On the support of irreducible weight modules

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Abstract

For a reductive complex finite-dimensional Lie algebra L which does not contain E_8 as a direct summand we study the support of the irreducible weight modules. In particular we show that any such non-dense module is a quotient of generalized Verma module.

1 Introduction

Let L be a finite dimensional reductive Lie algebra over C, H be a Cartan subalgebra, Δ be the root system, π be a basis of Δ and Q be a free abelian group generated by π.

For an L-module, V, set $V_\lambda := \{v \in V |\ hv = \lambda(h)v,\ \text{for all } h \in H\}$, where $\lambda \in H^*$. If V_λ is non-zero then λ is a *weight* of V and V_λ is the corresponding weight-space. We denote by $P(V)$ the set of all weights of V. An L-module is called weight if $V = \bigoplus V_\lambda$, $\lambda \in P(V)$. Clearly, if V is indecomposable module then $P(V) \subseteq \lambda + Q$, for some $\lambda \in H^*$.

A weight module V is said to be *torsion-free* if for all $\alpha \in \Delta$ the action of $X_\alpha : V \to V$ is injective, and to be *dense* if $P(V) = \lambda + Q$ for some $\lambda \in P(V)$. The classification of all irreducible torsion free sl(2)-modules is standard [B] and is obtained in [BFL] for the Lie algebra sl(3). The question remains open in other cases. Some classes of irreducible torsion free modules were studied in [BL].

Clearly, any torsion-free module is dense. The converse is also true for irreducible modules with finite-dimensional weight spaces [Fe], though it is not the case for modules with infinite-dimensional weight spaces [F].

It follows from [Fe] that any irreducible weight L-module with finite dimensional weight spaces is either torsion-free (in which case L is of type A_n or C_n) or the irreducible quotient of a generalized Verma module (GVM) induced from a torsion free module for parabolic subalgebra. This result is no longer valid for modules with infinite dimensional weight spaces. For such modules an analogous result for algebras of type A_n, B_n, C_n, D_n was obtained in [F].

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In particular, it was shown that irreducible weight module V for classical Lie algebras is either dense or is the quotient of GVM induced from a dense module for a parabolic subalgebra.

In this paper we re-examine the proof of Futorny's result for the classical Lie algebras and generalize it for any reductive algebra which does not contain E_8 as a direct summand.

We reduce the problem to some combinatorics on the root system (Lemma 3.3). In the case of algebras of type E_6, E_7 the needed computer software was developed. We present the result in the case of E_6 which can be checked by direct calculations. The case of E_7 is left out since it would take a lot of space even to sketch the result.

The structure of the paper is the following. In section 2 we recall the definition and basic properties of GVM 's and study the support of non-dense irreducible modules in section 3. The main result is Theorem 3.1 which provides a characterization of the support of a non-dense irreducible module. As a corollary we obtain that every such module is a unique irreducible quotient of a GVM .

2 Notation and definitions

We use [CF] as our main reference on GVM .

Let $\pi = \{\alpha_1, \ldots, \alpha_n\}$, $\Delta^+ = \Delta^+(\pi)$ (respectively $\Delta^- = \Delta^-(\pi)$) be a set of all positive (resp. negative) roots in Δ with respect to π. Let S denotes an arbitrary subset of π; Q_S be a subgroup of Q generated by S; $\Delta_S := \Delta \cap Q_S$, ${}_+\mathcal{N}_S := \bigoplus L_\alpha$, ${}_-\mathcal{N}_S := \bigoplus L_{-\alpha}$ with $\alpha \in \Delta_S^+$; $\Delta_S^+ := \Delta^+ \cap Q_S$; Q_+^S be a monoid generated by $\Delta^+ \setminus \Delta_S$.

Let $\mathcal{H}^S := \{h \in \mathcal{H} |\ \alpha(h) = 0 \text{ for } \alpha \in S\}$ and $\mathcal{M}_S :={}_- \mathcal{N}_S \bigoplus \mathcal{H} \bigoplus_+ \mathcal{N}_S$. Further, let $\mathcal{N}_S^+ := \bigoplus L_\alpha$ and $\mathcal{N}_S^- := \bigoplus L_{-\alpha}$ where $\alpha \in \Delta^+ \setminus \Delta_S$. Then the subalgebra $\mathcal{B}_S := \mathcal{M}_S \bigoplus \mathcal{N}_S^+$ is a parabolic subalgebra of L with $\mathcal{M}_S$ as a Levi factor.

Let V be an irreducible weight $\mathcal{M}_S$-module. Consider V as a $\mathcal{B}_S$-module on which $\mathcal{N}_S^+$ acts trivially. An L-module

$$M(S,V) = \mathcal{U}(L) \otimes_{\mathcal{U}(\mathcal{B}_S)} V$$

is called *GVM* , associated with L, H, π, S, V [L,R].

DEFINITION 2.1 *Let V be a weight L-module and $S \subset \pi$, $\lambda \in P(V)$. An element $v \in V_\lambda$ is called S-semiprimitive (with respect to π) if $\mathcal{N}_S^+ v = 0$.*

LEMMA 2.2 *If V is irreducible L-module, $\xi \in Q_S$ and $v \in V_\lambda$ is S-semiprimitive element then any $v' \in V_\lambda$ is S-semiprimitive.*

Proof. Since V is irreducible then V_λ is irreducible module over the centralizer $C(H)$ of H in $\mathcal{U}(H)$ [D]. Thus, $v' = uv$ for some $u \in C(H)$ and $\mathcal{N}_S^+ v' = \mathcal{N}_S^+ uv = 0$ implying (ii). □

Let W (resp. W_S) be the Weyl group of Δ (resp. Δ_S). The main properties of GVM are collected in the following statement.

Proposition 2.3 *Let $S \subset \pi$ and V be an irreducible weight $\mathcal{M}_S$-module.*

(i) There is canonical isomorphism $M(wS, V) \to M(S, V)$ for any $w \in W_S$.

(ii) $M(S, V)$ has a unique maximal submodule and thus a unique irreducible quotient $L(S, V)$.

(iii) If $\dim V_\lambda = \dim M(S, V)_\lambda < \infty$ for some $\lambda \in P(V)$ then $\dim M(S, V)_\mu < \infty$ for all $\mu \in P(M(S, V))$.

(iv) $P(M(S, V)) = P(V) + Q^-_{\overline{S}}$.

(v) If N is irreducible weight L-module, N_λ contains S-semiprimitive element for some $\lambda \in P(N)$ and $\overline{N} = \sum_{\xi \in Q_S} N_{\lambda+\xi}$ then $\overline{N}$ is irreducible $\mathcal{M}_S$-module and $N \simeq L(S, \overline{N})$.

(vi) If $\mathcal{M}_S$-module V contains a T-semiprimitive element $v \in V_\lambda$ (with respect to S) for some $T \subset S$ and $\lambda \in P(V)$ then $\overline{V} = \sum_{\xi \in Q_S} V_{\lambda+\xi}$ is irreducible $\mathcal{M}_T$-module and we have the following chain of epimorphisms:

$$M(T, \overline{V}) \to M(S, V) \to L(S, V) \simeq L(T, \overline{V}).$$

Proof. Since $\mathcal{M}_S \simeq \mathcal{M}_{wS}$ and $w\mathcal{N}^+_S = \mathcal{N}^+_S$ for any $w \in W_S$ then (i) follows. The statements (ii)-(iv) are standard (cf. [CF]). The statement (v) follows from Lemma 3.3, (ii), (iii), while the statement (vi) follows from (v) and Lemma 3.3,(i). $\square$

3 Non-dense irreducible modules

The main result of the paper is the following

Theorem 3.1 *Let L be a reductive complex finite dimensional Lie algebra which does not contain E_8 as a direct summand and let V be an irreducible weight non-dense L-module. Then either V has the highest weight and hence $V \simeq L(\emptyset, \mathbf{C})$ or there exists $\lambda \in P(V)$, a basis π' and a subset $S \subset \pi', S \neq \emptyset$ such that*

(i) $\mathcal{M}_S$-module $N = \sum_{\phi \in Q_S} V_{\lambda+\phi}$ is dense,

(ii) $V \simeq L(S, N)$,

(iii) $P(V) = \lambda + Q_S + Q^-_{\overline{S}}$.

To prove the theorem we will need the following result.

Proposition 3.2 *Let L be a reductive complex Lie algebra which does not contain E_8 as a direct summand and let V be an irreducible non-dense L-module. Then there exist a basis π', a root $\alpha \in \pi'$ and $\lambda \in P(V)$ such that V_λ contains a non-trivial S-semiprimitive element, where $S = \pi' \setminus \{\alpha\}$.*

If V is a non-dense L-module then there exist $\mu \in \lambda + Q$ and $\alpha \in \Delta$ such that $V_\mu = 0$, and $V_{\mu-\alpha} \neq 0$. A triple (v, α, A), $v \in V_{\mu-\alpha}$, $v \neq 0$, $\alpha \in A \subset \Delta$ is called *μ-admissible* if $X_\beta v = 0$ for all $\beta \in A$. Note that if $V_{\mu-\alpha} \neq 0$ then $(v, \alpha, \{\alpha\})$ is μ-admissible for any $0 \neq v \in V_{\mu-\alpha}$. Denote by $\mathrm{add}(A)$ an additive closure of the set $A \subseteq \Delta$.

LEMMA 3.3 *Let V be a non-dense L-module, $\mu \notin P(V)$, $0 \neq v \in V_{\mu-\alpha}$, $A \subset \Delta$, $\alpha \in A$.*

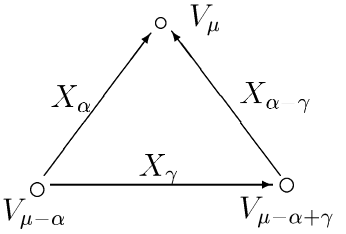
(i) Let (v, α, A) be μ-admissible and $\gamma \in \Delta$ such that $(\alpha - \gamma) \in \Delta$. Then either (v, α, A') or (v'', α'', A'') is μ-admissible where $A' = A \cup \{\gamma\}$, $v'' = X_\gamma v$, $\alpha'' = \alpha - \gamma$, $A'' = \{\beta \in A |\ \gamma + \beta \notin \Delta \cup \{0\},\ \text{or}\ \gamma + \beta \in A\} \cup \{\alpha - \gamma\}$.

(ii) If (v, α, A) is μ-admissible then $(v, \alpha, \mathrm{add}(A))$ is also μ-admissible.

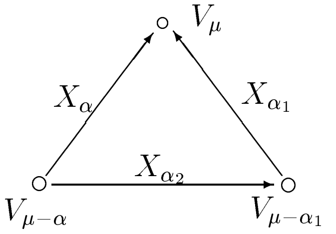
(iii) If (v, α, A) is μ-admissible then (v, α, A') is also μ-admissible, where $\alpha \in A' \subset A$.

(iv) If $(v, \alpha, \{\alpha\})$ is μ-admissible and $\alpha = \alpha_1 + \alpha_2$, $\alpha_1, \alpha_2 \in \Delta$ then either $(v', \alpha_1, \{\alpha_1\})$, $v' = X_{\alpha_2} v$ is μ-admissible, or $(v, \alpha_2, \{\alpha_2\})$ is $(\mu - \alpha_1)$-admissible.

Proof. The statements (ii), (iii) are evident. We prove the statement (i) (see picture 1). If $X_\gamma v = 0$ then (v, α, A') is μ-admissible. Let $X_\gamma v \neq 0$ then $v'' = X_\gamma v \neq 0$ and $v'' \in V_{\mu-(\alpha-\gamma)}$. If $\beta \in A''$ we have $0 = [X_\beta X_\gamma] v = X_\beta X_\gamma v - X_\gamma X_\beta v = X_\beta v''$ implying that (v'', α'', A'') is μ-admissible. The proof of (iv) is similar to the proof of (i) (see picture 2). $\square$



Picture 1.



Picture 2.

The transformations (i)-(iv) of one admissible triple to another are called admissible. It follows from Lemma 3.3 that to prove Proposition 3.2 it is enough to show that for any non-dense L-module V and any μ-admissible triple $T = (v, \alpha, \{\alpha\})$ there exists a sequence of triples $T_1, \ldots, T_{l+1}$ such that $T_1 = T$, T_{l+1} is a final element and T_{i+1} is received by admissible transformation from T_i, $i = 1, \ldots, l$. This follows from a more general result (Proposition 3.4) which is based on the combinatorics of Δ and is proved below for simple algebras except E_8 separately.

Define a subset $\Omega \subseteq \Delta \times 2^\Delta$ as follows: $(\alpha, A) \in \Omega$ if and only if $\alpha \in A$, $A \subseteq \Delta$. The Weyl group W of Δ acts on the set Ω as follows: $w(\alpha, A) = (w\alpha, wA)$, $w \in W$, $\alpha \in \Omega$. An element $(\alpha, A) \in \Omega$ is called final if there exists a basis π', and $\beta \in \pi'$, such that $A \supseteq \Delta^+ \setminus \Delta_S$, where $S = \pi' \setminus \{\beta\}$. If $F_1 = (\alpha_1, A_1)$, $F_2 = (\alpha_2, A_2) \in \Omega$ and there exists $w \in W$ such that $\alpha_1 = w\alpha_2$ and $A_1 \subseteq \mathrm{add}(wA_2)$ then we will write $F_1 \sqsubseteq F_2$. If $F_1 \sqsubseteq F_2$ and $F_2 \sqsubseteq F_1$, $F_1, F_2 \in \Omega$ then the elements F_1, F_2 are called W-equivalent and we write $F_1 \sim F_2$ in this case. We will denote the equivalence class of F by $\tilde{F}$.

Consider an oriented graph $\Gamma = (\Gamma_0, \Gamma_1 \cup \Gamma_2)$, where the set of vertices Γ_0 is a set of equivalent classes $\Omega/\sim$ and the set of arrows $\Gamma_1 \cup \Gamma_2 \subseteq \Gamma_0 \times \Gamma_0$ is defined by the following conditions:

(i) if $F_1 \sqsubseteq F_2$, $F_1, F_2 \in \Omega$ then $(\tilde{F}_2, \tilde{F}_1) \in \Gamma_1$,

(ii) if $F_1 = (\alpha, A)$, $F_2 = (\alpha, \mathrm{add}(A))$, $F_1, F_2 \in \Omega$ then $(\tilde{F}_1, \tilde{F}_2) \in \Gamma_1$,

(iii) if $F = (\alpha, \{\alpha\}) \in \Omega$, α is a long root and $\alpha = \alpha_1 + \alpha_2$, where α_1, α_2 are short roots then $(\tilde{F}, \tilde{F}_i) \in \Gamma_1$, where $F_i = (\alpha_i, \{\alpha_i\}) \in \Omega$, $i = 1, 2$,

(iv) if $F = (\alpha, A) \in \Omega$, $\gamma \in \Delta$ such that $\alpha - \gamma \in \Delta$ then $(\tilde{F}, \tilde{F}_1), (\tilde{F}, \tilde{F}_2) \in \Gamma_2$, where $F_1 = D^1_\gamma(F) := (\alpha, A \cup \{\gamma\})$, $F_2 = D^2_\gamma(F) := (\alpha'', A'')$, $\alpha'' = \alpha - \gamma$, $A'' = \{\beta \in A |\ \beta + \gamma \in A$ or $\beta + \gamma \notin \Delta \cup \{0\}\} \cup \{\alpha''\}$.

(v) if $(\tilde{F}_1, \tilde{F}_2), (\tilde{F}_2, \tilde{F}_3) \in \Gamma_1$ then $(\tilde{F}_1, \tilde{F}_3) \in \Gamma_1$,

(vi) if $(\tilde{F}_1, \tilde{F}_2) \in \Gamma_2$ and $(\tilde{F}_2, \tilde{F}_3) \in \Gamma_1$ then $(\tilde{F}_1, \tilde{F}_3) \in \Gamma_2$.

If $A'' \supseteq A$ in (iv) then the root γ is called a "good" root.

Consider the graph universal covering: $\Pi : \hat{\Gamma} \to \Gamma$. A vertex $\tilde{F} \in \Gamma_0$ is called *primitive* if there exists a finite subgraph $\hat{T} = (\hat{T}_0, \hat{T}_1 \cup \hat{T}_2) \subseteq \hat{\Gamma}$ such that:

(i) $\tilde{F} \in \Pi(\hat{T}_0)$,

(ii) every zero out-valency vertex (sink) of the graph $\hat{T}$ is a final element,

(iii) if $(\tilde{F}, \tilde{F}_1) \in \hat{T}_2$ then there exists $\gamma \in \Delta$ and $\tilde{F}_2 \in \hat{T}_0$ such that $(\tilde{F}, \tilde{F}_2) \in \hat{T}_2$ and $F_1 \sqsubseteq D^i_\gamma(F)$, $F_2 \sqsubseteq D^j_\gamma(F)$, $\{i, j\} = \{1, 2\}$.

In this case the graph $\tilde{T}$ is called admissible for the primitive vertex a.

PROPOSITION 3.4 *Any vertex $\tilde{F} \in \Gamma_0$ is primitive.*

Note that Proposition 3.4 immediately implies Proposition 3.2. Indeed, let V be an irreducible non-dense L-module. Then there exists $\mu \notin P(V)$ and $\alpha \in \Delta$ such that $V_{\mu-\alpha} \neq 0$. Consider $0 \neq v \in V_{\mu-\alpha}$. Then the triple $(v, \alpha, \{\alpha\})$ is μ-admissible. Since $(\alpha, \{\alpha\}) \in \Gamma_0$ is semiprimitive by Proposition 3.4, we conclude by Lemma 3.3 that V contains a $\pi' \setminus \{\beta\}$-semiprimitive element for some basis π' and some $\beta \in \pi'$.

4 Plane realization of the Weyl group

Let $A = (a_{ij})$, $i, j = 1, ..., n$ be a matrix with indeterminates a_{ij}. Consider a partition of all elements of the matrix into three parts $\{a_{ij}\} = A^1 \cup A^0 \cup A^*$, where $A^0 = \{a_{i,i}\}_{i=1,...,n}$ and $a_{ij} \in A^1$ if and only if $a_{ji} \in A^1$.

Denote by G_A the symmetric permutation group of the set $\{a_{ij}\}$, and let $G^{1,0}_A$ be a subgroup of G_A generated by the elements g^r_{ij}, g^c_{ij}, where g^r_{ij} (resp. g^c_{ij}) permute a_{ik} and a_{jk}, $k = 1, ..., n$ (resp. a_{ki} and a_{kj}) if and only if $a_{ik}, a_{jk} \in A^0 \cup A^1$ (resp. $a_{ki}, a_{kj} \in A^0 \cup A^1$).

Since the group $G^{1,0}_A$ permutes only the elements of $A^0 \cup A^1$ we will identify it with the permutation group of $A^0 \cup A^1$. Let $G^{0,1}_A = \{g \in G^{1,0}_A |$ if $a \in A^0$ then $a^g \in A^0\}$

Clearly, $G^0_A = \{g \in G^{0,1}_A |\ a^g = a, \forall a \in A^1\}$ is a normal subgroup in $G^{0,1}_A$. Thus $G^1_A = G^{0,1}_A / G^0_A$ is a permutation group of the set A^1. Suppose now that $|A^1| = |\Delta|$ and ϕ is a bijection $\phi : A^1 \to \Delta$ such that $\phi(a_{ij}) = -\phi(a_{ji})$, $a_{ij} \in A^1$. Then the group G^1_A can be viewed as a permutation group of the root system Δ with respect to the function ϕ.

A pair $(A^1 \cup A^0 \cup A^*, \phi)$ is called a *plane realization* of the Weyl group W if $W \subseteq G^1_A$ as a permutation group.

It is easy to see that the elements $g^r_{ij}, g^c_{ij} \in G^{1,0}$ commute and $g^r_{ij} g^c_{ij} \in G^{0,1}_A$. Denote by σ_{ij} the image of $g^r_{ij} g^c_{ij}$ in G^1_A.

5 Proof of Proposition 3.4

Since the Weyl group acts transitively on the set of short roots and since any long root is a sum of two short roots it is enough to prove Proposition 3.4 for an element $\tilde{F} \in \Gamma_0$, where $F = (\alpha, \{\alpha\})$ and $\alpha \in \Delta$ is a short root.

The proof of Proposition 3.4 is based on the combinatorics of Δ and the following algorithm applied for each simple algebra separately.

We define a certain subset $\tilde{M} \in \Gamma_0$ and choose a set $M \subseteq \Omega$ of representatives of $\tilde{M}$. Then we define an injective function $d : M \to \mathcal{N}^m$, $m > 0$. Consider a natural lexicographical order on the set $\mathcal{N}^m$: $(a_1, \dots, a_m) > (b_1, \dots, b_m)$ if and only if there exists $k \leq m$ such that $a_i = b_i, i < k$ and $a_k > b_k$. With respect to this order and function d we define an order on the set M as follows: $F' > F''$, $F', F'' \in M$ if and only if $d(F') > d(F'')$. Moreover, it follows from the construction of M that $(\alpha, \{\alpha\})$ is the minimum element for some short root α with respect to this order.

Further, if $F = (\alpha, A) \in M$ is not final then we can either choose an element $F_1 \in M$ such that $(F, F_1) \in \Gamma_1$ and $d(F_1) > d(F)$ or choose elements $F_1, F_2 \in M$ such that $(\tilde{F}, \tilde{F_1}) \in \Gamma_2$, $(\tilde{F}, \tilde{F_2}) \in \Gamma_2$ $F_1 \sqsubseteq D^i_\gamma(F)$, $F_2 \sqsubseteq D^j_\gamma(F)$, $\{i, j\} = \{1, 2\}$ for some proper γ and $d(F_1) > d(F)$, $d(F_2) > d(F)$. Then the maximum element of M is final and hence Proposition 3.4 follows.

Define a partial order on the set of roots Δ with respect to the basis $\pi = \{\alpha_1, \dots, \alpha_n\}$ as follows:

$$\beta_1 \geq \beta_2, \quad \Longleftrightarrow \quad b^1_i \geq b^2_i,\ i = 1, \dots, n, \quad \text{where } \beta_j = \sum_{i=1}^n b^j_i \alpha_i, j = 1, 2.$$

5.1 Case A_n

Denote $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$, $j \geq i$, where $\pi = \{\alpha_1, \dots, \alpha_n\}$ is a basis of the root system Δ.

Consider the following plane realization of W:

$$A = (a_{ij}),\ i, j = 1, \dots, n+1,\ A^* = \emptyset,\ A^0 = \{a_{i,i}\}_{i=1,\dots,n+1},$$

$$A^1 = \{a_{ij}\}_{i \neq j}, \quad \phi(a_{ij}) = \begin{cases} \alpha_{j,i-1}, & i > j \\ -\alpha_{i,j-1}, & i < j \end{cases}$$

and hence

$$\phi(A) = (\phi(a_{ij})) = \begin{pmatrix} 0 & -\alpha_{1,1} & -\alpha_{1,2} & \dots & \dots & -\alpha_{1,n} \\ \alpha_{1,1} & 0 & -\alpha_{2,2} & \dots & \dots & -\alpha_{2,n} \\ \alpha_{1,2} & \alpha_{2,2} & 0 & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & -\alpha_{n,n} \\ \alpha_{1,n} & \alpha_{2,n} & \dots & \dots & \alpha_{n,n} & 0 \end{pmatrix}$$

In this case the Weyl group W is generated by the permutations $s_i = \sigma_{i,i+1}, i = 1,\dots,n$.

Let

$$P_i^j = \{\gamma \in \Delta^+ | \ \gamma \geq \alpha_{ij}\}, \quad i = 1,\dots,n, \ j = i,\dots,n, \qquad P_0^0 = \emptyset,$$
$$P_{i_1,\dots,i_k}^{j_1,\dots,j_k} = P_{i_1}^{j_1} \cup \ldots \cup P_{i_k}^{j_k}.$$

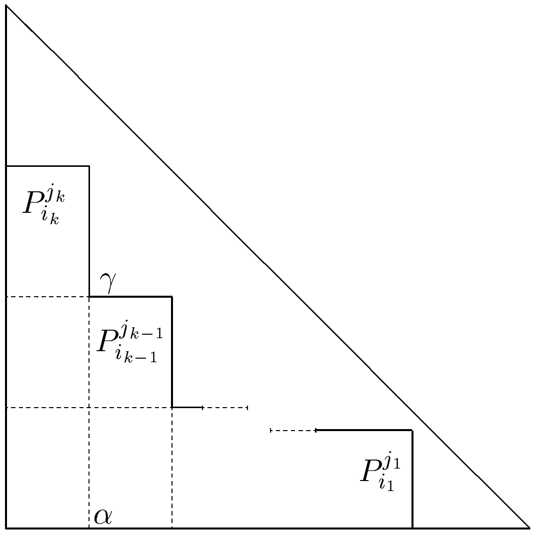
Set

$$M = \{(\alpha_{i_k+1,n}, P_{i_1,\dots,i_k}^{j_1,\dots,j_k} | \ 2 \leq k \leq n+1, i_1 > \ldots > i_k \geq 0,$$
$$n \geq j_1 > \ldots > j_k, i_p \leq j_p, p = 1,\dots,k, \text{ if } i_k = 0 \text{ then } j_k = 0\}$$

and define a function $d : M \to \mathcal{N}^{2n+4}$ by the following rule:

$$d(F) = (i_1, \overline{j}_1, \dots, i_{k-1}, \overline{j}_{k-1}, i_k+1, \overline{n}, i_k, \overline{j}_k, 0, \dots, 0), \qquad F \in M,$$

where $\overline{j} = n+1-j$. Note that $F \in M$ is final if and only if $i_p = j_p \neq 0$ for some $p \leq k$ and $s_k P_i^j = P_i^j$, $k \neq i,j$; $s_k \in W$. One can picture the sets P_i^j, M on the following diagram, which is a part of the $\phi(A)$ consisting of all positive roots.



Clearly, $(\alpha_{1,n}, P_{1,0}^{n,0}) = (\alpha_{1,n}, \{\alpha_{1,n}\})$ is the minimal element in M.

Let $F = (\alpha, A) = (\alpha_{i_k+1,n}, P_{i_1,\dots,i_k}^{j_1,\dots,j_k}) \in M$ be a non final element. We choose a root γ and elements $F_1, F_2 \in M$ such that $(\tilde{F}, \tilde{F}_1) \in \Gamma_2$, $(\tilde{F}, \tilde{F}_2) \in \Gamma_2$ and $F_1 \sqsubseteq D_\gamma^1(F)$, $F_2 \sqsubseteq D_\gamma^2(F)$ and $d(F_1) > d(F)$, $d(F_2) > d(F)$.

Take a "good" root $\gamma = \alpha_{i_k+1,j_{k-1}-1} \in \Delta^+ \setminus A$. Then $D_\gamma^1(F) = (\alpha, A \cup \{\gamma\})$, $D_\gamma^2(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{i_k+1,n} - \alpha_{i_k+1,j_{k-1}-1} = \alpha_{j_{k-1},n}$.

Depending on the indices we define the element $F_1 \in M$ as follows:

$$F_1 = \begin{cases} (\alpha_{1,n}, P^{j_1,\dots,j_{k-2},j_k,0}_{i_1,\dots,i_{k-2},i_k+1,0}), & \text{if } i_k+1 = i_{k-1},\ j_k = j_{k-1}-1, \\ (\alpha_{i_k+1,n}, P^{j_1,\dots,j_{k-1}-1,j_k}_{i_1,\dots,i_{k-1},i_k}), & \text{if } i_k+1 = i_{k-1},\ j_k < j_{k-1}-1, \\ (\alpha_{1,n}, P^{j_1,\dots,j_{k-1},j_k,0}_{i_1,\dots,i_{k-1},i_k+1,0}), & \text{if } i_k+1 < i_{k-1},\ j_k = j_{k-1}-1, \\ (\alpha_{i_k+1,n}, P^{j_1,\dots,j_{k-1},j_{k-1}-1,j_k}_{i_1,\dots,i_{k-1},i_k+1,i_k}), & \text{otherwise.} \end{cases}$$

It is easy to check using the plane realization of the Weyl group that $F_1 \sim D^1_\gamma(F)$ and $d(F_1) > d(F)$.

Since $i_{k-1} < j_{k-1}$ then $\alpha'' \notin P^{j_{k-1}}_{i_{k-1}}$. If $\alpha'' \in A$ then $\alpha'' \in P^{j_{l-1}}_{i_{l-1}}$ and $\alpha'' \notin P^{j_l}_{i_l}$ for some $l < k$. Set $F_2 = (\alpha_{i_l+1,n}, P^{j_1,\dots,j_l}_{i_1,\dots,i_l})$. If $\alpha'' \notin A$ then we set $F_2 = (\alpha_{1,n}, P^{n,0}_{i_1+1,0})$. Again it is easy to check that $F_2 \sqsubseteq D^2_\gamma(F)$ and $d(F_2) > d(F)$.

5.2 Case B_n

Set

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j, \quad 1 \le i \le j \le n,$$

$$\Delta_1 = \{\pm\alpha_{ij} |\ 1 \le i \le j \le n\}, \quad \Delta_1^+ = \Delta_1 \cap \Delta^+,$$

$$\beta_{ij} = \alpha_{i,n} + \alpha_{j,n}, \quad 1 \le i < j \le n,$$

$$\Delta_2 = \{\pm\beta_{ij} |\ 1 \le i < j \le n\}, \quad \Delta_2^+ = \Delta_2 \cap \Delta^+, \quad \Delta = \Delta_1 \cup \Delta_2.$$

Define the plane realization of the Weyl group W as follows:

$$A = (a_{ij})_{i,j=1,\dots,2n},\ A^* = \{a_{ij} |\ 2 \le i \le 2n, 2n+2-i \le j \le 2n, i \ne j\},$$

$$A^0 = \{a_{i,i}\}_{i=1,\dots,2n}, \quad A^1 = \{a_{ij}\} \setminus (A^0 \cup A^*),$$

$$\phi(a_{ij}) = \begin{cases} \alpha_{j,i-1}, & 1 \le j < i \le n+1 \\ \beta_{j,2n+2-i}, & n+1 < i \le 2n,\ 1 \le j < 2n+2-i \end{cases}$$

$$\phi(a_{ij}) = -\phi(a_{ji}), \quad \text{if } i < j \text{ and } a_{ji} \in A^1.$$

and hence

$$\phi(A) = (\phi(a_{ij})) = \begin{pmatrix} 0 & -\alpha_{1,1} & \dots & \dots & -\alpha_{1,n} & -\beta_{1,n} & \dots & -\beta_{1,2} \\ \alpha_{1,1} & 0 & & & \vdots & \vdots & & * \\ \vdots & & \ddots & & \vdots & -\beta_{n-1,n} & * & * \\ \alpha_{1,n-1} & \dots & \alpha_{n-1,n-1} & 0 & -\alpha_{n,n} & * & & \\ \alpha_{1,n} & \dots & \alpha_{n-1,n} & \alpha_{n,n} & 0 & * & & \\ \beta_{1,n} & \dots & \beta_{n-1,n} & * & * & 0 & & \vdots \\ \vdots & & * & & & & \ddots & * \\ \beta_{1,2} & * & * & \dots & & & * & 0 \end{pmatrix}$$

The Weyl group W is generated by permutations $s_1=\sigma_{1,2}$; $s_i=\sigma_{i,i+1}\sigma_{2n+1-i,2n+2-i}$, $i=2,\dots,n-1$; $s_n=\sigma_{n+1,n+2}\sigma_{n,n+1}\sigma_{n+1,n+2}$.

Set

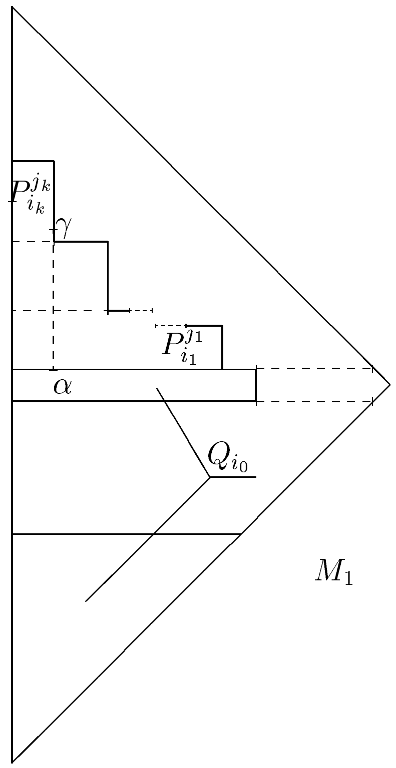
$$\begin{aligned}
P_i^j &= \{\gamma\in\Delta_1^+ \mid \alpha_{ij}\le\gamma<\alpha_{1,n}\}, \quad i=1,\dots,n-1, j=i,\dots,n-1, \qquad P_0^0=\emptyset,\\
Q_i &= \{\alpha_{1,n},\dots,\alpha_{i,n}\}\cup\{\gamma\in\Delta_2^+ \mid \gamma\ge\beta_{i-1,i}\}, \quad i=1,\dots,n,\\
R_i^j &= \{\gamma\in\Delta_1^+ \mid \alpha_{jj}\le\gamma<\alpha_{1,i}\}, \quad j=1,\dots,n-1, i=j,\dots,n-1,\\
S_i &= \{\gamma\in\Delta_2^+ \mid \gamma\ge\beta_{in}\}, \quad i=1,\dots,n-1,
\end{aligned}$$

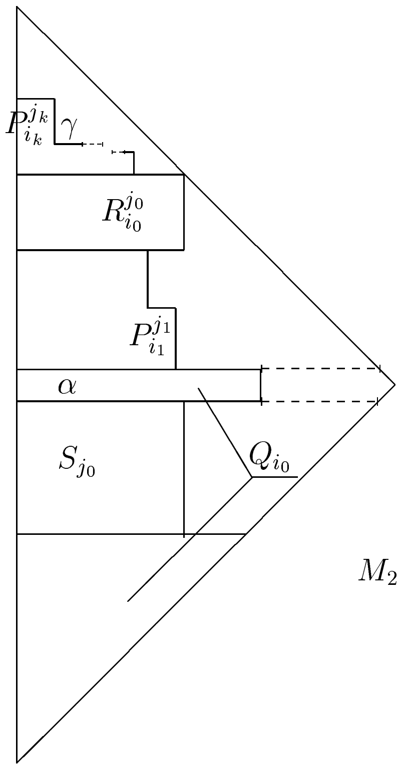
$$P_{i_1,\dots,i_k}^{j_1,\dots,j_k}=P_{i_1}^{j_1}\cup\ldots\cup P_{i_k}^{j_k}, \qquad T_i^j=Q_i\cup S_j\cup R_i^j.$$

Define the set $M=M_1\cup M_2$, where:

$$\begin{aligned}
M_1=\{(\alpha_{i_k+1,n},\, Q_{i_0}\cup P_{i_1,\dots,i_k}^{j_1,\dots,j_k}) \mid\ & n\ge i_0\ge i_1>\ldots>i_k\ge 0,\\
& n\ge j_1>j_2>\ldots>j_k\ge 0,\ i_l<j_l, l=1,\dots,k,\ k\ge 1,\}\\
M_2=\{(\alpha_{i_k+1,n},\, T_{i_0}^{j_0}\cup P_{i_1,\dots,i_k}^{j_1,\dots,j_k}) \mid\ & i_0\ge j_0\ge i_1>\ldots>i_k\ge 0,\\
& j_1>j_2>\ldots j_k, i_l\le j_l, l=1,\dots,k, k\ge 1,\}
\end{aligned}$$

One can picture the sets P_i^j, Q_i, R_i^j, S_i, M_1, M_2 on the following diagrams, each of which is a part of the $\phi(A)$ consisting of all positive roots.





Note that all these subsets are additively closed, the additive closure of the set $\{\alpha_{1,n},\dots,\alpha_{i,n}\}$ is Q_i and that $F=(\alpha,A)\in M$ is final if $A\supseteq Q_n$ or $A\supseteq Q_i\cup S_i\cup P_j^j$, $0<j\le i$.

Analogously to the case of A_n define the following function d:

$$d(F) = (i_0, 0, i_1, \overline{j}_1, \ldots, i_{k-1}, \overline{j}_{k-1}, i_k+1, \overline{n}, i_k, \overline{j}_k, 0, \ldots, 0), \quad F \in M_1,$$

$$d(F) = (i_0, j_0, i_1, \ldots, i_{k-1}, \overline{j}_{k-1}, i_k+1, \overline{n}, i_k, \overline{j}_k, 0, \ldots, 0), \quad F \in M_2,$$

where $\overline{j} = n+1-j$. Clearly, $(\alpha_{1,n}, Q_1 \cup P_0^0) = (\alpha_{1,n}, \{\alpha_{1,n}\})$ is the minimal element in M.

Consider two cases: 1) $F \in M_1$ and 2) $F \in M_2$ separately.

1. Let $F = (\alpha, A) = (\alpha_{i_k+1,n}, Q_{i_0} \cup P^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}) \in M_1$ be a non final element.

Consider a "good" root $\gamma = \alpha_{i_k+1,j_{k-1}-1} \in \Delta^+ \setminus A$. Then $D^1_\gamma(F) = (\alpha, A \cup \{\gamma\})$, $D^2_\gamma(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{i_k+1,n} - \alpha_{i_k+1,j_{k-1}-1} = \alpha_{j_{k-1},n}$.

If $i_{k-1} = j_k = i_k + 1 = j_{k-1} - 1$ then $A \cup \{\gamma\} \supseteq Q_{i_0} \cup P^{j_k}_{j_k}$. In this case we set $F_1 = (\alpha_{1,n}, T^{j_k}_{i_0} \cup P^0_0) \in M_2$. Since $(\alpha_{i_k+1}, Q_{i_0} \cup P^{j_k}_{j_k}) \in \Omega$ can be transformed into F_1 by the following element of the Weyl group: $w = (s_1 \cdot \ldots \cdot s_{j_k-1}) \cdot (s_n \cdot s_{n-1} \cdot \ldots \cdot s_{i_0+1}) \cdot (s_n \cdot \ldots \cdot s_{i_0+2}) \cdot \ldots \cdot (s_n \cdot s_{n-1}) \cdot s_n$ therefore $F_1 \sqsubseteq (\alpha, A \cup \{\gamma\})$.

If the above equality for indexes does not hold then analogously to the case A_n and depending on the indexes we define the element $F_1 \in M_1$ as follows:

$$F_1 = \begin{cases} (\alpha_{1,n}, Q_{i_0} \cup P^{j_1,\ldots,j_{k-2},j_k,0}_{i_1,\ldots,i_{k-2},i_k+1,0}), & \text{if } i_k+1 = i_{k-1}, \quad j_k = j_{k-1}-1; \\ (\alpha_{i_k+1,n}, Q_{i_0} \cup P^{j_1,\ldots,j_{k-1}-1,j_k}_{i_1,\ldots,i_{k-1},i_k}), & \text{if } i_k+1 = i_{k-1}, \quad j_k < j_{k-1}-1; \\ (\alpha_{1,n}, Q_{i_0} \cup P^{j_1,\ldots,j_{k-1},j_k,0}_{i_1,\ldots,i_{k-1},i_k+1,0}), & \text{if } i_k+1 < i_{k-1}, \quad j_k = j_{k-1}-1; \\ (\alpha_{i_k+1,n}, Q_{i_0} \cup P^{j_1,\ldots,j_{k-1},j_{k-1}-1,j_k}_{i_1,\ldots,i_{k-1},i_k+1,i_k}), & \text{otherwise.} \end{cases}$$

It is easy to check using the plane realization of the Weyl group that $F_1 \sim D^1_\gamma(F)$ and $d(F_1) > d(F)$.

If $\alpha'' \in A$ then $i_{l-1} \geq j_{k-1} > i_l$ for some $l < k$. Set $F_2 = (\alpha_{i_l+1,n}, Q_{i_0} \cup P^{j_1,\ldots,j_l}_{i_1,\ldots,i_l}) \in M_1$. If $\alpha'' \notin A$ then we set $F_2 = (\alpha_{1,n}, Q_{i_0+1} \cup P^0_0) \in M_1$. Again we obtain that $F_2 \sqsubseteq D^2_\gamma(F)$ and $d(F_2) > d(F)$.

2. Let $F = (\alpha_{i_k+1,n}, T^{j_0}_{i_0} \cup P^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}) \in M_2$ be a non final element. Take a "good" root

$$\gamma = \begin{cases} \alpha_{i_k+1,j_{k-1}-1}, & \text{if } j_{k-1} \neq i_0, \\ \alpha_{i_k+1,j_0-1}, & \text{if } j_{k-1} = i_0. \end{cases}$$

Then $D^1_\gamma(F) = (\alpha, A \cup \{\gamma\})$, $D^2_\gamma(F) = (\alpha'', A \cup \{\alpha''\})$, where

$$\alpha'' = \alpha - \gamma = \begin{cases} \alpha_{j_{k-1},n}, & \text{if } j_{k-1} \neq i_0, \\ \alpha_{j_0,n}, & \text{if } j_{k-1} = i_0. \end{cases}$$

If $i_{k-1} = j_k = i_k + 1 = j_{k-1} - 1$ then $F_1 = (\alpha_{1,n}, T^{j_0}_{i_0} \cup P^{j_1,\ldots,j_{k-2},j_k,0}_{i_1,\ldots,i_{k-2},j_k,0})$ is a final element. If the above equality for indexes does not hold then we define the element $F_1 \in M_2$ as follows:

$$F_1 = \begin{cases} (\alpha_{1,n}, T^{j_0}_{i_0} \cup P^{j_1,\ldots,j_{k-2},j_k,0}_{i_1,\ldots,i_{k-2},i_k+1,0}), & \text{if } i_k+1 = i_{k-1}, \quad j_k = j_{k-1}-1; \\ (\alpha_{i_k+1,n}, T^{j_0}_{i_0} \cup P^{j_1,\ldots,j_{k-1}-1,j_k}_{i_1,\ldots,i_{k-1},i_k}), & \text{if } i_k+1 = i_{k-1}, \quad j_k < j_{k-1}-1; \\ (\alpha_{1,n}, T^{j_0}_{i_0} \cup P^{j_1,\ldots,j_{k-1},j_k,0}_{i_1,\ldots,i_{k-1},i_k+1,0}), & \text{if } i_k+1 < i_{k-1}, \quad j_k = j_{k-1}-1; \\ (\alpha_{i_k+1,n}, T^{j_0}_{i_0} \cup P^{j_1,\ldots,j_{k-1},j_{k-1}-1,j_k}_{i_1,\ldots,i_{k-1},i_k+1,i_k}), & \text{otherwise.} \end{cases}$$

It follows that $F_1 \sim D^1_\gamma(F)$ and $d(F_1) > d(F)$.

If $\alpha'' \in A$ then $i_{l-1} \geq j_{k-1} > i_l$ for some $l < k$ and we set $F_2 = (\alpha_{i_l+1,n}, T^{j_0}_{i_0} \cup P^{j_1,\ldots,j_l}_{i_1,\ldots,i_l}) \in M_2$. If $\alpha'' \notin A$ then we set $F_2 = (\alpha_{1,n}, Q_{i_0+1} \cup P^0_0) \in M_1$. Again $F_2 \sqsubseteq D^2_\gamma(F)$ and $d(F_2) > d(F)$.

5.3 Case C_n

Denote

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j, \quad 1 \leq i \leq j \leq n,$$

$$\Delta_1 = \{\pm a_{ij} |\ 1 \leq i \leq j \leq n\}, \quad \Delta_1^+ = \Delta_1 \cap \Delta^+,$$

$$\beta_{ij} = \alpha_{i,n-1} + \alpha_{j,n}, \quad 1 \leq i \leq j < n,$$

$$\Delta_2 = \{\pm\beta_{ij} |\ 1 \leq i \leq j < n\}, \quad \Delta_2^+ = \Delta_2 \cap \Delta^+, \quad \Delta = \Delta_1 \cup \Delta_2.$$

Define the plane realization of W of root system Δ as follows:

$$A = (a_{ij})_{i,j=1,\ldots,2n},\ A^* = \{a_{ij} |\ 2 \leq i \leq 2n, 2n+2-i \leq j \leq 2n, i \neq j\},$$

$$A^0 = \{a_{ii}\}_{i=1,\ldots,2n}, \quad A^1 = \{a_{ij}\} \setminus (A^0 \cup A^*),$$

$$\phi(a_{ij}) = \begin{cases} \alpha_{j,i-1}, & 1 \leq j < i \leq n, \\ \beta_{j,j}, & i = n+1,\ j \leq n-1, \\ \alpha_{n,n}, & i = n+1,\ j = n, \\ \alpha_{j,n}, & i = n+2,\ j \leq n-1, \\ \beta_{j,2n+2-i}, & n+3 \leq i \leq 2n,\ 1 \leq j \leq 2n+1-i, \end{cases}$$

$$\phi(a_{ij}) = -\phi(a_{ji}), \quad \text{if } i < j \text{ and } a_{ji} \in A^1.$$

Hence

$$\phi(A) = \begin{pmatrix} 0 & -\alpha_{1,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\beta_{1,2} \\ \alpha_{1,1} & 0 & & & & \vdots & \vdots & & * \\ \vdots & & & \ddots & & \vdots & \vdots & * & * \\ \alpha_{1,n-1} & \cdots & \cdots & \alpha_{n-1,n-1} & 0 & -\alpha_{n,n} & * & * & \\ \beta_{1,1} & \cdots & \cdots & \beta_{n-1,n-1} & \alpha_{n,n} & 0 & * & * & \\ \alpha_{1,n} & \cdots & \cdots & \alpha_{n-1,n} & * & * & 0 & & \vdots \\ \beta_{1,n-1} & \cdots & \beta_{n-2,n-1} & * & * & & & & \vdots \\ \vdots & & * & * & & & & \ddots & \\ \beta_{1,2} & * & * & \cdots & & & & & \end{pmatrix}$$

The Weyl group W is generated by permutations $s_1 = \sigma_{1,2}$; $s_i = \sigma_{i,i+1}\sigma_{2n+1-i,2n+2-i}$, $i = 2, \ldots, n-1$; $s_n = \sigma_{n+1,n+2}\sigma_{n,n+1}\sigma_{n+1,n+2}$.

Let

$$
\begin{aligned}
P_i^j &= \{\alpha_{1,n-i}, \ldots, \alpha_{1,n-j}\}, \quad j = 1, 2,\ i = j, \ldots, n-1,\\
Q_i &= \{\alpha_{1,n}, \ldots, \alpha_{i,n}\}, \quad i = 1, \ldots, n,\\
R_i &= \{\beta_{1,n-i}, \ldots, \beta_{1,n-1}\}, \quad i = 1, \ldots, n-2, \qquad R_0 = \emptyset,\\
S_i &= \{\beta \in \Delta_2^+ |\ \beta \geq \beta_{i,n-1}\}, \quad i = 1, \ldots, n-1,\\
T_i^j &= \{\alpha \in \Delta_1^+ |\ \alpha_{1,n-1} \geq \alpha \geq \alpha_{ij}\}, \quad i = 1, \ldots, n-1, j = i, \ldots, n-1, \quad T_0^0 = T_i^n = \emptyset\\
U_i &= \{\beta \in \Delta_2^+ |\ \beta_{1,i+1} \geq \beta \geq \beta_{i,n-1}\}, \quad i = 1, \ldots, n-2.
\end{aligned}
$$

$$T_{i_1,\ldots,i_k}^{j_1,\ldots,j_k} = T_{i_1}^{j_1} \cup \ldots \cup T_{i_k}^{j_k}.$$

Define the set $M = M_1 \cup \ldots \cup M_9$:

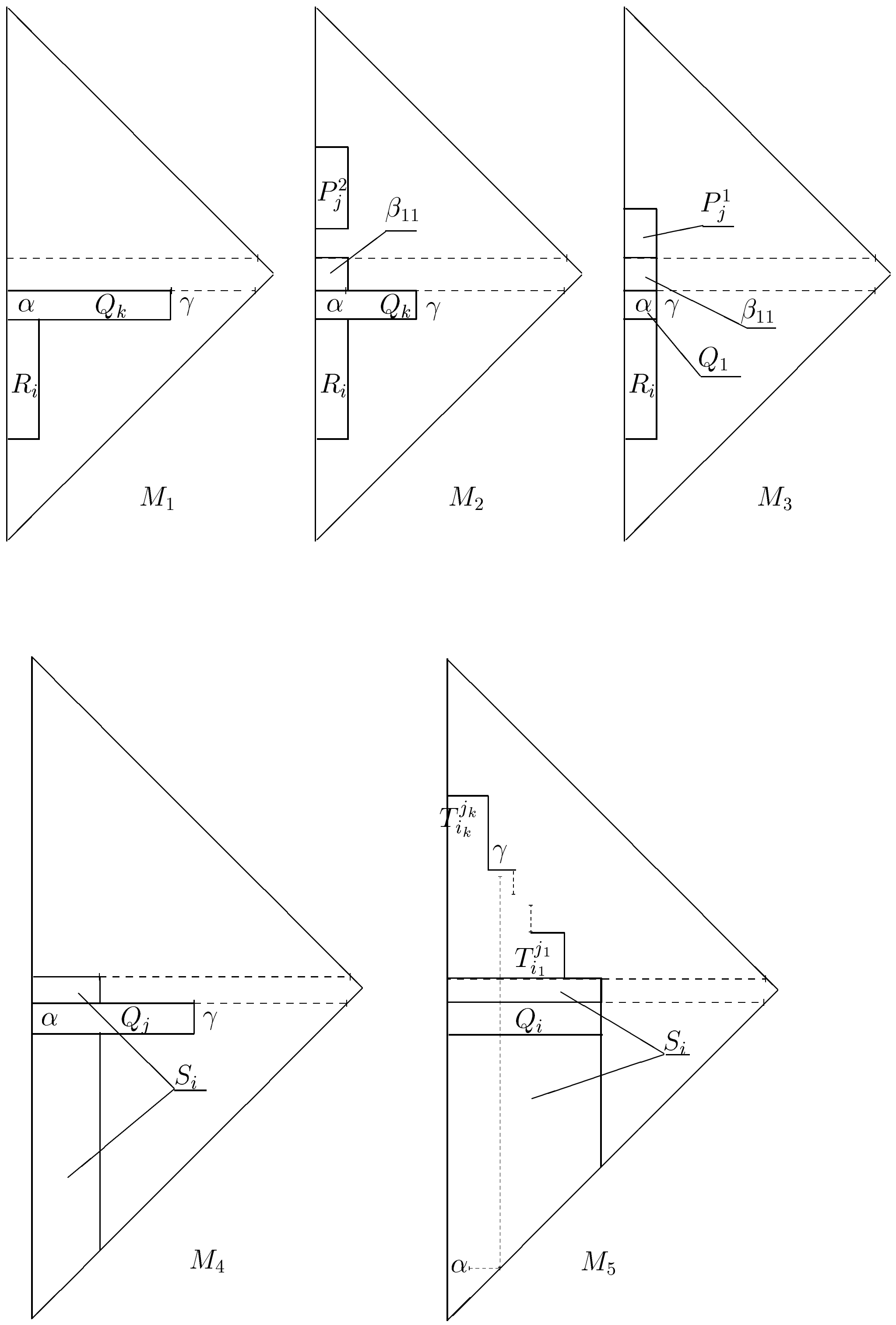
$$
\begin{aligned}
M_1 &= \{(\alpha_{1,n}, R_i \cup Q_k) |\quad 0 \leq i \leq n-2, 1 \leq k < n\},\\
M_2 &= \{(\alpha_{1,n}, R_i \cup P_j^2 \cup Q_k \cup \{\beta_{1,1}\}) |\quad 1 \leq j-1 \leq i < n-2, 1 \leq k < n\},\\
M_3 &= \{(\alpha_{1,n}, R_i \cup P_j^1 \cup Q_1 \cup \{\beta_{1,1}\}) |\quad 0 \leq j-1 \leq i < n-2\},\\
M_4 &= \{(\alpha_{1,n}, S_i \cup Q_j) |\quad 0 < i \leq j < n\},\\
M_5 &= \{(\beta_{1,i_k+1}, S_i \cup Q_i \cup T_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}) |\\
&\qquad\qquad i \geq i_1 > \ldots > i_k, j_1 > \ldots > j_{k-1} \geq j_k, k \geq 1\},\\
M_6 &= \{(\alpha_{1,j_k-1}, S_i \cup Q_i \cup T_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}) |\\
&\qquad\qquad 1 \leq i_1 < \ldots < i_k, i \leq j_1 < \ldots < j_k \leq n, k \geq 2\},\\
M_7 &= \{(\alpha_{1,n}, S_i \cup U_j \cup Q_k \cup T_i^j) |\quad 1 \leq i < j \leq k < n\},\\
M_8 &= \{(\alpha_{i_k+1,n-1}, S_i \cup U_j \cup Q_j \cup T_{i,i_1,\ldots,i_k}^{j,j_1,\ldots,j_k}) |\\
&\qquad\qquad j > i \geq i_1 > \ldots > i_k, j_1 > \ldots > j_k, k \geq 1\},\\
M_9 &= \{(\alpha_{i_k+1,n-1}, S_i \cup Q_n \cup T_{i_1,\ldots,i_k}^{j_1,\ldots,j_k}) |\\
&\qquad\qquad i \geq i_1 > \ldots > i_k, j > j_1 > \ldots > j_k, k \geq 2\}.
\end{aligned}
$$

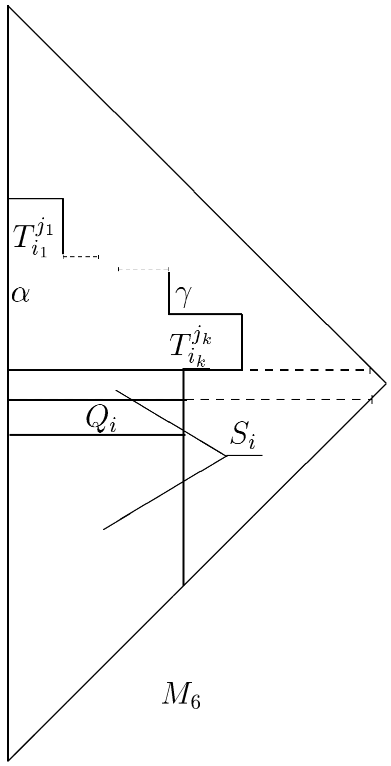
Define the function d as follows:

$$
\begin{aligned}
d(F) &= (0, i, 0, 0, j, 1, 0, \ldots), \quad F \in M_1,\\
d(F) &= (0, i, j, 1, k, 1, 0, \ldots), \quad F \in M_2,\\
d(F) &= (0, i, j, 2, 1, 1, 0, \ldots), \quad F \in M_3,\\
d(F) &= (i, i, n, 1, j, 0, \ldots), \quad F \in M_4,\\
d(F) &= (i, i, i_1, \overline{j}_1, \ldots, i_{k-1}, \overline{j}_{k-1}, i_k + 1, \overline{n}, i_k, \overline{j}_k, 0, \ldots), \quad F \in M_5,\\
d(F) &= (i, i, n, 2, \overline{j}_1, i_1, \ldots, \overline{j}_{k-1}, i_{k-1}, \overline{j}_k + 1, 1, \overline{j}_k, i_k, 0, \ldots), \quad F \in M_6,\\
d(F) &= (i, j, k, 0, \ldots), \quad F \in M_7,\\
d(F) &= (i, j, n, i_1, \overline{j}_1, \ldots, i_{k-1}, \overline{j}_{k-1}, i_k + 1, \overline{n}, i_k, \overline{j}_k, 0, \ldots), \quad F \in M_8,\\
d(F) &= (i+1, i+1, n+1, i_1, \overline{j}_1, \ldots, i_{k-1}, \overline{j}_{k-1}, i_k + 1, \overline{n}, i_k, \overline{j}_k, 0, \ldots), \quad F \in M_9,
\end{aligned}
$$

where $\overline{j} = n + 1 - j$. Clearly, $(\alpha_{1,n}, Q_1)$ is the minimal element in M.

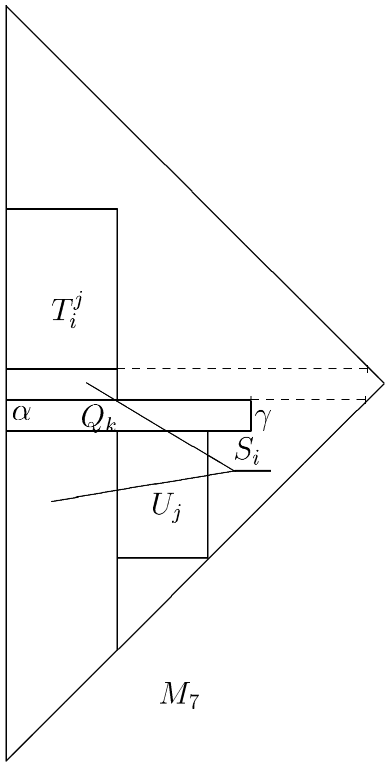
One can picture the sets P^i_j, Q_i, R_i, S_i, T^j_i, U_i, $M_1, \ldots, M_9$ on the following diagrams, each of which is a part of the $\phi(A)$ consisting of all positive roots.





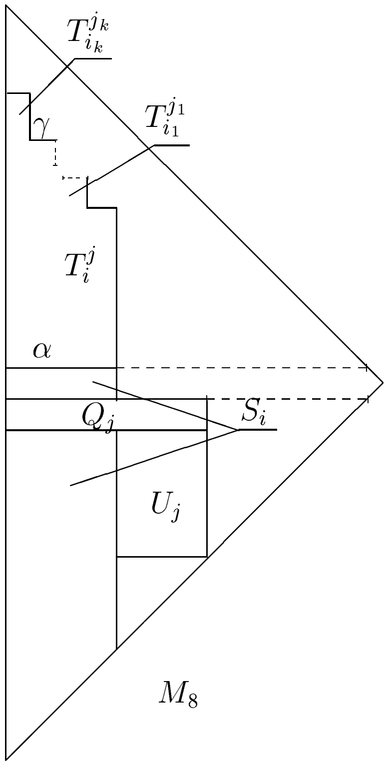
$T_{i_1}^{j_1}$
α
γ
$T_{i_k}^{j_k}$
Q_i
S_i

M_6



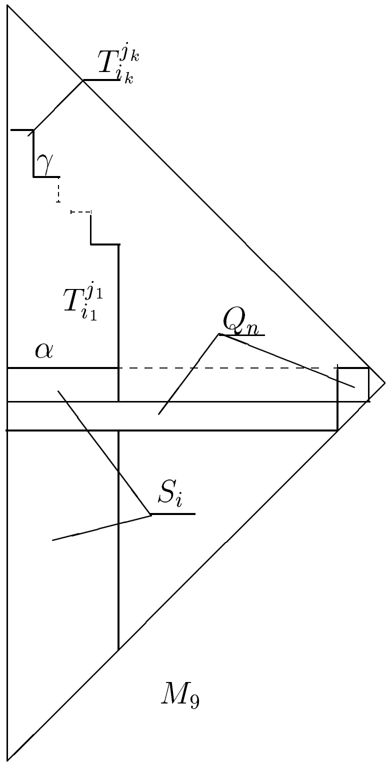
T_i^j
α
Q_k
γ
S_i
U_j

M_7



$T_{i_k}^{j_k}$
γ
$T_{i_1}^{j_1}$
T_i^j
α
Q_j
S_i
U_j

M_8



$T_{i_k}^{j_k}$
γ
$T_{i_1}^{j_1}$
Q_n
α
S_i

M_9

Note that $F=(\alpha,A)\in M$ is final if and only if $A\supseteq Q_i\cup S_j\cup T_k^k$, $0<k\le j\le i\le n$.

We will use the following formulas that can be easily checked using the plane realization of the Weyl group.

$$
\begin{aligned}
&(\alpha_{1,n},Q_n)\sim(\alpha_{1,n},S_1\cup Q_1)\sim(\alpha_{1,1},R_{n-3}\cup Q_1\cup\{\alpha_{1,1},\beta_{1,1}\}), && (1)\\
&(\alpha_{1,k},R_i\cup Q_1\cup\{\alpha_{1,k}\})\sim(\alpha_{1,n},R_{i+1}\cup Q_1),\\
&\qquad 0\le i<n-k-1\le n-2, && (2)\\
&(\alpha_{1,k},R_i\cup Q_1\cup\{\alpha_{1,k}\})\sim(\alpha_{1,n},R_i\cup Q_1\cup P_1^1\cup\{\beta_{1,1}\}),\\
&\qquad 0\le n-k-1\le i\le n-2, && (3)\\
&(\alpha_{1,k},R_i\cup Q_1\cup P_j^2\cup\{\alpha_{1,k}\})\sim(\alpha_{1,n},R_i\cup Q_1\cup P_{j-1}^1\cup\{\beta_{1,1}\}),\\
&\qquad 1\le n-i-1\le n-j\le k\le n-2, && (4)\\
&(\alpha_{1,k},R_i\cup Q_1\cup P_j^2\cup\{\alpha_{1,k}\})\sim(\alpha_{1,n},R_i\cup Q_1\cup P_j^1\cup\{\beta_{1,1}\}),\\
&\qquad 1\le n-i-1\le k<n-j\le n-2, && (5)\\
&(\alpha_{1,k},R_i\cup Q_1\cup P_j^2\cup\{\alpha_{1,k}\})\sim(\alpha_{1,n},R_{i+1}\cup Q_1\cup P_j^2\cup\{\beta_{1,1}\}),\\
&\qquad 1\le k<n-i-1\le n-j\le n-2, && (6)\\
&(\alpha_{1,n},S_1\cup Q_1)\sqsubseteq(\alpha_{1,n},\mathrm{add}(R_{n-3}\cup P_j^1\cup Q_1\cup\{\beta_{1,1},\alpha_{2,n}\})),\\
&\qquad 1\le j, && (7)\\
&(\alpha_{1,n},S_i\cup Q_n)\sim(\beta_{1,2},S_{i+1}\cup Q_{i+1}),\\
&\qquad 2\le i<n, && (8)\\
&(\alpha_{1,j},S_i\cup Q_i\cup\{\alpha_{1,j}\})\sim(\alpha_{1,n-1},S_i\cup Q_i\cup T_{1,n-1}^{n-1,n}),\\
&\qquad 1\le i\le j<n, && (9)\\
&(\beta_{1,j},S_i\cup Q_i)\sim(\alpha_{1,n},S_i\cup Q_i),\\
&\qquad 1\le i<j<n, && (10)\\
&(\beta_{1,i_k+1},S_i\cup Q_i\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k})\sim(\beta_{1,m},S_i\cup Q_i\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k}),\\
&\qquad i_k<m\le i_{k-1}, && (11)\\
&(\alpha_{1,j_k-1},S_{i_0}\cup Q_{i_0}\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k})\sim(\alpha_{1,m},S_{i_0}\cup Q_{i_0}\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k}),\\
&\qquad j_{k-1}\le m<j_k, && (12)\\
&(\alpha_{1,n-1},S_i\cup Q_i\cup T_j^j)\sim(\alpha_{1,n},S_i\cup Q_j\cup U_j\cup T_i^j),\\
&\qquad 1\le i<j<n, && (13)\\
&(\alpha_{1,j},S_i\cup Q_i\cup\{\alpha_{1,j}\})\sim(\alpha_{1,n-1},S_i\cup Q_n\cup T_{1,0}^{n-1,0}),\\
&\qquad 1\le j\le i\le n, && (14)\\
&(\alpha_{1,k},S_i\cup U_j\cup Q_j\cup T_i^j)\sim(\alpha_{1,n-1},S_i\cup U_j\cup Q_j\cup T_i^j),\\
&\qquad 1\le i<j\le k<n, && (15)\\
&(\alpha_{1,n-1},S_{i+1}\cup Q_{i+1}\cup T_{1,n-1}^{n-1,n})\sqsubseteq(\alpha_{m,n-1},\mathrm{add}(S_i\cup U_j\cup Q_j\cup\{\alpha_{m,n-1}\})),\\
&\qquad 1\le i<m\le j<n, && (16)\\
&(\alpha_{i_k+1,n-1},S_i\cup U_j\cup Q_p\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k})\sim(\beta_{m,n-1},S_i\cup U_j\cup Q_p\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k}),
\end{aligned}
$$

$$i_k < m \le i_{k-1},\ i \le j \le p, \tag{17}$$

$$(\alpha_{1,n-1}, S_{i+1} \cup Q_n \cup T_{1,0}^{n-1,0}) \sim (\alpha_{m,n-1}, \mathrm{add}(S_i \cup Q_n \cup \{\alpha_{m,n-1}\})),$$
$$1 \le i < m < n. \tag{18}$$

We will treat the sets M_i, $i = 1, \dots, 9$ separately.

1. Let $F = (\alpha, A) = (\alpha_{1,n}, R_i \cup Q_k) \in M_1$.

Consider a "good" root $\gamma = \alpha_{k+1,n} \in \Delta^+ \setminus A$. Then $D^1_\gamma(F) = (\alpha, A \cup \{\gamma\})$, $D^2_\gamma(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{1,n} - \alpha_{k+1,n} = \alpha_{1,k}$.

If $k+1 < n$ then we set $F_1 = D^1_\gamma(F) = (\alpha_{1,n}, R_i \cup Q_{k+1}) \in M_1$ otherwise we set $F_1 = (\alpha_{1,n}, S_1 \cup Q_1) \in M_4$. It follows from (1) that $F_1 \sqsubseteq D^1_\gamma(F)$ and $d(F_1) > d(F)$.

If $k + i < n - 1$ then we set $F_2 = (\alpha_{1,n}, R_{i+1} \cup Q_1) \in M_1$ otherwise we set $F_2 = (\alpha_{1,n}, R_i \cup P^1_1 \cup Q_1 \cup \{\beta_{1,1}\}) \in M_3$. Again by the formulas (2), (3) we have $F_2 \sqsubseteq D^2_\gamma(F)$ and $d(F_2) > d(F)$.

2. Let $F = (\alpha, A) = (\alpha_{1,n}, R_i \cup P^2_j \cup Q_k \cup \{\beta_{1,1}\}) \in M_2$.

Consider a "good" root $\gamma = \alpha_{k+1,n} \in \Delta^+ \setminus A$. Then $D^1_\gamma(F) = (\alpha, A \cup \{\gamma\})$, $D^2_\gamma(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{1,k}$.

If $k < n-1$ then we set $F_1 = D^1_\gamma(F) = (\alpha_{1,n}, R_i \cup P^2_j \cup Q_{k+1} \cup \{\beta_{1,1}\}) \in M_2$ otherwise we set $F_1 = (\alpha_{1,n}, S_1 \cup Q_1) \in M_4$. It follows from (1) that $F_1 \sqsubseteq D^1_\gamma(F)$ and $d(F_1) > d(F)$.

If $0 < n - k < i$ then we set $F_2 = (\alpha_{1,n}, R_i \cup P^1_j \cup Q_1 \cup \{\beta_{1,1}\}) \in M_3$ otherwise we set $F_2 = (\alpha_{1,n}, R_{i+1} \cup P^2_j \cup Q_1 \cup \{\beta_{1,1}\}) \in M_2$. It follows from (4), (5), (6) that $F_2 \sqsubseteq D^2_\gamma(F)$ and $d(F_2) > d(F)$.

3. Let $F = (\alpha, A) = (\alpha_{1,n}, R_i \cup P^1_j \cup Q_1 \cup \{\beta_{1,1}\}) \in M_3$.

Consider a non "good" root $\gamma = \alpha_{2,n} \in \Delta^+ \setminus A$. Then $D^1_\gamma(F) = (\alpha, A \cup \{\gamma\})$ and since $\alpha_{1,n} + \gamma = \beta_{1,2} \notin A$ we obtain that $D^2_\gamma(F) = (\alpha'', (A \setminus \{\alpha_{1,n-1}\}) \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{1,1}$.

If $i = n-3$ then we set $F_1 = (\alpha_{1,n}, S_1 \cup Q_1) \in M_4$ otherwise we set $F_1 = (\alpha_{1,n}, R_{i+1} \cup P^1_j \cup Q_1 \cup \{\beta_{1,1}\}) \in M_3$. It follows from (7) that $F_1 \sqsubseteq D^1_\gamma(F)$ and $d(F_1) > d(F)$.

If $i = n-3$ then we set $F_2 = (\alpha_{1,n}, S_1 \cup Q_1) \in M_4$ otherwise we set $F_2 = (\alpha_{1,n}, R_{i+1} \cup P^2_j \cup Q_1 \cup \{\beta_{1,1}\}) \in M_2$. It follows from (1), (7) that $F_2 \sqsubseteq D^2_\gamma(F)$ and $d(F_2) > d(F)$.

4. Let $F = (\alpha, A) = (\alpha_{1,n}, S_i \cup Q_j) \in M_4$.

Consider a "good" root $\gamma = \alpha_{j+1,n} \in \Delta^+ \setminus A$. Then $D^1_\gamma(F) = (\alpha, A \cup \{\gamma\})$ and we obtain that $D^2_\gamma(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{1,j}$.

If $j < n-1$ then we set $F_1 = (\alpha_{1,n}, S_i \cup Q_{j+1}) \in M_4$ otherwise we set $F_1 = (\beta_{1,2}, S_{i+1} \cup Q_{i+1}) \in M_5$. It follows from (8) that $F_1 \sqsubseteq D^1_\gamma(F)$ and $d(F_1) > d(F)$.

We set $F_2 = (\alpha_{1,n-1}, S_i \cup Q_i \cup T^{n-1,n}_{1,n-1}) \in M_6$. It follows from (9) that $F_2 \sqsubseteq D^2_\gamma(F)$ and $d(F_2) > d(F)$.

5. Let $F = (\alpha, A) = (\beta_{1,i_k+1}, S_{i_0} \cup Q_{i_0} \cup T^{j_1,\dots,j_k}_{i_1,\dots,i_k}) \in M_5$.

Consider a "good" root $\gamma = \alpha_{i_k+1,j_{k-1}-1} \in \Delta^+ \setminus A$. Then $D^1_\gamma(F) = (\alpha, A \cup \{\gamma\})$ and $D^2_\gamma(F) = (\alpha'', A \cup \{\alpha''\})$ where

$$\alpha'' = \alpha - \gamma = \begin{cases} \alpha_{1,n}, & j_{k-1} = n, \\ \beta_{1,j_{k-1}}, & j_{k-1} < n. \end{cases}$$

If $i_{k-1} = j_k = i_k + 1 = j_{k-1} - 1$ then $F_1 = (\beta_{1,2}, S_{i_0} \cup Q_{i_0} \cup T_{j_k}^{j_k}) \sqsubseteq D_\gamma^1(F)$ is a final element. If the above equality for indexes does not hold then we define the element $F_1 \in M_5$ as follows:

$$F_1 = \begin{cases} (\beta_{1,2}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-1},1}^{j_1,\dots,j_{k-1},j_k-1}), & \text{if } j_k = j_{k-1},\ i_k = 1, \\ (\beta_{1,2}, S_{i_0} \cup Q_{i_0} \cup P_{i_1,\dots,i_{k-2},i_{k-1,1}}^{j_1,\dots,j_{k-2},j_k,j_k}), & \text{if } i_k + 1 = i_{k-1},\ j_k = j_{k-1} - 1; \\ (\beta_{1,i_k+1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-1},i_k}^{j_1,\dots,j_{k-1}-1,j_k}), & \text{if } i_k + 1 = i_{k-1},\ j_k < j_{k-1} - 1; \\ (\beta_{1,2}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-1},i_k+1,1}^{j_1,\dots,j_{k-1},j_k,j_k}), & \text{if } i_k + 1 < i_{k-1},\ j_k = j_{k-1} - 1; \\ (\beta_{1,i_k+1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-1},i_k+1,i_k}^{j_1,\dots,j_{k-1},j_{k-1}-1,j_k}), & \text{otherwise.} \end{cases}$$

It follows from (11) that $F_1 \sqsubseteq D_\gamma^1(F)$ and $d(F_1) > d(F)$.

If $j_{k-1} > i_0$ then we set $F_2 = (\alpha_{1,n}, S_{i_0} \cup Q_{i_0}) \in M_4$. If $j_{k-1} \le i_0$ then there exists $l < k$ such that $i_l < j_{k-1} \le i_{l-1}$. We set $F_2 = (\beta_{1,i_l+1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_l}^{j_1,\dots,j_l}) \in M_5$. It follows from (10),(11) that $F_2 \sqsubseteq D_\gamma^2(F)$ and $d(F_2) > d(F)$.

6. Let $F = (\alpha, A) = (\alpha_{1,j_k-1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k}) \in M_6$.

Consider a "good" root $\gamma = \alpha_{i_{k-1}+1,j_k-1} \in \Delta^+ \setminus A$. Then $D_\gamma^1(F) = (\alpha, A \cup \{\gamma\})$ and $D_\gamma^2(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{1,i_{k-1}}$.

If $i_k = i_{k-1} + 1 = j_{k-1} = j_k - 1$ then we set $F_1 = (\alpha_{1,n-1}, S_{i_0} \cup Q_{i_0} \cup T_{i_k}^{i_k})$. In this case if $i_k \le i_0$ then F_1 is final, otherwise by (13) we obtain that $F_1 \sim (\alpha_{1,n}, S_{i_0} \cup U_{i_k} \cup Q_{i_k} \cup T_{i_0}^{i_k}) \in M_7$.

If the above equality for indexes does not hold then we define the element F_1 as follows:

$$F_1 = \begin{cases} (\alpha_{1,n-1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-2},i_k,n}^{j_1,\dots,j_{k-2},j_{k-1},n-1}), & \text{if } i_k = i_{k-1} + 1,\ j_{k-1} = j_k - 1; \\ (\alpha_{1,n-1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-1},i_k,n-1}^{j_1,\dots,j_{k-1},j_k-1,n}), & \text{if } i_k = i_{k-1} + 1,\ j_{k-1} = j_k - 1; \\ (\alpha_{1,j_k-1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-1}+1,i_k}^{j_1,\dots,j_{k-1},j_k}), & \text{if } i_k < i_{k-1} + 1,\ j_{k-1} = j_k - 1; \\ (\beta_{1,j_k-1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_{k-1},i_{k-1}+1,i_k}^{j_1,\dots,j_{k-1},j_k-1,j_k}), & \text{otherwise.} \end{cases}$$

It follows from (12) that $F_1 \sqsubseteq D_\gamma^1(F)$ and $d(F_1) > d(F)$.

If $i_{k-1} < i_0$ then we set $F_2 = (\alpha_{1,n-1}, S_i \cup Q_n \cup T_1^{n-1}) \in M_9$.

If $i_{k-1} \ge i_0$ then there exists $l < k$ such that $j_{l-1} \le i_{k-1} < j_l$, where $j_0 = i_0$. In this case we set $F_2 = (\alpha_{1,j_l-1}, S_{i_0} \cup Q_{i_0} \cup T_{i_1,\dots,i_l}^{j_1,\dots,j_l}) \in M_6$. It follows from (12), (14) that $F_2 \sqsubseteq D_\gamma^2(F)$ and $d(F_2) > d(F)$.

7. Let $F = (\alpha, A) = (\alpha_{1,n}, S_i \cup U_j \cup Q_k \cup T_i^j) \in M_7$.

Consider a "good" root $\gamma = \alpha_{k+1,n} \in \Delta^+ \setminus A$. Then $D_\gamma^1(F) = (\alpha, A \cup \{\gamma\})$ and $D_\gamma^2(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{1,k}$.

If $k = n - 1$ then we set $F_1 = (\beta_{1,2}, S_{i+1} \cup Q_{i+1} \cup T_1^n) \in M_5$ otherwise we set $F_1 = (\alpha_{1,n}, S_i \cup U_j \cup Q_{k+1} \cup T_i^j) \in M_7$. It follows from (8) that $F_1 \sqsubseteq D_\gamma^1(F)$ and $d(F_1) > d(F)$.

We set $F_2 = (\alpha_{1,n-1}, S_i \cup U_j \cup Q_j \cup T_{i,0}^{j,0}) \in M_8$ It follows from (15) that $F_2 \sqsubseteq D_\gamma^2(F)$ and $d(F_2) > d(F)$.

8. Let $F = (\alpha, A) = (\alpha_{i_k+1,n-1}, S_{i_0} \cup U_{j_0} \cup Q_{j_0} \cup T_{i_0,\dots,i_k}^{j_0,\dots,j_k}) \in M_8$.

Consider a "good" root $\gamma = \alpha_{i_k+1,j_{k-1}-1} \in \Delta^+ \setminus A$. Then $D_\gamma^1(F) = (\alpha, A \cup \{\gamma\})$ and $D_\gamma^2(F) = (\alpha'', A \cup \{\alpha''\})$ where $\alpha'' = \alpha - \gamma = \alpha_{j_{k-1},n-1}$

If $i_{k-1}=j_k=i_k+1=j_{k-1}-1$ then $F_1=(\alpha_{1,n-1}, S_{i_0}\cup U_{j_0}\cup Q_{j_0}\cup T^{j_k}_{j_k}\sqsubseteq D^1_\gamma(F)$ is a final element. If the above equality for indexes does not hold then we define the element $F_1\in M_8$ as follows:

$$F_1=\begin{cases}(\alpha_{1,n-1}, S_{i_0}\cup U_{j_0}\cup Q_{j_0}\cup T^{j_0,\dots,j_{k-2},j_k,0}_{i_0,\dots,i_{k-2},i_{k-1},0}), & \text{if } i_k+1=i_{k-1},\quad j_k=j_{k-1}-1;\\ (\alpha_{i_k+1,n-1}, S_{i_0}\cup U_{j_0}\cup Q_{j_0}\cup T^{j_0,\dots,j_{k-1}-1,j_k}_{i_0,\dots,i_{k-1},i_k}), & \text{if } i_k+1=i_{k-1},\quad j_k<j_{k-1}-1;\\ (\alpha_{1,n-1}, S_{i_0}\cup U_{j_0}\cup Q_{j_0}\cup T^{j_0,\dots,j_{k-1},j_k,0}_{i_0,\dots,i_{k-1},i_k+1,0}), & \text{if } i_k+1<i_{k-1},\quad j_k=j_{k-1}-1;\\ (\alpha_{i_k+1,n-1}, S_{i_0}\cup U_{j_0}\cup Q_{j_0}\cup T^{j_0,\dots,j_{k-1},j_{k-1}-1,j_k}_{i_0,\dots,i_{k-1},i_k+1,i_k}), & \text{otherwise.}\end{cases}$$

It follows that $F_1\sqsubseteq D^1_\gamma(F)$ and $d(F_1)>d(F)$.

If $j_{k-1}>i_0$ then we set $F_2=(\alpha_{1,n-1}, S_{i_0+1}\cup Q_{i_0+1}\cup T^{n-1,n}_{1,n-1})\in M_6$. If $j_{k-1}\le i_0$ then there exists $l<k$ such that $i_l<j_{k-1}\le i_{l-1}$. In this case we set $F_2=(\alpha_{i_l+1,n-1}, S_{i_0}\cup U_{j_0}\cup Q_{j_0}\cup T^{j_1,\dots,j_l}_{i_1,\dots,i_l})\in M_8$. It follows from (16),(17) that $F_2\sqsubseteq D^2_\gamma(F)$ and $d(F_2)>d(F)$.

9. Let $F=(\alpha,A)=(\alpha_{i_k+1,n-1}, S_{i_0}\cup Q_n\cup T^{j_1,\dots,j_k}_{i_1,\dots,i_k})\in M_9$.

Consider a "good" root $\gamma=\alpha_{i_k+1,j_{k-1}-1}\in\Delta^+\setminus A$. Then $D^1_\gamma(F)=(\alpha, A\cup\{\gamma\})$ and $D^2_\gamma(F)=(\alpha'', A\cup\{\alpha''\})$ where $\alpha''=\alpha-\gamma=\alpha_{j_{k-1},n-1}$

If $i_{k-1}=j_k=i_k+1=j_{k-1}-1$ then $F_1=(\alpha_{1,n-1}, S_{i_0}\cup Q_n\cup T^{j_k}_{j_k}\sqsubseteq D^1_\gamma(F)$ is a final element. If the above equality for indexes does not hold then we define the element $F_1\in M_9$ as follows:

$$F_1=\begin{cases}(\alpha_{1,n-1}, S_{i_0}\cup Q_n\cup T^{j_1,\dots,j_{k-2},j_k,0}_{i_1,\dots,i_{k-2},i_{k-1},0}), & \text{if } i_k+1=i_{k-1},\ j_k=j_{k-1}-1;\\ (\alpha_{i_k+1,n-1}, S_{i_0}\cup Q_n\cup T^{j_1,\dots,j_{k-1}-1,j_k}_{i_1,\dots,i_{k-1},i_k}), & \text{if } i_k+1=i_{k-1},\ j_k<j_{k-1}-1;\\ (\alpha_{1,n-1}, S_{i_0}\cup Q_n\cup T^{j_1,\dots,j_{k-1},j_k,0}_{i_1,\dots,i_{k-1},i_k+1,0}), & \text{if } i_k+1<i_{k-1},\ j_k=j_{k-1}-1;\\ (\alpha_{i_k+1,n-1}, S_{i_0}\cup Q_n\cup T^{j_1,\dots,j_{k-1},j_{k-1}-1,j_k}_{i_1,\dots,i_{k-1},i_k+1,i_k}), & \text{otherwise.}\end{cases}$$

It follows that $F_1\sqsubseteq D^1_\gamma(F)$ and $d(F_1)>d(F)$.

If $j_{k-1}>i_0$ then we set $F_2=(\alpha_{1,n-1}, S_{i_0+1}\cup Q_n\cup T^{n-1,0}_{1,0})\in M_9$. If $j_{k-1}\le i_0$ then there exists $l<k$ such that $i_l<j_{k-1}\le i_{l-1}$. In this case we set $F_2=(\alpha_{i_l+1,n-1}, S_{i_0}\cup Q_n\cup T^{j_1,\dots,j_l}_{i_1,\dots,i_l})\in M_9$. It follows from (17),(18) that $F_2\sqsubseteq D^2_\gamma(F)$ and $d(F_2)>d(F)$.

5.4 Case D_n

Denote

$$\alpha_{ij}=\alpha_i+\alpha_{i+1}+\ldots+\alpha_j,\quad 1\le i\le j\le n-1,$$

$$\alpha_{in}=\alpha_i+\alpha_{i+1}+\ldots+\alpha_{n-2}+\alpha_n, i=1,\ldots,n-2,n,$$

$$\Delta_1=\{\pm a_{ij}|\ 1\le i\le j\le n\},\quad \Delta_1^+=\Delta_1\cap\Delta^+,$$

$$\beta_{ij}=\alpha_{i,n-1}+\alpha_{j,n},\quad 1\le i<j\le n, i\le n-2,$$

$$\Delta_2=\{\pm\beta_{ij}|\ 1\le i<j\le n\},\quad \Delta_2^+=\Delta_2\cap\Delta^+,\quad \Delta=\Delta_1\cup\Delta_2.$$

Define the plane realization of W as follows:

$$A=(a_{ij})_{i,j=1,\dots,2n-1},\ A^*=\{a_{ij}|\ 2\le i\le 2n-1, 2n-i<j<2n, i\ne j\},$$

$$A^0=\{a_{ii}\}_{i=1,\dots,2n-1},\quad A^1=\{a_{ij}\}\setminus(A^0\cup A^*),$$

$$\phi(a_{ij})=\begin{cases}\alpha_{j,i-1}, & 1\le j<i\le n,\\ \alpha_{j,n}, & i=n+1,\ j\le n-2,\\ \alpha_{n,n}, & i=n+1,\ j=n-1,\\ \beta_{j,n}, & i=n+2,\ j\le n-2,\\ \beta_{j,2n+1-i}, & n+3\le i\le 2n-1,\ 1\le j\le 2n-i,\end{cases}$$

$$\phi(a_{ij})=-\phi(a_{ji}),\quad \text{if } i<j \text{ and } a_{ji}\in A^1.$$

The Weyl group W is generated by permutations $s_1=\sigma_{1,2}$; $s_i=\sigma_{i,i+1}\sigma_{2n-i,2n+1-i}$, $i=2,\dots,n-1$; $s_n=\sigma_{n-1,n+1}\sigma_{n,n+2}$.

Let

$$\begin{aligned}
P_i^j&=\{\alpha_{1,n-j},\dots,\alpha_{1,n-i}\},\quad j=1,2,\ i=j,\dots,n-1,\\
Q_i^k&=\{\alpha_{1,k},\dots,\alpha_{i,k}\},\quad i=1,\dots,n-2,\ Q_{n-1}^k=Q_{n-2}^k\cup\{\alpha_{k,k}\},\ k=n-1,n\\
R_i&=\{\beta_{1,n},\beta_{1,n-2},\dots,\beta_{1,n-i}\},\quad i=2,\dots,n-2,\quad R_1=\{\beta_{1,n}\},\ R_0=\emptyset,\\
S_i&=\{\beta\in\Delta_2^+|\ \beta\ge\beta_{i,n}\},\quad i=1,\dots,n-2,\\
T_i^j&=\{\alpha\in\Delta_1^+|\ \alpha_{1,n-2}\ge\alpha\ge\alpha_{ij}\},\quad i=1,\dots,n-2,j=i,\dots,n-1,\quad T_0^0=T_i^n=\emptyset\\
U_i&=\{\beta\in\Delta_2^+|\ \beta_{1,i}>\beta\ge\beta_{i,n}\},\quad i=2,\dots,n-2.
\end{aligned}$$

$$T_{i_1,\dots,i_k}^{j_1,\dots,j_k}=T_{i_1}^{j_1}\cup\ldots\cup T_{i_k}^{j_k}.$$

Define the set $M=M_1\cup\ldots\cup M_9$:

$$\begin{aligned}
M_1&=\{(\alpha_{1,k},R_i\cup Q_j^k)|\quad 0\le i\le n-2,1\le j<n-2,k=n-1,n\},\\
M_2&=\{(\alpha_{1,n},R_i\cup P_j^2\cup Q_k^n\})|\quad 1\le j-1\le i<n-2,1\le k\le n-2\},\\
M_3&=\{(\alpha_{1,n},R_i\cup P_j^1\cup Q_1^n\})|\quad 1\le j-1\le i\le n-2\},\\
M_4&=\{(\alpha_{1,k},S_i\cup Q_j^k)|\quad 0\le i\le j\le n-2,\ k=n-1,n\},\\
M_5&=\{(\beta_{1,i_k+1},S_i\cup Q_i^{m_1}\cup Q_j^{m_2}\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k})|\quad i\ge j\ge i_1>\ldots>i_k,\\
&\qquad j_1>\ldots>j_{k-1}\ge j_k,k\ge 2,\{m_1,m_2\}=\{n-1,n\}\},\\
M_6&=\{(\alpha_{1,j_k-1},S_i\cup Q_i^{m_1}\cup Q_j^{m_2}\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k})|\quad 1\le i_1<\ldots<i_k,\\
&\qquad i\le j_1<\ldots<j_k\le j\le n,\{m_1,m_2\}=\{n-1,n\},k\ge 2\},\\
M_7&=\{(\alpha_{1,m_1},S_i\cup U_j\cup Q_j^{m_1}\cup Q_i^{m_2}\cup T_i^j)|\quad 1\le i<j\le n-2,\{m_1,m_2\}=\{n-1,n\}\},\\
M_8&=\{(\alpha_{i_k+1,m_2},S_i\cup U_j\cup Q_j^{m_1}\cup Q_i^{m_2}\cup T_{i,i_1,\dots,i_k}^{j,j_1,\dots,j_k})|\\
&\qquad j>i\ge i_1>\ldots>i_k,j_1>\ldots>j_k,k\ge 2,\{m_1,m_2\}=\{n-1,n\}\},\\
M_9&=\{(\alpha_{i_k+1,m_2},S_i\cup Q_{n-1}^{m_1}\cup Q_i^{m_2}\cup T_{i_1,\dots,i_k}^{j_1,\dots,j_k})|\\
&\qquad i\ge i_1>\ldots>i_k,j_1>\ldots>j_k,k\ge 2,\{m_1,m_2\}=\{n-1,n\}\},
\end{aligned}$$

The rest of the proof in this case is absolutely analogous to the proof in the case C_n.

5.5 Case G_2

Let α_1 and α_2 be a simple long and short roots respectively. Denote $\alpha_{ij} = i \cdot \alpha_1 + j \cdot \alpha_2$. Then

$$\Delta^+ = \{\alpha_{1,0}, \alpha_{0,1}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}\}.$$

The element $F = (\alpha, A) \in \Omega$ is final if and only if there exists $w \in W$ such that $wA \supseteq \mathbf{F}_k$, for some $k = 1, 2$, where $\mathbf{F}_1 = \{\alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}\}$, $\mathbf{F}_2 = \{\alpha_{0,1}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3}\}$. In this case we write $F \succ \mathbf{F}_k$. We will use the notation $(\{\overline{\alpha}_{i_1,j_1}, \ldots, \alpha_{i_k,j_k}\})$ instead of $(\alpha_{i_1,j_1}, \{\alpha_{i_1,j_1}, \ldots, \alpha_{i_k,j_k}\})$.

Define a set $M = \{F^1, \ldots, F^8\}$ in the table below and the function $d : M \to \mathcal{N}$, $d(F^i) = i$. For every non final element $F^i \in M$ we define γ^i and two elements $F_1^i, F_2^i \in M$ such that $d(F_k^i) > d(F^i)$ and $F_k^i \sim D_{\gamma^i}^k(F^i)$, $k = 1, 2$. Note that the elements F^6, F^7, F^8 are final.

i	F^i	γ	F_1^i F_2^i	$D_\gamma^1(F^i)$ $D_\gamma^2(F^i)$
1	$(\overline{\alpha}_{1,2})$	$\alpha_{1,3}$	F^2 F^7	$(\overline{\alpha}_{1,2}, \alpha_{1,3})$ $(\overline{\alpha}_{0,-1}, \alpha_{1,2})$
2	$(\overline{\alpha}_{1,2}, \alpha_{1,3})$	$\alpha_{2,3}$	F^3 F^7	$(\overline{\alpha}_{1,2}, \alpha_{1,3}, \alpha_{2,3})$ $(\overline{\alpha}_{-1,-1}, \alpha_{1,2})$
3	$(\overline{\alpha}_{1,2}, \alpha_{1,3}, \alpha_{2,3})$	$\alpha_{1,1}$	F^5 F^4	$(\overline{\alpha}_{1,2}, \alpha_{1,3}, \alpha_{2,3}, \alpha_{1,1})$ $(\overline{\alpha}_{0,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3})$
4	$(\overline{\alpha}_{1,2}, \alpha_{1,1}, \alpha_{1,3}, \alpha_{2,3})$	$\alpha_{0,1}$	F^6 F^5	$(\overline{\alpha}_{1,2}, \alpha_{1,3}, \alpha_{1,1}, \alpha_{0,1}, \alpha_{2,3})$ $(\overline{\alpha}_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3})$
5	$(\overline{\alpha}_{1,2}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{2,3})$	$\alpha_{1,3}$	F^8 F^7	$(\overline{\alpha}_{1,2}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{1,3}, \alpha_{2,3})$ $(\overline{\alpha}_{0,-1}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,3})$
6	$(\overline{\alpha}_{1,2}, \alpha_{0,1}, \alpha_{1,1}, \alpha_{1,3}, \alpha_{2,3})$			$\succ \mathbf{F}_2$
7	$(\overline{\alpha}_{1,1}, \alpha_{0,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,3})$			$\succ \mathbf{F}_2$
8	$(\overline{\alpha}_{1,2}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{1,3}, \alpha_{2,3})$			$\succ \mathbf{F}_1$

5.6 Case F_4

Let $\mathbf{e}_1$, $\mathbf{e}_2$ (respectively $\mathbf{e}_3$, $\mathbf{e}_4$) simple short (resp. long) roots. Set $\gamma_{ijkl} = i \cdot \mathbf{e}_1 + j \cdot \mathbf{e}_2 + k \cdot \mathbf{e}_3 + l \cdot \mathbf{e}_4$, $\Delta^+ = \Delta_1^+ \cup \Delta_2^+$, where Δ_1^+ (resp. Δ_2^+) is a set of all short (resp. long) positive roots with respect to basis $\pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then

$$\begin{aligned} \Delta_1^+ = \{ \; & \gamma_{1234}, \gamma_{1231}, \gamma_{1221}, \gamma_{1121}, \gamma_{1111}, \gamma_{1110}, \\ & \gamma_{0121}, \gamma_{0111}, \gamma_{0110}, \gamma_{0011}, \gamma_{0010}, \gamma_{0001}\}; \end{aligned} \tag{19}$$

$$\begin{aligned} \Delta_2^+ = \{ \; & \gamma_{2342}, \gamma_{1342}, \gamma_{1242}, \gamma_{1222}, \gamma_{1220}, \gamma_{1122}, \\ & \gamma_{1120}, \gamma_{1100}, \gamma_{1000}, \gamma_{0122}, \gamma_{0120}, \gamma_{0100}\}. \end{aligned} \tag{20}$$

Let α_i (resp. β_i) denotes the root number i's element in the list (19) (resp. (20)).

Let $A_i = \{\alpha \in \Delta_1^+ | \ \alpha \geq \alpha_i\}$ and $B_i = \{\beta \in \Delta_2^+ | \ \beta \geq \beta_i\}$. An element $F = (\alpha, A) \in \Omega$ is final if and only if there exists $w \in W$ such that $wA \supseteq \mathbf{F}_k$ for some $k = 1, \ldots, 4$, where $\mathbf{F}_1 = \{A_6 \cup B_9\}$, $\mathbf{F}_2 = \{A_9 \cup B_{12}\}$, $\mathbf{F}_3 = \{A_{11} \cup B_{11}\}$, $\mathbf{F}_4 = \{A_{12} \cup B_{10}\}$. In this case we write $F \succ \mathbf{F}_k$.

Define a set $M = M_1 \cup M_2$, where $M_1 = \{F^1, \ldots, F^{84}\}$ is listed in the table below and $M_2 = \{(\alpha, \mathbf{F}_i) | \ i = 1, \ldots, 4, \alpha \in \mathbf{F}_i\} = \{F^{85}, \ldots, F^k\}$ with an arbitrary fixed order on M_2. The function $d : M \to \mathcal{N}$ is defined as follows: $d(F^i) = i$, $F_i \in M$. Clearly M_2 consists of only final elements. For every element $F^i \in M_1$ we define γ^i and two elements $F_1^i, F_2^i \in M$ such that $d(F_k^i) > d(F^i)$ and $F_k^i \sqsubseteq D_{\gamma^i}^k(F^i)$, $k = 1, 2$.

i	F^i	γ^i	F_1^i	F_2^i
1.	$(\alpha_1, \{\alpha_1\})$	β_1	F_2	F_2
2.	$(\alpha_1, \{\alpha_1, \beta_1\})$	α_2	F_4	F_3
3.	$(\alpha_1, \{\mathbf{A}_2, \beta_5\})$	β_1	F_4	F_6
4.	$(\alpha_1, \{\mathbf{A}_2, \beta_1\})$	β_2	F_6	F_5
5.	$(\alpha_4, \{\alpha_1, \alpha_3, \alpha_4, \alpha_8, \mathbf{B}_2\})$	β_7	F_6	F_6
6.	$(\alpha_1, \{\mathbf{A}_2, \mathbf{B}_2\})$	β_3	F_7	F_{14}
7.	$(\alpha_1, \{\mathbf{A}_2, \mathbf{B}_3\})$	β_4	F_8	F_{11}
8.	$(\alpha_1, \{\mathbf{A}_2, \mathbf{B}_4\})$	β_6	F_9	F_{11}
9.	$(\alpha_1, \{\mathbf{A}_2, \mathbf{B}_6\})$	β_{10}	F_{10}	F_{11}
10.	$(\alpha_1, \{\mathbf{A}_2, \mathbf{B}_{10}\})$	α_3	F_{12}	F_{17}
11.	$(\alpha_4, \{\alpha_1, \alpha_3, \alpha_4, \mathbf{B}_2, \beta_4\})$	β_3	F_{16}	F_{14}
12.	$(\alpha_1, \{\mathbf{A}_3, \mathbf{B}_{10}\})$	α_4	F_{13}	F_{17}
13.	$(\alpha_1, \{\mathbf{A}_4, \mathbf{B}_{10}\})$	α_7	F_{15}	F_{17}
14.	$(\alpha_4, \{\alpha_1, \alpha_3, \alpha_4, \alpha_8, \mathbf{B}_2, \beta_4\})$	β_3	F_{16}	F_{16}
15.	$(\alpha_1, \{\mathbf{A}_7, \mathbf{B}_{10}\})$	α_5	F_{29}	F_{40}
16.	$(\alpha_4, \{\alpha_1, \alpha_3, \alpha_4, \mathbf{B}_4\})$	β_6	F_{17}	F_{39}
17.	$(\alpha_4, \{\alpha_1, \alpha_3, \alpha_4, \mathbf{B}_6\})$	β_7	F_{18}	F_{38}
18.	$(\alpha_3, \{\alpha_1, \alpha_3, \alpha_4, \mathbf{B}_6, \beta_7\})$	β_8	F_{19}	F_{38}
19.	$(\alpha_3, \{\alpha_1, \alpha_3, \alpha_4, \mathbf{B}_5, \mathbf{B}_6, \beta_8\})$	β_{12}	F_{20}	F_{28}
20.	$(\alpha_1, \{\mathbf{A}_2, \alpha_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	α_3	F_{21}	F_{23}
21.	$(\alpha_1, \{\mathbf{A}_3, \alpha_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	α_4	F_{22}	F_{23}
22.	$(\alpha_1, \{\mathbf{A}_4, \alpha_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	α_5	F_{25}	F_{24}
23.	$(\alpha_4, \{\mathbf{A}_4, \alpha_9, \mathbf{B}_5, \mathbf{B}_{10}, \beta_{12}\})$	α_6	F_{26}	F_{27}
24.	$(\alpha_7, \{\mathbf{A}_7, \alpha_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	β_{11}	F_{41}	F_{41}
25.	$(\alpha_1, \{\mathbf{A}_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	α_7	F_{29}	F_{28}
26.	$(\alpha_4, \{\mathbf{A}_4, \alpha_6, \alpha_9, \mathbf{B}_5, \mathbf{B}_{10}, \beta_{12}\})$	β_7	F_{30}	F_{42}
27.	$(\alpha_8, \{\mathbf{A}_8, \mathbf{A}_6, \mathbf{B}_{10}, \beta_7, \beta_9\})$	α_{10}	F_{66}	$\succ \mathbf{F}_1$
28.	$(\alpha_4, \{\alpha_1, \alpha_3, \alpha_4, \mathbf{B}_5, \mathbf{B}_6, \beta_8\})$	β_7	F_{31}	F_{38}
29.	$(\alpha_1, \{\mathbf{A}_7, \mathbf{A}_5, \mathbf{B}_{10}\})$	α_8	F_{32}	F_{40}
30.	$(\alpha_4, \{\mathbf{A}_4, \alpha_6, \alpha_9, \mathbf{B}_7, \mathbf{B}_{10}, \beta_{12}\})$	α_5	F_{33}	F_{41}
31.	$(\alpha_1, \{\mathbf{A}_2, \alpha_6, \mathbf{B}_5, \beta_7, \beta_8, \beta_9\})$	β_6	F_{34}	F_{38}
32.	$(\alpha_1, \{\mathbf{A}_8, \mathbf{B}_{10}\})$	α_{10}	F_{35}	F_{44}
33.	$(\alpha_4, \{\mathbf{A}_6, \alpha_9, \mathbf{B}_7, \mathbf{B}_{10}, \beta_{12}\})$	α_7	F_{42}	$\succ \mathbf{F}_1$
34.	$(\alpha_1, \{\mathbf{A}_2, \alpha_6, \mathbf{B}_9\})$	α_3	F_{36}	F_{38}
35.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{B}_{10}\})$	α_{12}	$\succ \mathbf{F}_4$	F_{42}
36.	$(\alpha_1, \{\mathbf{A}_3, \alpha_6, \mathbf{B}_9\})$	α_4	F_{37}	F_{38}
37.	$(\alpha_1, \{\mathbf{A}_4, \alpha_6, \mathbf{B}_9\})$	α_5	$\succ \mathbf{F}_1$	F_{38}
38.	$(\alpha_4, \{\alpha_1, \alpha_3, \alpha_4, \alpha_7, \mathbf{B}_4, \beta_{10}\})$	β_6	F_{40}	F_{39}
39.	$(\alpha_7, \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \mathbf{B}_6, \beta_8\})$	β_{10}	F_{40}	F_{76}
40.	$(\alpha_2, \{\mathbf{A}_2, \alpha_5, \alpha_8, \mathbf{B}_{10}\})$	α_3	F_{41}	F_{49}
41.	$(\alpha_2, \{\mathbf{A}_3, \alpha_5, \alpha_8, \mathbf{B}_{10}\})$	α_4	F_{42}	F_{43}
42.	$(\alpha_2, \{\mathbf{A}_5, \alpha_8, \mathbf{B}_{10}\})$	α_7	F_{44}	F_{43}

i	F^i	γ^i	F_1^i	F_2^i
43.	$(\alpha_6, \{\mathbf{A}_3, \alpha_5, \alpha_6, \alpha_8, \mathbf{B}_{10}\})$	β_5	F_{45}	F_{49}
44.	$(\alpha_2, \{\mathbf{A}_8, \mathbf{B}_{10}\})$	β_5	F_{46}	F_{66}
45.	$(\alpha_6, \{\mathbf{A}_7, \alpha_6, \mathbf{B}_5, \mathbf{B}_{10}\})$	β_7	F_{47}	F_{49}
46.	$(\alpha_2, \{\mathbf{A}_8, \mathbf{B}_5, \mathbf{B}_{10}\})$	β_7	F_{48}	F_{53}
47.	$(\alpha_6, \{\mathbf{A}_7, \alpha_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	α_5	F_{62}	F_{50}
48.	$(\alpha_2, \{\mathbf{A}_8, \mathbf{B}_7, \mathbf{B}_{10}\})$	β_{11}	F_{51}	F_{53}
49.	$(\alpha_6, \{\mathbf{A}_4, \alpha_6, \alpha_8, \alpha_{10}, \mathbf{B}_{10}\})$	α_5	F_{68}	$\succ \mathbf{F}_4$
50.	$(\alpha_{12}, \{\mathbf{A}_7, \mathbf{A}_5, \alpha_{12}, \mathbf{B}_{11}\})$	α_8	F_{66}	$\succ \mathbf{F}_2$
51.	$(\alpha_2, \{\mathbf{A}_8, \mathbf{B}_{11}\})$	α_6	F_{52}	F_{53}
52.	$(\alpha_2, \{\mathbf{A}_8, \mathbf{A}_6, \mathbf{B}_{11}\})$	α_9	F_{54}	F_{60}
53.	$(\alpha_8, \{\mathbf{A}_8, \mathbf{B}_5, \mathbf{B}_{10}\})$	α_9	F_{55}	F_{66}
54.	$(\alpha_2, \{\mathbf{A}_9, \mathbf{B}_{11}\})$	α_{11}	F_{56}	F_{58}
55.	$(\alpha_8, \{\mathbf{A}_8, \alpha_9, \mathbf{B}_5, \mathbf{B}_{10}\})$	β_{12}	F_{60}	F_{66}
56.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_{11}\})$	α_{12}	$\succ \mathbf{F}_4$	F_{57}
57.	$(\alpha_2, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_{11}\})$	α_{11}	$\succ \mathbf{F}_3$	F_{58}
58.	$(\alpha_3, \{\mathbf{A}_9, \mathbf{B}_{11}\})$	β_8	F_{59}	F_{60}
59.	$(\alpha_3, \{\mathbf{A}_9, \mathbf{B}_8, \mathbf{B}_{11}\})$	β_{12}	$\succ \mathbf{F}_2$	F_{60}
60.	$(\alpha_8, \{\mathbf{A}_8, \alpha_9, \mathbf{B}_5, \mathbf{B}_{10}, \beta_{12}\})$	α_{12}	F_{61}	F_{62}
61.	$(\alpha_4, \{\mathbf{A}_{10}, \mathbf{A}_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	β_9	F_{64}	F_{63}
62.	$(\alpha_6, \{\mathbf{A}_7, \mathbf{A}_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	β_8	F_{73}	F_{74}
63.	$(\alpha_7, \{\mathbf{A}_{10}, \mathbf{A}_6, \mathbf{B}_7, \mathbf{B}_{10}\})$	α_9	F_{67}	F_{66}
64.	$(\alpha_4, \{\mathbf{A}_{10}, \mathbf{A}_6, \mathbf{B}_7, \mathbf{B}_{10}, \beta_9\})$	α_{11}	F_{65}	F_{66}
65.	$(\alpha_4, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_8, \mathbf{B}_{10}\})$	β_9	$\succ \mathbf{F}_1$	F_{67}
66.	$(\alpha_{10}, \{\mathbf{A}_{10}, \mathbf{B}_{10}\})$	α_{12}	$\succ \mathbf{F}_4$	F_{68}
67.	$(\alpha_7, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_7, \mathbf{B}_{10}\})$	β_{11}	F_{69}	$\succ \mathbf{F}_4$
68.	$(\alpha_6, \{\mathbf{A}_6, \alpha_8, \alpha_{10}, \mathbf{B}_{10}\})$	β_5	F_{70}	F_{71}
69.	$(\alpha_7, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_{11}\})$	α_{11}	$\succ \mathbf{F}_3$	F_{77}
70.	$(\alpha_6, \{\mathbf{A}_6, \alpha_8, \alpha_{10}, \mathbf{B}_5, \mathbf{B}_{10}\})$	β_7	F_{72}	F_{71}
71.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_5, \mathbf{B}_{10}\})$	β_{11}	F_{72}	$\succ \mathbf{F}_4$
72.	$(\alpha_6, \{\mathbf{A}_6, \alpha_8, \alpha_{10}, \mathbf{B}_7, \mathbf{B}_{10}\})$	β_8	F_{73}	F_{74}
73.	$(\alpha_6, \{\mathbf{A}_7, \mathbf{A}_6, \mathbf{B}_8, \mathbf{B}_{10}\})$	β_9	$\succ \mathbf{F}_1$	F_{74}
74.	$(\alpha_9, \{\mathbf{A}_7, \mathbf{A}_6, \alpha_9, \mathbf{B}_7, \mathbf{B}_{10}\})$	β_{11}	F_{75}	F_{76}
75.	$(\alpha_8, \{\mathbf{A}_8, \mathbf{A}_6, \mathbf{B}_{11}\})$	β_{12}	F_{78}	F_{77}
76.	$(\alpha_{10}, \{\mathbf{A}_6, \alpha_8, \alpha_{10}, \mathbf{B}_8, \mathbf{B}_{10}\})$	α_7	F_{83}	$\succ \mathbf{F}_1$
77.	$(\alpha_{10}, \{\mathbf{A}_{10}, \mathbf{A}_6, \mathbf{B}_{11}\})$	α_{12}	$\succ \mathbf{F}_4$	F_{79}
78.	$(\alpha_8, \{\mathbf{A}_8, \mathbf{A}_6, \mathbf{B}_{11}, \beta_{12}\})$	α_9	F_{81}	F_{80}
79.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_{11}\})$	β_{12}	F_{81}	$\succ \mathbf{F}_3$
80.	$(\alpha_{12}, \{\mathbf{A}_8, \mathbf{A}_6, \alpha_{12}, \mathbf{B}_{11}, \beta_{12}\})$	α_{10}	$\succ \mathbf{F}_4$	$\succ \mathbf{F}_2$
81.	$(\alpha_5, \{\mathbf{A}_9, \mathbf{B}_8, \mathbf{B}_{11}\})$	β_9	$\succ \mathbf{F}_1$	F_{82}
82.	$(\alpha_8, \{\mathbf{A}_9, \mathbf{B}_8, \mathbf{B}_{11}\})$	β_{12}	$\succ \mathbf{F}_2$	F_{83}
83.	$(\alpha_{10}, \{\mathbf{A}_{10}, \mathbf{A}_6, \mathbf{B}_8, \mathbf{B}_{10}\})$	α_{12}	$\succ \mathbf{F}_4$	F_{84}
84.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_9, \mathbf{B}_7, \mathbf{B}_{10}, \beta_9\})$	β_{12}	$\succ \mathbf{F}_1$	$\succ \mathbf{F}_3$

5.7 Case E_6

Let $\pi = \{\mathbf{e}_1, \dots, \mathbf{e}_6\}$ be a basis. Denote $\gamma_{ijklmn} = i\cdot\mathbf{e}_1 + j\cdot\mathbf{e}_2 + k\cdot\mathbf{e}_3 + l\cdot\mathbf{e}_4 + m\cdot\mathbf{e}_5 + n\cdot\mathbf{e}_6$. Then

$$
\begin{aligned}
\Delta^+ = \{\ & \gamma_{123212}, \gamma_{123211}, \gamma_{122211}, \gamma_{122111}, \gamma_{122101}, \gamma_{112211}, \gamma_{112111}, \gamma_{112101}, \gamma_{111111}, \\
& \gamma_{111110}, \gamma_{111101}, \gamma_{111100}, \gamma_{111001}, \gamma_{111000}, \gamma_{110000}, \gamma_{100000}, \gamma_{012211}, \gamma_{012111}, \\
& \gamma_{012101}, \gamma_{011111}, \gamma_{011110}, \gamma_{011101}, \gamma_{011100}, \gamma_{011001}, \gamma_{011000}, \gamma_{010000}, \gamma_{001111}, \\
& \gamma_{001110}, \gamma_{001101}, \gamma_{001100}, \gamma_{001001}, \gamma_{001000}, \gamma_{000110}, \gamma_{000100}, \gamma_{000010}, \gamma_{000001}\}.
\end{aligned} \tag{21}
$$

is the set of all positive roots with respect to π. Let α_i denotes the i's element in the list (21).

Set $A_i = \{\alpha \in \Delta^+ |\ \alpha \geq \alpha_i\}$. An element $F = (\alpha, A) \in \Omega$ is final if and only if there exists $w \in W$ such that $wA \supseteq \mathbf{F}_k$ for some $k = 1, \dots, 6$, where $\mathbf{F}_1 = A_{16}$, $\mathbf{F}_2 = A_{26}$, $\mathbf{F}_3 = A_{32}$, $\mathbf{F}_4 = A_{34}$, $\mathbf{F}_5 = A_{35}$, $\mathbf{F}_6 = A_{36}$. In this case we write $F \succ \mathbf{F}_k$.

Define a set $M = M_1 \cup M_2$, where $M_1 = \{F^1, \dots, F^{326}\}$ is listed in the table below and $M_2 = \{(\alpha, \mathbf{F}_i)|\ i = 1, \dots, 6, \alpha \in \mathbf{F}_i\} = \{F^{327}, \dots, F^k\}$ with an arbitrary fixed order on M_2. The function $d : M \to \mathcal{N}$ is defined as follows: $d(F^i) = i$, $F_i \in M$. Clearly M_2 consists of only final elements. For every element $F^i \in M_1$ we define γ^i and two elements $F^i_1, F^i_2 \in M$ such that $d(F^i_k) > d(F^i)$ and $F^i_k \sqsubseteq D^k_{\gamma^i}(F^i)$, $k = 1, 2$.

i	F^i	γ^i	F_1^i	F_2^i
1.	$(\alpha_1, \{\mathbf{A}_1\})$	α_2	F_2	F_2
2.	$(\alpha_2, \{\mathbf{A}_2\})$	α_3	F_3	F_6
3.	$(\alpha_3, \{\mathbf{A}_3\})$	α_4	F_4	F_6
4.	$(\alpha_4, \{\mathbf{A}_4\})$	α_5	F_5	F_6
5.	$(\alpha_5, \{\mathbf{A}_5\})$	α_8	F_7	F_{15}
6.	$(\alpha_2, \{\mathbf{A}_2, \alpha_9\})$	α_3	F_9	F_9
7.	$(\alpha_3, \{\mathbf{A}_5, \mathbf{A}_6\})$	α_{11}	F_8	F_{15}
8.	$(\alpha_2, \{\mathbf{A}_5, \mathbf{A}_7\})$	α_{10}	F_{10}	F_{15}
9.	$(\alpha_3, \{\mathbf{A}_3, \alpha_7\})$	α_4	F_{15}	F_{11}
10.	$(\alpha_1, \{\mathbf{A}_5, \mathbf{A}_9\})$	α_{13}	F_{12}	F_{25}
11.	$(\alpha_{18}, \{\mathbf{A}_6, \alpha_{11}, \alpha_{18}\})$	α_{17}	F_{15}	F_{24}
12.	$(\alpha_1, \{\mathbf{A}_8, \mathbf{A}_9\})$	α_{13}	F_{13}	F_{25}
13.	$(\alpha_1, \{\mathbf{A}_{11}\})$	α_{13}	F_{14}	F_{25}
14.	$(\alpha_1, \{\mathbf{A}_{13}\})$	α_{19}	F_{16}	F_{30}
15.	$(\alpha_4, \{\mathbf{A}_4, \mathbf{A}_6\})$	α_5	F_{25}	F_{17}
16.	$(\alpha_1, \{\mathbf{A}_{13}, \mathbf{A}_{17}\})$	α_{22}	F_{18}	F_{30}
17.	$(\alpha_{27}, \{\mathbf{A}_4, \alpha_{10}, \alpha_{27}\})$	α_9	F_{19}	F_{25}
18.	$(\alpha_1, \{\mathbf{A}_{13}, \mathbf{A}_{18}\})$	α_{20}	F_{20}	F_{30}
19.	$(\alpha_{18}, \{\mathbf{A}_6, \alpha_9, \alpha_{10}, \alpha_{18}\})$	α_7	F_{21}	F_{25}
20.	$(\alpha_1, \{\mathbf{A}_{13}, \mathbf{A}_{20}\})$	α_{24}	F_{22}	F_{30}
21.	$(\alpha_{17}, \{\mathbf{A}_4, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{17}\})$	α_6	F_{25}	F_{44}
22.	$(\alpha_1, \{\mathbf{A}_{13}, \mathbf{A}_{19}, \mathbf{A}_{20}\})$	α_{24}	F_{23}	F_{30}
23.	$(\alpha_1, \{\mathbf{A}_{13}, \mathbf{A}_{22}\})$	α_{24}	F_{24}	F_{30}
24.	$(\alpha_1, \{\mathbf{A}_{24}\})$	α_{29}	F_{26}	F_{30}
25.	$(\alpha_5, \{\mathbf{A}_5, \mathbf{A}_6\})$	α_8	F_{27}	F_{57}
26.	$(\alpha_1, \{\mathbf{A}_{24}, \mathbf{A}_{27}\})$	α_{31}	F_{28}	F_{30}
27.	$(\alpha_4, \{\mathbf{A}_5, \mathbf{A}_7\})$	α_9	F_{30}	F_{57}
28.	$(\alpha_1, \{\mathbf{A}_{24}, \mathbf{A}_{29}\})$	α_{36}	F_{29}	F_{30}
29.	$(\alpha_1, \{\mathbf{A}_{31}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{30}
30.	$(\alpha_4, \{\mathbf{A}_5, \mathbf{A}_9\})$	α_{10}	F_{31}	F_{57}
31.	$(\alpha_4, \{\mathbf{A}_5, \mathbf{A}_{10}\})$	α_{14}	F_{32}	F_{55}
32.	$(\alpha_1, \{\mathbf{A}_8, \mathbf{A}_{10}\})$	α_{13}	F_{33}	F_{55}
33.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{11}\})$	α_{13}	F_{34}	F_{55}
34.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}\})$	α_{20}	F_{35}	F_{47}
35.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{17}\})$	α_{27}	F_{36}	F_{47}
36.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{18}\})$	α_{27}	F_{37}	F_{47}
37.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{20}\})$	α_{27}	F_{38}	F_{47}
38.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{27}\})$	α_{36}	F_{39}	F_{44}
39.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{19}, \mathbf{A}_{27}\})$	α_{31}	F_{40}	F_{44}

i	F^i	γ^i	F_1^i	F_2^i
40.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{27}\})$	α_{29}	F_{41}	F_{44}
41.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{29}\})$	α_{31}	F_{42}	F_{44}
42.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{24}, \mathbf{A}_{29}\})$	α_{36}	F_{43}	F_{44}
43.	$(\alpha_1, \{\mathbf{A}_{10}, \mathbf{A}_{31}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{44}
44.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{27}\})$	α_{15}	F_{45}	F_{47}
45.	$(\alpha_6, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{27}\})$	α_{16}	F_{46}	F_{47}
46.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{27}\})$	α_{28}	F_{48}	F_{50}
47.	$(\alpha_{13}, \{\mathbf{A}_{10}, \mathbf{A}_{13}\})$	α_{14}	F_{49}	F_{57}
48.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{21}, \mathbf{A}_{27}\})$	α_{28}	F_{142}	F_{50}
49.	$(\alpha_{11}, \{\mathbf{A}_{12}, \mathbf{A}_{13}\})$	α_{15}	F_{50}	F_{55}
50.	$(\alpha_8, \{\mathbf{A}_{14}\})$	α_{16}	F_{51}	F_{55}
51.	$(\alpha_5, \{\mathbf{A}_{15}\})$	α_{23}	F_{52}	F_{54}
52.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{17}\})$	α_{19}	F_{53}	F_{54}
53.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{18}\})$	α_{20}	F_{74}	F_{54}
54.	$(\alpha_{15}, \{\mathbf{A}_{15}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{55}
55.	$(\alpha_{17}, \{\mathbf{A}_5, \mathbf{A}_{10}, \mathbf{A}_{17}\})$	α_{18}	F_{56}	F_{57}
56.	$(\alpha_{18}, \{\mathbf{A}_5, \mathbf{A}_{10}, \mathbf{A}_{18}\})$	α_{19}	F_{58}	F_{61}
57.	$(\alpha_6, \{\mathbf{A}_5, \mathbf{A}_6, \alpha_{24}\})$	α_7	F_{59}	F_{63}
58.	$(\alpha_{18}, \{\mathbf{A}_5, \mathbf{A}_{10}, \mathbf{A}_{18}, \alpha_{19}\})$	α_{20}	F_{60}	F_{75}
59.	$(\alpha_7, \{\mathbf{A}_5, \mathbf{A}_7, \alpha_{22}\})$	α_8	F_{61}	F_{63}
60.	$(\alpha_7, \{\mathbf{A}_8, \mathbf{A}_{20}, \alpha_{21}\})$	α_{10}	F_{62}	F_{75}
61.	$(\alpha_8, \{\mathbf{A}_8, \alpha_{20}\})$	α_{11}	F_{64}	F_{75}
62.	$(\alpha_7, \{\mathbf{A}_8, \mathbf{A}_{21}\})$	α_{14}	F_{65}	F_{67}
63.	$(\alpha_{27}, \{\mathbf{A}_5, \mathbf{A}_6, \alpha_{12}, \alpha_{18}, \alpha_{27}\})$	α_{17}	F_{75}	F_{75}
64.	$(\alpha_8, \{\mathbf{A}_8, \alpha_{11}, \alpha_{20}\})$	α_{13}	F_{67}	F_{75}
65.	$(\alpha_6, \{\mathbf{A}_{11}, \mathbf{A}_{21}\})$	α_{12}	F_{66}	F_{67}
66.	$(\alpha_6, \{\mathbf{A}_{12}, \mathbf{A}_{21}\})$	α_{28}	F_{68}	F_{96}
67.	$(\alpha_8, \{\mathbf{A}_8, \alpha_{11}, \alpha_{13}, \alpha_{20}\})$	α_{19}	F_{69}	F_{70}
68.	$(\alpha_6, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{27}\})$	α_{33}	F_{71}	F_{96}
69.	$(\alpha_6, \{\mathbf{A}_{11}, \mathbf{A}_{17}, \alpha_{24}\})$	α_{12}	F_{101}	F_{105}
70.	$(\alpha_{10}, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \alpha_{19}\})$	α_{12}	F_{72}	F_{75}
71.	$(\alpha_6, \{\mathbf{A}_{12}, \mathbf{A}_{28}\})$	α_{33}	F_{74}	F_{96}
72.	$(\alpha_{12}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \alpha_{18}\})$	α_{14}	F_{73}	F_{75}
73.	$(\alpha_{14}, \{\mathbf{A}_{14}, \mathbf{A}_{17}\})$	α_{15}	F_{163}	F_{166}
74.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{20}\})$	α_{24}	F_{76}	F_{84}
75.	$(\alpha_{27}, \{\mathbf{A}_5, \mathbf{A}_7, \mathbf{A}_{17}, \alpha_{23}, \alpha_{27}\})$	α_{18}	F_{77}	F_{91}
76.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{19}, \mathbf{A}_{20}\})$	α_{27}	F_{78}	F_{157}
77.	$(\alpha_{20}, \{\mathbf{A}_{18}, \alpha_8, \alpha_{12}, \alpha_{20}\})$	α_9	F_{86}	F_{79}
78.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{19}, \mathbf{A}_{27}\})$	α_{24}	F_{80}	F_{84}

i	F^i	γ^i	F_1^i	F_2^i
79.	$(\alpha_{26}, \{\mathbf{A}_8, \mathbf{A}_{18}, \alpha_{21}, \alpha_{22}, \alpha_{26}, \alpha_{29}\})$	α_{25}	F_{81}	F_{86}
80.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{22}, \mathbf{A}_{27}\})$	α_{29}	F_{82}	F_{158}
81.	$(\alpha_{15}, \{\mathbf{A}_{19}, \alpha_{10}, \alpha_{11}, \alpha_{13}, \alpha_{15}, \alpha_{28}\})$	α_9	F_{96}	F_{166}
82.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{29}\})$	α_{24}	F_{83}	F_{84}
83.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{24}, \mathbf{A}_{29}\})$	α_{36}	F_{85}	F_{158}
84.	$(\alpha_7, \{\mathbf{A}_{15}, \mathbf{A}_{20}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{86}
85.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{31}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{158}
86.	$(\alpha_{20}, \{\mathbf{A}_{20}, \alpha_8, \alpha_{12}\})$	α_{21}	F_{96}	F_{87}
87.	$(\alpha_{24}, \{\mathbf{A}_8, \mathbf{A}_9, \mathbf{A}_{17}, \alpha_{12}, \alpha_{20}, \alpha_{24}, \alpha_{27}\})$	α_{18}	F_{88}	F_{89}
88.	$(\alpha_{24}, \{\mathbf{A}_8, \mathbf{A}_{27}, \alpha_{12}, \alpha_{24}\})$	α_{19}	F_{90}	F_{91}
89.	$(\alpha_{28}, \{\mathbf{A}_{13}, \mathbf{A}_{27}, \alpha_{12}, \alpha_{22}, \alpha_{24}, \alpha_{28}\})$	α_{10}	F_{170}	F_{201}
90.	$(\alpha_{13}, \{\mathbf{A}_{19}, \mathbf{A}_{27}, \alpha_{13}, \alpha_{23}\})$	α_{11}	F_{106}	F_{91}
91.	$(\alpha_{13}, \{\mathbf{A}_5, \mathbf{A}_{18}, \alpha_{11}, \alpha_{13}, \alpha_{21}, \alpha_{23}, \alpha_{34}\})$	α_{14}	F_{95}	F_{92}
92.	$(\alpha_{32}, \{\mathbf{A}_{11}, \mathbf{A}_{18}, \alpha_{12}, \alpha_{23}, \alpha_{25}, \alpha_{32}\})$	α_{19}	F_{93}	F_{127}
93.	$(\alpha_{29}, \{\mathbf{A}_{10}, \mathbf{A}_{19}, \mathbf{A}_{20}, \alpha_{14}, \alpha_{22}, \alpha_{29}\})$	α_{11}	F_{94}	F_{95}
94.	$(\alpha_{27}, \{\mathbf{A}_{22}, \mathbf{A}_{27}, \alpha_{12}, \alpha_{14}\})$	α_{28}	F_{96}	F_{116}
95.	$(\alpha_{26}, \{\mathbf{A}_8, \mathbf{A}_{18}, \alpha_{11}, \alpha_{21}, \alpha_{22}, \alpha_{24}, \alpha_{26}, \alpha_{28}\})$	α_{20}	F_{96}	F_{96}
96.	$(\alpha_{21}, \{\mathbf{A}_{18}, \alpha_8, \alpha_{10}, \alpha_{11}, \alpha_{20}, \alpha_{21}\})$	α_{23}	F_{97}	F_{98}
97.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_{20}, \alpha_{11}, \alpha_{19}, \alpha_{23}\})$	α_{27}	F_{101}	F_{99}
98.	$(\alpha_{28}, \{\mathbf{A}_{21}, \alpha_{11}, \alpha_{13}, \alpha_{28}\})$	α_{27}	F_{151}	F_{100}
99.	$(\alpha_{13}, \{\mathbf{A}_8, \mathbf{A}_{10}, \mathbf{A}_{18}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{23}\})$	α_{11}	F_{102}	F_{103}
100.	$(\alpha_{19}, \{\mathbf{A}_{10}, \mathbf{A}_{19}, \mathbf{A}_{27}, \alpha_{15}, \alpha_{16}, \alpha_{33}\})$	α_{22}	F_{202}	F_{190}
101.	$(\alpha_6, \{\mathbf{A}_{10}, \mathbf{A}_{11}, \mathbf{A}_{17}, \alpha_{24}\})$	α_{12}	F_{119}	F_{105}
102.	$(\alpha_{13}, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{18}, \alpha_{21}, \alpha_{22}, \alpha_{23}\})$	α_{14}	F_{129}	F_{106}
103.	$(\alpha_{32}, \{\mathbf{A}_{11}, \mathbf{A}_{20}, \alpha_{14}, \alpha_{15}, \alpha_{21}, \alpha_{24}, \alpha_{25}, \alpha_{32}\})$	α_{39}	F_{104}	F_{116}
104.	$(\alpha_{35}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{19}, \alpha_{21}, \alpha_{24}, \alpha_{25}, \alpha_{28}, \alpha_{35}\})$	α_{20}	F_{201}	F_{204}
105.	$(\alpha_{18}, \{\mathbf{A}_{11}, \mathbf{A}_{18}, \alpha_{36}\})$	α_{20}	F_{106}	F_{107}
106.	$(\alpha_{20}, \{\mathbf{A}_{11}, \mathbf{A}_{20}, \alpha_{31}\})$	α_{24}	F_{108}	F_{126}
107.	$(\alpha_{25}, \{\mathbf{A}_{11}, \mathbf{A}_{18}, \alpha_{24}, \alpha_{25}, \alpha_{36}\})$	α_{14}	F_{127}	F_{109}
108.	$(\alpha_7, \{\mathbf{A}_8, \mathbf{A}_{20}, \alpha_{22}, \alpha_{24}, \alpha_{29}\})$	α_{10}	F_{110}	F_{115}
109.	$(\alpha_{16}, \{\mathbf{A}_{13}, \mathbf{A}_{22}, \alpha_{14}, \alpha_{16}\})$	α_{12}	F_{151}	F_{199}
110.	$(\alpha_{20}, \{\mathbf{A}_8, \mathbf{A}_{20}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{29}\})$	α_{24}	F_{111}	F_{123}
111.	$(\alpha_{20}, \{\mathbf{A}_8, \mathbf{A}_{20}, \alpha_{13}, \alpha_{21}, \alpha_{22}, \alpha_{24}, \alpha_{29}\})$	α_{26}	F_{119}	F_{112}
112.	$(\alpha_{27}, \{\mathbf{A}_8, \mathbf{A}_{10}, \mathbf{A}_{27}, \alpha_{13}, \alpha_{22}, \alpha_{24}, \alpha_{29}\})$	α_{33}	F_{113}	F_{116}
113.	$(\alpha_{20}, \{\mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_8, \alpha_{11}, \alpha_{13}, \alpha_{19}, \alpha_{22}, \alpha_{31}\})$	α_{24}	F_{114}	F_{123}
114.	$(\alpha_{20}, \{\mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_8, \alpha_{11}, \alpha_{13}, \alpha_{19}, \alpha_{22}, \alpha_{24}, \alpha_{29}\})$	α_{33}	F_{121}	F_{115}
115.	$(\alpha_{29}, \{\mathbf{A}_5, \mathbf{A}_{20}, \alpha_{11}, \alpha_{13}, \alpha_{19}, \alpha_{29}, \alpha_{31}\})$	α_8	F_{116}	F_{126}
116.	$(\alpha_{20}, \{\mathbf{A}_{13}, \mathbf{A}_{19}, \mathbf{A}_{20}, \alpha_{24}, \alpha_{29}\})$	α_{22}	F_{117}	F_{126}
117.	$(\alpha_{24}, \{\mathbf{A}_{24}, \mathbf{A}_{27}\})$	α_{25}	F_{118}	F_{120}

i	F^i	γ^i	F_1^i	F_2^i
118.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_{24}, \alpha_{29}\})$	α_{15}	F_{119}	F_{120}
119.	$(\alpha_6, \{\mathbf{A}_{12}, \mathbf{A}_{17}, \alpha_{24}\})$	α_{27}	F_{121}	F_{128}
120.	$(\alpha_{29}, \{\mathbf{A}_{24}, \mathbf{A}_{29}\})$	α_{30}	F_{122}	F_{124}
121.	$(\alpha_6, \{\mathbf{A}_{12}, \mathbf{A}_{17}, \alpha_{24}, \alpha_{27}\})$	α_{29}	F_{137}	F_{128}
122.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_{29}, \alpha_{24}, \alpha_{31}\})$	α_{13}	F_{124}	F_{123}
123.	$(\alpha_{28}, \{\mathbf{A}_8, \mathbf{A}_{10}, \mathbf{A}_{20}, \alpha_{22}, \alpha_{24}, \alpha_{28}, \alpha_{29}\})$	α_{27}	F_{128}	F_{192}
124.	$(\alpha_{31}, \{\mathbf{A}_{31}\})$	α_{32}	F_{125}	$\succ \mathbf{F}_6$
125.	$(\alpha_9, \{\mathbf{A}_{10}, \mathbf{A}_{31}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{126}
126.	$(\alpha_{21}, \{\mathbf{A}_{11}, \mathbf{A}_{27}, \alpha_{21}, \alpha_{31}\})$	α_{10}	F_{128}	F_{127}
127.	$(\alpha_{21}, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{18}, \alpha_{21}, \alpha_{22}, \alpha_{31}\})$	α_{20}	F_{128}	F_{192}
128.	$(\alpha_{12}, \{\mathbf{A}_{12}, \mathbf{A}_{17}, \alpha_{24}\})$	α_{14}	F_{129}	F_{168}
129.	$(\alpha_{12}, \{\mathbf{A}_{12}, \mathbf{A}_{17}, \alpha_{14}, \alpha_{24}\})$	α_{15}	F_{130}	F_{168}
130.	$(\alpha_{12}, \{\mathbf{A}_{12}, \mathbf{A}_{17}, \alpha_{14}, \alpha_{15}, \alpha_{24}\})$	α_{23}	F_{131}	$\succ \mathbf{F}_1$
131.	$(\alpha_7, \{\mathbf{A}_{14}, \mathbf{A}_{18}, \alpha_{26}\})$	α_{35}	F_{132}	F_{135}
132.	$(\alpha_6, \{\mathbf{A}_{12}, \mathbf{A}_{17}, \alpha_{14}, \alpha_{15}, \alpha_{21}, \alpha_{24}, \alpha_{27}\})$	α_{29}	F_{137}	F_{133}
133.	$(\alpha_{10}, \{\mathbf{A}_{12}, \mathbf{A}_{17}, \alpha_{14}, \alpha_{15}, \alpha_{21}, \alpha_{24}, \alpha_{27}\})$	α_{35}	F_{134}	F_{135}
134.	$(\alpha_7, \{\mathbf{A}_{14}, \mathbf{A}_{18}, \alpha_{26}, \alpha_{33}, \alpha_{35}\})$	α_{39}	F_{137}	F_{136}
135.	$(\alpha_8, \{\mathbf{A}_{14}, \mathbf{A}_{18}, \alpha_{26}\})$	α_{19}	F_{138}	$\succ \mathbf{F}_1$
136.	$(\alpha_7, \{\mathbf{A}_{14}, \mathbf{A}_{18}, \alpha_{26}, \alpha_{33}, \alpha_{35}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{139}
137.	$(\alpha_2, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \alpha_{36}\})$	α_{14}	F_{140}	F_{185}
138.	$(\alpha_8, \{\mathbf{A}_{14}, \mathbf{A}_{19}, \alpha_{26}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{185}
139.	$(\alpha_{17}, \{\mathbf{A}_{14}, \mathbf{A}_{18}, \alpha_{33}, \alpha_{35}\})$	α_{20}	F_{148}	F_{180}
140.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{19}, \alpha_{36}\})$	α_{32}	F_{141}	F_{155}
141.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{19}, \mathbf{A}_{21}, \alpha_{36}\})$	α_{28}	F_{142}	F_{155}
142.	$(\alpha_1, \{\mathbf{A}_{14}, \mathbf{A}_{28}\})$	α_{19}	F_{144}	F_{143}
143.	$(\alpha_{10}, \{\mathbf{A}_{14}, \mathbf{A}_{28}\})$	α_{15}	F_{145}	F_{151}
144.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{19}, \mathbf{A}_{28}\})$	α_{32}	F_{146}	F_{155}
145.	$(\alpha_{10}, \{\mathbf{A}_{15}, \mathbf{A}_{28}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{148}
146.	$(\alpha_1, \{\mathbf{A}_{14}, \mathbf{A}_{22}, \mathbf{A}_{28}\})$	α_{29}	F_{147}	F_{155}
147.	$(\alpha_1, \{\mathbf{A}_{14}, \mathbf{A}_{28}, \mathbf{A}_{29}\})$	α_{31}	F_{149}	F_{155}
148.	$(\alpha_{20}, \{\mathbf{A}_{10}, \mathbf{A}_{13}, \mathbf{A}_{20}, \alpha_{14}, \alpha_{15}, \alpha_{28}, \alpha_{33}\})$	α_{21}	F_{151}	F_{180}
149.	$(\alpha_1, \{\mathbf{A}_{14}, \mathbf{A}_{24}, \mathbf{A}_{28}, \mathbf{A}_{29}\})$	α_{31}	F_{150}	F_{155}
150.	$(\alpha_1, \{\mathbf{A}_{14}, \mathbf{A}_{28}, \mathbf{A}_{31}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{152}
151.	$(\alpha_{28}, \{\mathbf{A}_{28}, \alpha_{11}, \alpha_{13}\})$	α_{33}	F_{163}	F_{153}
152.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{28}, \mathbf{A}_{31}\})$	α_{32}	F_{154}	F_{155}
153.	$(\alpha_{23}, \{\mathbf{A}_8, \mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_{13}, \alpha_{23}, \alpha_{33}\})$	α_{22}	F_{168}	F_{166}
154.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{28}, \mathbf{A}_{31}\})$	α_{30}	F_{156}	F_{193}
155.	$(\alpha_8, \{\mathbf{A}_{14}, \mathbf{A}_{19}\})$	α_{16}	F_{157}	F_{185}
156.	$(\alpha_2, \{\mathbf{A}_{14}, \mathbf{A}_{30}, \mathbf{A}_{31}\})$	α_{32}	F_{184}	F_{193}

i	F^i	γ^i	F_1^i	F_2^i
157.	$(\alpha_5, \{\mathbf{A}_{15}, \mathbf{A}_{19}\})$	α_{23}	F_{158}	F_{163}
158.	$(\alpha_4, \{\mathbf{A}_{15}, \mathbf{A}_{19}, \mathbf{A}_{20}\})$	α_{21}	F_{159}	F_{163}
159.	$(\alpha_4, \{\mathbf{A}_{15}, \mathbf{A}_{19}, \mathbf{A}_{21}\})$	α_{25}	F_{160}	F_{163}
160.	$(\alpha_2, \{\mathbf{A}_{15}, \mathbf{A}_{21}, \mathbf{A}_{22}\})$	α_{28}	F_{161}	F_{188}
161.	$(\alpha_2, \{\mathbf{A}_{15}, \mathbf{A}_{21}, \mathbf{A}_{22}, \alpha_{28}\})$	α_{23}	F_{162}	F_{163}
162.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{27}\})$	α_{29}	F_{164}	F_{195}
163.	$(\alpha_{15}, \{\mathbf{A}_{15}, \mathbf{A}_{17}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{166}
164.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{29}\})$	α_{24}	F_{165}	F_{191}
165.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{29}\})$	α_{36}	F_{167}	F_{169}
166.	$(\alpha_{20}, \{\mathbf{A}_{14}, \alpha_{19}, \alpha_{20}\})$	α_{18}	F_{180}	F_{168}
167.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{31}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{169}
168.	$(\alpha_{20}, \{\mathbf{A}_{12}, \mathbf{A}_{20}, \alpha_{32}\})$	α_{21}	F_{170}	F_{171}
169.	$(\alpha_3, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{29}\})$	α_{34}	F_{172}	F_{197}
170.	$(\alpha_{21}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \alpha_{31}\})$	α_{25}	F_{173}	F_{174}
171.	$(\alpha_{36}, \{\mathbf{A}_{12}, \mathbf{A}_{18}, \alpha_{24}, \alpha_{25}, \alpha_{27}, \alpha_{36}\})$	α_{29}	F_{192}	F_{199}
172.	$(\alpha_2, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{28}, \mathbf{A}_{29}\})$	α_{30}	F_{175}	F_{197}
173.	$(\alpha_{21}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \alpha_{25}, \alpha_{31}\})$	α_{28}	F_{176}	F_{178}
174.	$(\alpha_{28}, \{\mathbf{A}_{12}, \mathbf{A}_{28}, \alpha_{36}\})$	α_{33}	F_{177}	F_{180}
175.	$(\alpha_2, \{\mathbf{A}_{15}, \mathbf{A}_{24}, \mathbf{A}_{30}\})$	α_{25}	F_{179}	F_{193}
176.	$(\alpha_{20}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_{24}, \alpha_{32}\})$	α_{35}	F_{181}	F_{190}
177.	$(\alpha_{33}, \{\mathbf{A}_{12}, \mathbf{A}_{33}, \alpha_{24}\})$	α_{34}	F_{205}	$\succ \mathbf{F}_5$
178.	$(\alpha_{26}, \{\mathbf{A}_8, \mathbf{A}_{21}, \alpha_{13}, \alpha_{14}, \alpha_{22}, \alpha_{23}, \alpha_{26}, \alpha_{29}\})$	α_{15}	F_{190}	$\succ \mathbf{F}_1$
179.	$(\alpha_2, \{\mathbf{A}_{15}, \mathbf{A}_{25}, \mathbf{A}_{30}\})$	α_{32}	F_{182}	F_{198}
180.	$(\alpha_{20}, \{\mathbf{A}_{14}, \mathbf{A}_{17}, \alpha_{19}, \alpha_{20}\})$	α_{18}	F_{190}	F_{362}
181.	$(\alpha_{20}, \{\mathbf{A}_8, \mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_{13}, \alpha_{14}, \alpha_{22}, \alpha_{30}, \alpha_{33}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{183}
182.	$(\alpha_1, \{\mathbf{A}_{15}, \mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{184}
183.	$(\alpha_{19}, \{\mathbf{A}_5, \mathbf{A}_{28}, \alpha_{11}, \alpha_{13}, \alpha_{15}, \alpha_{19}\})$	α_8	F_{185}	F_{201}
184.	$(\alpha_2, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}\})$	α_{32}	$\succ \mathbf{F}_3$	F_{193}
185.	$(\alpha_{17}, \{\mathbf{A}_{12}, \mathbf{A}_{19}\})$	α_{22}	F_{186}	F_{190}
186.	$(\alpha_{17}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{20}\})$	α_{21}	F_{187}	F_{190}
187.	$(\alpha_{17}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{21}\})$	α_{23}	F_{188}	F_{190}
188.	$(\alpha_{17}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22}\})$	α_{23}	F_{189}	F_{190}
189.	$(\alpha_{17}, \{\mathbf{A}_{23}\})$	α_{28}	F_{191}	F_{205}
190.	$(\alpha_{20}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{20}\})$	α_{21}	F_{199}	F_{192}
191.	$(\alpha_{17}, \{\mathbf{A}_{23}, \mathbf{A}_{27}\})$	α_{33}	F_{193}	F_{205}
192.	$(\alpha_{36}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{20}, \alpha_{36}\})$	α_{13}	F_{194}	F_{200}
193.	$(\alpha_{17}, \{\mathbf{A}_{23}, \mathbf{A}_{28}\})$	α_{33}	F_{195}	F_{205}
194.	$(\alpha_{36}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{19}, \mathbf{A}_{20}, \alpha_{36}\})$	α_{27}	F_{196}	F_{199}
195.	$(\alpha_{17}, \{\mathbf{A}_{23}, \mathbf{A}_{33}\})$	α_{30}	F_{197}	F_{207}

i	F^i	γ^i	F^i_1	F^i_2
196.	$(\alpha_{24}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{27}, \alpha_{24}, \alpha_{29}\})$	α_{22}	F_{251}	F_{199}
197.	$(\alpha_{17}, \{\mathbf{A}_{23}, \mathbf{A}_{29}, \mathbf{A}_{33}\})$	α_{34}	F_{198}	F_{207}
198.	$(\alpha_{17}, \{\mathbf{A}_{30}, \mathbf{A}_{33}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{207}
199.	$(\alpha_{21}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{21}\})$	α_{23}	F_{205}	F_{200}
200.	$(\alpha_{35}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{21}, \alpha_{35}\})$	α_{28}	F_{201}	F_{202}
201.	$(\alpha_{35}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_{35}\})$	α_{28}	F_{287}	F_{202}
202.	$(\alpha_{29}, \{\mathbf{A}_{11}, \mathbf{A}_{19}, \mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_{14}, \alpha_{15}, \alpha_{29}\})$	α_{22}	F_{203}	F_{204}
203.	$(\alpha_{29}, \{\mathbf{A}_{21}, \mathbf{A}_{29}, \alpha_{14}, \alpha_{15}\})$	α_{30}	F_{205}	F_{251}
204.	$(\alpha_{22}, \{\mathbf{A}_{21}, \mathbf{A}_{22}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{25}, \alpha_{28}, \alpha_{35}\})$	α_{23}	F_{218}	F_{248}
205.	$(\alpha_{20}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22}\})$	α_{24}	F_{206}	F_{223}
206.	$(\alpha_{20}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22}, \alpha_{24}\})$	α_{26}	F_{207}	F_{208}
207.	$(\alpha_{7}, \{\mathbf{A}_{13}, \mathbf{A}_{21}, \mathbf{A}_{22}, \alpha_{14}, \alpha_{23}\})$	α_{27}	F_{209}	F_{218}
208.	$(\alpha_{27}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{27}, \alpha_{24}\})$	α_{28}	F_{210}	F_{251}
209.	$(\alpha_{7}, \{\mathbf{A}_{13}, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{27}, \alpha_{14}, \alpha_{23}\})$	α_{28}	F_{211}	F_{252}
210.	$(\alpha_{27}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{24}\})$	α_{29}	F_{226}	F_{287}
211.	$(\alpha_{7}, \{\mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{14}, \alpha_{23}\})$	α_{35}	F_{213}	F_{212}
212.	$(\alpha_{8}, \{\mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{14}, \alpha_{23}\})$	α_{12}	F_{243}	F_{251}
213.	$(\alpha_{7}, \{\mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{14}, \alpha_{23}, \alpha_{35}\})$	α_{31}	F_{215}	F_{214}
214.	$(\alpha_{10}, \{\mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{14}, \alpha_{23}, \alpha_{35}\})$	α_{33}	$\succ \mathbf{F}_5$	F_{218}
215.	$(\alpha_{6}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{19}, \mathbf{A}_{33}, \alpha_{24}, \alpha_{25}, \alpha_{29}\})$	α_{30}	F_{217}	F_{216}
216.	$(\alpha_{9}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{19}, \mathbf{A}_{33}, \alpha_{24}, \alpha_{25}, \alpha_{29}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{252}
217.	$(\alpha_{3}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{33}, \alpha_{31}, \alpha_{32}\})$	α_{15}	F_{220}	F_{219}
218.	$(\alpha_{14}, \{\mathbf{A}_{13}, \mathbf{A}_{21}, \mathbf{A}_{22}, \alpha_{14}, \alpha_{23}\})$	α_{12}	F_{232}	F_{223}
219.	$(\alpha_{17}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{33}, \alpha_{31}\})$	α_{29}	F_{221}	F_{285}
220.	$(\alpha_{3}, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{33}, \alpha_{31}, \alpha_{32}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{222}
221.	$(\alpha_{17}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{31}\})$	α_{30}	F_{255}	F_{226}
222.	$(\alpha_{4}, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{33}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{224}
223.	$(\alpha_{27}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{21}, \mathbf{A}_{27}, \alpha_{23}\})$	α_{28}	F_{232}	F_{225}
224.	$(\alpha_{5}, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{33}\})$	α_{25}	F_{227}	F_{243}
225.	$(\alpha_{31}, \{\mathbf{A}_{23}, \mathbf{A}_{27}, \alpha_{31}\})$	α_{29}	F_{228}	F_{254}
226.	$(\alpha_{27}, \{\mathbf{A}_{12}, \mathbf{A}_{28}, \mathbf{A}_{29}, \alpha_{24}\})$	α_{31}	F_{229}	F_{232}
227.	$(\alpha_{5}, \{\mathbf{A}_{15}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{33}\})$	α_{26}	F_{230}	F_{243}
228.	$(\alpha_{31}, \{\mathbf{A}_{23}, \mathbf{A}_{29}, \alpha_{31}\})$	α_{24}	F_{262}	F_{232}
229.	$(\alpha_{27}, \{\mathbf{A}_{12}, \mathbf{A}_{28}, \mathbf{A}_{29}, \alpha_{24}, \alpha_{31}\})$	α_{33}	F_{231}	F_{251}
230.	$(\alpha_{5}, \{\mathbf{A}_{15}, \mathbf{A}_{25}, \mathbf{A}_{33}\})$	α_{26}	$\succ \mathbf{F}_2$	F_{243}
231.	$(\alpha_{27}, \{\mathbf{A}_{12}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{24}, \alpha_{31}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{233}
232.	$(\alpha_{28}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{28}\})$	α_{30}	F_{234}	F_{287}
233.	$(\alpha_{11}, \{\mathbf{A}_{13}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{23}, \alpha_{31}\})$	α_{12}	F_{235}	F_{272}
234.	$(\alpha_{28}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{30}\})$	α_{32}	F_{237}	F_{236}

i	F^i	γ^i	F_1^i	F_2^i
235.	$(\alpha_{11}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{31}\})$	α_{15}	F_{245}	F_{240}
236.	$(\alpha_{33}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{33}, \alpha_{30}\})$	α_{34}	F_{239}	$\succ \mathbf{F}_5$
237.	$(\alpha_{20}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{24}, \alpha_{30}\})$	α_{33}	F_{238}	F_{248}
238.	$(\alpha_{20}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{33}, \alpha_{24}, \alpha_{30}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{240}
239.	$(\alpha_{33}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{33}, \alpha_{30}, \alpha_{34}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{241}
240.	$(\alpha_{11}, \{\mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{33}, \alpha_{23}, \alpha_{30}\})$	α_{12}	F_{242}	F_{248}
241.	$(\alpha_{22}, \{\mathbf{A}_{12}, \mathbf{A}_{22}, \mathbf{A}_{33}, \alpha_{30}, \alpha_{34}\})$	α_{23}	F_{244}	F_{248}
242.	$(\alpha_{12}, \{\mathbf{A}_{23}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{14}\})$	α_{15}	F_{243}	F_{244}
243.	$(\alpha_{8}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{28}\})$	α_{30}	F_{245}	F_{252}
244.	$(\alpha_{30}, \{\mathbf{A}_{30}, \mathbf{A}_{33}\})$	α_{32}	F_{246}	$\succ \mathbf{F}_4$
245.	$(\alpha_{8}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{28}, \mathbf{A}_{29}\})$	α_{30}	F_{247}	F_{252}
246.	$(\alpha_{11}, \{\mathbf{A}_{13}, \mathbf{A}_{30}, \mathbf{A}_{33}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{248}
247.	$(\alpha_{8}, \{\mathbf{A}_{14}, \mathbf{A}_{30}\})$	α_{32}	F_{249}	F_{256}
248.	$(\alpha_{13}, \{\mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{28}, \alpha_{23}, \alpha_{30}\})$	α_{14}	F_{252}	F_{251}
249.	$(\alpha_{5}, \{\mathbf{A}_{14}, \mathbf{A}_{24}, \mathbf{A}_{30}\})$	α_{25}	F_{250}	F_{256}
250.	$(\alpha_{5}, \{\mathbf{A}_{25}, \mathbf{A}_{30}\})$	α_{26}	F_{255}	F_{253}
251.	$(\alpha_{24}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{19}, \mathbf{A}_{27}, \alpha_{24}, \alpha_{29}\})$	α_{22}	F_{262}	F_{254}
252.	$(\alpha_{13}, \{\mathbf{A}_{13}, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{27}, \alpha_{14}, \alpha_{23}\})$	α_{24}	F_{260}	F_{287}
253.	$(\alpha_{19}, \{\mathbf{A}_{25}, \mathbf{A}_{30}\})$	α_{32}	F_{274}	F_{256}
254.	$(\alpha_{28}, \{\mathbf{A}_{12}, \mathbf{A}_{24}, \mathbf{A}_{29}, \alpha_{15}, \alpha_{28}\})$	α_{21}	F_{283}	F_{257}
255.	$(\alpha_{17}, \{\mathbf{A}_{24}, \mathbf{A}_{30}, \mathbf{A}_{33}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{258}
256.	$(\alpha_{12}, \{\mathbf{A}_{14}, \mathbf{A}_{30}\})$	α_{15}	F_{259}	F_{260}
257.	$(\alpha_{23}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{27}, \alpha_{16}, \alpha_{31}, \alpha_{36}\})$	α_{26}	$\succ \mathbf{F}_1$	$\succ \mathbf{F}_6$
258.	$(\alpha_{18}, \{\mathbf{A}_{24}, \mathbf{A}_{30}, \mathbf{A}_{33}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{263}
259.	$(\alpha_{12}, \{\mathbf{A}_{15}, \mathbf{A}_{30}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{260}
260.	$(\alpha_{11}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{24}, \mathbf{A}_{27}, \alpha_{25}\})$	α_{15}	F_{261}	F_{262}
261.	$(\alpha_{11}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{24}, \mathbf{A}_{27}, \alpha_{15}, \alpha_{25}\})$	α_{29}	F_{263}	F_{298}
262.	$(\alpha_{24}, \{\mathbf{A}_{12}, \mathbf{A}_{24}, \mathbf{A}_{29}\})$	α_{25}	F_{264}	F_{272}
263.	$(\alpha_{11}, \{\mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{24}, \mathbf{A}_{29}, \alpha_{15}, \alpha_{25}\})$	α_{36}	F_{265}	F_{298}
264.	$(\alpha_{24}, \{\mathbf{A}_{12}, \mathbf{A}_{24}, \mathbf{A}_{29}, \alpha_{25}\})$	α_{26}	F_{266}	F_{272}
265.	$(\alpha_{8}, \{\mathbf{A}_{14}, \mathbf{A}_{21}, \mathbf{A}_{31}, \alpha_{26}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{267}
266.	$(\alpha_{13}, \{\mathbf{A}_{24}, \mathbf{A}_{29}, \alpha_{14}, \alpha_{15}, \alpha_{21}, \alpha_{23}\})$	α_{31}	F_{278}	F_{268}
267.	$(\alpha_{19}, \{\mathbf{A}_{14}, \mathbf{A}_{21}, \mathbf{A}_{31}\})$	α_{25}	F_{269}	F_{272}
268.	$(\alpha_{14}, \{\mathbf{A}_{24}, \mathbf{A}_{29}, \alpha_{14}, \alpha_{15}, \alpha_{21}, \alpha_{23}\})$	α_{25}	F_{270}	$\succ \mathbf{F}_1$
269.	$(\alpha_{19}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{31}\})$	α_{25}	F_{271}	F_{272}
270.	$(\alpha_{21}, \{\mathbf{A}_{14}, \mathbf{A}_{21}, \mathbf{A}_{24}, \alpha_{26}, \alpha_{29}, \alpha_{31}\})$	α_{23}	F_{300}	F_{283}
271.	$(\alpha_{19}, \{\mathbf{A}_{25}, \mathbf{A}_{31}\})$	α_{32}	F_{273}	F_{278}
272.	$(\alpha_{31}, \{\mathbf{A}_{12}, \mathbf{A}_{31}\})$	α_{32}	F_{278}	$\succ \mathbf{F}_6$
273.	$(\alpha_{18}, \{\mathbf{A}_{25}, \mathbf{A}_{28}, \mathbf{A}_{31}\})$	α_{35}	F_{275}	F_{274}

i	F^i	γ^i	F^i_1	F^i_2
274.	$(\alpha_{19}, \{\mathbf{A}_{25}, \mathbf{A}_{28}, \mathbf{A}_{31}\})$	α_{32}	F_{277}	F_{278}
275.	$(\alpha_{17}, \{\mathbf{A}_{25}, \mathbf{A}_{31}, \mathbf{A}_{33}\})$	α_{30}	F_{276}	F_{278}
276.	$(\alpha_{17}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}, \mathbf{A}_{33}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{277}
277.	$(\alpha_{19}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}\})$	α_{32}	$\succ \mathbf{F}_3$	F_{278}
278.	$(\alpha_{20}, \{\mathbf{A}_{31}, \alpha_{12}, \alpha_{14}, \alpha_{21}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{279}
279.	$(\alpha_{21}, \{\mathbf{A}_{31}, \alpha_{12}, \alpha_{14}, \alpha_{21}\})$	α_{10}	F_{280}	F_{287}
280.	$(\alpha_{21}, \{\mathbf{A}_{14}, \mathbf{A}_{21}, \mathbf{A}_{31}\})$	α_{23}	F_{281}	F_{283}
281.	$(\alpha_{23}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{31}\})$	α_{25}	F_{282}	F_{283}
282.	$(\alpha_{25}, \{\mathbf{A}_{25}, \mathbf{A}_{31}\})$	α_{26}	F_{285}	F_{284}
283.	$(\alpha_{28}, \{\mathbf{A}_{12}, \mathbf{A}_{24}, \mathbf{A}_{28}, \mathbf{A}_{29}, \alpha_{15}\})$	α_{33}	F_{287}	F_{312}
284.	$(\alpha_{28}, \{\mathbf{A}_{25}, \mathbf{A}_{28}, \mathbf{A}_{31}\})$	α_{30}	F_{286}	F_{298}
285.	$(\alpha_{21}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{33}, \alpha_{31}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{288}
286.	$(\alpha_{30}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}\})$	α_{32}	$\succ \mathbf{F}_3$	F_{298}
287.	$(\alpha_{35}, \{\mathbf{A}_{12}, \mathbf{A}_{19}, \mathbf{A}_{28}, \alpha_{35}\})$	α_{39}	F_{289}	$\succ \mathbf{F}_5$
288.	$(\alpha_{23}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{33}, \alpha_{31}\})$	α_{25}	F_{290}	F_{300}
289.	$(\alpha_{33}, \{\mathbf{A}_{14}, \mathbf{A}_{19}, \mathbf{A}_{33}, \alpha_{34}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{291}
290.	$(\alpha_{23}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{33}, \alpha_{25}, \alpha_{31}\})$	α_{30}	F_{292}	F_{298}
291.	$(\alpha_{34}, \{\mathbf{A}_{14}, \mathbf{A}_{19}, \mathbf{A}_{33}, \alpha_{34}\})$	α_{30}	F_{293}	F_{299}
292.	$(\alpha_{23}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{33}, \alpha_{25}, \alpha_{30}, \alpha_{31}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{294}
293.	$(\alpha_{30}, \{\mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{22}, \mathbf{A}_{33}, \alpha_{15}, \alpha_{30}\})$	α_{29}	F_{295}	F_{299}
294.	$(\alpha_{13}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{25}\})$	α_{14}	F_{296}	F_{297}
295.	$(\alpha_{23}, \{\mathbf{A}_{13}, \mathbf{A}_{23}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{16}\})$	α_{30}	F_{300}	F_{299}
296.	$(\alpha_{14}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{33}, \alpha_{30}\})$	α_{15}	F_{298}	F_{319}
297.	$(\alpha_{31}, \{\mathbf{A}_{23}, \mathbf{A}_{31}, \mathbf{A}_{33}, \alpha_{15}\})$	α_{32}	F_{303}	$\succ \mathbf{F}_6$
298.	$(\alpha_{14}, \{\mathbf{A}_{14}, \mathbf{A}_{21}, \mathbf{A}_{24}, \mathbf{A}_{27}, \alpha_{26}\})$	α_{25}	F_{300}	$\succ \mathbf{F}_1$
299.	$(\alpha_{24}, \{\mathbf{A}_{14}, \mathbf{A}_{24}, \mathbf{A}_{33}, \alpha_{16}, \alpha_{30}\})$	α_{31}	F_{301}	$\succ \mathbf{F}_1$
300.	$(\alpha_{12}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{27}, \alpha_{26}\})$	α_{15}	F_{302}	F_{307}
301.	$(\alpha_{31}, \{\mathbf{A}_{15}, \mathbf{A}_{19}, \mathbf{A}_{33}, \alpha_{23}, \alpha_{24}, \alpha_{29}, \alpha_{31}\})$	α_{32}	F_{303}	$\succ \mathbf{F}_6$
302.	$(\alpha_{14}, \{\mathbf{A}_{15}, \mathbf{A}_{25}, \mathbf{A}_{27}\})$	α_{16}	$\succ \mathbf{F}_1$	F_{304}
303.	$(\alpha_{22}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{28}, \mathbf{A}_{29}, \alpha_{16}, \alpha_{32}, \alpha_{35}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{305}
304.	$(\alpha_{25}, \{\mathbf{A}_{15}, \mathbf{A}_{25}, \mathbf{A}_{27}\})$	α_{26}	$\succ \mathbf{F}_2$	F_{307}
305.	$(\alpha_{23}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{28}, \mathbf{A}_{29}, \alpha_{16}, \alpha_{35}\})$	α_{25}	F_{306}	$\succ \mathbf{F}_5$
306.	$(\alpha_{25}, \{\mathbf{A}_{25}, \mathbf{A}_{33}, \alpha_{16}, \alpha_{31}\})$	α_{32}	F_{308}	$\succ \mathbf{F}_1$
307.	$(\alpha_{28}, \{\mathbf{A}_{23}, \mathbf{A}_{24}, \alpha_{15}, \alpha_{25}, \alpha_{28}, \alpha_{29}\})$	α_{27}	F_{309}	F_{310}
308.	$(\alpha_{23}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \alpha_{16}, \alpha_{35}\})$	α_{26}	$\succ \mathbf{F}_1$	F_{323}
309.	$(\alpha_{28}, \{\mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{28}, \mathbf{A}_{29}, \alpha_{15}, \alpha_{25}\})$	α_{33}	F_{311}	F_{312}
310.	$(\alpha_{36}, \{\mathbf{A}_{15}, \mathbf{A}_{25}, \mathbf{A}_{28}, \mathbf{A}_{29}, \alpha_{36}\})$	α_{41}	$\succ \mathbf{F}_2$	$\succ \mathbf{F}_6$
311.	$(\alpha_{33}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{26}\})$	α_{34}	F_{313}	$\succ \mathbf{F}_5$
312.	$(\alpha_{30}, \{\mathbf{A}_{15}, \mathbf{A}_{24}, \mathbf{A}_{28}, \alpha_{30}, \alpha_{31}\})$	α_{29}	F_{314}	$\succ \mathbf{F}_6$

i	F^i	γ^i	F_1^i	F_2^i
313.	$(\alpha_{33}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{26}, \alpha_{34}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{315}
314.	$(\alpha_{23}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{28}, \mathbf{A}_{31}, \alpha_{16}\})$	α_{30}	F_{316}	$\succ \mathbf{F}_1$
315.	$(\alpha_{30}, \{\mathbf{A}_{24}, \mathbf{A}_{30}, \mathbf{A}_{33}, \alpha_{15}, \alpha_{25}\})$	α_{32}	F_{317}	$\succ \mathbf{F}_4$
316.	$(\alpha_{30}, \{\mathbf{A}_{15}, \mathbf{A}_{30}, \mathbf{A}_{31}\})$	α_{32}	F_{318}	F_{324}
317.	$(\alpha_{29}, \{\mathbf{A}_{15}, \mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}, \mathbf{A}_{33}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{319}
318.	$(\alpha_{23}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}, \alpha_{16}\})$	α_{34}	F_{320}	F_{326}
319.	$(\alpha_{31}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{29}, \mathbf{A}_{33}, \alpha_{25}, \alpha_{31}\})$	α_{32}	F_{321}	$\succ \mathbf{F}_6$
320.	$(\alpha_{28}, \{\mathbf{A}_{15}, \mathbf{A}_{30}, \mathbf{A}_{31}, \mathbf{A}_{33}, \alpha_{32}\})$	α_{35}	$\succ \mathbf{F}_5$	F_{322}
321.	$(\alpha_{22}, \{\mathbf{A}_{14}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{28}, \alpha_{30}, \alpha_{31}, \alpha_{32}, \alpha_{35}\})$	α_{29}	F_{325}	F_{324}
322.	$(\alpha_{23}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}, \mathbf{A}_{33}, \alpha_{16}\})$	α_{34}	$\succ \mathbf{F}_4$	F_{323}
323.	$(\alpha_{30}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \alpha_{35}\})$	α_{32}	F_{325}	$\succ \mathbf{F}_4$
324.	$(\alpha_{33}, \{\mathbf{A}_{15}, \mathbf{A}_{30}, \mathbf{A}_{31}, \mathbf{A}_{33}\})$	α_{34}	$\succ \mathbf{F}_4$	$\succ \mathbf{F}_5$
325.	$(\alpha_{29}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}, \alpha_{35}\})$	α_{36}	$\succ \mathbf{F}_6$	F_{326}
326.	$(\alpha_{25}, \{\mathbf{A}_{25}, \mathbf{A}_{30}, \mathbf{A}_{31}, \alpha_{16}\})$	α_{26}	$\succ \mathbf{F}_1$	$\succ \mathbf{F}_3$

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