Coordinates of maximal roots of weakly nonnegative unit forms

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1 Introduction and main result

1.1. A unit form is a map $\chi : \mathbb{Z}^n \to \mathbb{Z}$, $x = (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^n \chi_{ij} x_i x_j$ where $\chi_{ij} \in \mathbb{Z}$. Unit forms appear in the representation theory of finite-dimensional algebras and similar structures. They occur as Tits forms or Euler characteristics (see e.g. [Ga], [Bo], [Ri]) and their weak positivity resp. nonnegativity frequently is related to finite resp. tame representation type.

We recall the basic definitions concerning unit forms. The unit form χ is said to be weakly positive resp. nonnegative if $\chi(x) > 0$ resp. $\chi(x) \geq 0$ for all $x \in C_n$, $x \neq 0$ where C_n is the cone in $\mathbb{Z}^n$ consisting of all $x = (x_1, \ldots, x_n)$ with $x_i \geq 0$. The vectors $x \in C_n$, $x \neq 0$ are called positive. Obviously $\mathbb{Z}^n$ is partially ordered by $x \geq y$ if $x - y \in C_n$. The vectors $x \in \mathbb{Z}^n$, $x \neq 0$ satisfying $\chi(x) = 1$ resp. $\chi(x) = 0$ are called 1-roots resp. 0-roots of χ.

1.2. Let us now review more precisely the connection between the representations of an algebra A over an algebraically closed field k and the weak definiteness of its Tits form χ_A. To give the definition of χ_A, we suppose that A is basic, the ordinary quiver of A is directed and A has up to isomorphism exactly n simple modules $S_1, \ldots, S_n$. Then $\chi_A(x) = \sum_{i,j=1}^n (\sum_{\nu=0}^2 (-1)^\nu \dim_k \mathrm{Ext}_A^\nu(S_i, S_j)) x_i x_j$. It is easy to see that this is a unit form in the above sense. The algebra A is called of finite representation type if there are only finitely many isomorphism classes of indecomposable finite-dimensional A-modules. In order to check finite representation type one may proceed as follows: Using covering theory (see [BG]) one may suppose that A is simply connected. This implies that A has a directed ordinary quiver and the Auslander-Reiten quiver of A has a postprojective component. For an algebra A of this kind it is shown in [Bo] that finite representation type is equivalent to the weak positivity of its Tits form χ_A.

But the connection between the representations of A and the form χ_A is even closer. Namely, the dimension vectors of the indecomposable A-modules are precisely the positive 1-roots of χ_A. Recall that the dimension vector $x = (x_1, \ldots, x_n)$ of an A-module X has as component x_i just the multiplicity of S_i in a composition series of X.

Concerning tame representation type the picture is not so complete. In [Pe1] it is shown that tame type of A implies the weak nonnegativity of the Tits form χ_A. Moreover there are interesting classes of tame algebras (see e.g. [Ri]) where the dimension vectors of the indecomposable A-modules are just the connected positive 1-roots and 0-roots of χ_A.

1.3. The above cited results show that informations about the possible coordinates x_i of positive 1-roots x of weakly positive unit forms χ are relevant at least for representation theory. A sharp bound for these coordinates is given by a theorem of Ovsienko ([Ov1], see also [Ri]) saying that always $x_i \leq 6$.

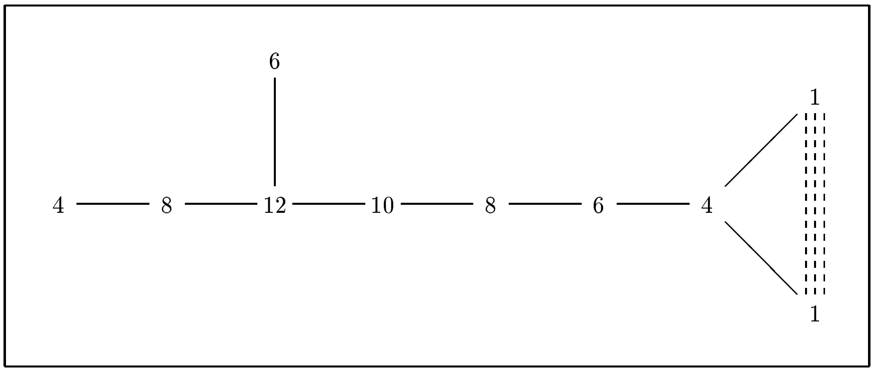
For weakly nonnegative forms the set of positive 1-roots is usually infinite. Hence it makes no sense to ask for bounds of the coordinates of positive 1-roots in general. On the other hand by a result due to Drozd (see e.g. [Ri]) weakly positive unit forms have only finitely many positive 1-roots. Using this fact, Ovsienko's theorem clearly may be reformulated as $x_i \leq 6$ for all maximal positive 1-roots x of a weakly positive unit form χ. This reformulation allows a reasonable generalization to weakly nonnegative forms which is actually the topic of this paper. Namely we want to answer the question whether there is a bound for the coordinates of the maximal positive 1-roots of weakly non-negative unit forms. Our main result is the following:

Main Theorem. *If $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly nonnegative unit form and $x = (x_1, \ldots, x_n)$ is a maximal positive 1-root of χ, then $x_i \leq 12$ for all $i = 1, \ldots, n$.*

We will prove this theorem in section 9 after establishing several preliminary results some of which should be of interest on their own.

Obviously we can and will restrict to that case that x is sincere. In this case , the requirement that the unit form χ is weakly nonnegative is not really necessary but follows from the maximality . Namely, it is shown in [HP] under the assumption $\chi_{ij} \geq -5$ for all $i < j$ and in [Ov2] in general that the existence of a maximal sincere positive 1-root forces a unit form to be weakly nonnegative.

Let us present here an example χ where the bound 12 is actually reached. For displaying the form χ we use the usual attached bigraph which will be reviewed more detailed in 2.4. To the points of the bigraph χ we attach the components of the only sincere positive 1-root of χ.



1.4. The question may arise whether weakly nonnegative unit forms with maximal positive 1-roots appear in practice. But in fact the question for bounds of maximal positive

1-roots of weakly nonnegative unit forms originated again from the representation theory of algebras where forms with this kind of 1-roots occur as Tits forms of certain algebras with sincere directing modules. We will give a more precise account on this class of forms together with some applications of our main theorem for the representation theory of finite-dimensional algebras in the final section of this paper.

The authors are grateful to CONACYT, DAAD, SFB 343 and the Volkswagenstiftung who supported stays and visits during which the results in this paper were achieved.

2 Preliminaries

2.1. In section 4 we will prove a reduction theorem allowing us to pass from general unit forms to so-called semigraphical forms which will be discussed in section 6. On the other hand the reduction theorem makes it necessary to increase slightly the class of forms we have to consider. Namely we will have to deal with semiunit forms where a semiunit form is a map $\chi : \mathbb{Z}^n \to \mathbb{Z}$, $x = (x_1, \ldots, x_n) \mapsto \sum_{\substack{i,j=1 \\ i \le j}}^{n} \chi_{ij} x_i x_j$ such that $\chi_{ij} \in \mathbb{Z}$ and $\chi_{ii} \in \{0, 1\}$. Obviously any unit form is also semiunit. For some definitions it will be convenient to use an even more general setup. We call a map χ as above an integral form if just $\chi_{ij} \in \mathbb{Z}$.

Given such a form we put $\chi_{ij} := \chi_{ji}$ for all $i < j$ and define a symmetric integral matrix A_χ with coefficients $(A_\chi)_{ij} := \chi_{ij}$ for $i \neq j$ and $(A_\chi)_{ii} := 2\chi_{ii}$. We denote by $e(1), \ldots, e(n)$ the canonical base vectors in $\mathbb{Z}$. The symmetric bilinear form $(-,-)_\chi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, $x \mapsto \frac{1}{2}(x X_\chi x^T)$ has the following properties:
a) $(x, x)_\chi = \chi(x)$ for all $x \in \mathbb{Z}^n$.
b) $(e(i), e(i))_\chi = \chi_{ii}$ and $2(e(i), e(j))_\chi = \chi_{ij}$ for all $i \neq j$.
c) $(x, y)_\chi = \frac{1}{2}(\chi(x + y) - \chi(x) - \chi(y))$ for all $x, y \in \mathbb{Z}^n$.
Whenever no confusion is possible, we omit the index χ.

The radical of χ is defined as $\operatorname{Rad}\chi := \{x \in \mathbb{Z} : x X_\chi = 0\}$ whereas the positive radical $\operatorname{Rad}^+\chi$ consists only of all positive x in $\operatorname{Rad}\chi$. The corank of χ is the rank of the free abelian group $\operatorname{Rad}\chi$.

2.2. Of course one may also consider integral forms $\chi : \mathbb{Z}^I \to \mathbb{Z}$ for arbitrary finite sets I. Usually we will identify forms which only differ by renaming the vertices. But let us present one example where this use of more general index sets is appropriate. Namely, if I is a subset of $\{1, \ldots, n\}$, then $\mathbb{Z}^I$ is embedded into $\mathbb{Z}^n$ in the canonical way. Obviously for an integral resp. semiunit resp. unit form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ the restriction $\chi|I : \mathbb{Z}^I \to \mathbb{Z}$ defined by $\chi|I(x) = \chi(x)$ is again an integral resp. semiunit resp. unit form.

Note that we will use the notation $x|I$ for the image of $x \in \mathbb{Z}^n$ under the canonical retraction of the above mentioned embedding $\mathbb{Z}^I \to \mathbb{Z}^n$.

A vector $x \in \mathbb{Z}^n$ is called sincere provided x does not lie in $\mathbb{Z}^I$ for any proper subset I of $\{1, \ldots, n\}$ or equivalently $x_i \neq 0$ for all $i = 1, \ldots, n$. A semiunit form χ is said to be sincere if there exists a sincere positive 1-root x of χ.

If x is a maximal positive 1-root of χ and we define I as the support $\operatorname{supp} x := \{i :$

$x_i \neq 0\}$ of x, then $x \in \mathbb{Z}^I$ and x is a maximal sincere positive 1-root of the restriction $\chi|I$. This shows that it is enough to prove our main theorem for maximal sincere positive 1-roots of weakly nonnegative semiunit forms.

2.3. If $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a unit form then we will use the well-known concept of reflections. The reflection $\sigma_i^\chi : \mathbb{Z}^n \to \mathbb{Z}^n$ with respect to $i \in \{1, \ldots, n\}$ is the linear map defined by $\sigma_i^\chi(x) = x - 2(e(i), x)_\chi e(i)$ and has the following properties:
a) $(\sigma_i^\chi)^2 = \mathrm{id}_{\mathbb{Z}^n}$.
b) $(\sigma_i^\chi(x), y)_\chi = (x, \sigma_i^\chi(y))_\chi$ for all $x, y \in \mathbb{Z}^n$.
c) $(\sigma_i^\chi(x), \sigma_i^\chi(y))_\chi = (x, y)_\chi$ for all $x, y \in \mathbb{Z}^n$, in particular $\chi(\sigma_i^\chi(x)) = \chi(x)$ for all $x \in \mathbb{Z}^n$.

2.4. The common way of visualizing integral forms uses bigraphs. The vertex set of the bigraph B_χ of a semiunit form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is just $\{1, \ldots, n\}$. The vertices i, j are connected by $|\chi_{ij} - \delta_{ij}|$ solid edges provided $\chi_{ij} - \delta_{ij} < 0$ and by $|\chi_{ij} - \delta_{ij}|$ dotted edges provided $\chi_{ij} - \delta_{ij} \geq 0$. In particular, if χ is a semiunit form, then the vertex i has no loop if $\chi_{ii} = 1$ and one solid loop if $\chi_{ii} = 0$.

3 Basic properties of weakly nonnegative semiunit forms

3.1. Throughout this section we suppose that $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly nonnegative semiunit form. This implies immediately that $\chi_{ij} \geq -2$. Moreover $0 \leq \chi(ne(i) + e(j)) = \chi_{jj} + n\chi_{ij}$ for all i, j such that $\chi_{ii} = 0$ shows $\chi_{ij} \geq 0$.

The following lemma is quite analogous to the corresponding one for the weakly positive situation (compare [Ri, 1.0.4]).

Lemma. *a) For a positive* 1*-root of a weakly nonnegative semiunit form* χ *the following assertions hold:*
a1) $2(e(i), x) \geq -2$ *for all* $i = 1, \ldots, n$.
a2) If $x_i > 0$, *then* $2(e(i), x) \leq 2$.
a3) $\chi_{ij} \leq 3$ *for all* $i \neq j$ *such that* $x_i \neq 0 \neq x_j$.
b) For a positive 0*-root of a weakly nonnegative semiunit form* χ *the following assertions hold:*
b1) $2(e(i), x) \geq 0$ *for all* $i = 1, \ldots, n$.
b2) If $x_i > 0$, *then* $2(e(i), x) = 0$.
b3) $\chi_{ij} \leq 2$ *for all* $i \neq j$ *such that* $x_i \neq 0 \neq x_j$.

Proof: a1) and a2) follow from applying χ to $x \pm e(i)$. To prove a3) by possibly interchanging i, j we may suppose $2(x, e(i) - e(j)) \leq 0$ and obtain $0 \leq \chi(x + (e(i) - e(j)) \leq \chi(x) + \chi(e(i) - e(j)) \leq 3 + \chi_{ij}$.

b1) and b2) follow from applying χ to $2x \pm e(i)$, whereas the proof of b3) is completely analogous to that of a3). $\square$

3.2. We call a weakly nonnegative semiunit form χ finitely sincere provided χ is sincere and there are only finitely many sincere positive 1-roots. Using the above lemma it turns out that the finitely sincere forms are exactly the forms possessing a maximal sincere positive 1-root.

Proposition. *For a weakly nonnegative semiunit form χ the following assertions are equivalent:*
a) There exists a maximal sincere positive 1*-root* x.
b) χ *is sincere and* $\mathrm{Rad}^+\chi = \emptyset$.
c) χ *is finitely sincere.*

Proof: $a) \Rightarrow b)$ and $c) \Rightarrow a)$ are obvious. For $b) \Rightarrow c)$ we assume that the set of sincere positive 1-roots is infinite. Hence we are able to find an infinite subset $\{x(1), x(2), \ldots\}$ satisfying $x(i) < x(i+1)$ for all $i \in \mathbb{N}$. By 3.1.a) we know $2(e(j), x(i)) \in \{0, \pm 1, \pm 2\}$ for all $i \in \mathbb{N}$ and $j = 1, \ldots, n$. Hence there exist $s < t$ such that $2(e(j), x(s)) = 2(e(j), x(t))$ for all $j = 1, \ldots, n$. Consequently $x(t) - x(s) \in \mathrm{Rad}^+\chi$. $\square$

Remark. If χ is a weakly nonnegative, finitely sincere semiunit form, then the bigraph B_χ of χ is connected and the points i such that $\chi_{ii} = 0$ are characterized by $\chi_{ij} \geq 0$ for all $j \neq i$.

Proof: If B_χ has two non-trivial connected components supported by the sets I_1, I_2 and x is a sincere positive 1-root, then $x = x_1 + x_2$ with $x_i = x|I_i$. Hence $1 = \chi(x_1) + \chi(x_2)$. Without loss of generality this means $\chi(x_1) = 1$ and $\chi(x_2) = 0$. Consequently all the vectors $x_1 + nx_2$ are sincere positive 1-roots.

To prove the second assertion we suppose $\chi_{ij} \geq 0$ for all $j \neq i$ and from the connectedness of B_χ derive the existence of some j such that actually $\chi_{ij} > 0$. Therefore the assumption $\chi_{ii} = 1$ would lead to $2(e(i), x) = 2x_i + \sum_{j \neq i} \chi_{ji} x_j \geq 3$ contradicting 3.1.a). $\square$

3.3. In [Ri, 1.0.(7)] it is shown that maximal sincere positive 1-roots of weakly positive forms have at most 2 exceptional vertices. This generalizes to our situation:

Lemma. *Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly nonnegative unit form with $n \geq 2$ and suppose that x is a maximal sincere positive* 1*-root of* χ. *Then one and only one of the following two situations occurs.*
a) There is exactly one exceptional vertex i (i.e. $x_i = 2$, $2(e(i), x) = 1$ and $2(e(j), x) = 0$ for all $j \neq i$).
b) There are exactly two exceptional vertices i_1, i_2 (i.e. $x_{i_1} = x_{i_2} = 1$, $2(e(i_1), x) = 2(e(i_2), x) = 1$ and $2(e(j), x) = 0$ for all $j \neq i_1, i_2$).

Proof: As χ is supposed to be a unit form, the vectors $\sigma_j(x)$ are all 1-roots as well. Hence $2(e(j), x) \geq 0$ for all j. We consider the equation $2 = 2\chi(x) = \sum_{j=1}^n x_j 2(e(j), x)$

and assume that there exists i such that $x_i = 1$, $2(e(i), x) = 2$ and $2(e(j), x) = 0$ for all $j \neq i$. Putting $\mu = x - e(i)$ and calculating $\chi(\mu) = 0$, from 3.1.b) we obtain $(e(j), \mu) = 0$ for all $j \neq i$ whereas immediately $(e(i), \mu) = (e(i), x) - \chi(e(i)) = 0$. Thus $n \geq 2$ yields $\mu \in \operatorname{Rad}^+\chi$. This contradiction shows that only the two alternatives of the lemma can occur. $\square$

3.4. **Lemma.** *Suppose $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a weakly nonnegative semiunit form and x is a sincere positive 1-root. For $r = 0, 1$ we put $I^r = \{i \in I : \chi_{ii} = r\}$ and $x^r = x|I^r$. Then one and only one of the following three situations occurs:*

a) $\chi(x^1) = 1$ and $\chi_{ij} = 0$ for all $i \in I^0$, $j \in I$.
b) $\chi(x^1) = 0$ and there exist $i, j \in I^0$ such that $x_i = x_j = \chi_{ij} = 1$. Additionally $s \in I^0$, $t \in I$ such that $\chi_{st} \neq 0$ implies $\{s, t\} = \{i, j\}$.
c) $\chi(x^1) = 0$ and there exist $i \in I^0$, $j \in I^1$ such that $x_i = x_j = \chi_{ij} = 1$. Additionally $s \in I^0$, $t \in I$ such that $\chi_{st} \neq 0$ implies $\{s, t\} = \{i, j\}$.

Moreover, if x is maximal and $I^0 \neq \emptyset \neq I^1$, then c) holds and $\operatorname{card}(I^0) = 1$.

Proof: By 3.1 we know that $\chi_{ij} \geq 0$ for all $i \in I^0$, $j \in I$. We may suppose $I = \{1, \ldots, n\}$ and $I^0 = \{1, \ldots, m\}$ for some $m \leq n$. Let us consider the equation $1 = (\sum_{i \in I^1} x_i^2 + \sum_{\substack{i,j \in I^1 \\ i<j}} \chi_{ij} x_i x_j) + \sum_{\substack{i,j \in I^0 \\ i<j}} \chi_{ij} x_i x_j + \sum_{\substack{i \in I^1 \\ j \in I^0}} \chi_{ij} x_i x_j$. The first summand of this sum is just $\chi(x^1)$ and the second $\chi(x^0)$. As also the third summand is nonnegative and integer, exactly one of these summands has to be 1 and the others have to be 0. This leads to the three cases using that x is sincere.

In case *a*) all the vectors $nx^0 + x^1$ and in case *b*) all the vectors $x^0 + nx^1$ are positive 1-roots bigger than x. Hence for x maximal only case *c*) is possible. Furthermore, if there would exist $k \in I^0$, $k \neq i$, then also the vectors $x + ne(k)$ would be bigger 1-roots than x. $\square$

3.5. We recall that a unit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is called critical resp. hypercritical if it is not weakly positive resp. weakly nonnegative but $\chi|J$ is weakly positive resp. weakly nonnegative for every proper subset J of I. Every critical unit form χ is positive semidefinit and its radical is generated by a sincere positive vector μ (see [Ri]) which in this paper will be called the characteristic vector of χ. For an arbitrary unit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ we denote a subset $J \subseteq I$ resp. the induced restriction $\eta := \chi|I$ as critical resp. hypercritical restriction provided that η is critical resp. hypercritical.

A weakly nonnegative semiunit form χ is called 0-sincere if there is a sincere vector $y \in \operatorname{Rad}^+\chi$. Note that by lemma 3.1.b2) it would be sufficient to require only that y is a positive sincere 0-root. That we can shift any vector into the positive cone by adding integer multiples of y shows that a 0-sincere form has to be positive semidefinite. As observed above, any critical unit form is 0-sincere with corank $= 1$.

If $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is an arbitrary weakly nonnegative unit form, we consider the union I^+ of all $\operatorname{supp}\mu$ where $\mu \in \operatorname{Rad}^+\chi$ and denote $\chi|I^+$ as χ^+. By construction the form χ^+ is 0-sincere. We will denote χ^+ as 0-sincere kernel of χ. Using the following lemma, we derive that a vector $x \in \mathbb{Z}^I$ is a 0-root of χ^+ if and only if $\operatorname{supp} x \subseteq I^+$ and $x \in \operatorname{Rad}\chi$. In particular this shows that $\operatorname{Rad}^+\chi = \operatorname{Rad}^+\chi^+$.

Lemma. *Suppose χ is a weakly nonnegative semiunit form and $x \in \operatorname{Rad}^+\chi$. If $\mu \in \mathbb{Z}^n$ is a 0-root of χ such that $\operatorname{supp}\mu \subseteq \operatorname{supp} x$, then $\mu \in \operatorname{Rad}\chi$.*

Proof: Assuming the existence of an index i such that $2(e(i), \mu) \neq 0$, we may choose $\varepsilon \in \{\pm 1\}$ such that $\varepsilon 2(e(i), \mu) \geq 1$. Putting $y = e(i) - 2\varepsilon\mu$ we observe $\chi(y) = \chi(e(i)) - \varepsilon 4(e(i), \mu) \leq 1 - 2 = -1$. On the other hand the requirement on the supports shows that $y + kx$ is positive for some $k \in \mathbb{N}$. Thus we arrive at the contradiction $0 \leq \chi(y + kx) = \chi(y) = -1$. □

4 The reduction theorem

4.1. Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form and pick $i \neq j \in \{1, \ldots, n\}$. The $\mathbb{Z}$-isomorphism $R_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n$ is defined on the canonical base vectors by $R_{ij}(e(k)) = e(k)$ for $k \neq j$ and $R_{ij}(e(j)) = e(j) - e(i)$. Hence for $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ the coordinates of $x' = R_{ij}(x)$ are $x'_k = x_k$ for $k \neq i$ and $x'_i = x_i - x_j$.

The map $\chi' = \chi R_{ij}^{-1}$ is called small reduction of χ with respect to (i, j). Note that, if χ is a unit form and $\chi_{ij} = -1$, then this is just a direct Gabrielov transformation. Immediate calculations show:
a) $\chi'_{kl} = \chi_{kl}$ if $j \notin \{k, l\}$.
b) $\chi'_{jk} = \chi_{jk} + \chi_{ik}$ if $j \neq k \neq i$.
c) $\chi'_{jj} = \chi_{jj} + \chi_{ii} + \chi_{ji}$ and $\chi'_{ji} = \chi_{ji} + 2\chi_{ii}$.

4.2. In the subsequent lemma we use the norm $|x| = \sum_{i=1}^n x_i$ to measure the size of vectors $x \in C_n$.

Lemma. *Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form and χ' be the small reduction of χ with respect to (i, j). Then the following assertions hold:*
a) $R_{ij}^{-1}(C_n) \subseteq C_n$, in particular R_{ij}^{-1} preserves the order on $\mathbb{Z}^n$.
b) $\chi R_{ij}^{-1}(x) = \chi'(x)$ for all $x \in \mathbb{Z}^n$, in particular R_{ij}^{-1} maps the set of positive 1- resp. 0-roots x' of χ' bijectively to the set of all positive 1- resp. 0-roots x of χ satisfying $x_i \geq x_j$.
c) If $x \in \mathbb{Z}^n$ is sincere positive and $x_i > x_j$, then $x' = R_{ij}(x)$ is sincere positive and $|x'| < |x|$.
d) If χ is a weakly nonnegative semiunit form and $\chi_{ij} < 0$ then χ' is also a weakly nonnegative semiunit form.
e) If χ is a weakly nonnegative semiunit form with $\chi_{ij} < 0$ and x is a maximal sincere positive 1-root of χ with $x_i > x_j$, then χ' is a weakly nonnegative semiunit form and $x' = R_{ij}(x)$ is a maximal sincere positive 1-root of χ' satisfying $|x'| < |x|$.

Proof: a), b) and c) are obvious. In d) the weak nonnegativity of χ' is clear. By 3.1 we know $\chi_{ii} = \chi_{jj} = 1$ and $\chi_{ij} \in \{-1, -2\}$. If $\chi_{ij} = -1$, then $\chi'_{jj} = 1 + 1 - 1 = 1$. If

$\chi_{ij} = -2$, then $\chi'_{jj} = 1 + 1 - 2 = 0$. Part e) is an immediate consequence of the previous parts. $\square$

Note that applying this lemma to a unit form will usually lead to a semiunit form. So just this lemma made it necessary to introduce semiunit forms.

4.3. Using the previous lemma, we can change to another weakly nonnegative, finitely sincere semiunit form with a maximal sincere 1-root of smaller norm x' provided in our given root x we find i, j such that $\chi_{ij} > 0$ and $x_i > x_j$. We will see now that the last restriction is not essential.

For an integer form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ and indices $i \neq j$ we consider the restriction $\chi' = (\chi R_{ij}^{-1})|J$ where $J = \{1, \ldots, i-1, i+1, \ldots, n\}$. The form χ' is called tightening of χ with respect to (i, j).

To formulate the properties of χ', let L_i be the subgroup of $\mathbb{Z}^n$ consisting of all $x = (x_1, \ldots, x_n)$ such that $x_i = x_j$. We observe that the map $\Delta : \mathbb{Z}^J \to L_i$ given by $(\Delta(x'))_k = x'_k$ for $k \neq i$ and $(\Delta(x'))_i = x'_j$ is an isomorphism. The following lemma is an immediate consequence of the previous considerations of this section.

Lemma. *Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form and χ' be the tightening of χ with respect to (i, j). Then the following assertions hold:*
a) $\Delta(C_J) \subseteq C_n$, in particular Δ is order preserving.
b) $\chi\Delta(x') = \chi'(x')$ for all $x' \in \mathbb{Z}^J$, in particular Δ maps the set of positive 1- resp. 0-roots x' of χ' bijectively to the intersection of L_i with the set of all positive 1- resp. 0-root of χ.
c) If $x \in \mathbb{Z}^n$ is sincere positive and $x_i = x_j$, then $x' = x|J$ is sincere positive and $|x'| < |x|$.
d) If χ is a weakly nonnegative semiunit form and $\chi_{ij} < 0$ then χ' is also a weakly nonnegative semiunit form.
e) If χ is a weakly nonnegative semiunit form with $\chi_{ij} < 0$ and x is a maximal sincere positive 1-root of χ with $x_i = x_j$, then χ' is a weakly nonnegative semiunit form and $x' = x|J$ is a maximal sincere positive 1-root of χ' satisfying $|x'| < |x|$.

4.4. 4.1 and 4.2 furnish the proof of our fundamental reduction theorem:

Theorem. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semiunit form and x a maximal sincere positive 1-root of χ. If we write I as the disjoint union of two subsets J and K, then there is a weakly nonnegative semiunit form $\chi' : \mathbb{Z}^{I'} \to \mathbb{Z}$, a maximal sincere positive 1-root x' of χ' and a monomorphism $\varphi : \mathbb{Z}^{I'} \to \mathbb{Z}^I$ with the following properties:*
a) I' is the disjoint union of K and a subset J'.
b) $\varphi(z) = z$ for all $z \in \mathbb{Z}^K$.
c) $\varphi(C_{I'}) \subseteq C_I$, in particular φ is order preserving.
d) $\chi\varphi(z') = \chi'(z')$ for all $z' \in \mathbb{Z}^{I'}$, in particular φ induces an injection from the set of positive 1- resp. 0-roots of χ' to the set of positive 1- resp. 0-roots of χ.
e) $\varphi(x') = x$.
f) $\chi_{ij} \geq 0$ for all $i, j \in J'$.

Proof: We apply 4.1 and 4.2 as long as we find vertices $i, j \in J$ such that $\chi_{ij} < 0$. This process has to stop since the norm of the considered maximal sincere positive 1-root always decreases. $\square$

The triple (χ', x', φ) in the above theorem is called a full reduction of the pair (χ, x) with respect to the subset J of I.

4.5. The reduction theorem usually decreases the number of variables occurring in the obtained form. We also need a process increasing the number of variables called doubling of vertices (see [D-Z]).

Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form. For $i \in \{1, \ldots, n\}$ we define a new integer form $\chi^{(i)} : \mathbb{Z}^{n+1} \to \mathbb{Z}$ by $\chi^{(i)}|\{1, \ldots, n\} = \chi$, $\chi^{(i)}_{n+1,n+1} = \chi_{ii}$ and $(e(n+1), e(j))_{\chi^{(i)}} = (e(i), e(j))_\chi$ for all $j = 1, \ldots, n$. We say that $\chi^{(i)}$ is obtained from χ by doubling the vertex i. Actually the bigraph of $\chi^{(i)}$ is constructed from the bigraph of χ by doubling the vertex i thus obtaining two vertices i and $n+1$. The edges between these two vertices depend on $\chi^{(i)}_{(n+1)i} = 2\chi_{ii}$. Clearly, if χ is a semiunit resp. unit form, then $\chi^{(i)}$ is a semiunit resp. unit form as well. To understand the relation of χ and $\chi^{(i)}$, it is suitable to introduce the surjective homomorphism $\pi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ given by $(\pi(x))_k = x_k$ for $k \neq i$ and $(\pi(x))_i = x_i + x_{n+1}$.

Lemma. *If $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is an integer form, then the form $\chi^{(i)}$ obtained by doubling the vertex i has the following properties:*
a) $\pi(C_{n+1}) = C_n$ and therefore π is order preserving. In addition, for $0 \le x \le y$ in $\mathbb{Z}^n$ and $y' \ge 0$ in $\mathbb{Z}^{n+1}$ such that $\pi(y') = y$ there exists $x' \in \mathbb{Z}^{n+1}$ such that $0 \le x' \le y'$ and $\pi(x') = x$.
b) $(x, y)_{\chi^{(i)}} = (\pi(x), \pi(y))_\chi$ for all $x, y \in \mathbb{Z}^{n+1}$. In particular, π maps the set of positive 1- resp. 0-roots of $\chi^{(i)}$ surjectively to the set of positive 1- resp. 0-roots of χ.
c) $\operatorname{Rad}\chi^{(i)} = \operatorname{Rad}\chi \oplus \mathbb{Z}(e(n+1) - e(i))$. In addition, a vector $x \in \mathbb{Z}^{n+1}$ lies in $\operatorname{Rad}^+\chi^{(i)}$ if and only if it can be written as $y + q(e(n+1) - e(i))$ where $y \in \operatorname{Rad}^+\chi$ and q is a nonnegative integer such that $q \le y_i$.
d) χ is weakly nonnegative if and only if $\chi^{(i)}$ is weakly nonnegative.
e) $x \in C_{n+1}$ is a maximal positive 1-root of $\chi^{(i)}$ if and only if $\pi(x)$ is a maximal positive 1-root of χ.

Proof: a), b), c) and one direction of d) are obvious. For the converse we pick $x \in C_{n+1}$ and from $\pi(x) \in C_n$ we obtain $\chi^{(i)}(x) = \chi(\pi(x)) \ge 0$. For e) we observe that for a 1-root y of $\chi^{(i)}$ such that $y > x$ also $\pi(y) > \pi(x)$ holds. Conversely if z is a 1-root of χ satisfying $z > \pi(x)$, then it is easy to find y such that $\pi(y) = z$ and $y > x$. $\square$

Remark. Obviously $e(n+1) - e(i) \in \operatorname{Rad}\chi^{(i)}$. If conversely $\chi' : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is an integer form such that $\chi'|\{1, \ldots, n\} = \chi$ and $e(n+1) - e(i) \in \operatorname{Rad}\chi'$, then $\chi' = \chi^{(i)}$.

5 2-layer 1-roots

Throughout this section we suppose that $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly nonnegative semiunit form.

5.1. **Lemma.** *Let μ be a positive 0-root of the weakly nonnegative semiunit form χ and $x \in \mathbb{Z}^n$. Suppose there is a nonnegative integer n such that $x + n\mu$ is positive and sincere. If $(x, \mu)_\chi = 0$, then $\mu \in \operatorname{Rad} \chi$.*

Proof: If we assume $\mu \notin \operatorname{Rad} \chi$, then there exists i such that by 3.1 $2(e(i), \mu) \geq 1$. Hence there is $l \in \mathbb{N}$ with $2(x + l\mu, e(i)) = 2(x, e(i)) + l2(\mu, e(i)) \geq \chi(x) + 2$. Putting $t = \max\{n, l\}$ and $y = x + t\mu$, we see that the sincere positive vector μ satisfies $2(y, e(i)) \geq \chi(x) + 2$. Observing that $(x, \mu) = 0$ implies $\chi(y) = \chi(x)$, we arrive at the contradiction $0 \leq \chi(y - e(i)) \leq \chi(x) + 1 - 2(e(i), y) \leq -1$. $\square$

5.2. A positive 1-root x of χ is called 2-layer if there exist positive 0-roots μ, μ' such that $\mu + \mu' = x$.

Proposition. *Suppose x is a positive 1-root of the weakly nonnegative semiunit form χ. If μ is a positive 0-root of χ such that $\mu \notin \operatorname{Rad} \chi$ and $x > \mu$, then $x - \mu$ is a positive 0-root as well. In particular x is 2-layer.*

Proof: Without loss of generality we may suppose that x is sincere. By 5.1 we obtain $(x, \mu) \neq 0$. The inequalities $0 \leq \chi(x - \mu) = \chi(x) - 2(x, \mu) = 1 - 2(x, \mu)$ and $0 \leq \chi(x + n\mu) = \chi(x) + 2n(x, \mu) = 1 + 2n(x, \mu)$ show that $2(x, \mu) = 1$. Hence $\chi(x - \mu) = \chi(x) + \chi(\mu) - 2(x, \mu) = 1 + 0 - 1 = 0$. $\square$

Corollary. *Suppose x is a maximal positive 1-root of a weakly nonnegative semiunit form χ. If there is a positive 0-root μ such that $x > \mu$, then x is 2-layer.*

5.3. **Lemma.** *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a 0-sincere weakly nonnegative semiunit form. Suppose there is $i \in I$ such that $\chi|(I \setminus \{i\})$ is a unit form. If the set U of all $y \in \operatorname{Rad}^+ \chi$ satisfying $y_i = 1$ is non-empty and finite, then $y_j \leq 6$ for all $y \in U$ and $j \in I$.*

Proof: We claim that $\chi' = \chi|(I \setminus \{i\})$ is weakly positive. If not, there is some positive μ' such that $i \notin \operatorname{supp} \mu'$ and $\chi(\mu) = 0$. From lemma 3.5 we know that $\mu' \in \operatorname{Rad}^+ \chi$ which contradicts the finiteness of U.

Let now $y \in U$. As $\chi(y - e(i)) = \chi(e(i)) = 1$, we obtain that $y - e(i)$ is a positive 1-root of the weakly positive unit form χ'. $\square$

Theorem. *If x is a maximal positive 2-layer 1-root of a weakly nonnegative semiunit form $\chi : \mathbb{Z}^n \to \mathbb{Z}$, then $x_i \leq 12$ for all $i = 1, \ldots, n$.*

Proof: Let $x = \mu + \mu'$ with μ, μ' positive 0-roots of χ. We inductively double all points $i \in I = \{1, \ldots, n\}$. Of course the resulting form $\bar{\chi} : \mathbb{Z}^{\bar{I}} \to \mathbb{Z}$ does not depend on the chosen ordering. We write $\bar{I} = \{1, \dots, 2n\}$ as $I = \{1^-, \dots, n^-\} \cup \{1^+, \dots, n^+\}$ where $i^- = i$ and $i^+ = i + n$ for all $i = 1, \dots, n$. Clearly the projection map $\pi : \mathbb{Z}^{\bar{I}} \to \mathbb{Z}^I$ sending $e(i^-)$ and $e(i^+)$ to $e(i)$ by induction has the properties listed in lemma 4.3.

We put $\mu^- = \sum_{i=1}^n \mu_i e(i^-)$, $\mu^+ = \sum_{i=1}^n \mu_i e(i^+)$ and $\bar{x} = \mu^- + \mu^+$. As $\pi(\bar{x}) = x$ part d) of lemma 4.3 tells us that $\bar{x}$ is a maximal positive 1-root of $\bar{\chi}$.

The set $I^- = \operatorname{supp} \mu^-$ and $I^+ = \operatorname{supp} \mu^+$ have the property that $I^- \cup I^+ = \operatorname{supp} \bar{x}$. Consequently $\chi' = \bar{\chi}|I^- \cup I^+$ is a finitely sincere semiunit form and $\bar{x}$ is maximal sincere positive 1-root of χ'.

Let now (η, z, φ) be a full reduction of $(\chi', \bar{x})$ with respect to I^- where $\eta : \mathbb{Z}^J \to \mathbb{Z}$ and J is the disjoint union of J^- and I^+. Putting $\lambda = z|J^-$, we see $z = \lambda + \mu^+$ yielding $\varphi(\lambda) = \mu^-$ and therefore $\eta(\lambda) = \bar{\chi}\varphi(\mu^-) = \bar{\chi}(\mu^-) = \chi(\mu) = 0$. Writing this down explicitly means $0 = \sum_{i \in J^-} \eta_{ii} \lambda_i^2 + \sum_{i<j} \eta_{ij} \lambda_i \lambda_j$. By construction all η_{ij} are nonnegative. Thus the sincerity of λ forces all η_{ij} to be 0. Without loss of generality there is $i^+ \in I^+$ such that $\eta_{i^+ i^+} = 1$. By lemma 3.4 we obtain that $\eta|I^+$ is a unit form and J^- consists of just one element ω. Moreover there is unique $i^+ \in I^+$ such that $\eta_{\omega i^+} = 1$ satisfying $\mu_{i^+}^+ = 1 = z_\omega$.

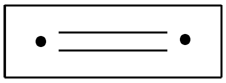
Putting $\eta' = \eta|I^+ = \bar{\chi}|I^+ = \chi|\operatorname{supp}(\mu')$, we obtain from 5.1 that $\mu^+ \in \operatorname{Rad}^+\eta'$. The vector μ^+ is contained in the set $U = \{y \in \operatorname{Rad}^+\eta' : y_{i^+} = 1\}$. We observe that $(e(j^+), y)_\eta = 0$ and $(e(\omega), y)_\eta = y_{i^+} = 1$ for all $y \in U$ and $j^+ \in I^+$. The assumption that the set U is infinite would lead to the existence of $y, y' \in U$ satisfying $y' < y$. But then $0 < y - y'$ would lie in $\operatorname{Rad}^+\eta$ contradicting the finite sincerity of η. Hence U is finite and by the above lemma we obtain $\mu_{j^+} \leq 6$ for all $j^+ \in I^+$. But this shows $\mu'_j \leq 6$ and by symmetry also $\mu_j \leq 6$ for all $j = 1, \dots, n$. $\square$

6 Semigraphical forms

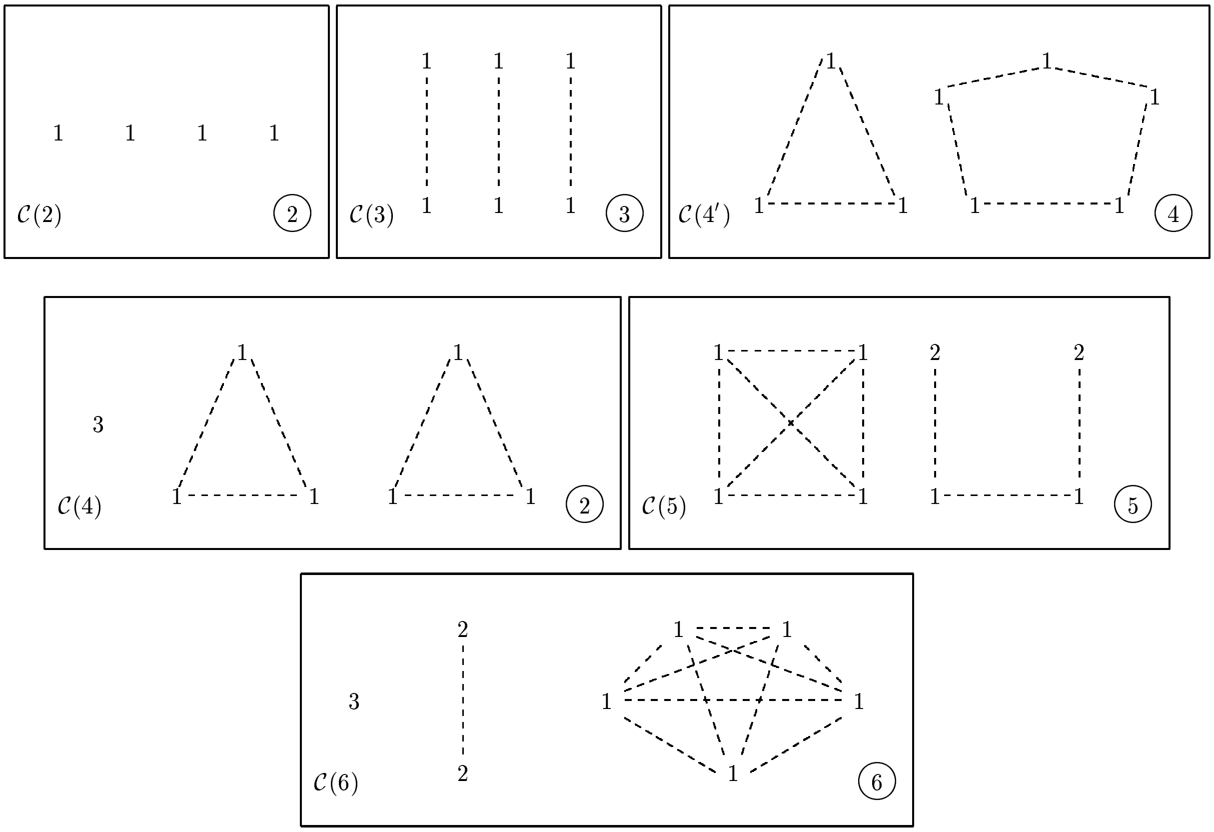
6.1. A semiunit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is called semigraphical if there exists $\omega \in I$ such that $\chi_{\omega i} < 0$ for all $i \in I$ with $i \neq \omega$ and $\chi_{ij} \geq 0$ for all $i, j \neq \omega$. An index ω as occurring in the definition is called center. The center is unique provided $\operatorname{card} I > 2$.

The graphical forms χ introduced in [Ri] are just the semigraphical unit forms such that $|\chi_{ij}| \leq 1$ for all $i, j \in I$. Graphical forms χ with $\operatorname{card} I > 2$ are usually visualized by their reduced bigraph $B'(\chi)$ which is just the full subbigraph of $B(\chi)$ supported by the edges different from the center. Note that $B'(\chi)$ does not contain any solid edge hence is a graph with dotted edges.

6.2. We observe that a weakly nonnegative semigraphical semiunit form by 3.1 is actually a unit form and investigate the critical semigraphical unit forms. At first we notice that the form $\mathcal{C}(1)$ given by the Kronecker bigraph



is obviously a critical semigraphical unit form. As shown in [Ri] there are up to isomorphism exactly 6 critical graphical forms namely $\mathcal{C}(2), \dots, \mathcal{C}(6)$ and $\mathcal{C}(4')$ whose reduced bigraphs are presented in the list below where we replace the vertices by the coefficients of the characteristic vector. The coefficient of the center ω is the encircled number in the right lower corner.



Lemma. *If $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a critical semigraphical unit form and* $\mathrm{card} I > 2$, *then χ is actually graphical. Hence the critical semigraphical forms are exactly the forms $\mathcal{C}(1), \ldots, \mathcal{C}(6)$ and $\mathcal{C}(4')$.*

Proof: As $\mathrm{card} I > 2$ clearly $\chi_{\omega i} = -1$ for all $i \neq \omega$. Hence we only have to show $\chi_{ij} \leq 1$ for all i, j different from ω. Observing that for $i \neq j$ the vector $e(i) - e(j)$ and the characteristic vector of χ are linearly independent, we obtain $1 \leq \chi(e(i) - e(j)) = 2 - \chi_{ij}$. $\square$

6.3. **Lemma.** *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semigraphical unit form with center ω satisfying $\chi_{\omega j} = -1$ for all $j \in I$ different from ω. We fix $i \in I$ different from ω and put $S_i = \{j \in I : \chi_{ij} > 0\}$.*

If x is a positive sincere vector, S is a subset of S_i and the inequality $x_i - 2(x, e(i)) \geq x_\omega - \sum_{j \in S} x_j$ holds, then $S = S_i$ and $\chi_{ij} = 1$ for all $j \in S_i$.

Proof: We calculate $x_i - 2(x, e(i)) = x_i - \sum_{j \in S_i, j \neq i} \chi_{ij} x_j - 2x_i + x_\omega = x_\omega - \sum_{j \in S} x_j - \Delta$ where $\Delta = \sum_{j \in S} (\chi_{ij} - 1) x_j + \sum_{j \in S_i \setminus S} \chi_{ij} x_j \geq 0$. By assumption it follows that $\Delta = 0$ which implies $S_i \setminus S = \emptyset$ and $\chi_{ij} = 1$ for all $j \in S$ since x is sincere. $\square$

Theorem. *Suppose $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a weakly nonnegative, finitely sincere, semigraphical form with center ω. Let x be a maximal sincere positive 1-root of χ such that x_ω is maximal among all those 1-roots. If x has only 1 exceptional vertex, then $x_\omega \leq 7$.*

Proof: Note that card$I \geq 3$. Let i be the exceptional vertex of x, thus $2(e(i), x) = 1$, $x_i = 2$ and $2(e(j), x) = 0$ for all $j \neq i$. Without loss of generality we may assume $x_\omega \geq 7$ which shows $\omega \neq i$. We will carry out the proof by showing several claims.

Claim 1: $\chi_{\omega j} = -1$ holds for all $j \neq \omega$.
Assuming that this is false furnishes a point j such that $\chi_{\omega j} = -2$ and consequently $\chi(e(\omega) + e(j)) = 0$. In the case $j \neq i$ we obtain $(e(\omega) + e(j), x) = 0$ and therefore $e(\omega) + e(j) \in \mathrm{Rad}^+\chi$ by 5.1. Using proposition 3.2 this yields a contradiction. For $j = i$ because of $x - 2e(i) - 2e(\omega) > 0$ we get the contradiction $0 \leq \chi(x - 2e(i) - 2e(\omega)) = 1 + 0 - 4(x, e(i) + e(\omega)) = -1$.

Claim 2: $\chi_{ij} \leq 1$ holds for all $j \neq \omega$.
Assuming $\chi_{ij} \geq 2$ for some $j \neq i, \omega$ would give $\chi(e(i) - e(j)) \leq 0$. Since $x - 2(e(i) - e(j)) \geq 0$, we would arrive at the contradiction $0 \leq \chi(x - 2(e(i) - e(j))) = \chi(x) + 4\chi(e(i) - e(j)) - 4(x, e(i) - e(j)) \leq 1 + 0 - 2 = -1$.

Claim 3: $\chi_{i_1 j_2} > 0$ for all $j_1, j_2 \neq \omega$ satisfying $\chi_{ij_1} > 0 < \chi_{ij_2}$.
Of course for $j_1 = j_2$ nothing has to be proved. Assuming that the claim fails for some $j_1 \neq j_2$, we first deal with the case that j_1, j_2 are both different from i. Using claim 2 and claim 1, we get $y := \sigma_\omega \sigma_{j_1} \sigma_{j_2} \sigma_i(x) = \sigma_\omega(x - e(i) + e(j_1) + e(j_2)) = x - e(i) - e(\omega) + e(j_1) + e(\omega) + e(j_2) + e(\omega) = x + e(\omega) + e(j_1) + e(j_2) - e(i)$. This is a contradiction to the maximal choice of x since y is a sincere positive 1-root of χ satisfying $y_\omega = x_\omega + 1$. In the case $j_1 = i$ we can use the same vector y for a similar argument.

Claim 4: Considering $S_i := \{j : \chi_{ij} > 0\}$, the following assertions hold:
i) $x_j = 1$ for all $j \in S_i, j \neq i$.
ii) $\chi_{jk} = 1$ for all $j, k \in S_i$.
iii) If $j \in S_i$ and $k \in I$ such that $\chi_{jk} > 0$, then $k \in S_i$.
iv) card$S_i = x_\omega - 2$.
Using claim 2, we calculate $1 = x_i - 2(e(i), x) = x_i - \sum_{j \in S_i} \chi_{ij} x_j - x_i + x_\omega = x_\omega - \sum_{j \in S_i} x_j$. By claim 3 we obtain $S_i \subseteq S_j$ for an arbitrary $j \neq i$.Observing $1 \leq x_j = x_j - 2(e(j), x) = x_j - \sum_{k \in S_j, k \neq j} \chi_{jk} x_k - 2x_j + x_\omega \leq x_\omega - \sum_{k \in S_j} x_k \leq x_\omega - \sum_{k \in S_i} x_k = 1$, by application of lemma 6.3 we obtain i), ii), iii). From the equation $1 = x_\omega - 1 - \sum_{k \in S_i}$ follows card$S_i = x_\omega - 2$ which is iv).

Proceeding with the proof we see that because of $x_\omega \geq 7$ the form χ cannot be weakly positive. Thus there exists a critical restriction $\chi|J$ which has to be one of the $\mathcal{C}(i)$, $2 \leq i \leq 6$ or $\mathcal{C}(4')$.

Claim 5: $S_i \subseteq J$.
If μ is the characteristic vector of $\chi|J$ then by 5.1 we get $0 \neq 2(\mu, x) = \sum_{j \in I} \mu_j 2(e(j), x) = \mu_i$ and thus $i \in J$. For all $j \in S_i$, $j \neq i$ we have that $\sigma_j \sigma_i(x) = x - e(i) + e(j)$ is still a positive sincere vector. Again using 5.1 we get $0 \neq 2(\mu, \sigma_j \sigma_i(x)) = \sum_{k \in I} \mu_k 2(e(k), x - e(i) + e(j)) = \mu_j$ where the last equality is obtained by going through all possibilities for k and applying claim 4.

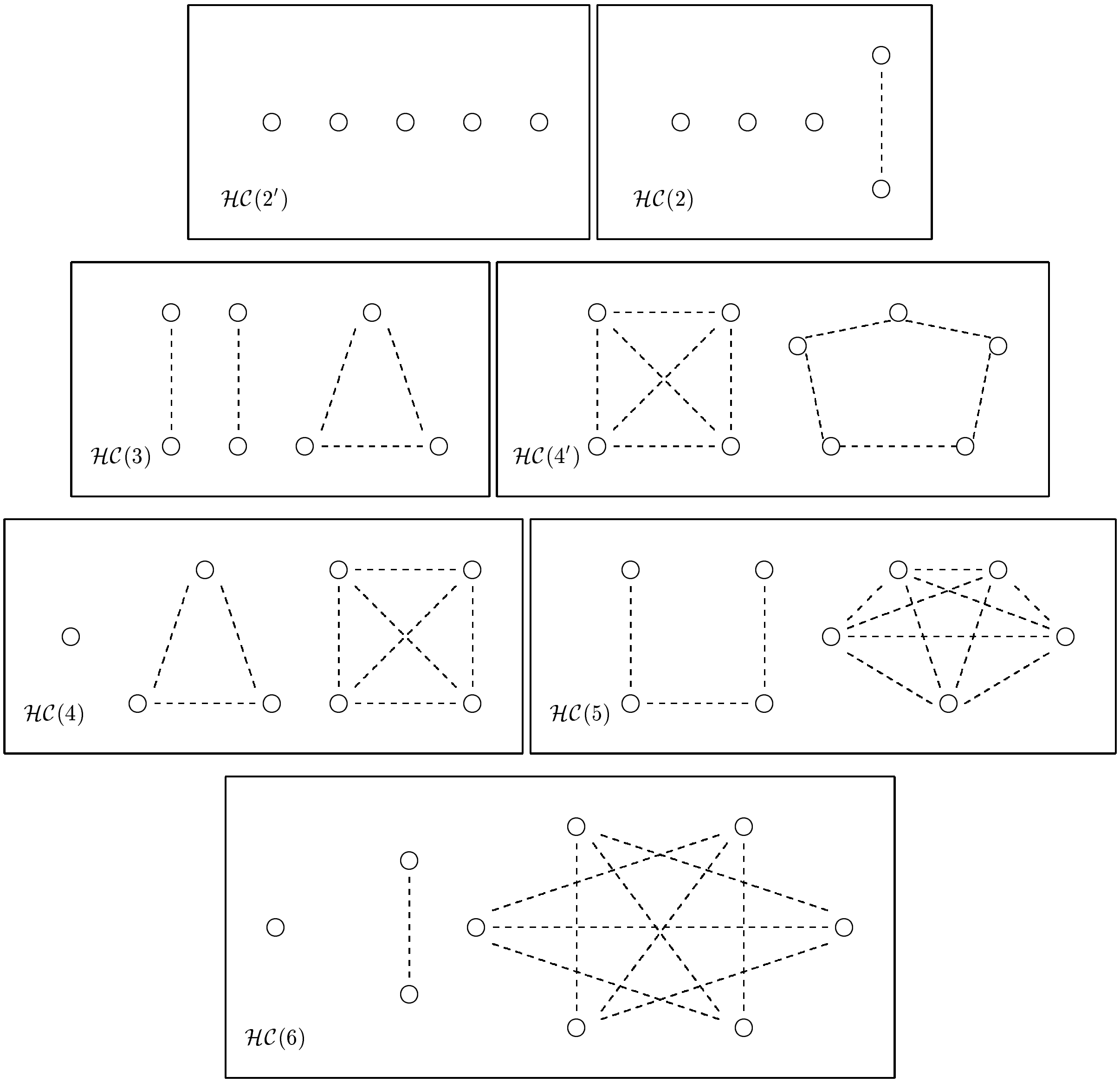
Using again claim 4, we see that the bigraph of $\chi|J$ contains as full subbigraph a full graph on $x_\omega - 2 \geq 5$ vertices with single dotted edges. Hence $\chi|J$ can be identified with

$\mathcal{C}(6)$. We finish by deriving $x_\omega - 2 = 5$ and hence $x_\omega = 7$. $\square$

6.4. We will need the classification of the hypercritical semigraphical unit forms χ such that $-1 \le \chi_{ij}$ for all i, j. It is helpful to observe that they actually satisfy $\chi_{ij} \le 1$ for all i, j.

Namely, if $\chi_{ij} \ge 2$ for some i, j, then $\chi(e(i) - e(j)) \le 0$. Since χ is not weakly nonnegative, we find $0 \le v \in \mathbb{Z}^I$ such that $\chi(v) < 0$. Possibly interchanging i and j we may suppose that $(v, e(i)) \ge 0$. We obtain $\chi(v - v_i(e(i) - e(j))) = \chi(v) + \chi(e(i) - e(j)) - 2v_i(v, e(i) - e(j)) < 0$ contradicting the weak nonnegativity of $\chi | I \setminus \{i\}$.

Lemma. *If $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a hypercritical semigraphical unit form such that $-1 \le \chi_{ij}$ for all i, j, then χ is one of the graphical forms $\mathcal{HC}(2')$, $\mathcal{HC}(2)$, $\mathcal{HC}(3)$, $\mathcal{HC}(4')$, $\mathcal{HC}(4)$, $\mathcal{HC}(5)$, $\mathcal{HC}(6)$ whose reduced bigraphs are shown in the following list.*



Proof: Since $-1 \leq \chi_{ij}$, there has to be a critical restriction $\chi|J$ of type $\mathcal{C}(2)$, $\mathcal{C}(3)$, $\mathcal{C}(4')$, $\mathcal{C}(4)$, $\mathcal{C}(5)$ or $\mathcal{C}(6)$. By [Pe2] the set $I \setminus J$ consists exactly of one element s and for the characteristic vector μ of $\chi|J$ the inequality $2(\mu, e(s)) < 0$ holds.. For all $j \in J$ satisfying $\mu_j = 1$ we obtain $0 \leq \chi(\mu - e(j) + e(s)) = 2 + 2(\mu, e(s)) - \chi_{sj}$. Hence $2 \geq -2(\mu, e(s)) = \mu_\omega - \sum_{i \neq \omega} \mu_i \chi_{si} > 0$ and even $-2(\mu, e(s)) = 1$ provided there exists $j \in J$ such that $\chi_{sj} = 1$.

By a case by case inspection using these numerical conditions it is easy to see that the above bigraphs are the only possible candidates. Then one easily checks that these forms are actually hypercritical. $\square$

7 0-sincere unit forms

7.1. The following basic results on 0-sincere forms can be found in [DP]. For convenience of the reader we include a short sketch of the proof.

Proposition. *Suppose $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a 0-sincere unit form and let $\{J_1, \ldots, J_m\}$ be the set of all subsets J of I such that $\chi|J$ is critical. For all $i = 1, \ldots, m$ we denote by μ_i the characteristic vector of $\chi|J_i$. Then the following assertions hold:*
a) If $\mu \in \mathrm{Rad}^+\chi$ is sincere, then there exist $p \in \mathbb{N}$, $q_i \in \mathbb{N}_0$ such that $p\mu = \sum_{i=1}^m q_i \mu_i$.
b) $I = \cup_{i=1}^m J_i$.
c) For any $x \in \mathrm{Rad}\,\chi$ there exists $r \in \mathbb{Z}$ such that $rx \in \sum_{i=1}^m \mathbb{Z}\mu_i$.
d) $\sum_{i=1}^m \mathbb{Z}\mu_i$ is a subgroup of $\mathrm{Rad}\,\chi$ of finite index. In particular, χ is of corank $\leq m$.

Proof: Let $\mu \in \mathrm{Rad}^+\chi$ be sincere. We prove (a) by induction on $n = \mathrm{card} I$. Since χ is 0-sincere, we know $n \geq 0$. For $n = 2$ we obtain $\chi = \mathcal{C}(1)$ where the claim is obvious. For $n > 2$ there is some linear combination $0 \leq w = q\mu - p\mu_n$ with $p, q \in \mathbb{N}$ and $K := \mathrm{supp}\, w$ is a proper subset of I. Then by induction (a) holds for the sincere vector $w \in \mathrm{Rad}^+\chi|K$. Consequently, (a) and also (b) is proved.

For (c) we choose a sincere $\mu \in \mathrm{Rad}^+\chi$ and $a \in \mathbb{N}$ such that $w = a\mu$ lies in $\mathrm{Rad}^+\chi$ and is again sincere. Now we apply (a) to w and μ. Clearly, (d) follows from (c). $\square$

7.2. It does not seem to be obvious that a 0-sincere form of corank n has to have a 0-sincere subform of corank $n-1$. To establish this result, we need the following lemma.

Lemma. *Suppose $x^{(1)}, \ldots, x^{(r)} \in \mathbb{N}_0^n$ are $\mathbb{Q}$-linear independent vectors satisfying $\mathrm{supp}\,(\sum_{i=1}^r x^{(i)}) = \{1, \ldots, n\}$. Then there exist $\mathbb{Q}$-linear independent vectors $z^{(1)}, \ldots, z^{(r-1)} \in \mathbb{N}_0^n$ lying in $\sum_{i=1}^r \mathbb{Z}x^{(i)}$ such that $\mathrm{supp}\, \sum_{i=1}^{r-1} z^{(i)}$ is a proper subset of $\{1, \ldots, n\}$.*

Proof: We apply induction on r. For $r = 1$ nothing is to prove. In the case $r > 1$, if necessary, we replace $x^{(1)}$ by $\sum_{i=1}^r x^{(i)} = \{1, \ldots, n\}$ in order to establish that $x^{(1)}$ is sincere. By induction for $x^{(1)}, \ldots, x^{(r-1)}$ there are $z^{(1)}, \ldots, z^{(r-2)}$ as required. Since $\mathrm{supp}\, \sum_{i=1}^{r-2} z^{(i)}$ is a proper subset of $\{1, \ldots, n\}$, after possibly rearranging indices and vertices, we may suppose that there is some $l \in \mathbb{N}$, $1 \leq l \leq n$ such that for all $i = 1, \ldots, r-2$ we have $z_j^{(i)} = 0$ for all $j = 1, \ldots, l$ whereas for all $j > l$ there is an index i, $1 \leq i \leq r-2$ satisfying $z_j^{(i)} > 0$. As $x^{(1)}$ is sincere, the vectors $x^{(1)}, z^{(1)}, \ldots, z^{(r-2)}$ are $\mathbb{Q}$-linear independent. Because $x^{(1)}, \ldots, x^{(r)}$ are $\mathbb{Q}$-linear independent, without loss of generality we may suppose that even $x^{(r)}, x^{(1)}, z^{(1)}, \ldots, z^{(r-2)}$ are $\mathbb{Q}$-linear independent. Defining $\frac{p}{q}$ as the minimum of all quotients $\frac{x_i^{(r)}}{x_i^{(1)}}$, we obtain that $z = qx^{(r)} - px^{(1)} \geq 0$ and there is k, $1 \leq k \leq n$ such that $z_k = 0$.

Since we may freely replace $x^{(r)}$ by $x^{(r)} + Mz^{(i)}$ where M is any natural number, it is possible to shift the minimum k until $k \leq l$. Therefore without loss of generality we may even suppose $z_1 = 0$. Putting $z^{(r-1)} = z$, we obtain that $x^{(1)}, z^{(1)}, \ldots, z^{(r-1)}$ are still $\mathbb{Q}$-linear independent and $1 \notin \operatorname{supp} \sum_{i=1}^{r-1} z^{(i)}$. □

Proposition. *If $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a 0-sincere weakly nonnegative unit form of corank r, then there exists a proper subset J of I such that the restriction $\chi|J$ is 0-sincere and of corank $r-1$.*

Proof: By 1.2 we can find vectors $x^{(1)}, \ldots, x^{(r)} \in \mathbb{N}_0^n$ which are $\mathbb{Q}$-linear independent and lie in $\operatorname{Rad}\chi$. We apply the previous lemma and put $J := \operatorname{supp} \sum_{i=1}^{r-1} z^{(i)}$. Obviously the corank of $\chi|J$ is $\geq r-1$. But, since χ is positive semidefinite, $\operatorname{Rad}\chi|J$ is a subgroup of $\operatorname{Rad}\chi$. This subgroup has to be proper, as it does not contain any sincere vector. Consequently the corank of $\chi|J$ is $r-1$. □

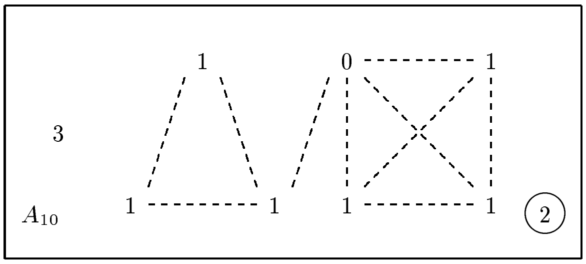
7.3. A 0-sincere weakly nonnegative unit form χ is said to be reduced provided $\chi_{ij} \leq 1$ for all vertices i, j. The reason for calling these forms reduced is the following assertion.

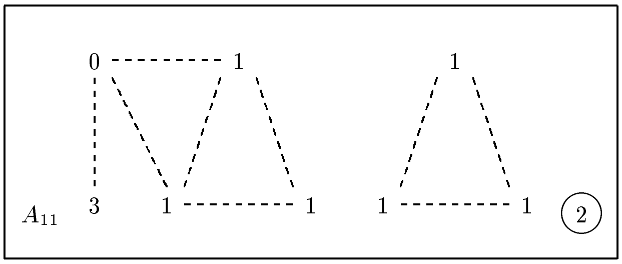
Lemma. *Suppose $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a 0-sincere unit form. Then χ is not reduced if and only if there exists a vertex i such that χ is obtained from $\chi|I \setminus \{i\}$ by doubling a point.*

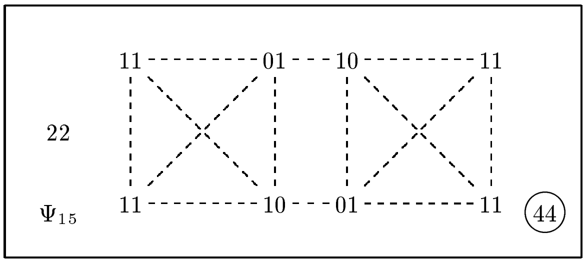
Proof: The sufficiency of the condition is clear. To prove necessity, we fix i, j such that $\chi_{ij} > 1$ and derive from 3.1 that $\chi_{ij} = 2$. Hence $0 = 2 - \chi_{ij} = \chi(e(i) - e(j))$ which by 5.3 and the remark at the end of 4.5 implies that χ is obtained by doubling a point. □

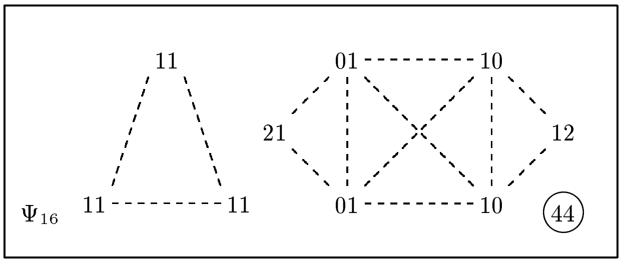
8 Graphical 0-sincere forms of small corank

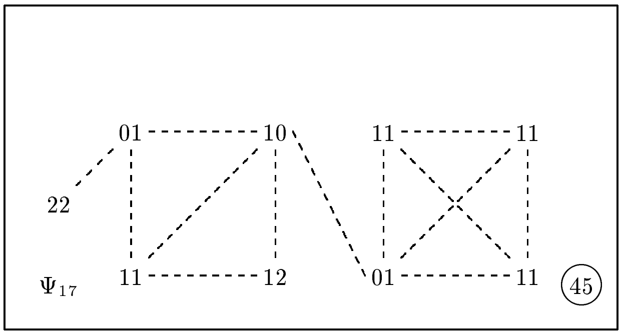
8.1. In [Za] (see also [Si]) the sincere partially ordered sets of polynomial growth where classified. Among others there occurred the sets A_{10}, A_{11} resp. $\Psi_{15}, \ldots, \Psi_{20}$. We want to denote the corresponding Tits forms by the same symbols. These forms are by construction graphical and are known to be positive semidefinite. In fact, more details about them can be found in [Ri] because the corresponding posets are domestic tubular resp. tubular. Let us as display the corresponding reduced bigraphs. We replace each vertex by a tuple of numbers which are the coefficients of the characteristic vectors of the critical restrictions. Since we know that these coefficients are integers between 0 and 6, we do not insert any separators between the different numbers. Again we provide the coefficients at the center inside a circle in the right lower corner.

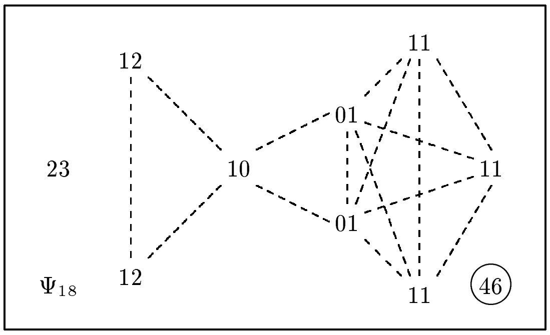


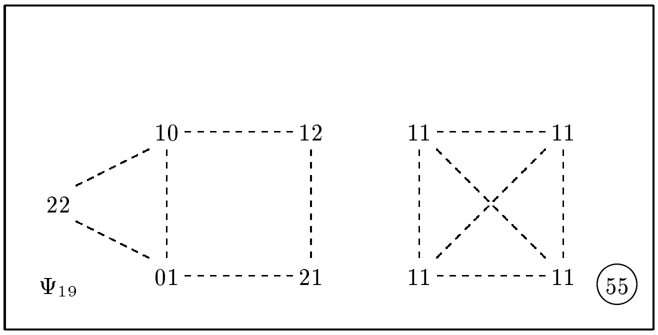


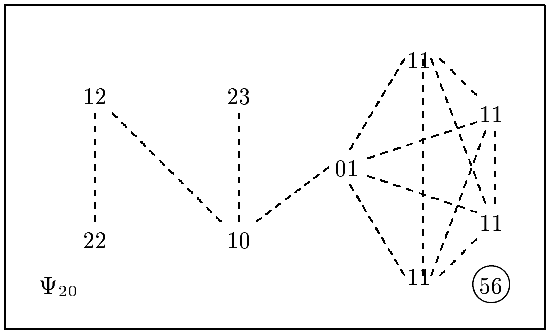












Lemma. *Let $\chi : \mathbb{Z}^K \to \mathbb{Z}$ be a reduced positive semidefinite semigraphical form such that there is no critical restriction of the form $\mathcal{C}(1), \dots, \mathcal{C}(3)$ or $\mathcal{C}(4')$. Suppose that J is a subset of K such that* $\mathrm{card}J = \mathrm{card}K - 1$.
a) If $\chi|J = \mathcal{C}(4)$, then χ coincides with one of A_{10} , A_{11} and is of corank $= 1$.
b) If $\chi|J = \mathcal{C}(5)$, then χ coincides with one of Ψ_{17}, Ψ_{19}, Ψ_{20} and is of corank $= 2$.
c) If $\chi|J = \mathcal{C}(6)$, then χ coincides with one of Ψ_{18}, Ψ_{20} and is of corank $= 2$.
d) If $\chi|J = A_{10}$ or $= A_{11}$ and μ is the characteristic vector of the unique critical restriction (of type $\mathcal{C}(4)$), then χ is of corank $= 2$ and either χ coincides with one of Ψ_{15}, Ψ_{16}, Ψ_{17},

Ψ_{18} *or* $\operatorname{Rad}\chi$ *besides* μ *has a generator* v *with coefficients* $v_s = 1$ *and* $v_t = -1$ *where* s, t *are the 2 vertices not belonging to* $\operatorname{supp}\mu$. *In particular, in this second case the form* χ *is not 0-sincere.*

Proof: The statements about the coranks can be found in [Ri] with the exception of the non-0-sincere forms appearing in d) which have to be calculated explicitly.

For a),b),c) let us denote by k the vertex in $K \setminus J$ and by μ the characteristic vector of $\chi|J$. Since $\mu \in \operatorname{Rad}(\chi)$, we get $0 = 2(e(k), \mu) = \mu_\omega - \sum_{j\in S}\mu_j$ where $S := \{j \in J : \chi_{kj} = 1\}$. The rest of the proof consists of an inspection of the few cases left possible by this condition

To show d) we observe that we can apply a) to the restriction of χ to the union of $\mathcal{C}(4)$ and the additional vertex. Hence there are again only few cases left to examine. $\square$

8.2. **Theorem.** *Let* $\chi : \mathbb{Z}^I \to \mathbb{Z}$ *be a reduced* 0*-sincere semigraphical form such that there is no critical restriction of the form* $\mathcal{C}(1), \ldots, \mathcal{C}(3)$ *or* $\mathcal{C}(4')$. *Then the corank of* χ *is* 2 *if and only if* χ *is one of the forms* $\Psi_{15}, \ldots, \Psi_{20}$.

Proof: That the forms $\Psi_{15}, \ldots, \Psi_{20}$ have the desired properties follows from [Ri] which was already stated in the proof of the above lemma.

For the converse we choose a critical restriction $\chi|J$. Since χ is of corank $= 2$ the set J cannot exhaust I and we can apply the above lemma to the union K of J with an arbitrary vertex. In the case that $\chi|J$ coincides with $\mathcal{C}(5)$ or $\mathcal{C}(6)$ the parts b),c) of the lemma show that $\chi|K = \chi$ since corank 2 is already reached. If $\chi|J$ is of shape $\mathcal{C}(4)$, then part a) shows that $\chi|K$ is still of corank $= 1$; therefore K is still a proper subset of I, and we can add another point to obtain a subset K'.

Changing the notation for K to J and for K' to K we are able to apply part d) of the above lemma. Thus it remains to exclude that $\chi|K$ is a form of corank $= 2$ which is not 0-sincere. But, since χ is 0-sincere, the last condition shows that in this case K would still be a proper subset of I leading to the contradiction that χ would be of corank ≥ 3. $\square$

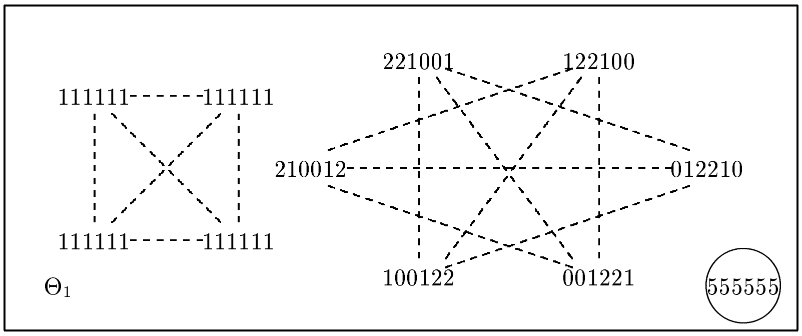
Remark. The forms χ appearing in the above list have exactly two critical restrictions with characteristic vectors $\mu^{(1)}$, $\mu^{(2)}$. An inspection shows that there are vertices $i \in \operatorname{supp}\mu^{(1)} \setminus \operatorname{supp}\mu^{(2)}$ and $j \in \operatorname{supp}\mu^{(2)} \setminus \operatorname{supp}\mu^{(1)}$ such that $\mu_i^{(1)} = \mu_j^{(2)} = 1$. In particular, this means that any $\mu \in \operatorname{Rad}^+\chi$ can be written as $\mu = n_1\mu^{(1)} + n_2\mu^{(2)}$ where n_1, n_2 are nonnegative integers which are not both 0.

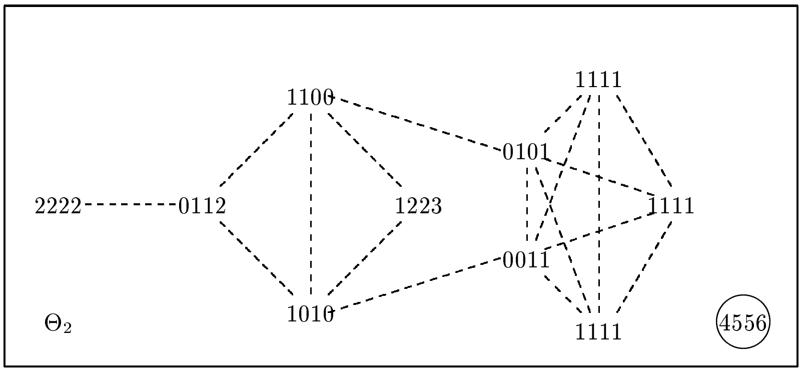
8.3. Fortunately we do not need the corresponding complete classification of forms of corank $= 3$ but only will use certain properties of these forms which we will establish in this final part of this section. Let us start out with a ad hoc definition. We denote a 0-sincere graphical form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ as triangular provided there are three critical restrictions I_1, I_2, I_3 such that for all $i \neq j \in \{1, 2, 3\}$ the restriction $\chi|I_i \cup I_j$ is 0-sincere of corank $= 2$.

Lemma. *Let $\chi : \mathbb{Z}^K \to \mathbb{Z}$ be a reduced positive semidefinite semigraphical form such that there is no critical restriction of the form $\mathcal{C}(1), \dots, \mathcal{C}(3)$ or $\mathcal{C}(4')$. Suppose that J is a subset of K such that* $\mathrm{card} J = \mathrm{card} K - 1$*. $\chi|J$ is of corank $= 2$, then χ is a 0-sincere form of corank $= 3$ which is either triangular or one of the forms Θ_1, Θ_2 whose reduced bigraphs are shown below.*

Proof: Since $\chi|J$ is of corank $= 2$, we can apply theorem 8.2 to obtain that $\chi|J$ is one of the forms $\Psi_{15}, \dots, \Psi_{20}$. We denote by k the vertex in $K \setminus J$, choose a critical restriction J' of J and apply lemma 8.1,a),b) or c) to the restriction of χ to $J' \cup \{k\}$. By inspection of all possibilities how k can be connected to the vertices of $J \setminus J'$ we obtain the result. $\square$

Let us present the reduced bigraphs of Θ_1, Θ_2. We again replace the vertices by the coefficients of the characteristic vectors of the critical restrictions.





Theorem. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a reduced 0-sincere semigraphical form such that there is no critical restriction of the form $\mathcal{C}(1), \dots, \mathcal{C}(3)$ or $\mathcal{C}(4')$. If the corank of χ is 3 then χ is either triangular or one of the forms Θ_1, Θ_2.*

Proof: By proposition 7.2 there is a restriction $\chi|J$ of χ which is of corank $= 2$. We choose a vertex $k \in I \setminus K$ and apply the above lemma to $K := J \cup \{k\}$. Since the lemma tells us that the corank of $\chi|K$ is 3, we get $K = I$ and the theorem is proved. $\square$

9 Proof of the main theorem

9.1. We assume the existence of a weakly nonnegative, finitely sincere semiunit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ which has a maximal sincere positive 1-root x with a coordinate $x_i > 12$ and try to establish a contradiction.

Applying 4.4 we replace χ, x by χ', x' where (χ', x', ϕ) is a full reduction of (χ, x) with respect to $I \setminus \{i\}$. Obviously card$I' \geq 2$. If there exists $j \in I'$ such that $\chi'_{ij} \geq 0$, then $\chi'_{kj} \geq 0$ for all $k \in I'$ which by remark 3.2 implies $\chi'_{jj} = 0$. This shows that $e(j)$ is a positive 0-root with the property $x' > e(j)$. By corollary 5.2 the 1-root x' is 2-layer which by theorem 5.3 yields the contradiction $x'_i \leq 12$.

Hence in the sequel we only have to deal with the case that χ' is a semigraphical unit form with center i. In order to simplify notations we replace χ' by χ, x' by x and i by ω. If χ has a critical restriction $\chi|J$ which is of shape $\mathcal{C}(1), \ldots, \mathcal{C}(3)$ or $\mathcal{C}(4')$, then the characteristic vector of this restriction is a 0-root μ satisfying $x > \mu$. Again we obtain a contradiction by corollary 5.2 and theorem 5.3.

9.2. Let us sum up to which extend we already reduced our original problem. To produce a contradiction, we dispose now of a weakly non-negative, finitely sincere semigraphical form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ with center ω having a maximal sincere positive 1-root x with $x_\omega > 12$ and having the property that any critical restriction is of shape $\mathcal{C}(4)$, $\mathcal{C}(5)$ or $\mathcal{C}(6)$. Since $\mathcal{C}(1)$ does not occur, we know that $\chi_{i\omega} = 1$ for all $i \in I$ different from ω.

Among all forms χ as above we consider one where the number of vertices card$I$ is minimal. 4.5 shows that this choice implies that χ is not obtained from any subform by doubling a vertex. By 3.4 the 1-root x has one or two exceptional vertices. If x has only one exceptional vertex applying theorem 6.3 we arrive at the contradiction $x_\omega \leq 7$.

Thus there are exactly two exceptional vertices $i \neq j$ which have the property that $x_i = x_j = 2(x, e(i)) = 2(x, e(j)) = 1$ and $2(x, e(k)) = 0$ for all $k \neq i, j$. Because of $x_\omega > 12$ we see $\omega \neq i, j$.

9.3. We observe that the form $\chi|I \setminus \{i, j\}$ is weakly positive. Namely, if we assume otherwise, then there exists a critical restriction of this form having a characteristic vector μ which yields the contradiction $\chi(\mu + x) = \chi(x) = 1$. Using this observation we can prove:

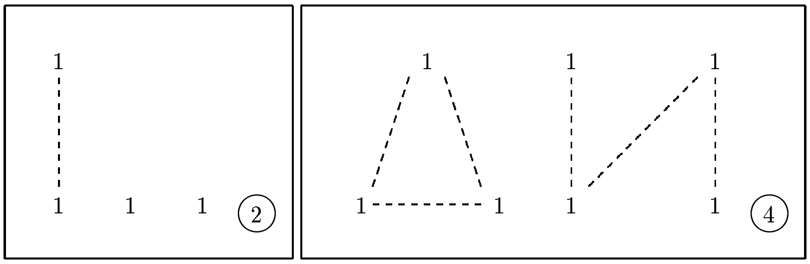
Lemma. *If $0 \leq z \in \mathbb{Z}^I$ satisfies $z_k \leq 1$ for all $k \in I$, $k \neq \omega$, $z_i = z_j = 1$ and $\chi(z) \leq 2$, then $z_\omega > 6$.*

Proof: Assuming $z_\omega \leq 6$, we obtain $x - z > 0$ and therefore $0 < \chi(x - z) = \chi(z) - 1 \leq 1$. Thus $x - z$ is a positive 1-root of the weakly positive form $\chi|I \setminus \{i, j\}$ which shows $(x - z)_\omega \leq 6$ and finally the contradiction $x_\omega \leq 12$. □

We can apply this lemma immediately to show that $\chi_{ij} \in \{2, 3\}$. By 3.1 we know that $0 \leq \chi_{ij} \leq 3$. The lemma shows that $\chi_{ij} \in \{0, 1\}$ is impossible using the vector $z := e(\omega) + e(i) + e(j)$ which satisfies $\chi(z) \leq 2$.

9.4. Let us consider the unit form $\bar{\chi} : \mathbb{Z}^I \to \mathbb{Z}$ defined by $\bar{\chi}(y) = \chi(y) - y_i y_j$ for all $y \in \mathbb{Z}^I$. Clearly x is a sincere positive 0-root of $\bar{\chi}$. We want to show that $\bar{\chi}$ is weakly nonnegative and consequently a 0-sincere unit form.

If we assume that $\bar{\chi}$ is not weakly nonnegative, then there has to be a critical restriction $\bar{\chi}|J$ where J has to contain both i and j. If $\chi_{ij} = 3$, we have a contradiction to 6.4. If $\chi_{ij} = 2$, we see from 6.4 that the reduced bigraph of $\bar{\chi}|J$ contains a subbigraph K satisfying $i, j \in K$ of the following two shapes:



Defining vectors $z \in \mathbb{Z}^{K \cup \{\omega\}}$ as shown in the pictures above and considering them as elements of $\mathbb{Z}^I$, we obtain $\bar{\chi}(z) = 1$ resp. $\chi(z) = 2$. This is a contradiction to the previous lemma.

9.5. As next step we convince ourselves that $\bar{\chi}_{st} \leq 1$ for all $s, t \in I$. Since $\bar{\chi}$ is 0-sincere, by 3.1 we only have to show that $\chi_{st} = 2$ is leads to a contradiction for $s \neq t$ such that $\{s, t\} \neq \{i, j\}$.

If $\{s, t\} \cap \{i, j\} = \emptyset$, then by 7.3 the forms $\bar{\chi}$ and also χ would be obtained form the restriction to $I \setminus \{t\}$ by doubling the point s.

If the intersection of $\{s, t\}$ and $\{i, j\}$ is just $t = j$, then we consider the positive 1-root $y := x - e(i)$ of $\eta = \chi|I \setminus \{i\}$. If η would be weakly positive, then we would get $x_\omega = y_\omega \leq 6$. Hence η has a critical restriction with a characteristic vector v. By 7.3 the form $\bar{\chi}$ is obtained from $\chi|I \setminus \{j\}$ by doubling s. Hence also η is obtained from $\eta|I \setminus \{i, j\}$ by doubling s. Consequently the vector $w := v - v_j e(j) + v_j e(s)$ is a 0-root of η and also of χ which gives the contradiction $\chi(x+w) = \chi(x) + 2(w, x) = \chi(x) + 2w_i(e(i), x) + 2w_j(e(j), x) = \chi(x) = 1$.

9.6. Before we continue, let us recall that because of the positive definiteness of $\bar{\chi}$ any 0-root of $\bar{\chi}$ lies in $\operatorname{Rad} \bar{\chi}$. Let us consider the restriction ξ of χ to $I' := I \setminus \{i\}$ which as a restriction of $\bar{\chi}$ as well is a positive semidefinite reduced semigraphical form. The vector $y = x - e(i) = \sigma_i(x)$ is a positive 1-root of χ and therefore a sincere positive 1-root of ξ. As $y_\omega > 12$, we know that ξ is not weakly positive and consequently the 0-sincere kernel ξ^+ of ξ is non-trivial. We want to establish that ξ^+ is of corank ≤ 2.

Assume that the corank of ξ^+ is at least 3. Then by 7.3 the form ξ^+ has a 0-sincere restriction ζ whose corank is precisely 3. We will obtain a contradiction using again corollary 5.2 and theorem 5.3 by constructing a positive 0-root μ such that $x > \mu$.

In fact, by theorem 8.3 the form ζ is either triangular or coincides with ϑ_1 or ϑ_2. If $\zeta = \vartheta_1$, the positive 0-root z defined by $z_\omega = 5$ and $z_k = 1$ for all other vertices allows to apply corollary 5.2 and theorem 5.3 immediately. If ζ happens to be triangular, we denote by μ_1, μ_2, μ_3 the characteristic vectors of the three critical restrictions I_1, I_2, I_3

such that for all $s \neq t$ the union $I_s \cup I_t$ is 0-sincere of corank $= 2$ and consequently by theorem 8.2 is one of the forms $\Psi_{15}, \ldots, \Psi_{20}$. Looking at these forms, we find that $\{(\mu_s - \mu_t)_k \mid k \in I, k \neq \omega\} = \{-1, 0, 1\}$ and $(\mu_s - \mu_t)_\omega \in \{-2, 0, 2\}$. In case that $\zeta = \vartheta_2$ the same holds for the the first three vectors μ_1, μ_2, μ_3 displayed in 8.3 although this form is not triangular.

We claim that at least 2 of the 3 numbers $2(e(i), \mu_1), 2(e(i), \mu_2), 2(e(i), \mu_3)$ are equal. Namely, otherwise there exist s, t satisfying $2(e(i), \mu_s) - 2(e(i), \mu_t) \geq 2$ and therefore also $2(x, \mu_s - \mu_t) = 2(y + e(i), \mu_s - \mu_t) = 2(e(i), \mu_s - \mu_t) \geq 2$. On the other hand $x - (\mu_s - \mu_t) > 0$ yielding the contradiction $0 \leq \chi(x - (\mu_s - \mu_t)) = \chi(x) + \chi(\mu_s - \mu_t) - 2(x, \mu_s - \mu_t) \leq -1$.

Supposing now that without loss of generality $2(e(i), \mu_1) = 2(e(i), \mu_2)$ and $(\mu_1 - \mu_2)_\omega \geq 0$, we obtain that $\mu_1 - \mu_2 \in \operatorname{Rad} \chi$. Putting $d := \min\{x_k \mid (\mu_1 - \mu_2)_k = -1\}$, it follows that $z := x + d(\mu_1 - \mu_2)$ is a positive 1-root of χ satisfying $z_\omega > 12$ but z is not sincere. By the subsequent lemma we arrive at a contradiction to the minimal choice of card$I$.

Lemma. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly nonnegative semiunit form. Suppose x is a maximal sincere positive 1-root of χ and $\mu \in \operatorname{Rad} \chi$. If $x + \mu > 0$ and $J := \operatorname{supp}(x + \mu)$, then $x + \mu$ is a maximal sincere positive 1-root of $\chi|J$.*

Proof: Let $y \geq x + \mu$. Then $y - \mu > x$ and $\chi(y - \mu) = 1$. Hence $y - \mu = x$. $\square$

9.7. It remains to deal with the case that the corank of ξ^+ is 1 or 2. But, before doing so, we want to establish that also $\bar{\chi}$ does not admit a critical restriction of type $\mathcal{C}(1), \ldots, \mathcal{C}(3)$ or $\mathcal{C}(4')$. It is clear that $\mathcal{C}(1)$ cannot occur. If one of the others would occur, then the support of its characteristic vector z had to contain i and j because otherwise it would even be a critical restriction of χ. Therefore z would be a positive 1-root of χ for which lemma 9.3 would apply furnishing the contradiction $z_\omega > 6$.

9.8. Let us now suppose that $\xi^+ : \mathbb{Z}^K \to \mathbb{Z}$ is of corank $= 2$ and therefore classified in theorem 8.2. Assume that there exists $k \in I' \setminus K$. Then by lemma 8.3 the restriction of ξ to $K \cup \{k\}$ is a 0-sincere form of corank $= 3$ and consequently ξ^+ is of corank ≥ 3, a contradiction. Hence $I = K \cup \{i\}$.

If $\bar{\chi}_{ij} = 2$, then $\bar{\chi}$ is obtained from ξ^+ by doubling j and $u := x - e(i) + e(j)$ is a sincere 0-root of ξ^+ such that $u_j = 2$. If $\mu^{(1)}$ and $\mu^{(2)}$ are the characteristic vectors of the 2 critical restrictions of ξ^+, then $u = p\mu^{(1)} + q\mu^{(2)}$ where $p, q \in \mathbb{N}$. Up to symmetry we only need to consider the cases $\mu_j^{(1)} = 1$ and $\mu_j^{(1)} = 0$. In both cases we obtain $x \geq \mu^{(1)}$ which yields the usual contradiction to theorem 5.3 by corollary 5.2 since $\mu^{(1)}$ is a positive 0-root of χ.

If $\bar{\chi}_{ij} = 1$, then again by lemma 8.3 we know that $\bar{\chi}$ is one of the 0-sincere semigraphical forms of corank $= 3$ which are classified in theorem 8.3. If $\bar{\chi} = \vartheta_2$, then the vector z defined by $z_\omega = 5$ and $z_k = 1$ for all other vertices is a 1-root of χ thus leading to a contradiction via lemma 9.3. In the other cases we always find characteristic vectors $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$ of critical restrictions generating the radical of $\bar{\chi}$ such that $x = p_1\mu^{(1)} + p_2\mu^{(2} + p_3\mu^{(3}$ where p_1, p_2, p_3 are nonnegative integers and at least $p_1, p_2 > 0$. Since $x_i = 1$, we may assume that that $\mu_i^{(1)} = 0$. Hence we end up with the same contradiction as above as this $\mu^{(1)}$ is a 0-root of χ such that $x \geq \mu^{(1)}$.

9.9. Finally, we have to manage the case that that ξ^+ is of corank $= 1$ or in other words is critical. we denote by μ the corresponding characteristic vector. This case splits into two subcases namely the first one that ξ^+ is of shape $\mathcal{C}(5)$ or $\mathcal{C}(6)$ and the second one that ξ^+ is of shape $\mathcal{C}(4)$.

In the first subcase we see from parts b) and c) of lemma 8.1 that $K = I'$. If $\bar{\chi}_{ij} = 2$ then $\bar{\chi}$ is obtained from ξ^+ by doubling j and we consider again the sincere positive 0-root $u = x - e(i) + e(j)$ of ξ^+. Because $u_j = 2$ we get $u = p\mu$ where $p \in \{1, 2\}$ yielding the contradiction $x_\omega = u_\omega \leq 2\mu_\omega \leq 12$.

If $\bar{\chi}_{ij} = 1$, the again form lemma 8.1 we obtain that $\bar{\chi}$ is one of the forms $\Psi_{17}, \ldots, \Psi_{20}$. Using $x_i = 1$, the existence of a critical restriction avoiding i whose characteristic vector μ satisfies $x \geq \mu$ is easy to derive.

Let us consider the second subcase. If $I' \setminus K$ happens to be empty, then for $\bar{\chi}_{ij} = 2$ we can argue as in the first subcase and for $\bar{\chi}_{ij} = 1$ by part a) of lemma 8.1 the form $\bar{\chi}$ is not 0-sincere, a contradiction.

If $I' \setminus K$ is non-empty, we fix a vertex k in this set. Let us first analyze the situation that $I' \setminus K = \{k\}$. Then either $\bar{\chi}$ is not 0-sincere if $\bar{\chi}_{ij} = 2$ or $\bar{\chi}$ is one of the forms occurring in part d) of lemma 8.1. For $\bar{\chi}$ among $\Psi_{15}, \Psi_{16}, \Psi_{17}, \Psi_{18}$ we obtain a contradiction as in the first subcase. Otherwise $\bar{\chi}$ again fails to be 0-sincere.

Supposing now that there are elements in $I' \setminus K$ different from k, we observe that for any $k \neq l \in I' \setminus K$ by part d) of 8.3 using that ξ^+ is of corank $= 1$ we find a radical vector $\nu^{(l)}$ such that $\nu^{(l)}_l = 1$ and $\nu^{(l)}_k = -1$. Moreover, it is easy to see that the vectors $\nu^{(l)}$ together with the characteristic vector μ of ξ^+ form a basis of $\operatorname{Rad}\xi$. This shows that ξ is not 0-sincere which remains true for $\bar{\chi}$ if $\bar{\chi}_{ij} = 2$. To solve the case $\bar{\chi}_{ij} = 1$, we apply part d) of lemma 8.1 to $\bar{\chi}|K \cup \{k, i\}$. If this form is not 0-sincere, we obtain another radical vector $\nu^{(i)}$ as above and $\bar{\chi}$ is not 0-sincere. If $\bar{\chi}$ is among $\Psi_{15}, \Psi_{16}, \Psi_{17}, \Psi_{18}$, the argument is slightly more subtle. Let us denote by μ' the second characteristic vector for $\bar{\chi}|K \cup \{k, i\}$ besides μ. One sees that $\mu'_i = \mu'_k = 1$ and the vectors $\nu^{(l)}$ together with μ and μ' form a basis of $\operatorname{Rad}\bar{\chi}$. Writing x as linear combination of these base vectors and observing $x_1 = 1$, it follows that $x_k \leq 0$ which is the final contradiction finishing the proof.

10 Applications to finite-dimensional algebras

10.1. For more details concerning the notions used in this section we refer to [Ri]. We suppose that Λ is a finite-dimensional basic algebra over an algebraically closed field k. Let us write $A = k[Q]/I$ where Q is the ordinary quiver of A and I is an admissible ideal of the path algebra $k[Q]$. To consider the Tits form χ_A as defined in the introduction we demand that Q is directed i.e. does not admit oriented cycles.

By A-mod we denote the category of all finite-dimensional left A-modules which we identify with the representations of Q satisfying I. We do not distinguish between a module X and its isomorphism class.

On the indecomposable modules X in A-mod we consider the relation $\prec$ which is defined as the transitive closure of the relation given by putting $X \prec Y$ if there is non-zero non-isomorphism $X \to Y$. An indecomposable A-module X is said to be directing if $X \not\prec X$.

For a given indecomposable module X we define $\Lambda(X)$ to be the subalgebra of Λ induced by the support of X. In the natural way X will be considered as $\Lambda(X)$-module which is directing provided X is a directing Λ-module. We say that X is properly directing if X is a directing Λ-module but it is neither a postprojective nor a preinjective $\Lambda(X)$-module.

Proposition. *Let Λ be a tame algebra. If X is an indecomposable properly directing Λ-module, then* $\dim_k X(i) \leq 12$ *holds for all vertices* $i \in Q_0$.

Proof: Without loss of generality we my suppose $\Lambda = \Lambda(X)$. Hence Λ is a tame tilted algebra having the sincere directing module X. Consequently χ_A is a weakly non-negative unit form and $\mathbf{dim}\, X$ is a sincere positive 1-root of χ_A. Thus by our main theorem it remains to show that χ_A is finitely sincere. By [Pe3, 3.2] A is domestic in at most two 1-parameters.

If A would be domestic in less than two 1-parameters, then A would be representation-finite or tame concealed. Thus X would be postprojective or preinjective.

Consequently A has to be domestic in exactly two 1-parameters. By [Pe3, 1.6] and [Pe3, 2.1] it follows that χ_A is finitely sincere. $\square$

Corollary. *If X is an indecomposable properly directing module over the algebra Λ whose ordinary quiver has n vertices, then* $\dim_k X \leq 2n + 180$.

Proof: Again we may assume $\Lambda = \Lambda(X)$. If $n \geq 20$, then Λ belongs to the list presented in [Pe4] and one checks directly that $\dim_k X \leq 2n - 2$.

For $n \leq 19$ we assume that $\mathbf{dim}_X$ is a maximal positive 1-root of χ_A which consequently has an exceptional vertex i satisfying $\dim_k X(i) \leq 2$. Therefore we obtain $\dim_k X \leq 2n - \sum j \neq i |\dim X(j) - 2| \leq 2n + 18(12-2)$. $\square$

References

[Bo] K. Bongartz: *Algebras and quadratic forms,* J. London Math. Soc. (2) 28 (1983), 461-469.

[BG] K. Bongartz; P. Gabriel: *Covering spaces in representation theory,* Invent. Math. 65 (1982), 331-378.

[DP] A. Dean, J.A. de la Peña: *Algorithms for weakly non-negative quadratic forms,* To appear in Linear Alg. Appl.

[D-Z] P. Dräxler; Yu.A. Drozd; N.S. Golovachtchuk; S.A. Ovsienko; M.M. Zeldych: *Towards classification of sincere weakly positive unit forms,* To appear in European J. of Comb.

[Ga] P. Gabriel: *Unzerlegbare Darstellungen I,* Manuscr. Math. 6 (1972), 71-103.

[HP] D. Happel, J.A. de la Peña: *Quadratic forms with a maximal sincere root,* to appear in the proceedings of ICRA VII.

[Ov1] S. Ovsienko: *Integral weakly positive forms,* In Schur matrix problems and quadratic forms, Kiev (1978), 3-17.

[Ov2] S. Ovsienko: *Maximal roots of sincere weakly nonnegative forms,* lecture at the workshop on *Quadratic Forms in the Representation Theory of Finite-Dimensional Algebras*, Bielefeld, November 9-12, 1995.

[Pe1] J.A. de la Peña: *On the representation type of one point extensions of tame concealed algebras,* Manuscr. Math. 61 (1988), 183-194.

[Pe2] J.A. de la Peña: *Algebras with hypercritical Tits form,* in: Topics in Algebra, Banach Center Publ. 26, part I (1990), 353-369.

[Pe3] J.A. de la Peña: *Tame algebras with sincere directing modules,* J. Algebra 161 (1993), 171-185.

[Pe4] J.A. de la Peña: *The families of two-parametric tame algebras with sincere directing modules,* CMS Proceedings 14 (1993), 361-392.

[Ri] C.M. Ringel: *Tame algebras and integral quadratic forms,* Springer LNM 1099 (1984).

[Si] D. Simson, *Linear representations of partially ordered sets and vector space categories,* Amsterdam (1992).

[Za] A.G. Zavadskij: *Sincere partially ordered sets of finite growth,* Inst. Mat. Akad. Nauk USSR, Preprint 81.27, Kiev (1981), 30-42 (in Russian).

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