϕ. The first one is the standard metric on an infinite tree by letting $a > 0$ be the distance between any two neighboring vertices. Denote by d_a the induced metric on Γ^∞. For this metric one has the simple equality $\delta(d_a) = \frac{1}{a}$. The second metric ψ on $\mathcal{K}(F,S)$ is the metric induced by the original metric on X.

Let $\mathcal{E}$ be the set of probability measures on Γ^∞ which are invariant and ergodic for the shift map. Choose $\mu \in \mathcal{E}$. In §5 we show that the metric ψ on $\mathcal{K}(F,S)$ induced by the X-metric is essentially the natural metric $d_{\alpha(\mu)}$ for almost all vertices of $\mathcal{K}(F,S)$ with respect to μ. The value $\alpha(\mu)$ is the limit of the "average" X-distance between the adjacent vertices of $\mathcal{K}(F,S)$. The existence of this limit is guaranteed by Kingman's subadditive ergodic theorem. Let $h(\mu)$ be the measure entropy of the shift map on Γ^∞. Assume that $h(\mu) > 0$. We then show that

$$\kappa(F) \geq \frac{h(\mu)}{\alpha(\mu)}, \mu \in \mathcal{E}. \tag{0.1}$$

Moreover in the above inequality we may replace $\alpha(\mu)$ by $\frac{\alpha_m(\mu)}{m}, m = 1, ...,$ where $\alpha_m(\mu)$ is the μ-average

X-distance of the vertices having graph distance m to the origin O (these vertices correspond to the words of length m in F). These inequalities give a computable way to find lower bounds for $\kappa(F)$. In particular we have the inequality

$$\kappa(F) \geq \sup_{\mu \in \mathcal{E}, h(\mu) > 0} \frac{h(\mu)}{\alpha(\mu)}. \tag{0.2}$$

The works of Bowen [**Bow**] and Ruelle [**Rue**] suggest that in the case F is Fuchsian, or more generally when F is geometrically finite free group of hyperbolic isometries, then equality holds in (0.2). The maximal measure is the Bowen-Margulis-Ruelle measure which should be closely related to the Patterson-Sullivan measure.

Most of our paper is devoted to the various connections between the geometrical notions of groups as above and the corresponding notions in SFT. It seems that many results of these paper (including inequality (0.1-0.2)) can be extended to nonfree (Fuchsian) groups G if the limit set $\Lambda(G,O)$ can be given by SFT. See for example [**Seri**] and the references therein. We are postponing this discussion to the near future.

We now survey briefly the content of our paper. §1 introduces the basic notions of the paper. We consider an infinite locally finite tree $\mathcal{T}$ with the standard metric. This metric induces a metric on the boundary $\partial\mathcal{T}$. Let F be a finitely generated free group of isometries acting on $\mathcal{T}$. We then discuss the limit set $\Lambda(F,\mathcal{T})$. In §2 we consider the Hausdorff dimension $\delta(F,\mathcal{T})$ of $\Lambda(F,\mathcal{T})$. Assume that G is a finitely generated subgroup of F. We then consider the problem when $\delta(G,\mathcal{T})=\delta(F,\mathcal{T})$. We show that the above equality holds iff G has a finite index in F. §3 considers the classical case of the modular group $PSL(2,\mathbf{Z})$. Let $\mathcal{T}$ be the Bruhat-Tits tree corresponding to the modular group. Assume that G is a free finitely generated subgroup of the modular group. We then discuss the relations between the (standard) Hausdorff dimension $\delta(G)$ and the corresponding Hausdorff dimension $\delta(G,\mathcal{T})$. In §4 we show how a metric ϕ on an infinite locally finite tree $\mathcal{T}(\Gamma)$, induced by a given finite digraph Γ, gives rise to the metric d_ϕ on $\partial\mathcal{T}(\Gamma)=\Gamma^\infty$. In §5 we show that the metric ϕ on $\mathcal{T}(\Gamma)$ gives rise to a set of functions satisfying the conditions of Kingman's subadditive ergodic theorem. Denote by $\alpha(\mu)\geq 0$ the limit value of these functions with respect to an ergodic measure $\mu\in\mathcal{E}$. It then follows that the metric ϕ is essentially the natural metric $d_{\alpha(\mu)}$ on $\mathcal{T}(\Gamma)$. We then prove the inequalities (0.1-0.2). In the last section we give some arguments based on Ruelle's results in favor of the conjecture that for finitely generated free Fuchsian groups equality holds in (0.2). We then define and discuss an analog of the Patterson-Sullivan measure μ on Γ^∞. We consider in detail the case of the modular group. Finally we show that for any group G of discrete isometries containing a free subgroup on two generators and which acts freely and properly discontinuously on a locally compact separable matric space $\kappa(G)>0$.

§1. Limit sets of free groups acting on infinite trees

Let $\mathcal{T}=(V,E)$ be an infinite locally finite tree. Here V is the set of vertices and $E\subset V\times V$ is a set of (undirected) edges. We suppose that $\mathcal{T}$ does not have leaves, i.e. the degree $deg(v)$ of any vertex $v\in V$ is greater than one. Introduce the standard metric on the set of vertices V by letting

$$dg(u,u):=0, u\in V;\;\; dg(u,v):=h>0, u,v\in V,(u,v)\in E.$$

For any $u,v\in V, u\neq v$ set $dg(u,v):=kh$ where k is the length of the unique path in $\mathcal{T}$ between u and v. In what follows we assume that $h=1$ unless stated otherwise.

Assume that $\mathcal{T}$ is an infinite locally finite tree with no leaves. The standard compactification of $\mathcal{T}$ is as follows. Fix a point O (the origin). Then every point $a\in\partial\mathcal{T}$ is an infinite path $a=(v_i)_0^\infty, v_0=O$, i.e. $v_{i+1}\neq v_{i-1}, i=1,...,$. Note that a is a global geodesic. Since $\mathcal{T}$ is locally finite it follows that $\partial\mathcal{T}$ is a compact totally disconnected (Cantorian) set. We now define a usual metric on $\partial\mathcal{T}$. Fix $0<\tau<1$. (Later we show that there is a convenient choice of τ to make our formulas simpler.) For $a,b\in\partial\mathcal{T}$ we let $dist(a,b)=\tau^k$ where k is the length of the maximal common subchain of a and b starting form O. We call the metric $dist(\cdot,\cdot)$ the *chordal* metric on $\partial\mathcal{T}$. It is possible to extend this metric to $\mathcal{T}\cup\partial\mathcal{T}$ as follows. Identify each $u\in V$ with a unique path $c=(v_i)_0^k, v_0=O, v_k=u$. Let $d=(w_i)_0^m, 1\leq m\leq\infty$ be either finite or infinite path starting from O. Then $dist(c,c)=0$ and $dist(c,d)=\tau^k, c\neq d$

where k is the length of the maximal common subchain of c and d starting from O. Let $dist'$ be the induced metric on $\mathcal{T}\cup\partial\mathcal{T}$ with respect the origin O'. It then follows that the corresponding metrics $dist, dist'$ are equivalent.

$$\frac{dist(a,b)}{K}\leq dist'(a,b)\leq K dist(a,b), a,b\in\mathcal{T}\cup\partial\mathcal{T},$$

for some $K\geq 1$.

Let F be a finitely generated free group with a minimal set of generators $S=\{f_1,...,f_r\}$. Then one associates with the pair (F,S) the Cayley tree $\mathcal{K}(F,S)=(V,E), E=F$, where each vertex $v\in V$ is expressed as a reduced word

$$w=f_{i_1}^{\epsilon_1}\cdots f_{i_k}^{\epsilon_{i_k}}, \epsilon_j=\pm 1, j=1,...,k.$$

Clearly $\mathcal{K}(F,S)$ is a $2r$ regular tree. Then $\partial\mathcal{K}(F,S)$ with respect to $O=e$ is the set of all infinite reduced products

$$w = f_{i_1}^{\epsilon_1}\cdots f_{i_j}^{\epsilon_{i_j}}\cdots, \epsilon_j = \pm 1, j = 1, ...,. \tag{1.1}$$

Assume that $\mathcal{T}$ is a locally finite infinite tree with no leaves. Let $Aut(\mathcal{T})$ be the group of isometries of $\mathcal{T}$ under the graph metric dg. Assume that $F \subset Aut(\mathcal{T})$ is a subgroup of isometries that acts freely. It then follows that F acts properly discontinuously on $\mathcal{T}$. Furthermore F is a free group. Assume that F is finitely generated and let $S = \{f_1, ..., f_r\}$ be a minimal generating set. Define $\Lambda(F,S) \subset \partial\mathcal{T}$ to be the limit set of (F,S) as follows. Consider an infinite product w of the form (1.1). Then $c(w) \in \partial\mathcal{T}$ is the induced chain given by the walk on $\mathcal{T}$ generated by the chains connecting $f_{i_1}^{\epsilon_1}\cdots f_{i_{j-1}}^{\epsilon_{i_{j-1}}}O$ to $f_{i_1}^{\epsilon_1}\cdots f_{i_j}^{\epsilon_{i_j}}O$ for $j = 1, ...,$. (That is, delete from the above infinite walk starting from O all edges on which the walk traverses an even number of times.) The above set of chains in $\partial\mathcal{T}$ is denoted by $\Lambda(F,S)$ - the limit set of F. It is a compact set in $\partial\mathcal{T}$. We now show how to present $\Lambda(F,S)$ in term of subshifts.

Recall that a standard representation of SFT on the alphabet of n letters $< n >=< 1, ..., n >$ is as follows. Let $\Gamma \subset < n > \times < n >$ be a digraph on the n vertices $< n >$. Denote by $A(\Gamma) = (a_{ij})_1^n$ an $n \times n$ $0-1$ matrix induced Γ. That is, $a_{ij} = 1 \iff (i,j) \in \Gamma$. Denote by $\rho(\Gamma)$ the spectral radius of Γ which is defined as the spectral radius of $A(\Gamma)$. Set $\Gamma^1 =< n >, \Gamma^2 = \Gamma$. For $k > 2$ let

$$\begin{aligned}\Gamma^k &= \{a : a = (a_i)_1^k, a_i \in < n >, i = 1, ..., n, (a_i, a_{i+1}) \in \Gamma, i = 1, ..., k-1\},\\ \Gamma^\infty &= \{a : a = (a_i)_1^\infty, a_i \in < n >, i = 1, ..., (a_i, a_{i+1}) \in \Gamma, i = 1, ..., \},\end{aligned}.$$

In what follows we assume that Γ is strongly connected. That is any two vertices in Γ connected by a directed path in Γ. Then Γ^∞ is a nonempty SFT induced by Γ. Γ^∞ is viewed as a topological space endowed with the Tychonov (coordinate-wise) topology. It is well known that Γ^∞ is a compact topological space under the Tychonov topology. Recall that

$$\sigma : \Gamma^\infty \to \Gamma^\infty, \sigma((a_i)_1^\infty) = (a_{i+1})_1^\infty, (a_i)_1^\infty \in \Gamma^\infty$$

is a shift operator.

Let F be a free group generated by the set $S = \{f_1, ..., f_r\}$. Identify f_i, f_i^{-1} with $i, i+r$ respectively for $i = 1, ..., r$. Set

$$\Gamma = \Gamma(F) := \{(i,j) : i, j \in < 2r >, |i-j| \neq r\}.$$

It then follows that every sequence $(a_i)_1^\infty \in \Gamma^\infty$ represents an infinite reduced word w of the form (1.1). Vice versa, any infinite reduced word w of the form (1.1) corresponds to a unique sequence $(a_i)_1^\infty \in \Gamma^\infty$. Thus

$$\Gamma^\infty = \Lambda(F,S) = \partial\mathcal{K}(F,S).$$

§2. On the Hausdorff dimension of subgroups of free groups

Let $\mathcal{T}$ be an infinite locally finite tree with no leaves. Assume that F is a finitely generated free group which acts freely as a group of isometries on $\mathcal{T}$. Then $\Lambda(F,S) \subset \partial\mathcal{T}$ and $\Lambda(F,S)$ is a compact metric space with respect to the chordal metric. By abusing the notation we let $dist : \Gamma^\infty \to \mathbf{R}_+$ the metric induced by the chordal metric on $\Lambda(F,S)$. Denote by $\delta(\Lambda(F,S))$ the Hausdorff dimension of $\Lambda(F,S)$ with respect to the chordal metric. It is not diffcult to show that $\Lambda(F,S)$ and $\delta(\Lambda((F,S))$ are independent of a specific choice of generators. See for example [**Fri**]. We thus let $\Lambda(F) := \Lambda(F,S), \delta(\Lambda(F)) := \delta(\Lambda(F,S))$ and no ambiguity will arise. In [**Fri**] we showed.

Theorem 2.1. *Let $\mathcal{T}$ be an infinite locally finite tree with no leaves. Let F be a finitely generated free group which acts freely on $\mathcal{T}$. Denote by $\Lambda(F) \subset \partial\mathcal{T}$ the limit set of F. Let $\delta(\Lambda(F))$ be the Hausdorff dimension of $\Lambda(F)$ with respect to the chordal metric on $\partial\mathcal{T}$. Then there exists a digraph $\Phi(F)$ so that.*

$$\delta(\Lambda(F)) = \frac{\log \rho(\Phi(F))}{\log \tau^{-1}}.$$

Let the assumptions of Theorem 2.1 hold. Assume that $G \subset F$ is a finitely generated subgroup. Then Theorem 2.1 yields

$$\delta(\Lambda(G)) = \frac{\log \rho(\Phi(G))}{\log \tau^{-1}}.$$

Set

$$\delta(G,F) \stackrel{\mathrm{def}}{=} \frac{\delta(\Lambda(G))}{\delta(\Lambda(F))}$$

to be the relative Hausdorff dimension of the $\Lambda(G)$ in $\Lambda(F)$. It then follows that

$$\delta(G,F) = \frac{\log \rho(\Phi(G))}{\log \rho(\Phi(F))}. \tag{2.2}$$

That is, the relative Hausdorff dimension of the limit set of G in the limit set of F is independent of the value of the parameter τ.

Let F be a free finitely generated group on r generators $S = \{f_1, ..., f_r\}$. Assume that $G \subset F$ is a finitely generated subgroup. We view F and G as a groups of isometries of the tree $\mathcal{K}(F,S)$. We now adress the equalities $\Lambda(F) = \Lambda(G)$ and $\delta(G,F) = 1$. Note that the first equality yields the second equality. We show that either of these conditions yield that G has a finite index in F, i.e. $[F:G] < \infty$. To prove these results we recall some standard facts about the subgroups of free groups - the Schreier method. Consult for example with [**Hal**, Ch. 7].

Let G be a subgroup of a free group F on r generators $S = \{f_1, ..., f_r\}$. It is well known that G is free. In particular, for any Schreier system of the representation of F in terms of right cosets of G one obtains a set of free generators Y which posses significant factors. That is for each $y \in Y$ we may select a factor from the reduced form of y and y^{-1}:

$$y = a_1 \cdots a_i \cdots a_t, \ \ y^{-1} = a_t^{-1} \cdots a_i^{-1} \cdots a_1^{-1},$$
$$a_1, ..., a_t \in \{f_1, f_1^{-1}, ..., f_r, f_r^{-1}\},$$

selecting a_i from y and a_i^{-1} from y^{-1} in such a way that in any product

$$zw, z \neq w^{-1}, z, w \in Y \cup Y^{-1},$$

the cancellation does not reach the significant factor in z or w. Furthermore, it is possible to choose Y to have the central significant factors. That is, if y has an odd length then a_i is the central term while if y has even length then a_i is one of the two central terms. In what follows we assume that the set of free generators of Y of G posseses the central significant factors.

Theorem 2.3. *Let F be a finitely generated free group. Assume that G is a finitely generated subgroup. Let F and G act as a group of isometries on $\mathcal{K}(F,S)$-the Cayley tree of F with respect to a minimal set of generators S. Then $\Lambda(F) = \Lambda(G)$ if and only if $[F:G] < \infty$, i.e. G has a finite index in F.*

Proof. Let $F = < f_1, ..., f_r >$. Assume first that $[F:G] < \infty$. It then follows that G is a finitely generated free subgroup. Assume that $Y = \{y_1, ..., y_n\}$ is a set of the free generators of G obtained from a Shreirer system of right cosets induced by any lexicographic order on the $\{f_1, f_1^{-1}, ..., f_r, f_r^{-1}\}$: $F = \cup_0^m G g_i, g_0 = e$. Set

$$L_1 = \max_{1 \le j \le n} l(y_j), L_2 = \max_{1 \le i \le m} l(g_i), L = \max(L_1, L_2).$$

Let $w \in \Lambda(F)$ given by (1.1). It is enough to show that for any given $k \geq 1$ there exists $u = u(k) \in \Lambda(G)$ so that $dist(w,u) < \tau^k$. Let $v = a_1 \cdots a_{k+2L}$ be a reduced word of length $k + 2L$ which coincides with the first $k + 2L$ part of w. Then $v \in G g_j$. Thus $v' = v g_j^{-1} = a_1 \cdots a_{k'} \in G$ is a reduced word of length $k' \geq k + L$. Rewrite v' as a reduced word

$$x = x_1 x_2 \cdots x_p, x_i \in Y \cup Y^{-1}.$$

Let $y = x_1 \cdots x_p \cdots$ be an infinite reduced word in $Y \cup Y^{-1}$ whose first $p-th$ part coincides with x. Rewrite y as an infinite reduced product $v = b_1 b_2 \cdots$ in $< f_1, f_1^{-1}, ..., f_r, f_r^{-1} >$. Use the fact that each y_i, y_i^{-1} posseses a significant factor to deduce that $a_i = b_i, i = 1, ..., k+1$. Hence $dist(w, v) < \tau^k$.

Assume now G is finitely generated by the set of generators Y chosen as above. Suppose furthermore that $\Lambda(G) = \Lambda(F)$. Let L_1 be defined as above. Let $h_0 = e, ..., h_N$, be the set of all reduced words of length $2L_1$ at most. It is enough to show that $F = \cup_{0 \le i \le N} G h_i$. Let u any finite reduced word. Extend it an infinite reduced word $w = uu' \in \Lambda(F)$. Let $y = x_1 \cdots$ be an infinite reduced word in $Y \cup Y^{-1}$ so that when we rewrite y as an infinite reduced word in $f_1, f_1^{-1}, ..., f_r, f_r^{-1}$ we get w. In the product $x_1 \cdots$ we mark the significant factor a_i of each x_i. Let k be the minimal k so that $w = w' a_k w''$ and $l(w' a_k) \ge l(u)$. It then follows that $u = x_1 \cdots x_k h_j$ for some $0 \le j \le N$. Hence $[F : G] < \infty$ $\diamond$

Assume that F is a finitely generated free group and G is its subgroup which is not finitely generated. We suspect that $\Lambda(G) \ne \Lambda(F)$.

Let $\Gamma \subset < n > \times < n >$ be a non-acyclic graph. There is a standard way to express Γ^∞ as $\Gamma(m)^\infty, m = 1, ...,$. Here $\Gamma(1) = \Gamma$ and $\Gamma(m) \subset \Gamma^m \times \Gamma^m, 1 < m$. See for example [**Fri**]. Set

$$\Gamma(m) = \{(a, b) : a = (a_i)_1^m, b = (b_i)_1^m \in \Gamma^m, a_{i+1} = b_i, i = 1, ..., m-1, \}, m = 2, ..., .$$

Clearly, any $a \in \Gamma^\infty$ induces a unique element $a' \in \Gamma(m)^\infty$ and vice versa. Let $\phi_m : \Gamma^\infty \to \Gamma(m)^\infty$ be the above isomorphism. A subset $X \subset \Gamma^\infty$ is called a strict subshift of a finite type (SSFT) if there exists an $m \ge 1$ and a strict subset $\Delta \subset \Gamma(m)$, viewed as a spanning subdigraph on Γ^m vertices, so that $\phi_m(X) = \Delta^\infty$. Given $0 < \tau < 1$ then the standard metric on Γ^∞ is as follows.

$$d(a, a) = 0, d(a, b) = d(b, a) = \tau^k, a = (a_i)_1^\infty, b = (b_i)_1^\infty, a_i = b_i, i = 1, ..., k, a_{k+1} \ne b_{k+1}.$$

See [**Fri**] for more general natural metrics on Γ^∞. If Γ^∞ is the SFT induced by a free group on r generators then the above metric is the chordal metric induced on the corresponding Cayley tree. The above metric on Γ^∞ induces the unique metric on $\Gamma(m)^\infty$ so that $\phi_m : \Gamma^\infty \to \Gamma(m)^\infty$ is an isometry. Let $X \subset \Gamma^\infty$ be a closed set. Then we let $\delta(X)$ to be the Hausdorff dimension of X with respect the above metric. ¿From the results of [**Fri**] one can find $\delta(\Gamma^\infty)$ and hence $\delta(X)$ for any SSFT of Γ^∞.

Theorem 2.4. *Let $\Gamma \subset < n > \times < n >$ be a non-acyclic digraph. Assume that the Γ^∞ is endowed with the above metric. Then*

$$\delta(\Gamma^\infty) = \frac{\log \rho(\Gamma)}{\log \frac{1}{\tau}}.$$

Let $X = \Delta^\infty \subset \Gamma(m)^\infty$ be a SSFT. Then

$$\delta(X) = \frac{\log \rho(\Delta)}{\log \frac{1}{\tau}}.$$

Assume furthermore that Γ is strongly connected digraph. Then $\delta(X) < \delta(\Gamma^\infty)$.

Proof. The formula for $\delta(\Gamma^\infty)$ is given in [**Fri**]. To obtain the formula for $\delta(X)$ on needs the following observation. Let $\Gamma' = \Gamma(m)$. View Γ' as a digraph on $< N > \times < N >, N = |\Gamma^m|$. Introduce on Γ'^∞ a metric d' of the above form with $\tau' = \tau$. It then follows that the metric d' is equivalent to the metric d on $\Gamma(m)^\infty$ induced by the above metric on Γ^∞. That is,

$$\tau^m d(a, b) \le d'(a, b) \le \tau^{-m} d(a, b), a, b \in \Gamma(m)^\infty.$$

Hence $\delta'(X) = \delta(X)$, i.e. the Hausdorff dimension with respect to the equivalent metrics remain the same. Use the first formula for $\Gamma = \Delta$ to deduce the formula for $\delta(X)$.

Assume now that Γ is strongly connected. We now show that $\Gamma(m)$ is strongly connected. Observe that $a = (a_i)_1^m, b = (b_1)_1^m \in \Gamma^m$ are two paths of length m in Γ. Connect a_m to b_1 by some path. This will induce a path in $\Gamma(m)$ from a to b. That is $A(\Gamma(m))$ is an irreducible matrix. As $\Delta \subset \Gamma(m)$ is a strict subdigraph we deduce that $A(\Delta) \leq A(\Gamma(m))$ and the two matrices are different. Hence $\rho(\Delta) < \rho(\Gamma(m))$. Clearly

$$\delta(\Gamma^\infty) = \delta(\Gamma(m)^\infty) \Rightarrow \rho(\Gamma) = \rho(\Gamma(m)).$$

Therefore, $\delta(X) < \delta(\Gamma^\infty)$. $\diamond$

In fact, Theorem 2.1 was derived in **[Fri]** by skillful use of the formula for $\delta(\Gamma^\infty)$ given by Theorem 2.4.

Theorem 2.5 *Let F be a free group on $r > 1$ generators $S = \{f_1, ..., f_r\}$ and suppose that G is a finitely generated subgroup of F. Assume that F, G act as the groups of isometries on the Cayley tree $\mathcal{K}(F, S)$. Then $\delta(\Lambda(G)) = \delta(\Lambda(F))$ if and only if G has a finite index.*

Proof. Assume first that $[F : G] < \infty$. Then Theorem 2.4 yields that $\Lambda(F) = \Lambda(G) \Rightarrow \delta(\Lambda(F)) = \delta(\Lambda(G))$.

We now claim that the equality $\delta(\Lambda(F)) = \delta(\Lambda(G))$ is equivalent to the following condition. There exists $L \geq 0$ with the following property. Let $W_L = \{w_0, ..., w_N\}, N = N(L)$ be the set of all reduced words of length of length L at most. Then

$$F = \cup_{u,v \in W_L} uGv. \tag{2.6}$$

Assume first that (2.6) holds. The arguments of the proof of Theorem 2.3 yield that

$$\Lambda(F) = \cup_{w \in W_L} w\Lambda(G). \tag{2.7}$$

It then follows that

$$\delta(\Lambda(F)) = \max_{w \in W_L} \delta(w\Lambda(G)) = \delta(\Lambda(G)).$$

Vice versa, if (2.7) holds for some L then $\delta(\Lambda(G)) = \delta(\Lambda(F))$. Clearly, if (2.7) holds then (2.7) holds for any $L' > L$. Assume that (2.7) holds for $L' > 2L_1$ where L_1 is given in the proof of Theorem 2.3. Then arguments of the proof of Theorem 2.3 yield the condition (2.6).

Assume now that the equality (2.7) does not hold for $L = L_1$ where L_1 is given in the proof of Theorem 2.3. Let

$$x \in \Lambda(F) \backslash (\cup_{w \in W_L} w\Lambda(G)).$$

Let $w_i \in F$ be $i - th$ first part of x. The compactness argument yields that there exists $p > L$ so that that any reduced infinite word corresponding to $uv, u \in W_L, v \in \Lambda(G)$ does not have the word $w = w_p$ as its first part. Clearly, we may assume $p = l(w) > L + 1$. Let $w = (c_i)_1^p \in \Gamma^p$. Set

$$\Delta = \{((a_i)_1^{p-1}, (a_i)_2^p) : (a_i)_1^{p-1}, (a_i)_2^p \in \Gamma^{p-1}, (a_i)_1^p \neq (c_i)_1^p\}.$$

Form the definition of Δ we obtain Δ is a strict subdigraph of $\Gamma(p-1)$. It is left to show the condition

$$\phi_m(\cup_{u \in W_L} u\Lambda(G)) \subset \Delta^\infty \tag{2.8}$$

is equivalent to the following one. Consider any infinite word $uv, u \in W_L, v = x_1 x_2 \cdots \in \Lambda(G)$ and rewrite as $uv = s \in \Lambda(F)$. Set $s = (s_i)_1^\infty \in \Gamma^\infty$. Then $(s_i)_{i=j}^{j+p-1} \neq (c_i)_1^p, j = 1, ...,$. Assume to the contrary that $(s_i)_{i=j}^{j+p-1} = (c_i)_1^p$ for some $j \geq 1$. We now obtain a contradiction according to each of the following case.
(i) $u = e$. Then either the s_j is equal to the significant factor of $x_l = y_l^{\pm 1}$ or the element s_j appears in between the significant factor of x_l and x_{l+1}. Assume first s_j comes from the uncancelled part of x_l. Set $v' = x_l x_{l+1} \cdots$. Now choose a corresponding world $u' \in F, l(u') < l(x_l)$ so that $u'v'$ written as an infinite word in $\Lambda(F)$ starts with w. This contradicts the definiton of w.
(ii) It is impossible to reduce to case (i). That is, $uv = u'v'$, u is reduced, the element $u'v'$ is reduced where $v' = x'_k x_{k+1} x_{k+2} \cdots$ and $l(v') \leq L$. Here x'_k is x_k with some deleted beginning part and in the product $x'_k x_{k+1}$ x'_k is not cancelled completely. If the word w starts from some part of of u' then by cancelling out the word preceding the start of w in $u'v'$ we deduce that w is in the beginning of $u''v$ which is impossible by our assumption. Thus w has to start either from the beginning of x'_k or in between the beginning of x'_k

and the significant part of x_{k+1}. Let $v'' = x_k x_{k+1} \cdots$. Then w starts somewhere in the uncancelled part of x_k. Obviously, there exists a reduced word $\tilde{u}, l(\tilde{u}) < L$ so that w is the beginning of $\tilde{u}v$ contrary to our assumption. We thus proved the validity of (2.8). Hence

$$\delta(\Lambda(G)) = \max_{u \in W_L} \delta(u\Lambda(G)) = \delta(\cup_{u \in W_L} u\Lambda(G)) = \delta(\phi_m(\cup_{u \in W_L} u\Lambda(G)) \leq \delta(\Delta^\infty).$$

As Γ is strongly connected ($r > 1$) from Theorem 2.4 we deduce that $\delta(\Lambda(G)) < \delta(\Lambda(F))$.

We thus demonstrated that the equality $\delta(\Lambda(G)) = \delta(\Lambda(F))$ is equivalent to the condition (2.7). As (2.7) is equivalent to (2.6) it is left to show that (2.6) yields $[F : G] < \infty$. Rewrite (2.6) in the form

$$F = \cup_{u,v \in W_L} (uGu^{-1})(uv).$$

That is F is a finite union of right cosets of the subgroups uGu^{-1}. B.H. Neumann's lemma claims that some uGu^{-1} has a finite index in F [**Neu**]. Hence $[F : G] < \infty$. $\diamond$

§3. The modular group

Consider the modular group $PSL(2,\mathbf{Z})$ which acts as a discrete group of Möbius transformations on the upper half plane H^2. It is well known that $PSL(2,\mathbf{Z})$ is an amalgamated product $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ where the generators of the groups $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/3\mathbf{Z}$ are S and T respectively.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, S(z) = -\frac{1}{z},$$
$$T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, T(z) = -\frac{1}{z+1}.$$

Let $D \subset H^2$ be the standard Dirichlet domain

$$D = \{z : 0 < \Im z, -\frac{1}{2} \leq \Re z \leq \frac{1}{2}, 1 \leq |z|\}$$

associated with $PSL(2,\mathbf{Z})$.

There are two possible ways to obtain two infinite graphs for $PSL(2,\mathbf{Z})$. The first one is the standard way associated with the idea of the space tiling. We tile the upper half plane with the copies of D.

$$H^2 = \cup_{f \in PSL(2,\mathbf{Z})} f(D).$$

Fix a point O in the interior of D - $Int(D)$. Then the vertices V of the infinite graph $\mathcal{G}$ is the orbit of O.

$$Orb(O) = \{z : z = f(O), f \in PSL(2,\mathbf{Z})\}.$$

Two vertices $f(O), g(O)$ are connected, i.e. $(f(O), g(O)) \in E$, iff the two tiles $f(D), g(D)$ have a common boundary of dimension 1. That is $f(D) \cap g(D)$ is a geodesic of a positive length. It then follows that $\mathcal{G} = (V, E)$ is an infinite regular trivalent graph, i.e. $deg(v) = 3, v \in V$. Note that $\mathcal{G}$ is not a tree since it has a cycle of length 6 corresponding to the six tiles surrounding the following corner of D

$$\zeta = \cos\frac{\pi}{3} + \sqrt{-1}\sin\frac{\pi}{3} \in H^2.$$

The second way is essentially the Bruhat-Tits tree associated with $PSL(2,\mathbf{Z})$ although it seems to be known before in the case of $PSL(2,\mathbf{Z})$. Consult with [**Ser**]. Let

$$H((1,0)) = \{z : \Im z \geq 1\}$$

be a closed horoball in H^2. Set

$$U = H^2 \setminus \cup_{f \in PSL(2,\mathbf{Z})} f(D).$$

Let Q be the domain bounded by the following three horocycles

$$\Im(z) = 1, |z - \frac{\sqrt{-1}}{2}| = \frac{1}{2}, |z - (1 + \frac{\sqrt{-1}}{2})| = \frac{1}{2}$$

which correspond to the horoballs $H((1,0)), H((0,1)), H((1,1))$ respectively. It then follows that

$$U = \cup_{f \in PSL(2,\mathbf{Z})} f(Q).$$

Two domains $f(Q), g(Q)$ are said to be neighbors if their boundary intersect (necessary at one point). Equivalently, $f(Q), g(Q)$ are neighbors if both of them are bounded by two common horocycles. Pick up $O \in Int(D) \cap Q$. Then the Bruhat-Tits tree is $\mathcal{T} = (V, E')$ where $V = Orb(O)$ and $(f(O), g(O)) \in E'$ iff $f(Q), g(Q)$ are neighbors. Note that $\mathcal{T}$ is a regular trivalent tree. Thus $\mathcal{G}$ and $\mathcal{T}$ are two trivalent graphs on the common set of vertices $V = orb(O)$ on which $PSL(2,\mathbf{Z})$ acts properly discontinously as a group of isometries. The limit set of $PSL(2,\mathbf{Z})$ considered as a Fuchsian group is

$$S^1 = \mathbf{R} \cup \infty = \partial H^2.$$

It is the set of the acummulation points of $Orb(O)$ viewed as the points of the Riemann sphere lying in H^2. Each such a point $x \in S^1$ can given as an infinite reduced product (1.1) in terms of the generators S, T. We now show how to associate with the infinite product (1.1) a full two shift Γ^∞ with $\Gamma =< 2 > \times < 2 >$. Let

$$w((a_i)_1^\infty) = ST^{a_1} ST^{a_2} \cdots ST^{a_i} \cdots, (a_i)_1^\infty, a_i \in < 2 >, i = 1, ...,. \tag{3.1}$$

Set

$$w((a_i)_1^k) = ST^{a_1} \cdots ST^{a_k}, a_i \in < 2 >, i = 1, ..., k.$$

Then $w((a_i)_1^\infty)$ represents the limit point of the sequence $w((a_i)_1^k)O, k = 1, ...,$. Finally, all the reduced infinite products in terms of S, T are either of the form $w((a_i)_1^\infty)$ or $Sw((a_i)_1^\infty)$. By considering the action of $PSL(2,\mathbf{Z})$ on $\mathcal{T}$ it follows that to all points

$$x \in S^1 \setminus \cup_{f \in PSL(2,\mathbf{Z})} f(\infty)$$

corresponds a unique infinite product which is either w given by (3.1) or Sw. Any point of the form $f(\infty)$ have two representations. For example, the point ∞ has the following two representations

$$w = w((a_i)_1^\infty), w' = Sw((b_i)_1^\infty), a_i = 1, b_i = 2, i = 1, ...,.$$

Thus the two representation of $f(\infty)$ would correspond to the two infinite reduced words induced by fw and fw'. An analogous situation holds for the binary representation of $x \in [0,1]$.

$$x = \sum_1^\infty \frac{a_i - 1}{2^i}, (a_i)_1^\infty \in \Gamma^\infty. \tag{3.2}$$

That is, except the two constant sequences $(a_i)_1^k, a_1 = a_2 = \cdots$, all eventually constant sequences

$$(a_i)_1^\infty \in \Gamma^\infty, a_i = C, i \geq k > 1, a_{k-1} \neq C$$

represent numbers $x \in [0,1]$ which have two representations. Note that the standard metric on $[0,1]$ induces the following metric on Γ^∞ using (3.2).

$$dist(a,b) = |\sum_1^\infty \frac{a_i}{2^i} - \sum_1^\infty \frac{b_i}{2^i}|, a = (a_i)_1^\infty, b = (b_i)_1^\infty \in \Gamma^\infty.$$

Clearly, $\delta'(\Gamma^\infty)$-the Hausdorff dimension of Γ^∞ in the above metric is equal to 1-the standard dimension of the interval $[0,1]$. Let $d(\cdot,\cdot)$ be the metric on Γ^∞ introduced in §2 with $\tau = \frac{1}{2}$.

$$d(a,b) = \max_{i\ge 1} \frac{|a_i - b_i|}{2^{i-1}}, a = (a_i)_1^\infty, b = (b_i)_1^\infty \in \Gamma^\infty.$$

Clearly, $dist(a,b) \le d(a,b)$. Theorem 2.4 yields that $\delta(\Gamma^\infty)$-the Hausdorff dimension of Γ^∞ in the metric $d(\cdot,\cdot)$ is also 1. The equality $\delta(\Gamma^\infty) = \delta'(\Gamma^\infty)$ raises the following problem which is the primary motivation of this paper.

Let $F \subset PSL(2,\mathbf{Z})$ be the standard free subgroup of index 6 on the following two generators.

$$A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, z \to f_1(z) = z + 2;$$
$$A_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, z \to f_2(z) = \frac{z}{2z+1}.$$

The fundamental domain $D(F)$ of F is:

$$D(F) = \{z : 0 < \Im z, -1 \le \Re z \le 1, 1 \le |2z+1|, 1 \le |2z-1|\}.$$

$D(F)$ has two cusps. Let $\Lambda'(F)$ be the limit set of F as a Fuchsian group. Clearly, $\Lambda'(F) = S^1$. Let G be a finitely generated subgroup of F. Denote by $\Lambda'(G) \subset S^1$ the limit set of G viewed as Fuchsian group. Let $\delta'(\Lambda'(G))$ be the Hausdorff dimension of $\Lambda'(G)$. Set

$$\delta'(G,F) := \frac{\delta'(\Lambda'(G))}{\delta'(\Lambda'(F))} = \delta'(\Lambda'(G)).$$

On the other hand, F and G act the group of isometries on the Bruhat-Tits tree $\mathcal{T}$. In §2 we defined the relative Hausdorff dimension $\delta(G,F)$. If we choose the value $\tau = \frac{1}{2}$ for the definition of the chordal metric on $\partial\mathcal{T} = \Lambda(F)$ it the follows that $\delta(\Lambda(G)) = \delta(G,F)$.

Problem 3.3. *Let F be a free group on two generators of index 6 of the modular group $PSL(2,\mathbf{Z})$. Assume that G is a finitely generated subgroup of F. Is the following equality hold?*

$$\delta'(G,F) = \delta(G,F).$$

Note that if the answer to Problem 3.3 is positive then (2.2) yield that

$$\delta'(G) = \frac{\log \rho(\Phi(G))}{\log 2} \tag{3.4}$$

for the digraph $\Phi(G)$ induced by the action of G on $\mathcal{T}$. A weaker version of Problem 3.3 is the validity of (3.4) for any finitely generated subgroup G of F for some digraph $\Phi(G)$. We now show that Problem 3.3 has an affirmative answer in the following simple case.

Theorem 3.5. *Let F be a free subgroup on two generators of $PSL(2,\mathbf{Z})$ of index 6. Assume that G is a finitely generated subgroup of F. Then*

$$\delta'(G,F) = 1 \iff \delta(G,F) = 1.$$

That is $\delta'(\Lambda'(G)) = \delta'(\Lambda'(F))$ iff $[F:G] < \infty$.

Proof. Assume first that $\delta(G,F) = 1$. The arguments of Theorem 2.5 yield that $[F:G] < \infty$. Hence

$$\Lambda'(G) = S^1 \Rightarrow \delta'(\Lambda'(G)) = 1 = \delta'(G,F).$$

Assume now $\delta'(\Lambda(G)) = 1$. As G is finitely generated it is geometrically finite, e.g. [**Bea2**, §10.1]. Let $D(G)$ be any convex fundamental domain corresponding to G. Since G is free $D(G)$ is formed by $2k$ nonintersecting geodesics in H^2. Suppose first that $\partial D(G) \cap S^1$ consists of a finite number of points. That is, $D(G)$ has only cusps and hence of finite volume. It then follows that

$$[F : G] = \frac{vol(D(G))}{vol(D(F))} < \infty.$$

Assume now that $\partial D(G) \cap S^1$ contains an open interval (α, β). Then $(\alpha, \beta) \cap \Lambda'(G) = \emptyset$. Hence G is of the second kind. The result of Beardon [**Bea1**] imply that $\delta'(\Lambda'(G)) < 1$. See also [**Nic**, Th. 9.3.7]. This contradicts our assumption that $\delta'(\Lambda'(G)) = 1$. Hence $[F : G] < \infty$. $\diamond$

The above results and problems can be generalized as follows. Let F be a free finitely generated Fuchsian group on $r > 1$ generators $S = \{f_1, ..., f_r\}$. Let $D(F)$ a fundamental domain of F with $2r$ sides. Assume that $O \in Int(D(F))$. Then the Cayley tree $\mathcal{K}(F, S) = (V, E)$ corresponds to the geometrical tiling induced by $D(F)$. That is, let $V = Orb(O)$ under the action of F. Then $(f(O), g(O)) \in E$ iff $f(D(F)), g(D(F))$ have a common 1 dimensional side. Let $\Lambda'(F) \subset S^1$ be the limit set of F. Assume that G is a finitely generated subgroup of F. Let $\Lambda'(G)$ be the limit set of G. We then can define the relative Hausdorff $\delta'(G, F)$ as above. Thus, Problem 3.3 can be raised in this setting too. We are skeptical that the answer to this problem is positive. However we conjecture an analogous result to Theorem 3.5

Conjecture 3.6. *Let F be a finitely generated free Fuchsian group. Assume that G is finitely generated subgroup of F. Then*

$$\delta'(G, F) = 1 \iff [F : G] < \infty.$$

We conclude this section with the discussion of the Bruhat-Tits tree $\mathcal{T}$ for $PSL(2, \mathbf{Z})$. Each horoball appearing in the construction of $\mathcal{T}$ is of the form

$$f(H((1, 0))), f \in PSL(2, \mathbf{Z}).$$

Note that $f(H(1, 0))$ is tangent to S^1 at the point $f(\infty)$. We then assign to each horoball $f(H((1, 0)))$ an element $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}^2 \backslash 0$ so that $f(\infty) = \frac{\alpha_1}{\alpha_2}$. Clearly, this rule applies to the horoball $H((1, 0))$. Then all the horoballs of radius $\frac{1}{2}$ are of the form $H((n, 1)), n \in \mathbf{Z}$. Assume we already defined two tangent horoballs $H(\alpha)$ and $H(\beta)$ which are different from $H((1, 0))$. Let $H(\mu)$ be the unique horoball who is tangent to the horoballs $H(\alpha), H(\beta)$ and has a smaller radius then $H(\alpha)$ and $H(\beta)$. Then

$$\mu = \alpha + \beta.$$

Let $\mu = (\mu_1, \mu_2)$. Then the diameter of the horoball $H(\mu)$ is $\frac{1}{\mu_2^2}$. (I am indebted to J. Conway for showing me these formulas.) Consider a binary tree $\mathcal{T}'$ which is the following subtree of $\mathcal{T}$. The root of this tree is located at $O' \in Q_0$, where Q_0 is the domain bounded by $H(\alpha(O')), H(\beta(O')), H(\gamma(O'))$:

$$\alpha(O') = (0, 1), \beta(O') = (1, 1), \gamma(O') = \alpha(O') + \beta(O') = (1, 2).$$

Set $\Gamma =< 2 > \times < 2 >$. Then O' is connected to two vertices $(a_i)_1^1 \in \Gamma^1$. In general, all other vertices of $\mathcal{T}'$ are represented by $(a_i)_1^k \in \Gamma^k, k = 2, ...,$. Each $v = (a_i)_1^k$ connected to one vertex (the parent) of the form $u = (a_i)_1^{k-1}$ and two vertices (the children) of the form $w = (a_i)_1^{k+1}$. To each vertex $v = (a_i)_1^k \in \Gamma^k, k = 1, ...,$ we attach three vectors $\alpha(v), \beta(v)), \gamma(v)$ where

$$\gamma(v) = \alpha(v) + \beta(v).$$

Let u be the parent of $v = (a_i)_1^k$. Assume first that $a_k = 1$, that is v is to the left (and below) of u. Then

$$\alpha(v) = \alpha(u), \beta(v) = \gamma(u).$$

If $a_k = 2$, that is v is to the right (and below) of u then

$$\alpha(v) = \gamma(u), \beta(v) = \beta(u).$$

Note that the vertes v is located in the domain bounded by the three horoballs

$$H(\alpha(v)), H(\beta(v)), H(\gamma(v)).$$

Consider the sequence $(a_i)_1^\infty \in \Gamma^\infty$. Set

$$\alpha((a_i)_1^k) = (\alpha_1((a_i)_1^k), \alpha_2((a_i)_1^k)),$$
$$\beta((a_i)_1^k) = (\beta_1((a_i)_1^k), \beta_2((a_i)_1^k)),$$
$$\gamma((a_i)_1^k) = (\gamma_1((a_i)_1^k), \gamma_2((a_i)_1^k)), k = 1, ...,.$$

Then the rate of the approach of the sequence of the vertices in $\mathcal{T}'$ represented by $(a_i)_1^k, k = 1, ...,$ is the rate of convergence of the sequence $\frac{1}{\gamma_2((a_i)_1^k)^2}, k = 1, ...,$. Observe that $(a_i)_1^\infty$ represents uniquely the corresponding point on $[0, 1]$ (the interval on the real line between the two horoballs $H((0, 1))$ and $H(1, 1))$ unless the sequence $(a_i)_1^\infty$ is virtually constant. If the sequence $(a_i)_1^\infty$ is not virtually constant then the three sequences

$$\frac{1}{\alpha_2((a_i)_1^k)^2}, \frac{1}{\beta_2((a_i)_1^k)^2}, \frac{1}{\gamma_2((a_i)_1^k)^2}, k = 1, ...,$$

have the same rate of convergence to 0. The fastest rate of convergence to zero is achieved is for the alternating sequence $a_i \neq a_{i+1}, i = 1, ...,$. In that case $\alpha_2((a_i)_1^k), k = 1, ...,$ is a Fibonacci sequence. Thus the maximum rate of convergence of the diameters of the corresponding horoballs is $(\frac{2}{1+\sqrt{5}})^{2k}, k = 1, ...,$. The minimum rate of convergence is for the constant sequence. Then the diameters of the corresponding decreasing sequence of horoballs is $\frac{1}{(k+1)^2}, k = 1, ...,$. Assuming the standard normal distribution we see that for "most" of the sequences $(a_i)_1^\infty$ the corresponding rate of convergence is r^n for some $1 < r \leq (\frac{2}{1+\sqrt{5}})^2$. It is of interest to find the value of r.

Finally we remark that the sequence $\alpha(v), \beta(v)$ on the binary tree $\mathcal{T}'$ is a natural generalization of the deterministic Fibonacci sequence.

§4. Metrics on subshifs of finite types

Let $\Gamma \subset < n > \times < n >$ be a strongly connected digraph. Associate with Γ the following infinite tree $\mathcal{T}(\Gamma) := \mathcal{T} = (V, E)$. Let O be the root of the tree. The degree of O is n. We label the edges from O by $i = 1, ..., n$. Thus $(a_1) \in < n >$ represents a vertex connected to O by the edge a_1. A vertex v with $dg(v, O) = m > 1$ is given by the sequence $v = (a_i)_1^m \in \Gamma^m$. The vertex v is connected to $u = (a_i)_1^{m-1}$ so that $dg(u, O) = m - 1$. The vertex v is connected to all vertices $w = (a_i)_1^{m+1} \in \Gamma^{m+1}$. Let $deg(i), i \in < n >$ be the outdegree of the vertex i. (That is it is the number of all edges of the form $(i, j) \in \Gamma$.) It then follows $deg((a_i)_1^m)) = deg(a_m) + 1$ in $\mathcal{T}$. Thus, if $\Gamma \subset < 2r > \times < 2r >$ is digraph represents the free group on r generators then $\mathcal{T}$ is the regular $2r$-tree. We now introduce the distance function $d : V \times V \to \mathbf{R}_+$ as follows. First let

$$0 < dist((a_i)_1^m, O) = \phi_m(a_1, ..., a_m), m = 1, ...,. \tag{4.1}$$

Our next assumption is

$$dist((a_i)_1^{p+q}, (a_i)_1^p) = dist((a_i)_{p+1}^{p+q}, O) = \phi_q(a_{p+1}, ..., a_{p+q}), 1 \leq p, q. \tag{4.2}$$

Then the triangle inequality

$$dist((a_i)_1^{p+q}, O) \leq dist((a_i)_1^{p+q}, (a_i)_1^p) + dist((a_i)_1^p, O)$$

is equivalent to the inequality

$$\phi_{p+q}(a_1,...,a_{p+q}) \le \phi_p(a_1,...,a_p) + \phi_q(a_{p+1},..,a_{p+q}), p, q = 1, ...,. \tag{4.3}$$

The next assumption is the cancellation rule

$$a_1 = b_1 \Rightarrow dist((a_i)_1^p, (b_j)_1^q) = dist((a_i)_2^p, (b_j)_2^q). \tag{4.4}$$

In the case that $a_1 \ne b_1$ a simple condition is.

$$a_1 \ne b_1 \Rightarrow dist((a_i)_1^p, (b_j)_1^q) = \phi_p(a_1,...,a_p) + \phi_q(b_1,...,b_q). \tag{4.5}$$

Assume that $\Gamma \subset < 2r > \times < 2r >$ corresponds to the subshift induced by the free group on r generators. For each $i \in < 2r >$ let $\bar{i} \in < 2r >$ be the unique number so that $|\bar{i} - i| = r$. We then replace condition (4.5) by

$$a_1 \ne b_1 \Rightarrow dist((a_i)_1^p, (b_j)_1^q) = \phi_{p+q}(\bar{b}_q, \bar{b}_{q-1}, ..., \bar{b}_1, a_1, ..., a_p). \tag{4.5f}$$

Furthermore we assume the equality

$$\phi_p(a_1,...,a_p) = \phi_p(\bar{a}_p,...,\bar{a}_1), p = 1, ...,. \tag{4.6}$$

We discuss a few examples that motivate the above definitions.

First, assume that $\phi_p(a_1,...,a_p) = p$. It then follows that the induced metric on $\mathcal{T}$ is the standard metric $dg(\cdot,\cdot)$ on $\mathcal{T}$.

Our next example is as follows. Let $C = (c_{ij})_1^n \in M_n(\mathbf{R}_+)$ be a nonnegative $n \times n$ matrix. Recall that $\Gamma(C) \subset < n > \times < n >$ is the digraph induced by C. That is

$$(i,j) \in \Gamma(C) \iff c_{ij} > 0.$$

Suppose that $\Gamma \subset \Gamma(C)$. We then let

$$\phi_1(i) = t, i \in < n >, \max_{1 \le i,j} c_{ij} \le t, \phi_p(a_1,...,a_p) = t + \sum_{i=1}^{p-1} c_{a_i a_{i+1}}, (a_i)_1^p \in \Gamma^p, p = 2, ...,. \tag{4.7}$$

Consider the special linear group $SL(N,\mathbf{C})$ of $N \times N$ complexed valued matrices. Let I denote the identity matrix. For $A \in SL(N,\mathbf{C})$ let $A^* \in SL(N,\mathbf{C})$ denote the conjugate transpose of A. Recall that the spectral norm $||A||$ is given by the formula $||A||^2 = \rho(AA^*) = \rho(A^*A)$, where $\rho(B)$ is the spectral radius of $B \in SL(N,\mathbf{C})$. Note that $A \in SL(N,\mathbf{C}) \Rightarrow ||A|| \ge 1$. Let $SU(N,\mathbf{C}) \subset SL(N,\mathbf{C})$ be the special unitary group, i.e. the maximal compact subgroup of $SL(N,\mathbf{C})$. It is easy to show that $B \in SL(N,\mathbf{C}), ||B|| = 1 \iff B \in SU(N,\mathbf{C})$. Assume next that $A_1,...,A_n \in SL(N,\mathbf{C})$. Suppose furthermore that the following conditions are satisfied.

$$A_i \notin SU(N,\mathbf{C})), i = 1,...,n, A_{a_1} \cdots A_{a_m} \notin SU(N,\mathbf{C}), (a_i)_1^m \in \Gamma^m, m = 1, ...,. \tag{4.8}$$

The above conditions hold if $A_1,...,A_n$ generate a torsion free and discrete semigroup (in the standard topology). We then let

$$\phi_p(a_1,...,a_p) = 2\log ||A_{a_1} \cdots A_{a_p}||, (a_i)_1^p \in \Gamma^p, p = 1, ...,. \tag{4.9}$$

It then follows that (4.1) holds. Furthermore the submultiplicativity of the norm yields (4.3). Assume that (4.8) does not hold. It then follows that (4.1) is replaced by

$$0 \le \phi_m(a_1,...,a_m), m = 1, ...,. \tag{4.1'}$$

Then the conditions $(4.1)'$ and $(4.2)-(4.5)$ yield that $dist(\cdot,\cdot)$ is a semimetric on $\mathcal{T}=(V,E)$. Set

$$V(v)=\{u: u\in V, dist(u,v)=0\}.$$

For each $v\in V$ we identify the vertices $V(v)$ with one vertex v'. We thus obtain a new graph $\mathcal{T}'$ which is not a tree unless $dist(\cdot,\cdot)$ is a metric. Clearly, $dist(\cdot,\cdot)$ is a metric on $\mathcal{T}'$.

The following inverse problem arises naturally. Let $\phi_p, p=1,...,$ be defined as above and assume that $(4.1)'$ and (4.3) hold. Does there exist a separable Hilbert space $\mathcal{H}$ and n bounded linear operators

$$A_i:\mathcal{H}\to\mathcal{H}, i=1,...,n,$$

so that (4.9) holds. Here we assume that $||A||$ is the operator norm on $\mathcal{H}$.

We now consider $(4.5f)$ and (4.6). Let $F=<A_1,...,A_r>, A_1,...,A_r\in SL(N,\mathbf{C})$ be a free group. Recall that for $i\in <r>, \bar{i}=i+r$ and $A_{\bar{i}}=A_i^{-1}$. Set

$$\begin{aligned} dist((a_i)_1^p,(b_i)_1^q) &= \log ||A_{b_q}^{-1}\cdots A_{b_1}^{-1}A_{a_1}\cdots A_{a_p}|| + \log ||A_{a_p}^{-1}\cdots A_{a_1}^{-1}A_{b_1}\cdots A_{b_q}|| \\ &= \log ||A_{\bar{b}_q}\cdots A_{\bar{b}_1}A_{a_1}\cdots A_{a_p}|| + \log ||A_{\bar{a}_p}\cdots A_{\bar{a}_1}A_{b_1}\cdots A_{b_q}||, 0\le p,q. \end{aligned} \tag{4.10}$$

The above definition implies $(4.5f)$ and (4.6). As for any $B\in SL(2,\mathbf{C})$ we have the equality $||B^{-1}||=||B||$ we deduce that for $SL(2,\mathbf{C})$ (4.9) is equivalent to (4.10).

In the classical case set let $A_1,A_2\in SL(2,\mathbf{Z})$ be defined as in §3. That is, A_1,A_2 generate the free group $F\subset PSL(2,Z)$ which is a subgroup of the modular group of index 6. Let A_1,A_2 represent the Möbius transformations $f_1(z),f_2(z)$ as in §3. Set $O=\sqrt{-1}\in H^2$. It then follows that

$$\phi_p(a_1,...,a_p)=hdist(f_{a_1}\cdots f_{a_p}O,O) \tag{4.11}$$

where $hdist(u,v)$ is the hyperbolic distance between the two points u,v in the upper half plane H^2 and ϕ_p is given by (4.9). Indeed recall that $H^2\equiv SL(2,\mathbf{R})/SO(2,\mathbf{R})$. Furthermore the hyperbolic distance between $A,B\in SL(2,\mathbf{R})$ viewed as representatives of the cosets $ASO(2,\mathbf{R}), BSO(2,\mathbf{R})$ is

$$hdist(A,B)=\log ||B^{-1}A|| + \log ||A^{-1}B||.$$

(Note that the above equality applies to $H^3\equiv SL(2,\mathbf{C})/SU(2,\mathbf{C})$.) It is left to show that $O=\sqrt{-1}$ represents the coset $SO(2,\mathbf{R})$. Let

$$T=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, z\to h(z)=-\frac{1}{z}.$$

Note that O is the only fixed point of h in the upper half plane. As $TSO(2,\mathbf{R})=SO(2,\mathbf{R})$ we deduce that O represents the coset $SO(2,\mathbf{R})$ in the homogenous space $SL(2,\mathbf{R})/SO(2,\mathbf{R})$.

Let $\Gamma\subset <n>\times<n>$ be a strongly connected graph. In what follows we assume that $\rho(\Gamma)>1$. (Otherwise Γ^∞ has a finite number of points.) Let $\mathcal{T}$ be the associated tree. Assume furthermore that the metric on $\mathcal{T}$ is given either by $(4.1)-(4.5)$ or by $(4.1)-(4.4),(4.5f),(4.6)$. In both of the cases the distance $dist(u,O)$ is defined by (4.1) and the functions $\phi_1...,$ satisfy the inequalities (4.3). In what follows we assume.

(4.12) The finiteness assumption. *For any $t>0$ the ball*

$$B(O,t)=\{v: v\in V, dist(v,O)\le t\}$$

is a finite set in V.

The finiteness assumption is equivalent to the existence of the following function $K(t), t>0$.

$$p>K(t), (a_i)_1^p\in\Gamma^p \Rightarrow \phi_p(a_1,...,a_p)>t.$$

We now attach to the families $\phi_m, m=0,...,$ geometric quantities ξ,η. For $t>0$ set

$$M(t)=\begin{array}{l}\{v: v=(a_i)_1^m\in\Gamma^m,\phi_m(a_1,...,a_m)\le t,\\ \phi_m(x_1,...,x_{m+k})>t,(x_i)_1^{m+k}\in\Gamma^{m+k},x_i=a_i,i=1,...,m,k=1,...,\}.\end{array}$$

Note that $M(t)\subset B(O,t)$. Let $|B(O,t)|,|M(t)|$ be the cardinalities of $B(O,t),M(t)$ respectively. Set

$$\xi=\frac{\log\rho(\Gamma)}{\limsup_{t\to\infty}\frac{\log|B(O,t)|}{t}},\quad \eta=\frac{\log\rho(\Gamma)}{\limsup_{t\to\infty}\frac{\log|M(t)|}{t}}. \tag{4.13}$$

Clearly, $\xi\le\eta$. The quantities ξ,η can be viewed as two versions of a geometric "average" distance between the neighboring vertices of the induced tree $\mathcal{T}(\Gamma)$. Indeed, assume that $\phi_m(a_1,...,a_m)=mh$. For $t=mh$ we will have that

$$|M(mh)|\approx\rho(\Gamma)^m,\quad |B(O,mh)|\approx\rho(\Gamma)^m.$$

Hence $\xi=\eta=h$.

We define the following metric on Γ^∞.

$$\begin{aligned}&d((a_i)_1^\infty,(b_i)_1^\infty)=1,a_1\ne b_1,\\ &d((a_i)_1^\infty,(b_i)_1^\infty)=\tau^{\phi_p(a_1,\dots,a_p)},a_i=b_i,i=1,...,p,a_{p+1}\ne b_{p+1},p\ge 1,\\ &\tau=\frac{1}{\rho(\Gamma)}.\end{aligned} \tag{4.14}$$

The finiteness assumption yields that Γ^∞ is compact in the above metric. Note that the metric considered in [**Fri**] is equivalent to the above metric with ϕ defined by (4.7). Let $\delta(\phi)$ be the Hausdorff dimension of Γ^∞ with respect to the metric (4.14).

Lemma 4.15. *Let $\Gamma\subset<n>\times<n>$ be a strongly connected digraph with $\rho(\Gamma)>1$. Let $\mathcal{T}$ be the associated infinite tree. Assume that $\phi_1,...,$ satisfy* (4.1), (4.3) *and suppose furthermore that the finiteness condition* (4.12) *hold. Let η be given by* (4.13). *Then*

$$\delta(\phi)\le\frac{1}{\eta}.$$

Proof. For $(a_i)_1^m\in\Gamma^m$ let

$$C((a_i)_1^m)=\{x: x=(x_i)_1^\infty\in\Gamma^\infty,x_i=a_i,i=1,...,m,\} \tag{4.16}$$

be the cylindrical set corresponding to $(a_i)_1^m$. Note that $C((a_i)_1^m)$ is open and closed set in the product topology on Γ^∞. Furthermore $diam(C((a_i)_1^m))$ (the diameter of $C((a_i)_1^m)$) is less than $\rho(\Gamma)^{-t}$ for $(a_i)_1^{m-1}\in M(t)$.

Fix $1>\epsilon>0$. Then

$$\cup_{(a_i)_1^m\in\Gamma^m,(a_i)_1^{m-1}\in N(-\frac{\log\epsilon}{\log\rho(\Gamma)}),m\ge 1}C((a_i)_1^m)$$

is a closed cover of Γ^∞ with sets of diameters less than ϵ. From the definitions of the Hausdorff dimension and η we deduce that $\delta(\phi)\le\frac{1}{\eta}$. $\diamond$

For the standard metric on the tree induced by Γ we showed in [**Fri**] that the equality holds in (4.15).

§5. A lower bound for the critical exponent

Let $\Gamma \subset <n> \times <n>$ be a strongly connected graph. Consider the SFT Γ^∞. We view Γ^∞ as a compact metric space with the standard metric defined in §2. Let $\mathcal{B}$ the Borel σ-algebra generated by the cylindrical sets (4.16). Denote by Π the set of probability measures on Γ^∞ which are invariant under the shift σ. Let $\mathcal{E} \subset \Pi$ be the set of ergodic measures. It is well known that $\mathcal{E}$ is the set of the extreme points of Π in the w^* topology, e.g.

[**Wal**, §6.2]. For each $\mu \in \Pi$ let $h(\mu)$ denote the measure

entropy of σ. As σ is expansive it follows that $h(\mu)$ is an upper semi-continuous function on Π [**Wal**, §8.2].

Consider the usual construction of families of ergodic invariant measures μ using Markov chains. Let $S = (s_{ij})_1^n$ be a stochastic matrix whose graph $\Gamma(S) = \Gamma$. That is $s_{ij} > 0 \iff (i,j) \in \Gamma$. Let $\pi = (\pi_1, ..., \pi_n)$ be the unique probability left eigen-vector of S. That is, π is positive vectors whose coordinates sum to one and $\pi S = \pi$. Define the probability measure on ν_S on $\mathcal{B}$ by its value on the cylindrical sets.

$$\nu_S(C((i))) = \pi_i, i \in <n>, \nu_S(C((a_i)_1^m)) = \pi_{a_1} s_{a_1 a_2} \cdots s_{a_{m-1} a_m}, (a_i)_1^m \in \Gamma^m, m > 1. \tag{5.1}$$

It is well known that ν_S is an invariant measure. As Γ was assumed to be strongly connected we deduce that the shift is ergodic with respect to ν_S, e.g. [**Wal**, Thm. 1.13]. S induces the following random walk Y on the tree $\mathcal{T}$. $Y(0)$ starts at O. At time m $Y(m)$ is at some vertex $(a_i)_1^m \in \Gamma^m$. We then let

$$\begin{aligned} &P(Y(1) = i) = \pi_i, i = 1, ..., n, \\ &P(Y(m) = (a_i)_1^m | Y(m-1) = (a_j)_1^{m-1}) = s_{a_{m-1} a_m}, (a_i)_1^m \in \Gamma^m, m = 2, ...,. \end{aligned}.$$

Recall that the topological entropy h_{top} of the shift is equal to $\log \rho(\Gamma)$. See for example [**Wal**]. Furthermore, our assumption that Γ is strongly connected implies that there exists a unique ergodic invariant measure μ so that the Kolmogorov-Sinai measure entropy $h(\mu)$ is equal to h_{top}. This is so called the Parry measure [**Par**]. This measure is ν_P where P is the unique stochastic matrix of the form

$$P = \rho(\Gamma)^{-1} D^{-1} A(\Gamma) D, D = diag(u_1, ..., u_n), u = (u_1, ..., u_n)^T > 0, Au = \rho(\Gamma) u.$$

See for example [**Wal**, Thm 8.10]. It can be shown that Parry measure is the w^*-limit of invariant probability measures uniformly distributed on the periodic points of period $p, p = 1, ...$

Lemma 5.2. *Let $\Gamma \subset <n> \times <n>$ be a strongly connected graph. Denote by ν_P the Parry measure on Γ^∞ induced by the Markov chain determined by the stochastic matrix P. Then*

$$\nu_P(C((a_i)_1^m)) = \rho(\Gamma)^{-m+1} \pi_{a_1} u_{a_1}^{-1} u_{a_m}, (a_i)_1^m \in \Gamma^m, m = 1, ...,.$$

In particular, there exist $0 < K_1 \le K_2$ so that

$$K_1 \rho(\Gamma)^{-m} \le \nu_P(C((a_i)_1^m)) \le K_2 \rho(\Gamma)^{-m}, (a_i)_1^m \in \Gamma^\infty, m = 1, ...,.$$

Proof. Assume that $(i,j) \in \Gamma$. Then the definiton of $P = (p_{ij})_1^n$ yields that $p_{ij} = \rho(\Gamma)^{-1} u_i^{-1} u_j$. Hence

$$\nu_P(C((a_i)_1^m)) = \rho(\Gamma)^{-m+1} \pi_{a_1} \prod_{i=1}^{m-1} u_{a_i}^{-1} u_{a_{i+1}} = \rho(\Gamma)^{-m+1} \pi_{a_1} u_{a_1}^{-1} u_{a_m}, (a_i)_1^m \in \Gamma^m.$$

Set

$$K_1 = \rho(\Gamma) \min_{1 \le i,j \le n} \pi_i u_i^{-1} u_j, \quad K_2 = \rho(\Gamma) \max_{1 \le i,j \le n} \pi_i u_i^{-1} u_j.$$

and the lemma follows. ◇

In the case that $\Gamma \subset < 2r > \times < 2r >$ corresponds to the free group on r generators it follows that

$$\nu_P(C((a_i)_1^m)) = \frac{1}{2r}\frac{1}{(2r-1)^{m-1}}, (a_i)_1^m \in \Gamma^m, = 1, ..., .$$

Assume that the functions $\phi_m, m = 1, ...,$ satisfy the assumptions (4.1)-(4.4). Let $\psi_p : \Gamma^\infty \to \mathbf{R}_+, p = 0, ...,$ be defined as follows.

$$\begin{aligned} &\psi_0((a_i)_1^\infty) = 0, \\ &\psi_p((a_i)_1^\infty) = \phi_p(a_1, ..., a_p), p = 1, ..., (a_i)_1^\infty \in \Gamma^\infty. \end{aligned} \tag{5.3}$$

That is ψ_p is the random variable which describes the length of the path of the random variable X travelled in p units of time so that $X(p) = (a_i)_1^p$. As ψ_p is continuous functions it follows that ψ_p is μ measurable for any probability measure μ on $\mathcal{B}$ for $p = 0, ...,$. ¿From the inequality (4.3) we deduce inequality

$$0 \le \psi_{p+q} \le \psi_p + \psi_q \circ \sigma^p, p, q \ge 0,$$

for any $x \in \Gamma^\infty$. Assume that $\mu \in \Pi$. Kingman's subadditive ergodic theorem claims that
the sequence $\frac{\psi_m}{m}, m = 1, ...,$ converges $\mu - a.e.$ to ψ. Furthermore, $\psi(\sigma(x)) = \psi(x)$ $\mu - a.e.$ and

$$\lim_{m\to\infty} \int \frac{\psi_m}{m} d\mu = \int \psi d\mu.$$

If in addition $\mu \in \mathcal{E}$ then ψ is a constant function $\psi \equiv \alpha(\mu)$ $\mu - a.e.$. See for example [**Wal**, §10.2].

In the case that $\mu = \mu_S$ where S is a stochastic matrix then $\alpha(\mu)$ is the average distance travelled by the random walk Y induced by S in one unit of time. It is not difficult to show that in most of the cases the limit $\alpha(\mu)$ is dependent of the ergodic measure μ. Use the example (4.7) to deduce that $\alpha(\mu)$ depends on the choice of S.

Let X be a locally compact separable metric space with a metric
$dist(\cdot,\cdot)$. Assume that $F =< f_1, ..., f_r >$ is a free group of isometries of X which acts freely and properly discontinously. Let $O \in X$. Set

$$\phi_m(a_1, ..., a_m) = dist(O, f_{a_1} \cdots f_{a_m} O), (a_i)_1^m \in \Gamma^m, m = 1, ..., . \tag{5.4}$$

Clearly, (4.1)-(4.4), (4.5f) and (4.6) hold. Then the Poincaré series are given by

$$\sum_{m=1}^\infty \sum_{(a_i)_1^m \in \Gamma^m} e^{-s\phi_m(a_1,...,a_m)}. \tag{5.5}$$

Let $\kappa(F)$ be the critical exponent of (5.5). Set $\mathcal{T} = \mathcal{T}(\Gamma)$. Denote by $B(0, r)$ the closed ball of radius r centered at O as in §4. Then the standard argument shows that

$$\kappa(F) = \limsup_{r\to\infty} \frac{\log |B(O, r)|}{\log r}. \tag{5.6}$$

Consult for example with [**Nic**]. It now follows that $\kappa(F)$ is
independent of the choice of O.

Theorem 5.7. *Let X be a locally compact separable metric space. Assume that $F =< f_1, ..., f_r >$ is a free group of isometries of X which acts freely and properly discontinuously on X. Denote by $\Gamma \subset < 2r > \times < 2r >$ the graph associated with the free group F. Let $\phi_m, m = 1, ...,$ be defined by (5.4) and let $\psi_1, ...,$ be given by (5.3). Assume that μ is an*
invariant ergodic measure with respect to the shift on Γ^∞. Then $\{\frac{\psi_m}{m}\}_1^\infty$ converges almost everywhere to a
constant $\alpha(\mu)$ with respect to μ. Let $h(\mu)$ be the measure

entropy of the shift. Assume that $h(\mu) > 0$. Then

$$\kappa(F) \geq \frac{h(\mu)}{\alpha(\mu)}.$$

In particular

$$\kappa(F) \geq \sup_{\mu \in \mathcal{E}, h(\mu) > 0} \frac{h(\mu)}{\alpha(\mu)}.$$

Proof. Let $\mu \in \mathcal{E}, h(\mu) > 0$. Fix $\epsilon > 0$. The Kingman ergodic theorem yields that there exists $n = n(\epsilon)$ so that

$$\begin{aligned} &\mu(A(m,\epsilon)) \geq 1 - \epsilon, m \geq n(\epsilon), \\ &A(m,\epsilon) = \{x : x = (x_i)_1^\infty \in \Gamma^\infty, \psi_m(x) \leq m(\alpha(\mu) + \epsilon)\}. \end{aligned} \tag{5.8}$$

Since σ is expansive one can compute the measure entropy $h(\mu)$ with respect to a fixed partition corresponding to the first $2r$ cylinders $\mathcal{A} = \{C(1), ..., C(2r)\}$. Thus

$$h(\mu) \leq \frac{1}{m} \sum_{w \in \Gamma^m} -\mu(C(w)) \log \mu(C(w)). \tag{5.9}$$

Observe that

$$(a_i)_1^m \notin B(O, m(\alpha(\mu) + \epsilon)) \iff \phi_m(a_1, ..., a_m) > m(\alpha(\mu) + \epsilon) \iff \psi_m(x) > m(\alpha(\mu) + \epsilon), \forall x \in C((a_i)_1^m).$$

Set

$$\begin{aligned} &B(m,\epsilon) = \{u : u \in \Gamma^m, C(u) \subset A(m,\epsilon)\}, \\ &M(m,\epsilon) = |B(m,\epsilon)|. \end{aligned}$$

Clearly

$$|B(O, m(\alpha(\mu) + \epsilon))| \geq |\Gamma^m \cap B(O, m(\alpha(\mu) + \epsilon))| = M(m,\epsilon).$$

We now estimate $M(m,\epsilon)$. The convexity of the function $x \log x$ yield.

$$\frac{1}{M(m,\epsilon)} \sum_{u \in B(m,\epsilon)} \mu(C(u)) \log \mu(C(u)) \geq \frac{\sum_{u \in B(m,\epsilon)} \mu(C(u))}{M(m,\epsilon)} \log \frac{\sum_{u \in B(m,\epsilon)} \mu(C(u))}{M(m,\epsilon)} =$$
$$\frac{\sum_{u \in B(m,\epsilon)} \mu(C(u))}{M(m,\epsilon)} \log \sum_{u \in B(m,\epsilon)} \mu(C(u)) - \frac{\sum_{u \in B(m,\epsilon)} \mu(C(u))}{M(m,\epsilon)} \log M(m,\epsilon).$$

Hence

$$\sum_{u \in B(m,\epsilon)} \mu(C(u)) \log \mu(C(u)) \geq$$
$$(\sum_{u \in B(m,\epsilon)} \mu(C(u))) \log \sum_{u \in B(m,\epsilon)} \mu(C(u)) - (\sum_{u \in B(m,\epsilon)} \mu(C(u))) \log M(m,\epsilon).$$

Set $B^c(m,\epsilon) = \Gamma^m \backslash B(m,\epsilon)$ and assume the nontrivial assumption that

$$|B^c(m,\epsilon)| = 2r(2r-1)^{m-1} - M(m,\epsilon) > 0.$$

We then have a similar estimate

$$\sum_{v \in B^c(m,\epsilon)} \mu(C(v)) \log \mu(C(v)) \geq$$
$$(\sum_{v \in B^c(m,\epsilon)} \mu(C(v))) \log \sum_{v \in B^c(m,\epsilon)} \mu(C(v)) - (\sum_{v \in B^c(m,\epsilon)} \mu(C(v))) \log(2r(2r-1)^{m-1} - M(m,\epsilon)).$$

Assume that $\epsilon < \frac{1}{e}$. The assumption (5.8) yields.

$$-\sum_{w\in\Gamma^m} \mu(C(w)) \log \mu(C(w)) \leq \log M(m,\epsilon) + \epsilon \log(2r(2r-1)^{m-1} - M(m,\epsilon)) - (1-\epsilon)\log(1-\epsilon) - \epsilon \log \epsilon.$$

Use (5.9) to deduce that

$$h(\mu) \leq \frac{\log M(m,\epsilon) + \epsilon \log(2r(2r-1)^{m-1} - M(m,\epsilon)) - (1-\epsilon)\log(1-\epsilon) - \epsilon\log\epsilon}{m} \leq$$
$$\frac{\log |B(O, m(\alpha(\mu)+\epsilon))|}{m} + \frac{\epsilon \log 2r(2r-1)^{m-1}}{m} - \frac{(1-\epsilon) log(1-\epsilon) + \epsilon \log \epsilon}{m}.$$

Let $m \to \infty$ and use the characterization (5.6) to deduce

$$h(\mu) \leq (\alpha(\mu)+\epsilon)\kappa(F) + \epsilon \log(2r-1).$$

As ϵ is an arbitrary positive number and $h(\mu) > 0$ we deduce (0.1). Then (0.1) implies (0.2). The proof of the theorem is completed. $\diamond$

Assume that F is a group of isometries of $\mathbf{H}^n$ which satisfy the assumptions of Theorem 5.7. As $\kappa(F) \leq n-1$ it follows that

$$\alpha(\mu) \geq \frac{h(\mu)}{n-1}, \mu \in \mathcal{E}.$$

Assume in addition that $n = 2$, i.e. F is a finitely generated free Fuchsian group. Then $\delta(F) = \kappa(F)$, e.g. [**Nic**]. Hence

$$\delta(F) \geq \sup_{\mu\in\mathcal{E}, h(\mu)>0} \frac{h(\mu)}{\alpha(\mu)}. \tag{5.10}$$

Lemma 5.11. *Let Γ be a strongly connected graph and denote by $\mathcal{T}$ the associated infinite tree. Assume that $\phi_1, ...,$ satisfy* (4.1), (4.3)*. Let $\psi_0, ...,$ be defined by* (5.3)*. Let $\mu \in \Pi$. Set $\alpha_m(\mu) = \int \psi_m d\mu, m = 1, ...,$. Then*

$$\alpha_{l+m}(\mu) \leq \alpha_l(\mu) + \alpha_m(\mu), l, m = 1, ...,$$
$$\alpha(\mu) = \lim_{l\to\infty} \frac{\alpha_l(\mu)}{l} \leq \frac{\alpha_m(\mu)}{m}, m = 1, ..., .$$

Proof. Integrate the inequality $\psi_{l+m} \leq \psi_l + \psi_m \circ \sigma^l$ with respect to $d\mu$ and use the invariance of μ under the shift σ to deduce $\alpha_{l+m}(\mu) \leq \alpha_l(\mu) + \alpha_m(\mu)$. Hence $\frac{\alpha_l(\mu)}{l}, l = 1, ...,$ converges to $\alpha(\mu)$ and $\alpha(\mu) \leq \frac{\alpha_m(\mu)}{m}$. $\diamond$

Consider the invariant ergodic measure ν_S given by (5.1). Recall that [**Wal**, §4.8]

$$h(\nu_S) = -\sum_{1\leq i,j\leq n} \pi_i s_{ij} \log s_{ij}. \tag{5.12}$$

Corollary 5.13 *Let the assumptions of Theorem 5.7 hold. Assume that ν_S is an ergodic measure given by* (5.1)*. Then*

$$\alpha(\nu_S) \leq \frac{1}{m} \sum_{(a_i)_1^m \in \Gamma^m} \pi_{a_1} s_{a_1 a_2} \cdots s_{a_{m-1} a_m} \phi(a_1, ..., a_m) = \frac{\alpha_m(\nu_S)}{m}, m = 1, ...,$$
$$\kappa(F) \geq \frac{m h(\nu_S)}{\alpha_m(\nu_S)}, m = 1, ..., .$$

In particular for the Parry measure ν_P we have the following.

$$\alpha(\nu_P) \le \frac{\sum_{(a_i)_1^m \in \Gamma^m} \phi(a_1, ..., a_m)}{m2r(2r-1)^{m-1}} = \frac{\alpha_m(\nu_P)}{m}, m = 1, ...,$$
$$\kappa(F) \ge \frac{m \log(2r-1)}{\alpha_m(\nu_P)}, m = 1, ..., .$$

Assume that Γ is a strongly connected graph, $\phi_m, m = 1, ...,$ is a sequence of functions satisfying (4.1) and (4.3). Suppose furthemore that the finiteness assumption 4.12 holds. Consider the Poincaré series (5.5). Let $\kappa(\phi)$ be the critical exponent of (5.5). The standard argument, e.g. [**Nic**, §1.6], yields the characterization (5.6) for $\kappa(\phi)$. The arguments of the proof of Theorem 5.7 yield.

Theorem 5.14. *Let Γ be a strongly connected graph and denote by $\mathcal{T}$ the associated infinite tree. Suppose that $\phi_1, ...,$ satisfy (4.1), (4.3) and assume the finiteness condition (4.12). Let $\psi_0, ...,$ be defined by (5.3). Denote by $\mathcal{E}$ the set of invariant ergodic measures*

of the shift on Γ^∞. Let $\mu \in \mathcal{E}$. Then $\{\frac{\psi_m}{m}\}_1^\infty$ converges almost everywhere to a constant $\alpha(\mu)$.

Assume that the μ-entropy of the shift- $h(\mu)$ is positive. Then

$$\kappa(\phi) \ge \frac{h(\mu)}{\alpha(\mu)}.$$

More precisely, let $\alpha_m(\mu), m = 1, ...,$ be defined as in Lemma 5.11. Then

$$\kappa(\phi) \ge \frac{m h(\mu)}{\alpha_m(\mu)}, m = 1, ..., .$$

In particular

$$\kappa(\phi) \ge \sup_{\mu \in \mathcal{E}, h(\mu) > 0} \frac{h(\mu)}{\alpha(\mu)}. \tag{5.15}$$

§6. Remarks

Assume that F is a Fuchsian group that satisfies the conditions of Theorem

5.7. Then $\kappa(F) = \delta(F)$ and we conjecture that equality holds in

(0.2). The motivation for this conjecture are the results of Bowen [**Bow**] and Ruelle [**Rue**]. For the simplicity of the exposition we consider Ruelle's results.

Let $f : \mathbf{CP} \to \mathbf{CP}$ be a rational function on the Riemann sphere of degree $d > 1$. Denote by $J(f)$ the Julia set of f. Assume that $J(f)$ is a repeller. Set

$$q(z) = -\log |f'(z)|, z \in J(f).$$

Consider the topological pressure $P(tq)$ on $J(f)$ corresponding to the map $f : J(f) \to J(f)$. Then the Hausdorff dimension $\delta(J(f))$ of $J(f)$ is the unique solution of $P(tq) = 0$. Furthermore the

$\delta(J(f))$-Hausdorff measure is the Gibbs measure corresponding to $\delta(J(f))q$. Let Π be the set of f-invariant measures supported on $J(f)$ and denote by $\mathcal{E}$ the subset of ergodic measures. For $\mu \in \Pi$ we let $h(\mu)$ to be the μ-entropy of f. Recall the maximum characterization, e.g. [**Wal**, §9.3].

$$P(tq) = \sup_{\mu \in \mathcal{E}} h(\mu) + t \int q d\mu.$$

¿From the definition of q it follows that $\int q d\mu$ is the Lyapunov exponent $\lambda(\mu)$ of $f|_{J(f)}$. We then deduce that

$$\delta(J(f)) = \sup_{\mu \in \mathcal{E}, h(\mu)>0} \frac{h(\mu)}{\lambda(\mu)}.$$

Assume that F is a Fuchsian finitely generated free group. Then the limit set $\Lambda(F)$ has a Markov partition consisting of $2r$ intervals

which are obtained by the intersection of $2r$ geodesics, which form the fundamental domain in the unit disk, with the boundary of the unit disk-S^1. Moreover the shift map is a piecewise analytic map $f : S \to S$ given by the generators of F. (See an example below.) Thus one has the corresponding analog of $q = -\log|f'|$. It then follows that in order to prove the equality in (0.2) it is enough to show that $\alpha(\mu)$ is equal to the Lyapunov exponent $\lambda(\mu)$ for each ergodic measure μ. We denote by ρ the unique Gibbs measure on $\Lambda(F)$ given by the thermodynamics formalism as above.

We now show briefly that the geometric construction of the Patterson-Sullivan can carry over under the assumptions of Theorem 5.14. Assume that $\kappa(\phi) < \infty$ and the Poincaré series (5.5) diverge at $\kappa(\phi)$. Then the Patterson-Sullivan measure on Γ^∞ is constructed as follows. Let

$$\phi(s) = \sum_{m=1}^{\infty} \sum_{(a_i)_1^m \in \Gamma^m} e^{-s\phi_m(a_1,\ldots,a_m)}, s > \kappa(\phi).$$

We then define a probability measure μ_s on the Borel σ algebra generated by the cylindrical set of Γ^∞ by the equality

$$\mu_s(C(a_i)_1^m) = \frac{1}{\phi(s)} \sum_{p \geq m} \sum_{(b_i)_1^p \in \Gamma^p, b_i = a_i, i=1,\ldots,m} e^{-s\phi_p(b_1,\ldots,b_p)}. \tag{6.1}$$

Then the Patterson measure μ (with respect to the origin O) is any weak limit a sequence $\mu_{s_k}, \lim s_k = \kappa(\phi)^+$. In the case that (5.5) converges at the critical exponent $s = \kappa(\phi)$ one uses the Patterson trick [**Pat**] to modify the construction of the Patterson-Sullivan measure. In general μ is not an invariant under the shift map. We will demonstrate that for the case of a free subgroup on two generators of the modular group. To obtain a shift invariant measure from μ consider the sequence

$$\mu_k = \frac{1}{k+1} \sum_{i=0}^{k} (\sigma^*)^i \mu, k = 0, \ldots, . \tag{6.2}$$

Let ν be any weak limit of a subsequence of $\{\mu_k\}_0^\infty$. It is well known that σ is ν invariant. It is an interesting problem to give nontrivial conditions on the functions $\phi_m, m = 1, \ldots,$ so that the sequence (6.2) corresponding to a Patterson-Sullivan measure μ will converge w^* to the Gibbs measure ρ .

We now discuss the geometrical aspects of shift invariant measures. Assume that $F =< f_1, \ldots, f_r >$ is a free Fuchsian group on r generators. We view F as a discrete group of hyperbolic motions of either the upper half plane $\mathbf{H}^2$ or the unit disk D which ever is more convenient. Let $\Lambda(F) \subset S^1 = \mathbf{R} \cup \infty$ be the limit set of F. Assume that $Q = Q(F)$ is a fundamental domain of F bounded by $2r$ circles in the unit disk D which are orthogonal to S^1. Let $\Gamma = \Gamma(F) \subset < 2r > \times < 2r >$ be the induced graph by the free group. Then the cylindrical set $C((a_i)_1^m) \subset \Gamma^\infty$ corresponds to $\hat{C}((a_i)_1^m)$-the part of S^1 inside the complete geodesic line which bounds "from above" $f_{a_1} \circ \cdots f_{a_m} Q$.

Let λ be a finite measure on Γ^∞. Then λ induces a finite measure $\hat{\lambda}$ on the limit set $\Lambda(F)$ as follows. Set

$$\hat{\lambda}(\hat{C}((a_i)_1^m) \cap \Lambda(F)) = \lambda(C((a_i)_1^m)), (a_i)_1^m \in \Gamma^m.$$

Let $\phi_1, \ldots,$ be defined by (5.4). Then $\kappa(\phi) = \kappa(F)$. Let $\mu_s, s > \kappa(F)$ be the measure defined by (6.1). Thus $\mu_s(C(a_i)_1^m)), s > \kappa(F)$, is the measure of the part of the disk given by

$$\cup_{(b_i)_1^p \in \Gamma^p, p \geq m, b_i = a_i, i=1,\ldots,m} f_{b_1} \cdots f_{b_p} Q.$$

Alternatively, it can be viewed as $\hat{\mu}_s(\hat{C}((a_i)_1^m) \cap \Lambda(F))$. Let μ be the Patterson measure. It is known that $\hat{\mu}$ and hence μ are unique in this case, e.g. [**Nic**, Ch. 4]. Let ν_P be the Parry measure on Γ^∞. Then

$$\hat{\nu}_P(\hat{C}((a_i)_1^m) \cap \Lambda(F)) = \frac{1}{2r}\frac{1}{(2r-1)^{m-1}}, (a_i)_1^m \in \Gamma^m, m = 1, ...,.$$

We now discuss these measures in the following simple case.

Theorem 6.3. *Let $F =< f_1, f_2 >$ be the standard free subgroup of the modular group $PSL(2, \mathbf{Z})$ of index 6 generated by $f_1 = z + 2, f_2 = \frac{z}{2z+1}$. Set*

$$D(F) = \{z : 0 < \Im z, -1 \leq \Re z \leq 1, 1 \leq |\pm 2z + 1|\}.$$

Then the Patterson measure $\hat{\mu}$ (with respect to $i \in \mathbf{H}^2$) is given by $d\hat{\mu} = \frac{dx}{\pi(1+x^2)}, x \in \mathbf{R}$. Any shift invariant measure $\hat{\lambda}$ on $S^1 = \mathbf{R} \cup \infty$ is an invariant probability measure of the following map T.

$$\begin{aligned} T(x) &= x + 2, x \in (-\infty, -1], \\ T(x) &= \frac{x}{2x+1}, x \in (-1, 0], \\ T(x) &= \frac{x}{-2x+1}, x \in (0, 1], \\ T(x) &= x - 2, x \in (1, \infty]. \end{aligned} \tag{6.4}$$

T is C^1, expansive and intrinsically ergodic. The four intervals on which T is defined form a Markov partition for T. The Parry measure $\hat{\nu}_P$ on S^1 is the unique maximal measure with respect to the topological entropy of the shift.

The measure $\hat{\nu}_P$ is different from any Patterson measure in the class of $\hat{\mu}$.

Proof. Since $\Lambda(F) = S^1$ it is well known that the Patterson measure of F considered as a group of the discrete motions of D (with respect to the origin) must give the normalized Haar measure θ of S. Then any other other Patterson measure is of the form $f^*\theta, f \in F$. Hence $\frac{df^*\theta}{d\theta}$ is a continuous positive function on S. Pull back θ to $\mathbf{R}$ by $x \mapsto \frac{x-\sqrt{-1}}{1-\sqrt{-1}x}$ to deduce that $d\hat{\mu} = \frac{dx}{\pi(1+x^2)}, x \in \mathbf{R}$.

We now claim that the shift map on Γ^∞ is the map (6.4). This can be proved as follows. The fundamental domain $D(F)$ divides $\mathbf{R} \cup \infty$ to four intervals given in (6.4). Assume that x was in the interior of one of these intervals say $(1, \infty)$. Then x has an expansion $a = (a_i)_1^\infty \in \Gamma^\infty$ with a_1 fixed. It now follows that $\sigma(a)$ corresponds to $f_i^\epsilon(x)$ for some fixed $i = 1, 2, \epsilon = \pm 1$. It is straighforward to show that the corresponding f_i^ϵ given in (6.4) are the right ones. Clearly, T is continuous on $\mathbf{R} \cup \infty$. Note that T has two fixed points $0, \infty$. Furthermore, T is analytic in the interior of the four intervals. The "gluing" on the four points $-1, 0, 1, \pm\infty$ is C^1. (To verify the condition at ∞ map the upper half plane to the unit disk.) Since T is a realization of the shift map it is expansive. By the construction the four intervals given in (6.4) form the Markov partition of T. Thus $\lambda \in \Pi$ iff the measure $\hat{\lambda}$ is an invariant measure of T.

Finally as T has the unique maximal measure - the Parry measure - T is intrinsically ergodic [**Wei**].

We now show that $\hat{\nu}_P$ is different from any Patterson measure in the class of $\hat{\mu}$. Consider the interval $(2m, 2m+1], 0 \leq m \in \mathbf{Z}$. As $T^m(2m, 2m+1] = (0, 1]$ it follows that $(2m, 2m+1]$ is a cylindrical set of the form $C((a_i)_1^{m+1})$. Hence

$$\hat{\nu}_P((2m, 2m+1]) = \frac{1}{4}\frac{1}{3^m}, m = 0, ...,.$$

Clearly

$$\frac{1}{\pi(1+(2m+1)^2)} < \hat{\mu}((2m, 2m+1]) < \frac{1}{\pi(1+4m^2)}, m = 0, ...,.$$

Thus if $\frac{d\hat{\mu}}{d\hat{\nu}_P}$ exists then it is not a bounded function. Therefore $f^*\hat{\mu} \neq \hat{\nu}_P, \forall f \in F$. $\diamond$

Let X be a noncompact separable metric space with the metric $d(\cdot,\cdot)$, i.e. X has a countable dense set. Assume that G is a discrete group of isometries which acts freely and properly discontinuously on X. One then defines the Poincaré series as follows

$$\sum_{g\in G} e^{-sd(gO,O)}, O \in X.$$

Let $\kappa(G)$ be the critical exponent of the Poincaré series.

The standard argument, e.g. [**Nic**] gives the following formula for $\kappa(G)$. Let

$$N(O,t) = |\{g : d(gO,O) \le t\}|.$$

Then

$$\kappa(G) = \limsup_{t\to\infty} \frac{\log N(O,t)}{t}.$$

Theorem 6.5. *Let X be a noncompact separable metric space. Assume that G is a discrete group of isometries which acts freely and properly discontinuously on X. Suppose furthermore that G contains a free group on two generators. Then $\kappa(G) > 0$.*

Proof. Assume that $F \subset G$ is a free subgroup on two generators. Clearly the Poincaré series for F are majorized by the Poincaré series for G for the same value of s. Hence $\kappa(G) \ge \kappa(F)$. The distance function $d(\cdot,\cdot)$ restricted to the orbit of O under the action of F gives rise to the functions ϕ_m, ψ_m as §5. Use Theorem 5.7 to deduce that $\kappa(F) > 0$. $\diamond$

Assume that $G \subset SL(n, \mathbf{C})$. Then G satisfies the Tits alternative [**Tit**]. That is either G is virtually solvable or G contains a free subgroup on two generators. Recall that a Kleinian group G is virtually solvable iff it is elementary.

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