Combinatorics of Delaunay polytopes of the isodual lattice Q_{10}

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Abstract

The results of [9] are generalized and simplified for code lattices. As an example, the code lattice Q_{10}, mentioned and named in the paper [6], is considered. Q_{10} has two symmetric Delaunay polytopes P_5, P_3 and an asymmetric P_5', and is generated by P_5. P_5 is a symmetrization of the cut polytope $PCut_5$, i.e. it is the convex hull of all cuts and their complements in the complete graph K_5. The cuts and their complements are all circuits of the regular matroid R_{10} [12]. Besides P_5 is the convex hull of the unique 10-dimensional closed odd system of 16 pairs of opposite vectors of norm 5 spanning equiangular lines at angle $arccos\frac{1}{5}$. P_5 is also the convex hull of all codewords of the linear binary code $C_{10} = [10, 5, 4]$. The second symmetric Delaunay polytopes P_3 is the convex hull of a system of 40 pairs of opposite vectors of norm 3 with $0, \pm 1$ inner products. The asymmetric Delaunay polytope P_5' is the convex hull of an odd system of 32 vectors of norm 5 with $\pm 1, -3$ inner products.

9-dimensional facets of P_5 are convex hulls of vector representations of the well known strongly regular graphs: the 4×4 grid graph $L_2(4)$, the Shrikhande graph, the Petersen graph. Similarly, 4 regular subgraphs of the Shrikhande graph define 8-faces of P_5.

Lattices, graphs, odd systems and polytopal combinatorics interlace tightly in this study.

1 The lattice Q_{10}, the matroid R_{10}, the code [10,5,4], and the two-graph $\mathcal{T}(5, 10)$

A lattice is called *isodual* [6] if it is congruent to its dual. The lattice Q_{10} is an even isodual lattice of minimal norm 4. It is generated by indicator vectors of cuts and their complements in the complete graph K_5. Let $V_5 = \{1, 2, ..., 5\}$ and $E_5 = \{ij : 1 \le i < j \le 5\}$ be the sets of vertices and edges of K_5, respectively. For $S \subseteq V_5$, the *cut* $\delta(S) \subseteq E_5$ is the set of edges having exactly one vertex in the set S.

*The author has been supported by DFG– Sonderforschungsberich 343 “DISKRETE STRUKTUREN IN DER MATHEMATIK” Universität Bielefeld.

Let $c(S)$ be the (10-dimensional) indicator vector of the cut $\delta(S)$, i.e. the vector of the space $\mathbf{R}^{E_5}$ such that

$$c_{ij}(S) = \begin{cases} 1 & \text{if } |\{ij\} \cap S| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $c(V_5 - S) = c(S)$. Let j_{10} be the all-one vector. Denote by $c^*(S) = j_{10} - c(S)$ the indicator vector of the complement $\delta^*(S) = E_5 - \delta(S)$ of the cut $\delta(S)$.

Let $S\Delta T = (S - S \cap T) \cup (T - S \cap T)$ be the symmetric difference of the sets S and T. Using the identity $\delta^*(S) = E_5 \Delta \delta(S)$, it is easy to verify that

$$\delta(S\Delta T) = \delta(S)\Delta\delta(T) = \delta^*(S)\Delta\delta^*(T), \;\; \delta^*(S\Delta T) = \delta^*(S)\Delta\delta(T) = \delta(S)\Delta\delta^*(T). \quad (1)$$

The 30 nontrivial sets $\delta(S)$, $\delta^*(S)$ for $S \subseteq V_4$, $S \neq \emptyset$, form the family of cycles of the famous regular matroid R_{10} of rank 5 (see, for example [12]). Cocycles of R_{10}, i.e. cycles of the dual matroid R^*_{10}, are all minimal by inclusions subsets of E_5 having intersections of even cardinality with all cycles of R_{10}. These are all 15 circuits of length 4 of the graph K_5 and all 15 their complements. There is a permutation of the edge-set E_5 determining an isomorphism between the sets of all cycles and all cocycles of R_{10}. For example, let $m \in V_5$ and $\{ijkl\} = V_5 - \{m\}$. Let (ij, jk, kl, il), (ij, jl, kl, ik), (ik, jk, jl, il) be the three 4-circuits of K_5 on the 4-set $\{ijkl\}$. Then there is a permutation of E_5 such that these circuits are mapped into $\delta(\{m\}) = \{im, jm, km, lm\}$, $\delta^*(\{ij\}) = \{ij, kl, km, lm\}$, $\delta^*(\{kl\}) = \{ij, kl, im, jm\}$. So we have that R_{10} is permutationally equivalent (i.e. isomorphic) to R^*_{10} (but not self-dual). The automorphism group of R_{10} is doubly transitive [12].

Since the matroid R_{10} is a regular matroid of rank 5, it has a representation by a 5-dimensional subspace of the 10-dimensional binary space of the indicator vectors of all subset of the 10-set E_5. The vectors $c(S)$, $c^*(S)$, $S \subseteq V_4$, provide one of the representations of R_{10}.

Recall that each k-subspace of a binary n-space $\mathbf{B}^n$ forms a binary linear code $[n, k, d]$, where d is the minimal length of code words. Hence the 32 vectors $c(S)$ and $c^*(S)$ form the code $C_{10} = [10, 5, 4]$. This code has 1 word $c(\emptyset) = 0$ of length 0, 15 words of length 4, 15 words of length 6 and 1 word $c^*(\emptyset) = j_{10}$ of length 10. Similarly to the matroid R_{10}, this code is isomorphic to its dual, although is not self-dual.

The lattice Q_{10} is obtained from the code C_{10} by Construction A of [3]. This means that every lattice vector

$a = \sum_S (z_S c(S) + z'_S c^*(S)) \in Q_{10}$ with $z_S, z'_S \in \mathbf{Z}$ has the form

$$a = c(S) + 2f \text{ or } a = c^*(S) + 2f$$

for some S, where $f \in \mathbf{Z}^{10}$ is an integral vector. Since the code C_{10} is isomorphic to its dual, the dual lattice $Q^*_{10} = \{d \in \mathbf{R}^{E_5} : da \in \mathbf{Z}, a \in Q_{10}\}$ consists of the vectors $\frac{1}{2}b + f$, where b is a word of the dual code and $f \in \mathbf{Z}^{10}$. Hence Q^*_{10} is isomorphic to $\frac{1}{2}Q_{10}$, i.e. Q_{10} and Q^*_{10} are congruent, i.e. Q_{10} is isodual.

The convex hull of all codewords of C_{10} is a Delaunay polytope $P(C_{10})$ of the lattice Q_{10}.

Below we call the lattice obtained from a linear binary code C by Construction A of [3] by a *code* lattice and denote it as $L(C)$.

Note that endpoints of all vectors $c(S)$ and $c^*(S)$ are vertices of the 10-dimensional unit cube $\mathbf{B}^{10}$. Let take the center $\frac{1}{2}j_{10}$ as a new origin. We set

$$u(S) = \sqrt{2}(c(S) - \frac{1}{2}j_{10}),\ u^*(S) = \sqrt{2}(c^*(S) - \frac{1}{2}j_{10}).$$

Since $u(V_5 - S) = u(S)$, we can use only $S \subseteq V_5$ with $|S| \le 2$. In what follows, we denote $S = \{i\}$ and $S = \{ij\}$ simply by $S = i$ and $S = ij$. For example, $c(i) = c(\{i\})$, $u(ij) = u(\{ij\})$. Let

$$\mathcal{U}_{10} = \{u(S), u^*(S) : S = \emptyset, i \in V_5, ij \in E_5\}. \tag{2}$$

Then we have

$$u^*(S) = -u(S),\ \text{and}\ u(S)u(T) = 5 - |\delta(S\Delta T)|. \tag{3}$$

Since the amount $|\delta(S)|$ take only three values 0, 4 and 6, the vectors $v(S)$ and $v^*(S)$ have equal norm (= squared length) 5 for all S and ± 1 inner products. In other words, these vectors of $\mathcal{U}_{10}$ span 16 equiangular lines at angle $arccos\frac{1}{5}$ in the 10-dimensional space $\mathbf{R}^{E_5}$. It is well known that there is a one-to-one correspondence between sets of equiangular lines and two-graphs.

A two-graph is a set of *coherent* triples such that every 4-set contains an even number of coherent triples [14]. But it is more important for us that each two-graph is represented by a set of equiangular lines. If the acute angle between lines is equal to $arccos\frac{1}{m}$, then one can choose a vector of norm m along each line such that inner products of chosen vectors is equal to ± 1. If v_1, v_2, v_3 are vectors along 3 lines, then these 3 lines compose a coherent triple if $(v_1v_2)(v_2v_3)(v_3v_1) = -1$, where v_iv_j is the inner product of vectors v_i and v_j.

Sets of equiangular lines of a sufficient large size in a space of fixed dimension exist only if m is an odd number. There is a special bound on the maximal number $n(m,d)$ of equiangular lines at the angle $arccos\frac{1}{m}$ in a space of dimension d: $n(m,d) \le n_r(m,d)$, where $n_r(m,d) = \frac{d(m^2-1)}{m^2-d}$.

This bound is achieved if and only if the corresponding two-graph is *regular*. A two-graph is regular if every pair of points belongs to the same number of coherent triples. We denote a regular two-graph with parameters m and d by $\mathcal{T}(m,d)$. So, a regular two-graph $\mathcal{T}(m,d)$ is on $n_r(m,d)$ points, i.e. it relates to $n_r(m,d)$ equiangular lines at angle $arccos\frac{1}{m}$ in a d-dimensional space.

The two-graph related to the lattice Q_{10} has the parameters $m = 5$ and $d = 10$, and relates to 16 lines. For these parameters the above bound is tight. Hence this two-graph is a regular two-graph $\mathcal{T}(5,10)$. This two-graph is unique. The automorphism group of $\mathcal{T}(5,10)$ is the semidirect product of the additive group 2^{10} of the binary space $\mathbf{B}^{10}$ by the symplectic group $Sp(4,2)$ [15]. The automorphism group $2^{10} : Sp(4,2)$ of $\mathcal{T}(5,10)$ is also the automorphism group $2^{10} : Sym_6$ of the lattice Q_{10} [6], since $Sp(4,2) = Sym_6$.

2 Odd systems and related lattices

I repeat here some results of [7]. A set of vectors of an odd norm m with ± 1-inner products spanning equiangular lines is a special case of an odd system. An *odd system* $\mathcal{V}$ is a set of vectors v such that the inner product vv' of any (may be equal) two vectors of $\mathcal{V}$ is an odd integer. (We denote the inner product of two vectors v and v' by its juxtaposition vv'.) The inner product $v^2 = vv$ of a vector v with itself is called *norm* of v. Hence norms of all vectors of an odd system are odd. An odd system is called *uniform* (of norm m) if norms of all its vectors are equal (to m). As we use here only uniform odd systems, in what follows, we omit sometimes the word uniform.

We call an odd system *regular* if it represents a regular two-graph. We consider also *reduced* odd systems such that from two opposite vectors only one belongs to the odd system. Call an interchanging of a subset of vectors of a reduced odd system by its opposite *switching* of the odd system. Similarly, we call the operation of changing the sign of a vector v by *switching* v.

We call odd systems $\mathcal{V}$ and $\mathcal{V}'$ isomorphic if there is a bijection $\phi : \mathcal{V} \to \mathcal{V}'$ such that $\phi(v_1 v_2) = \phi(v_1)\phi(v_2)$. Obviously, isomorphic odd systems are equicardinal.

Let $\mathcal{U}$ be an odd system related to a two-graph (i.e. spanning equiangular lines). Since $vv' = \pm 1$ for $v, v' \in \mathcal{U}$, $v \neq \pm v'$, we can introduce a graph $G(\mathcal{U})$ with $\mathcal{U}$ as the set of its vertices. Two vertices v, v' of $G(\mathcal{U})$ are adjacent if and only if $vv' = -1$. If $\mathcal{U}$ relates to a regular two-graph, then $G(\mathcal{U})$ is a Taylor distance-regular graph of diameter 3 (see [2]).

Let $\mathcal{U}^0$ be a reduced odd system. A switching of $\mathcal{U}^0$ corresponds to a switching of $G(\mathcal{U}^0)$. All switching of $G(\mathcal{U}^0)$ form a switching class of the graph. Fix $v_0 \in \mathcal{U}^0$. By a switching, we can isolate v_0, i.e. $G(\mathcal{U}^0_{sw}) = \{v_0\} \cup H_0$, where $H_0 = G(\mathcal{U}(v_0))$ and

$$\mathcal{U}(v_0) = \mathcal{U}^0_{sw} - \{v_0\} = \{v \in \mathcal{U}_{10} : vv_0 = 1\}.$$

If $\mathcal{U}$ relates to a regular two-graph, then H_0 is a strongly regular graph.

Consider as an example the odd system $\mathcal{U}_{10}$, see (2). As a reduced odd system we can take the set

$$\mathcal{U}^0_{10} = \{u(\emptyset), u(i), i \in V_5, u(ij), ij \in E_5\}.$$

By (3), $u(\emptyset)$ is adjacent to $u(ij)$ and $u^*(i)$; $u(i)$ is adjacent to $u(j)$, $u^*(ij)$ and $u(jk)$, $j, k \neq i$; $u(ij)$ is adjacent to $u(ik)$, $u^*(kl)$, $u(k)$ and $u^*(i)$, and so on.

The switching class of $G(\mathcal{U}^0_{10})$ contains 3 strongly regular graphs:

Cle, the Clebsh graph with parameters $(v, k, \lambda, \mu) = (16, 10, 6, 6)$,

$L_2(4)$, the 4×4 grid graph with parameters (16,6,2,2), and

Shr, the Shrikhande graph with the same parameters (16,6,2,2) as $L_2(4)$.

For the regular two-graph $\mathcal{T}(5, 10)$ on 16 points, $H_0 = G(\mathcal{U}(v_0))$ is the triangular graph $T(6)$ and has parameters (15,8,4,4). If we take $v_0 = u(\emptyset)$, then $T(6)$ is induced by vectors $u^*(i)$, $i \in V_{15}$ and $u(ij)$, $ij \in E_5$. Clearly, the last 10 vertices $u(ij)$ induce the triangular graph $T(5)$. (We shall identify the vectors an vertices.)

If we switch any 5-circuit of $T(5)$, for example, the circuit induced by vertices $u(12)$, $u(23)$, $u(34)$, $u(45)$, $u(15)$, then we obtain the Petersen strongly regular graph Pe with parameters (10,3,0,1).

A lattice L of dimension n is a free Abelian group of rank n of vectors. A lattice is called *integral* if inner products of all its vectors are integral. An integral lattice is called *even* if norms of all its vectors are even. An even lattice L is called *doubly even* if $\frac{1}{\sqrt{2}}L$ is even. Norms of all vectors of a doubly even lattice are multiples of 4, and all inner products are even. Hence minimal norm of a nonzero vector of a doubly even lattice L is not less than 4. The set L_4 of all vectors of norm 4 of L is, up to the multiple $\sqrt{2}$, a *root system*. Hence below we call a vector of norm 4 a *root*.

Consider the following lattices generated by an odd system $\mathcal{V}$:

$$L^q(\mathcal{V}) = \{u : u = \sum_{v \in \mathcal{V}} z_v v, \ \sum_{v \in \mathcal{V}} z_v = q, \ z_v \in \mathbf{Z}\}, \ q = 0, 1.$$

Let $\tilde{L}^q(\mathcal{V})$ be defined as $L^q(\mathcal{V})$ with $\sum_{v \in \mathcal{V}} z_v \equiv q \pmod 2$. Clearly, $L^q(\mathcal{V}) \subseteq \tilde{L}^q(\mathcal{V})$, and $L^q(\mathcal{V}) = \tilde{L}^q(\mathcal{V})$ if $\mathcal{V}$ is symmetric. Let $\mathcal{V}$ be uniform and of norm 5. It is proved in [7] that $\tilde{L}^0(\mathcal{V})$ is a doubly even lattice, and the affine lattice $\tilde{L}^1(\mathcal{V}) = v + L^0(\mathcal{V})$ is a translation of $\tilde{L}^0(\mathcal{V})$. Of course this is true for $L^q(\mathcal{V})$, too.

Let $\mathcal{U}$ be a uniform symmetric odd system with ± 1 inner products. Recall that $\mathcal{U}(v_0) = \{v \in \mathcal{U} : vv_0 = 1\}$. Obviously, $2v_0 = v_0 - (-v_0) \in L^0(\mathcal{U})$ for any $v_0 \in \mathcal{U}$. By the definition of [4], the lattice $L^0(\mathcal{V})$ is a *gluing* of the lattice $L^0(\mathcal{U}(v_0))$ to the one-dimensional lattice generated by $2v_0$.

We have $L^0(\mathcal{U}_{10}) = \sqrt{2}Q_{10}$, i.e. $L^0(\mathcal{U}_{10}) =^2 Q_{10}$ in the denotations of [6]. Similarly, $L^0(\mathcal{U}_{10}(v_0)) = \sqrt{2}Q_9(5)$, for any $v_0 \in \mathcal{U}_{10}$, where $Q_9(5)$ is defined in [5], and $\frac{1}{2}Q_{10}$ is the gluing of $\frac{1}{2}Q_9(5)$ to v_0 [6].

$L^1(\mathcal{V})$ is an odd system and $u^2 \equiv 1 \pmod 4$ for all $u \in L^1(\mathcal{V})$. Let $L^1_k(\mathcal{V})$ be the set of all vectors of $L^1(\mathcal{V})$ of norm k. Obviously, $\mathcal{V} \subseteq L^1_5(\mathcal{V})$.

A *Delaunay polytope* of a lattice is the convex hull of all lattice points lying on an empty sphere of the lattice, and the lattice points on the empty sphere have full rank. A sphere is called *empty* if there is no lattice point strictly inside it. A Delaunay polytope P is called *symmetric* or *asymmetric* according to whether the antipode of each vertex of P *is* or *not is* (respectively) a vertex of P.

The convex hull $\mathrm{conv}L^1_5(\mathcal{V})$ of endpoints of all vectors from $L^1_5(\mathcal{V})$ is very often a Delaunay polytope of the lattice $L^0(\mathcal{V})$. In particular, it is so if $\mathcal{V}$ is not pillar, i.e. there is no vector e of the norm 1 such that $ev = \pm 1$ for all $v \in \mathcal{V}$. The conditions when $\mathrm{conv}L^1_5(\mathcal{V})$ is a Delaunay polytope is given in [7].

We prove below that $\mathrm{conv}L^1_5(\mathcal{U}_{10})$ is the Delaunay polytope $P(C_{10})$ of the lattice Q_{10}.

Definition. The uniform odd system $L^1_5(\mathcal{V})$ is called the *closure* of the odd system $\mathcal{V}$. The odd system $\mathcal{V}$ is called *closed* if $\mathcal{V} = L^1_5(\mathcal{V})$. We denote the closure of $\mathcal{V}$ by $\mathrm{cl}\mathcal{V}$. □

Let P be a Delaunay polytope of a lattice L. Let the center of P be the origin, and let the vectors in the endpoints in vertices of P form an odd system of norm 5. Let H be an affine space intersecting P. Then the set $\mathcal{V}$ of all the vectors with the endpoints in $P \cap H$ is closed, since it has the form $L^1_5(\mathcal{V})$, where $L^1(\mathcal{V}) \subseteq L \cap H$.

Let G be one of the strongly regular graphs $T(6)$, $L_2(4)$, *Shr*, *Pe*. Let $\mathcal{U}(G)$ be a 5-representation of G. We show below that the endpoints of $\mathcal{U}(G)$ are all vertices of a facet of the Delaunay polytope $P(C_{10})$ of the lattice Q_{10}, for $G = L_2(4)$, Shr, Pe, and the endpoints of $\mathcal{U}(T(6))$ are all vertices of the intersection of $P(C_{10})$ by an affine hyperplane.

Similarly, $\mathcal{U}(Cle)$ is closed, since the convex hull of $\mathcal{U}(Cle)$ is a Delaunay polytope of the lattice $L^0(\mathcal{U}(Cle))$.

3 Odd systems representing graphs

Let G be a graph with the set of vertices $V(G)$. Let $\mathcal{V}$ be a uniform integral (i.e. with integral inner products) system of vectors of norm m. Recall [2] that the map $v : V(G) \to \mathcal{V}$ is called (m,p,q)*-representation* of G if for $x, y \in V$ we have

$$v(x)v(y) = \begin{cases} m & \text{if } x = y, \\ p & \text{if } x \sim y, \\ q & \text{if } x \not\sim y, \end{cases}$$

where $x \sim y$ means that x and y are adjacent.

We are interested mainly in $(2t+1, -1, 1)$-representations. Up to the multiple $\frac{1}{\sqrt{2}}$ the representation is the same as $(t + \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$-representation considered by Neumaier [11].

Obviously the vectors $v(x)$, $x \in V(G)$, of a $(2t+1, -1, 1)$-representation form a uniform odd system of norm $m = 2t+1$. For brevity sake, call the representation by *m-representation*.

Let I and J be the unity matrix and the all-one matrix, respectively. Let $A(G)$ be the (0,1)-adjacency matrix of G. Then $B(G) = J - I - 2A(G)$ is the $(1, -1)$-adjacency matrix of G. If G has an m-representation, then the matrix $mI + B(G)$ is Gram matrix of vectors of the representation. Conversely, if this matrix is positive semidefinite, then the graph G has an m-representation. Since $B(G)$ is symmetric, its eigenvalues are real. Since $trB(G) = 0$, the matrix $B(G)$ has a negative eigenvalue. Let $-\rho(G)$ be the smallest eigenvalue of $B(G)$. We have the following well known fact [2], [11].

Proposition 1 *If $\rho(G) \le m$, then the graph G has an m-representation.* $\square$

Let $f(G)$ be the multiplicity of the eigenvalue $-\rho(G)$. Then the rank of the matrix $\rho(G)I + B(G)$ is equal to $|V| - f(G)$. Call dimension of the space spanned by an m-representation by *dimension* of the representation. We have

Proposition 2 *Let $\mathcal{V}$ be an m-representation of a graph G with minimal eigenvalue $-\rho(G)$ of multiplicity $f(G)$. Then*

(i) $dim\mathcal{V} = |V(G)|$ if $\rho(G) < m$,

(ii) $dim\mathcal{V} = |V(G)| - f(G)$ if $\rho(G) = m$. $\square$

In particular, the mentioned above 4 strongly regular graphs $G = Cle$, $L_2(4)$, Shr, $T(6)$, all have $\rho(G) = 5$, and $f(G) = 6$, for $G \neq T(6)$, $f(T(6)) = 5$. Hence all these graphs have a 5-representation of dimension 10. For the Petersen graph, the inequality $\rho(Pe) = 3 < 5$ implies that Pe has a 5-representation of dimension 10.

If G has an m-representation $\mathcal{V}$, then endpoints of vectors $v \in \mathcal{V}$ lie on a sphere of squared radius m and form a spherical distance space $(V(G), d)$ with the distance $d(x,y) = (v(x) - v(y))^2$ for $x, y \in V(G)$. The definition of the $(2t+1)$-representation implies for $x \neq y$

$d(x,y) = 4(t+1)$, if $x \sim y$, $d(x,y) = 4t$, if $x \not\sim y$.

If $\rho(G) < m$, the vectors of the m-representation $\mathcal{V}$ are linearly independent, and its endpoints span a simplex of affine dimension $|V(G)| - 1$. If $\rho(G) = m$, then there are dependencies of vectors in $\mathcal{V}$. Let $\sum_{x \in C} z_x v(x) = 0$ be a minimal dependency of $\mathcal{V}$ with $C \subseteq V(G)$. This dependency can be represented by the labeled graph $G(C)$ induced by the set C with the labels z_x. Of course, the set of vectors $\mathcal{V}(C) = \{v(x) : x \in C\}$ is an m-representation of the graph $G(C)$. Since the above dependency is minimal, dimension of the representation $\mathcal{V}(C)$ is equal to $|C| - 1$. By Proposition 2, the multiplicity of the eigenvalue m of the graph $G(C)$ equals 1, i.e. the eigenvalue $-\rho(G(C)) = -m$ is simple.

Conversely, let $G(C)$ be an induced subgraph of G such that its minimal eigenvalue $-\rho(G(C))$ is equal to $-\rho(G)$ and is simple. Then $\mathcal{V}(C)$ is a minimal dependency.

A typical example of a minimal dependency C for $t = 2$ is the union of two disjoint n-cycles C_1 and C_2 with $z_x = 1$ for $x \in C_1$, and $z_x = -1$ for $x \in C_2$. Any switching of C by a subset $X \subseteq C$, such that $\sum_{x \in X} z_x = 0$, is a minimal dependency, too.

Let G be a regular graph of valency k with $n = |V(G)|$ vertices and having a $(2t+1)$-representation. Then the sum

$$D(G) \equiv \sum_{x \in V(G)} d(x,y) = 4(k-t) + 4tn \tag{4}$$

does not depend on y. A proof of the following proposition can be found, for example, in [10](Prop.3.30).

Proposition 3 *If the vertices of a graph are represented by points $v(x)$ of Euclidean space and $D(G)$ does not depend on y, then the points $v(x)$, $x \in V(G)$, lie on a sphere of squared radius $r^2(G) = \frac{1}{2n} D(G)$.*

This Proposition and the equality (4) imply

Corollary 1 *Let G be a regular graph of valency k with n vertices. If G has a $(2t+1)$-representation $\mathcal{V}$, then the endpoints of $\mathcal{V}$ span a sphere of squared radius*

$$r^2(G) = 2t + 2\frac{k-t}{n}. \tag{5}$$

If $k < \frac{n}{2} + t$ (when $r^2(G) < 2t+1$), then endpoints of the $(2t+1)$-representation $\mathcal{V}$ span an affine space of dimension $\dim\mathcal{V} - 1$.

We apply this Corollary to the triangular graphs $T(5)$, $T(6)$, to the Petersen graph Pe and to the 3 strongly regular graphs in the switching class of $G(\mathcal{U}_{10}^0)$. Since the smallest eigenvalue of $B(G)$ is -5 for the 4 graphs above (excluding Pe), we have $t = 2$ and

$$r^2(Cle) = 5, r^2(T(5)) = r^2(T(6)) = 4 + \frac{4}{5}, r^2(L_2(4)) = r^2(Shr) = 4 + \frac{1}{2}, r^2(Pe) = 4 + \frac{1}{5}. \tag{6}$$

So, the endpoints of 5-representations of these graphs span 10-dimensional space for Cle, and 9-dimensional space for the other graphs. Note that the vertices of $T(5)$ and Pe span 9-dimensional simplices, since these graphs have 10 vertices.

4 Code lattices

Recall that by a *code* lattice $L(C)$ we call a lattice generated by a linear binary code $C \subseteq \mathbf{B}^n$. Let $P(C)$ be the *code polytope*, i.e. the convex hull of all binary vectors of C. So, $P(C)$ is a subpolytope of the cube $P(\mathbf{B}^n)$, and it is inscribed into the sphere circumscribing $P(\mathbf{B}^n)$.

Proposition 4 *The code polytope $P(C)$ is a Delaunay polytope of the code lattice $L(C)$. It is symmetric or asymmetric according to the code C contains or does not contain the all-one word j_n.*

Proof. Since $L(C)$ contains only integral vectors, the sphere circumscribing $P(C)$ is empty. Since C generates $L(C)$, $P(C)$ is full dimensional. Since C is linear, for $c \in C$, $c^* \equiv (j - c) \in C$ if and only if $j \in C$. It is easy to see that the vertices c and c^* are antipodal in $P(\mathbf{B}^n)$, and therefore they are antipodal in $P(C)$, too. □

All Delaunay polytopes of a lattice L form a partition of the space spanned by L. This partition is normal, i.e. it is face-to-face. This means that if two Delaunay polytopes of this partition are contiguous, then they share a common face.

Every symmetric Delaunay polytope of L is tightly related to minimal elements of a coset of the lattice L in $\frac{1}{2}L$. Here $\frac{1}{2}L$ is a lattice of all vectors $\frac{1}{2}a$ for $a \in L$. Obviously, $L \subseteq \frac{1}{2}L$. Since L and $\frac{1}{2}L$ are Abelian groups, the factor $\frac{1}{2}L/L$ is well defined. A coset M of L in $\frac{1}{2}L$ consists of all vectors of $\frac{1}{2}L$ equal to each other modulo L, i.e. $a - a' \in L$ for any $a, a' \in M$. Obviously, any vector of $\frac{1}{2}L$ having 0,1-coordinates in a basis of $\frac{1}{2}L$ represents a coset M. Hence the factor group $\frac{1}{2}L/L$ is a finite Abelian group of order 2^n, where n is the dimension of L.

For a subset $M \subseteq \mathbf{R}^n$, a vector $g \in M$ is called *minimal* (in M) if it has minimal norm among all vectors of M. Let M_{min} be the set of all minimal vectors of a coset M of $\frac{1}{2}L/L$. Let $g \in M_{min}$ and let $P_g(M)$ be the translation by g of the convex hull of endpoints of all vectors from M_{min}. Note that g is the center of $P_g(M)$. If we take g as a new origin, then the coset M consists of all the vectors with endpoints in the lattice points of L. Let $H(M)$ be the space spanned by M_{min}. Baranovskii [1] (see also [8]) proves the following

Theorem 1 *$P_g(M)$ is a symmetric Delaunay polytope of the lattice $L \cap H(M)$, and every symmetric Delaunay polytope of L is of the form $P_g(M)$ for some coset M of L in $\frac{1}{2}L$ and $g \in M_{min}$.* □

Since, by the definition of a Delaunay polytope, it is full dimensional, $P_g(M)$ is not, in general, a Delaunay polytope of L, but it is a symmetric face of the partition of the space $\mathbf{R}^n$ into the Delaunay polytopes of L.

It is not difficult to characterize the cosets of L in $\frac{1}{2}L$ for a code lattice $L(C)$. Recall that $a \equiv c$ modulo $2\mathbf{Z}^n$ for every $a \in L(C)$ and some $c \in C$.

Proposition 5 *Each coset of $L(C)$ in $\frac{1}{2}L(C)$ is uniquely determined by a vector $\frac{1}{2}c$, $c \in C$, and a coset $\mathcal{A}$ of C in $\mathbf{B}^n$.*

Proof. Each vector $a \in \frac{1}{2}L(C)$ has the form $a = a(c,f)$, where $a(c,f) = \frac{1}{2}c + f$, and $c \in C$, $f \in \mathbf{Z}^n$. By modulo $2\mathbf{Z}^n \subseteq L(C)$, we can consider $f \in \mathbf{B}^n$ only. Then $a(c,f) - a(c',f') \in L(C)$ if and only if $c = c'$ and $f - f' \in C$ by modulo $2\mathbf{Z}^n$. $\square$

Let $\mathcal{A}$ be a coset of C in $\mathbf{B}^n$. Denote by $M(c,\mathcal{A})$ the coset of $L(C)$ in $\frac{1}{2}L(C)$ determined by $c \in C$ and $\mathcal{A}$. Now we describe minimal vectors of $M(c,\mathcal{A})$.

Let E be the set of indices of vectors in $\mathbf{R}^n = \mathbf{R}^E$. Let $\{e_i : i \in E\}$ be an orthonormal basis of $\mathbf{R}^E$. A subset $X \subseteq E$ determines uniquely a vector (=word)

$$e(X) := \sum_{i \in X} e_i \in \mathbf{B}^n.$$

Clearly, each $b \in \mathbf{B}^n$ has the form $e(X)$ for some $X \subseteq E$. The set X is called the *support* of b and is denoted by $S(b)$. Since each $b \in \mathbf{B}^n$ is uniquely determined by its support, in what follows, we identify each coset $\mathcal{A}$ with the family of supports of vectors in $\mathcal{A}$.

For a set $A \in \mathcal{A}$ and $c \in C$ define

$$A_c := A \cap S(c),\ \ A^c := A - S(c),$$

such that $A = A_c \cup A^c$. Call a set $A \in \mathcal{A}$ *c-minimal* if A^c has minimal cardinality among all A'^c with $A' \in \mathcal{A}$. We set

$$c[X] := c - 2e(X) \text{ for } c \in C,\ X \subseteq S(c), \text{ and } \varepsilon(A) := \sum_{i \in A} \varepsilon_i e_i \text{ for } A \subseteq E,\ \varepsilon_i \in \{\pm 1\}.$$

Note that for $i \in X$, the ith coordinate $c_i[X]$ of $c[X]$ equals –1.

Proposition 6 *The minimal vectors of the coset $M(c,\mathcal{A})$ are vectors*

$$\frac{1}{2}c[A_c] + \sum_{i \in A^c} \varepsilon_i e_i, \tag{7}$$

for all c-minimal sets $A \in \mathcal{A}$ and arbitrary $\varepsilon_i \in \{\pm 1\}$.

Proof. By Proposition 5, each $a \in M(c,\mathcal{A})$ has the form $a = \frac{1}{2}c + e(A) + 2\sum_{i \in E} z_i e_i$ for $A \in \mathcal{A}$, $z_i \in \mathbf{Z}$. The norm of a is equal to

$$a^2 = \sum_{i \in S(c) - A_c} (\frac{1}{2} + 2z_i)^2 + \sum_{i \in A_c} (\frac{3}{2} + 2z_i)^2 + \sum_{i \in A^c} (1 + 2z_i)^2 + 4 \sum_{i \in E - (S(c) \cup A)} z_i^2.$$

For fixed A, the norm a^2 takes a minimal value if

$$z_i = 0,\ \text{for } i \in E - A,\ \ z_i = -1 \text{ for } i \in A_c,\ \ z_i = 0, -1 \text{ for } i \in A^c.$$

For these z_i, a takes the form (7). Since $(\varepsilon(A^c))^2 = |A^c|$ and $(c[X])^2 = c^2$ for any $X \subseteq S(c)$, the norm of the vector (7) takes a minimal value if A is c-minimal. $\square$

5 The cosets of Q_{10}

As an application of results of the previous section, we give a description of minimal vectors of cosets of Q_{10} in $\frac{1}{2}Q_{10}$. Recall that $E = E_5$ here, and the orthonormal basis is $\{e_{ij} : 1 \le i < j \le 5\}$.

At first, we describe the cosets of the code $C_{10} := [10, 4, 5]$ in $\mathbf{B}^{10}$. Since C_{10} contains 32 words, there are $2^{10}/2^5 = 32$ cosets, and each coset contains 32 sets. A coset can be represented by its sets of minimal cardinality. The cosets of C_{10} in $\mathbf{B}^{10}$ are determined by sets of cardinality ≤ 3.

It is convenient to describe a subset $A \subseteq E_5$ by the subgraph of the complete graph K_5 induced by the set of edges A. We have the following cosets and their representatives.

The empty set $\emptyset$ represents the coset $\mathcal{A}_0$ consisting of the supports of words of C_{10}.

There are 10 one-element subsets $\{ij\} \subseteq E_5$, each representing a coset $\mathcal{A}_{ij}$.

The $\binom{10}{2} = 45$ two-element subsets are partitioned into 15 families each containing three 2-sets. Let $V_5 = \{ijklm\}$. Then the following three 2-subsets represent the same coset: $\{ij, ik\}$, $\{il, im\}$, $\{jk, lm\}$. Since the first 2-set determines uniquely the other two, we denote this coset by $\mathcal{B}_{i;jk}$.

There are $\binom{10}{3} = 120$ 3-subsets of E_5. They are of the following four types.

a) 10 triangles $\Delta_{ijk} = \{ij, jk, ik\}$;

b) 20 fork $F_{i;m} = \{ij, ik, il\}$;

c) 30 two-paths with an edge $\Lambda_{jk} = \{ij, ik, lm\}$;

d) 60 three-path $(ijkl) = \{ij, jk, kl\}$.

Since $e(F_{i;m}) + e_{im} = c(i)$, the fork F_i belongs to the coset $\mathcal{A}_{im}$. Since $e(\Delta_{ijk}) + e_{lm} = e(\Lambda_{jk}) + e_{jk} = c^*(lm)$, the triangle Δ_{ijk} and Λ_{jk} belong to the cosets $\mathcal{A}_{lm}$ and $\mathcal{A}_{jk}$, respectively.

The 60 three-paths are partitioned into 6 families each containing 10 three-paths. These 6 families represent 6 distinct cosets $\mathcal{D}_p$, $1 \le p \le 6$. As an example, I write out in the lexicographic order 10 three-paths representing one of the cosets $\mathcal{D}_p$:

$$(ijkl), (iklm), (ilmj), (imjk), (jikm), (jkml), (jlik), (kljm), (kmil), (ljim).$$

It can be shown that the sets $A \subseteq E_5$ with $|A| \ge 4$ belong to one of the above cosets. So, we describe the 1 coset $\mathcal{A}_0$, the 10 cosets $\mathcal{A}_{ij}$, the 15 cosets $\mathcal{B}_q$ and the 6 cosets $\mathcal{D}_p$, i.e. all the 1+10+15+6=32 cosets of C_{10} in $\mathbf{B}^{10}$.

Now, using Proposition 6, we show in Table 1 below the minimal vectors of the cosets $M(c, \mathcal{A})$. The code words are $c(\emptyset) = 0$, j_{10}, $c(i)$, $c^*(i)$ for $i \in V_5$, $c(ij)$, $c^*(ij)$ for $ij \in E_5$. For $X \subseteq S(c(i))$, we set $c(i; X) := c(i)[X]$. Similarly is defined $c(ij; X)$.

There are three types of minimal vectors of $M(c, \mathcal{A})$, depending on $c = 0$, $c = i_{10}$ or $c \ne 0, j_{10}$.

Clearly, for $c = 0$, minimal c-sets of a coset $\mathcal{A}$ are sets of $\mathcal{A}$ of minimal cardinality. Hence, for $c = 0$, minimal vectors of $M(0, \mathcal{A})$ have the form $\varepsilon(A)$, for $A \in \mathcal{A}$ of minimal cardinality.

For $c = j_{10}$ all sets of $\mathcal{A}$ are c-minimal. Hence the minimal vectors of $M(j_{10}, \mathcal{A})$ are the 32 vectors $j_{10}[A]$, $A \in \mathcal{A}$.

It is easy to see that the inclusion $A \subseteq S(c)$ implies $S(c) - A \subseteq \mathcal{A}$ for any $A \in \mathcal{A}$. Then both A and $S(c) - A$ are c-minimal sets of $\mathcal{A}$, and $A^c = \emptyset$ for all c-minimal sets $A \in \mathcal{A}$. Hence the minimal vectors of the coset $M(c, \mathcal{A})$ are $\pm\frac{1}{2}c[A]$ for c-minimal sets $A \in \mathcal{A}$ with $|A| \leq \frac{1}{2}|S(c)|$. This is the case for all cosets $\mathcal{A}$ if c has length $|S(c)| = 6$, i.e. if $c = c(ij)$, $ij \in E_5$, or $c = c^*(i)$, $i \in V_5$.

If $c = c(i)$ or $c = c^*(ij)$, then $|A^c| = 0$ or $|A^c| = 1$ for c-minimal sets $A \in \mathcal{A}$. In this case, if $|A^c| = 1$, i.e. $A^c = \{ij\}$ for some $ij \in E_5$, then the minimal vectors of $M(c, \mathcal{A})$ are $\pm\frac{1}{2}c[A_c] \pm e_{ij}$.

All in all, the family of all minimal vectors of the cosets $M(c, \mathcal{A})$ for $c \neq 0, j_{10}$ consists of the vectors $\pm\frac{1}{2}c[X]$, for all $X \subseteq S(c)$, $|X| \leq \frac{1}{2}S(c)$, $c = c(i)$, $c^*(i)$, $c(ij)$, $c^*(ij)$, and the vectors $\pm\frac{1}{2}c(i; X) \pm e_{jk}$, $\pm\frac{1}{2}c^*(ij; X) \pm e_{ik}$, for similar X.

For each of these 6 types of minimal vectors, Table 1 shows only the number # of cosets $M(c, \mathcal{A})$, $c \neq 0, j_n$, having the corresponding minimal vector, but not the cosets itselves. Besides, for the cosets $M(0, \mathcal{A})$, the set A in $\varepsilon(A)$ of Table 1 means $A \in \mathcal{A}$ of minimal cardinality.

Table 1

Coset $M(c,\mathcal{A}) = \frac{1}{2}c + \mathcal{A}$	# of cosets	Minimal vectors of the coset	Norm of minimal vectors	# of minimal vectors in the coset	dim$P(c,\mathcal{A})$
$M(0, \mathcal{A}_0)$	1	0	0	1	0
$M(0, \mathcal{A}_{ij})$	10	$\varepsilon(A)$	1	2	1
$M(0, \mathcal{B}_{i;jk})$	15	$\varepsilon(A)$	2	12	6
$M(0, \mathcal{D}_p)$	6	$\varepsilon(A)$	3	80	10
	40	$\pm\frac{1}{2}c(i; X)$	1	2	1
	320	$\pm\frac{1}{2}c(ij; X)$	$\frac{3}{2}$	2	1
	160	$\pm\frac{1}{2}c^*(i; X)$	$\frac{3}{2}$	2	1
	80	$\frac{1}{2}c^*(ij; X)$	1	2	1
	120	$\frac{1}{2}c(i; X) \pm e_{jk}$	2	4	2
	240	$\frac{1}{2}c^*(ij; X) \pm e_{ik}$	2	8	4
$M(j_{10}, \mathcal{A})$	32	$\frac{1}{2}j_{10}[A], A \in \mathcal{A}$	$\frac{5}{2}$	32	10

6 Delaunay polytopes of Q_{10}

Table 1 shows that the lattice Q_{10} has exactly two types of symmetric Delaunay polytopes: the polytopes $P(0, \mathcal{D}_p)$ of squared radius 3, and the polytopes $P(j_{10}, \mathcal{A})$ of squared radius $\frac{5}{2}$ for all 32 cosets $\mathcal{A}$ of C_{10} in $\mathbf{B}^{10}$. The Delaunay polytope $P(j_{10}, \mathcal{A}_0)$ is the translation by the vector $\frac{1}{2}j_{10}$ of the code polytope $P(C_{10})$.

We call any polytope congruent to a polytope P by the polytope of *type* P. Denote by P_3 a representative of the Delaunay polytopes of the type $P(0, \mathcal{D}_p)$, and by $P_5 = P(C_{10})$

the representative of the Delaunay polytopes of type $P(j_{10}, \mathcal{A})$. Besides, Q_{10} has an assymetric Delaunay polytope the type of which we denote by P_5'.

All types of Delaunay polytopes of a lattice L are represented in the *star* of Delaunay polytopes having the origin $0 \in L$ as a vertex. If two Delaunay polytopes are contiguous, (i.e. are adjacent), then they share a common facet (i.e. a face of dimension $n-1$).

Each symmetric Delaunay polytope $P(c, \mathcal{A})$ is represented in the star N times, where N is the number of vertices of $P(c, \mathcal{A})$, since each vertex of $P(c, \mathcal{A})$ can be the center of the star. Hence the star of Q_{10} contains $6 \times 80 = 240$ Delaunay polytopes of the type P_3, and $32 \times 32 = 1024$ Delaunay polytopes of type P_5.

Each asymmetric Delaunay polytope P is represented in the star $2N$ times, since each vertex of P can be the center of the star, and besides for all these representatives P_v the polytope $-P_v$ belongs to the star, too.

A bound on the covering radius of a lattice is given in [8]. The covering radius of a lattice is equal to the maximal radius of Delaunay polytopes. The bound is based on the following fact. Let r_1 be the maximal radius of not full-dimensional symmetric Delaunay polytopes. Then for the squared radius r^2 of an asymmetric Delaunay polytope the following bound holds: $r^2 \leq \frac{4}{3}r_1^2$.

We apply this bound to the lattice Q_{10}. We see from Table 1 that $r_1^2 = 2$. Hence the squared radius of asymmetric Delaunay polytopes of Q_{10} is not greater then $\frac{8}{3} < 3$. Comparing this with the maximal squared radius 3 of symmetric Delaunay polytopes of Q_{10}, we obtain that $\sqrt{3}$ is the covering radius of Q_{10}.

Let x be the center of a Delaunay polytope $P(x)$ of the star of a code lattice $L(C)$ whose set of vertices $V(x)$ contains the origin 0. Recall that if $P(x)$ is symmetric, then x belongs to a coset $M(c, \mathcal{A})$ of $L(C)$ in $\frac{1}{2}L(C)$, and $2x \in L(C)$, i.e. $2x$ is a vertex of $P(x)$. We have

$$(a-x)^2 \geq x^2, \text{ i.e. } 2ax \leq a^2 \text{ for all } a \in L(C), \tag{8}$$

with equality for $a \in V(x)$. Here x^2 is the squared radius of the sphere circumscribing $P(x)$. We say that x is of *full rank* if the system of inequalities (8) satisfied by x as equality determines uniquely x. Clearly, x of full rank is the center of a Delaunay polytope.

We seek Delaunay polytopes of Q_{10} as follows. Let F be a facet of $P(x)$ such that the origin 0 is a vertex of F. If h is a vector orthogonal to F, then the center of the Delaunay polytope P_1 contiguous to $P(x)$ by the facet F lies on the line $x(t) = x + th$. Suppose that the center of F is given by $x(t_0)$ with $t_0 \geq 0$. Then the center of P_1 is given by $x(t_1)$ for some $t_1 > t_0$. For to determine that $x(t_1)$ is the center of P_1, we use the following useful proposition. It is proved in [9] for cut lattices which are special cases of code lattices. We repeat its proof for sake a completeness.

Proposition 7 *Let $x \in \mathbf{R}^E$ satisfy (8) for $a = 2e_i$ for all $i \in E$, and for $a = c$ for all $c \in C$. Then*

(i) $x_i \leq 1$ for all $i \in E$,

(ii) $x_i = 1$ if $2e_i$ satisfies (8) as equality,

(iii) if $x \geq 0$, then x satisfies (8) for all $a \in L(C)$.

Proof. (i) is implied by the inequality (8) for $a = 2e_i$.

(ii) If $a = 2e_i$ satisfies (8) as equality, then this equality takes the form $x_i = 1$.

(iii) If $a \in L(c)$, then $a = c+2f$ for $c \in C$ and $f \in \mathbf{Z}^n$. Set $y = c-x$. Since $x \geq 0$ and, by (i), $x_i \leq 1$ for all i, we have $|y_i| \leq 1$ for all $i \in E$. Hence $(a-x)^2 = (2f+y)^2 = 4f(f+y)+y^2$. But $f(f+y) = \sum_{i:f_i \neq 0} |f_i||f_i+y_i| \geq 0$. Therefore $(a-x)^2 \geq y^2 = (c-x)^2 \geq x^2$. $\square$

6.1 Delaunay polytope $P_5 = P(C_{10})$

So, we have to know facets of known Delaunay polytopes P_3 and P_5. Recall that every reduced odd subsystem $\mathcal{U} \subseteq \mathcal{U}_{10}$ generates a graph $G(\mathcal{U})$. Conversely, for a graph $G = G(\mathcal{U})$, denote by $\mathcal{U}(G) \subseteq \mathcal{U}_{10}$ the odd system representing the graph G. Denote by $P(G)$ the convex hull of the endpoints of all vectors of $\mathcal{U}(G)$.

We say that a vector $h \in \mathbf{R}^E$ *define* a facet F of the polytope P_5 if the set $V(h)$ of all vertices v of P_5 satisfying the equality $hv = 0$ spans the facet F. Then h is the unique up to a multiple solution of the system of equations $yv = 0$, $v \in V(h)$.

Proposition 8 *(i) The vector $h = e_{ij}$ defines a facet of P_5 of type $P(L_2(4))$, and by this facet the polytope $P_5 = P(j_{10}, \mathcal{A}_0)$ is contiguous to the Delaunay polytope $P(j_{10}, \mathcal{A}_{ij})$ of the same type P_5.*

(ii) The vector $h = e_{ij} + e_{ik} + e_{jl} - e_{kl}$ defines a facet of P_5 of type $P(Shr)$, and by this facet P_5 is contiguous to an asymmetric Delaunay polytope P'_5 with the center $x(P'_5) = \frac{1}{2}j_{10} - \frac{1}{2}h$.

(iii) The vector $h = j_{10} - 2(e_{ij} + e_{jk} + e_{kl})$ defines a facet of P_5 of type $P(Pe)$, and by this facet P_5 is contiguous to the Delaunay polytope $P(0, \mathcal{D}_p)$ of type P_3, where $\mathcal{D}_p$ is the coset of C_{10} in $\mathbf{B}^E$ containing the three-path $(ijkl)$.

Proof. (i) The vector $h = e_{ij}$ partitions the set of all vertices of P_5 into two subsets $V_0 = \{v : e_{ij}v = 0\}$ and $V_1 = \{v : e_{ij}v = 1\}$. Each set contains 16 vertices and induces the greed graph $L_2(4)$. Such a partition of vertices of a polytope is possible only if each part spans a facet. Using Proposition 7, we obtain that P_5 is contiguous by the facet spanned by V_0 to a polytope P with the center $x(P) = \frac{1}{2}j_{10} - e_{ij} = \frac{1}{2}j_{10}[ij] \in M(j_{10}, \mathcal{A}_{ij})$, i.e. $P = P(j_{10}, \mathcal{A}_{ij})$.

(ii) The vector $h = e_{ij} + e_{ik} + e_{jl} - e_{kl}$ partitions the set of all vertices of P_5 into two subsets $V'_0 = \{v : hv = 0\}$ and $V'_2 = \{v : hv = 2\}$. Each set contains 16 vertices and induces the Shrikhande graph Shr. By Proposition 7, we obtain that P_5 is contiguous by the facet spanned by V'_0 to a polytope P with the center $x(P) = \frac{1}{2}j_{10} - \frac{1}{2}h$. Hence P is inscribed into a sphere of squared radius 5. Using the symmetry of Q_{10} we find that this polytope is asymmetric and contains 32 vertices. We call it P'_5.

(iii) It is easy to verify that the system of the equations $hv = 0$ for $h = j_{10} - 2(e_{ij} + e_{jk} + e_{kl})$ is satisfied by a set of 10 vertices of P_5 (containing 0). These 10 vertices induce the Petersen graph and uniquely up to a multiple determine the vector h. This means that $P(Pe)$ is a simplicial facet of P_5. Using Proposition 7 we obtain that P_5 is contiguous by this facet to a polytope with the center $e_{ij}+e_{jk}+e_{kl}\in M(0,\mathcal{D}_p)$, where the coset $\mathcal{D}_p$ contains the path $(ijkl)$. $\square$

We denote the facets in (i), (ii), and (iii) of Proposition 8 by $P_{ij}(L_2(4))$, $P_{ij,kl}(Shr)$ and $P_{ijkl}(Pe)$, respectively. We set

$$h_{ij,kl}=e_{ij}+e_{ik}+e_{jl}-e_{kl}, \text{ and } h_{(ijkl)}=j_{10}-2(e_{ij}+e_{jk}+e_{kl}).$$

We can compute the number $N(G)$ of facets of P_5 of type $P(G)$, $G=L_2(4)$, Shr, Pe, as follows. Let $N_0(G)$ be the number of facets of type $P(G)$ containing the vertex $c(\emptyset)=0$. Since the vectors e_{ij}, $h_{ij,kl}$ and $h_{(ijkl)}$ are orthogonal to two opposite facets, and one of these facets contains the vertex $c(\emptyset)$, $N_0(G)$ is equal to the number of the vectors of the corresponding type.

There are 10 vectors of type e_{ij}. There are 5 unordered quadruples $ijkl$, and each quadruple provides $3\times 4=12$ vectors of type $h_{ij,kl}$. The number of vectors of type $h_{(ijkl)}$ is equal to the number of three-paths $(ijkl)$, i.e. it is equal to 60. Since the automorphism group of P_5 is transitive on vertices, and P_5 has 32 vertices, $N(G)=32\frac{N_0(G)}{|G|}$, where $|G|$ is the number of vertices of the graph G. Hence we have

$$N(L_2(4))=20,\ N(Shr)=120,\ N(Pe)=192.$$

In [2], p.105, 3 regular subgraphs (which are not line graphs) of Shr are described, namely, a 8-vertex graph G_8 (which is a subgraph of Cle), a 10-vertex graph G_{10} and a 12-vertex graph G^1_{12}. The graph G^1_{12} is obtained from Shr by deleting a 4-coclique. But Shr has two types of 4-cocliques. The cocliques of one of these types are those cocliques whose switching transforms Shr into $L_2(4)$. Deleting a coclique of this type provides a planar regular line graph G^0_{12} of the same valency as G^1_{12}.

For to show that $P(G)$, $G=G_{10}$, G^0_{12}, G^1_{12}, are faces of P_5, we give explicitly the sets $V(h)$ for the three h from Proposition 8. We set $V_5=\{ijklm\}$.

We begin with

$$V(h_{(ijkl)})=\left\{\begin{array}{ccccc} c(\emptyset) & c(ik) & c(j) & c(k) & c(jl) \\ c^*(m) & c^*(lm) & c^*(ij) & c^*(kl) & c^*(im)\end{array}\right\}.$$

The vertices of each of these two lines form a 5-circuit of the Petersen graph Pe, and the two vertices of each column are adjacent. Recall that if we switch the 5 vertices of any line, we obtain the triangular graph $T(5)$ which is a subgraph of $T(6)$ and Cle.

We formate the vertices of $V(e_{ij})$ in the following 4×4 square such that the 4 vertices in each line (row and column) are mutually adjacent.

$$V(e_{ij})=\left\{\begin{array}{llll} c(kl) & c^*(jm) & c^*(im) & c(m) \\ c(km) & c^*(jl) & c^*(il) & c(l) \\ c(lm) & c^*(jk) & c^*(ik) & c(k) \\ c(\emptyset) & c^*(i) & c^*(j) & c(ij)\end{array}\right\}.$$

The four vertices of an arbitrary line (row of $L_2(4)$ induce a clique, i.e. the complete subgraph K_4. The 8 vertices of any two parallel lines of $L_2(4)$ form a minimal dependency of the odd system representing $L_2(4)$.

Note that the 4 vertices on the main diagonal form a coclique of $L_2(4)$. When we switch the main diagonal of the square, we obtain the set $V(h_{ij,kl})$ of vertices of the Shrikhande graph. The set of 12 vertices common to $L_2(4)$ and Shr induce the graph G^0_{12}.

Proposition 9 *The intersection $P_{ij}(L_2(4)) \cap P_{ij,kl}(Shr)$ is a 8-dimensional face of type $P(G^0_{12})$, i.e. it is a facet of both $P_{ij}(L_2(4))$ and $P_{ij,kl}(Shr)$. All facets of $P(L_2(4))$ are of the type $P(G^0_{12})$.*

Proof. It is not difficult to verify that the system of equations $yv = 0$ for $v \in V(e_{ij}) \cap V(h_{ij,kl}) = V(G^0_{12})$ has the solution $\alpha e_{ij} + \beta h_{ij,kl}$. This means that this system defines a facet of both $P_{ij}(L_2(4))$ and $P_{ij,kl}(Shr)$.

Now we prove that $P(G^0_{12})$ is the unique type of facets of $P(L_2(4))$. Let F be a facet of $P(L_2(4))$. Suppose that F contains 4 vertices inducing the complete graph K_4. But K_4 has the same squared radius $r^2(K_4) = 4 + \frac{1}{2}$ as $L_2(4)$ (see (6)). It means that vertices of K_4 span a 3-dimensional simplex with the center in the center of $P(L_2(4))$. If the facet F contains K_4, then F go through the center of the sphere circumscribing $P(L_2(4))$. But a polytope can has at most one facet going through the center of the sphere circumscribing it. Hence F contains all 8 cliques of $L_2(4)$, i.e. F contains $L_2(4)$, a contradiction. Hence the set $V(F)$ of vertices of any facet of $P(L_2(4))$ does not contain a 4-clique of $L_2(4)$. Since every clique of $L_2(4)$ intersects every its coclique, we have that $V(F) \subseteq V(L_2(4)) - V_c$, where V_c is the set of vertices of a coclique. But the set $V(L_2(4)) - V_c$ induces G^0_{12}, i.e. it is the set of vertices of a facet of $P(L_2(4))$. Since no two facets can lie one in the other, we have $V(F) = V(L_2(4)) - V_c$. $\square$

Now we formate the set $V(h_{ij,kl})$ in an array such that a vertex v of this array is adjacent to the vertices neighbouring to v by the vertical, the horizontal and the diagonal parallel to the main diagonal. We suppose that this array lies on a torus. This means that the vertices of the upper line are adjacent to the corresponding vertices of the low line, and the vertices of the left column are adjacent to the corresponding vertices of the right column, and similarly for diagonals.

$$V(h_{ij,kl}) = \left\{ \begin{array}{llll} c(jl) & c^*(j) & c^*(kl) & c(k) \\ c(m) & c^*(im) & c^*(il) & c(l) \\ c(ik) & c^*(jm) & c^*(ij) & c(km) \\ c(\emptyset) & c^*(i) & c^*(jk) & c(lm) \end{array} \right\}.$$

The four vertices of any line (i.e. row, column or diagonal) of Shr induce a 4-circuit. The 8 vertices of any two parallel nonadjacent lines form a minimal dependency of the odd system representing Shr.

Each vertex v of the set $V_{ij,kl}$ determines uniquely the set $V(v)$ of those vertices of the facet $P_{ij,kl}(Shr)$ by that the facet intersects a facet of type $P(Pe)$. Namely, $V(v)$ is the subset of vertices of $P_{ij,kl}(Shr)$ which are in the column, the line and the diagonal where v stays. Moreover, the set $V(v)$ with the switched vertex v^* induces the Petersen graph. The graph on 9 vertices induced by $V(v)$ dos not depend on v. Denote this graph by Pe_9, since it the Petersen graph with a vertes deleted. Therefore each facet of type $P(Shr)$ is adjacent to 16 facets of type $P(Pe)$ by the face $P(Pe_9)$.

By Proposition 9 we know that facets of type $P(L_2(4))$ are adjacent only to facets of type $P(Shr)$. A similar fact we have for facets of type $P(Pe)$.

Proposition 10 *All facets of type $P(Pe)$ are adjacent only to facets of type $P(Shr)$ (by faces of type $P(Pe_9)$).*

Proof. We saw that each facet of type $P(Shr)$ is adjacent to 16 facets of type $P(Pe)$. Since $N(Shr) = 120$, there are 120×16 faces of type $P(Pe_9)$. But each facet of type $P(Pe)$ is simplicial. Hence it has exactly 10 faces of dimension 8. Since $N(Pe) = 192$, there are 192×10 8-dimensional faces of $P(Pe)$. Comparing this number with the number of faces $P(Pe_9)$ in all facets $P(Shr)$, we see that each 8-dimensional face of $P(Pe)$ is an intersection of adjacent facets of types $P(shr)$ and $P(Pe)$. □

Using the set $V(h_{ij,kl})$, we find that the intersection $V(h_{ij,kl}) \cap V(h_{il,kj})$ contains 12 vertices and spans the graph G^1_{12}. Similarly, the intersection

$$V(h_{ij,kl}) \cap V(h_{im,kj}) = \left\{ \begin{array}{lll} c(\emptyset) & c(jl) & c^*(im) \\ c^*(i) & c^*(jk) & c^*(ij) \\ c(ik) & c(l) & c^*(il) \\ & c(k) & \end{array} \right\}$$

induces the graph G_{10}. Here, the 3 vertices of the first and the third columns form triangles of G_{10}. Each vertex v of the second column is adjacent only to two vertices of the line, where v stays, and to the vertex $c(k)$.

As in Proposition 9, it is not difficult to verify that $P(G^1_{12})$ and $P(G_{10})$ have dimension 8, i.e. they are facets of $P(Shr)$. Hence we have

Proposition 11 *The intersections $P_{ij,kl}(Shr) \cap P_{il,kj}(Shr) = P(G^1_{12})$ and $P_{ij,kl}(Shr) \cap P_{im,kj}(Shr) = P(G_{10})$ are 8-dimensional faces of P_5 by which a facet of type $P(Shr)$ is contiguous to a facet of the same type.*

6.2 Delaunay polytope P'_5

The Delaunay polytope P'_5 contiguous to P_5 by the facet $P_{ij,kl}(Shr)$ has besides this facet the second facet $P'_{ij,kl}(Shr)$ which is orthogonal to the vector $h_{ij,kl}$ (and therefore is parallel to $P_{ij,kl}(Shr)$) and contains the following set of 16 vertices inducing the Shrikhande graph:

$$V'(h_{ij,kl}) = \left\{ \begin{array}{llll} c(jl) - 2e_{ij} & c^*(j) - 2e_{ik} & c^*(kl) - 2e_{ij} & c(k) - 2e_{ik} \\ c(m) + 2e_{kl} & c^*(im) - 2e_{jl} & c^*(il) + 2e_{kl} & c(l) - 2e_{jl} \\ c(ik) - 2e_{jl} & c^*(jm) - 2e_{ik} & c^*(ij) - 2e_{ij} & c(km) - 2e_{ik} \\ c(\emptyset) + 2e_{kl} & c^*(i) - 2e_{jl} & c^*(jk) + 2e_{kl} & c(lm) - 2e_{jl} \end{array} \right\}.$$

(Note that $vh_{ij,kl} = -2$ for $v \in V'(h_{ij,kl})$.

Recall that vertices v and v' are adjacent if $(v-v')^2=6$, and all the vertices of the type $c(jl)-2e_{ij}$ have the form $c(jl;ij)$. If we take $c(\emptyset)+2e_{kl}$ as the new origin, then all vertices obtain the form $c(Y;X)$.

All the vertices of $P'_{ij,kl}(Shr)$ are partitioned into 4 groups (which are cocliques of Shr) such that the vertices of the same group have either the form $c-e_{\alpha\beta}$, for $\alpha\beta\in\{ij,ik,jl\}$, with the same vector $e_{\alpha\beta}$, or the form $c+e_{kl}$. Call the vertices $c+e_{\alpha\beta}$ and c *matched*. Then the distance between the vertices of matched groups is equal to 8, excluding the matched pairs having distance $(2e_{\alpha\beta})^2=4$. The distance between vertices of non matched groups is not changed. This implies that each of the $2^4=16$ sets of 16 vertices of P'_5 containing precisely one of the matched groups induce the Shrikhande graph.

Recall that the center of P'_5 is $\frac{1}{2}j_{10}-\frac{1}{2}h_{ij,kl}=e_{kl}+\frac{1}{2}e(T)$, where $T=E_5-\{ij,ik,jl,kl\}$. We take the center of P'_5 as the new origin. Then the vectors $v=\sqrt{2}(c-e_{kl}-\frac{1}{2}e(T))$, c is a vertex of P'_5, form a closed odd system $\mathcal{V}_5$ of 32 vectors of norm 5 with $\pm1,-3$ inner products. The inner product -3 have the vectors corresponding to vertices at distance 8. Note that the vectors of $\mathcal{V}_5$ corresponding to a pair of matched vertices are not opposite, since $\mathcal{V}_5$ has no pair of opposite vectors.

Clearly, P'_5 is contiguous by the facet $P'_{ij,kl}(Shr)$ to a polytope of type P_5. The polytope P'_5 has two other types of facets. Similarly to P_5, the vector of the type $e_{\alpha\beta}$, for $\alpha\beta\in T$, partitions the set of vertices of P'_5 into two parallel facets of the same type $P(L'_2(4))$. $P(L'_2(4))$ contains 16 vertices which induce the graph $L_2(4)$ if we suppose that two vertices are adjacent if the distance between them is either 6 or 8. As in the case of P_5, the polytope P'_5 is contiguous by the facet $P'(L_2(4))$ to a polytope of the same type P'_5.

A facet of the third type $P(T)$ contains 18 vertices. It is defined by the vector of the type $h_{T,il,jk}=e(T)-2e_{il}-2e_{jk}$. We have

$$V(h_{T,il,jk})=\left\{\begin{array}{ll|ll|ll} c(\emptyset) & c(\emptyset)+2e_{kl} & c^*(il) & c^*(il)+2e_{kl} & c^*(jk) & c^*(jk)+2e_{kl}\\ c(jl) & c(jl;ij) & c(l) & c(l;jl) & c^*(jm) & c^*(jm;ik)\\ c(ik) & c(ik;ij) & c^*(im) & c^*(im;jl) & c(k) & c(k;ik) \end{array}\right\}.$$

Here two vertices of the form $c,c+2e_{kl}$ are adjacent only to four vertices written below them. It is easy to verify that P'_5 is contiguous by this facet to the polytope of the type P_3 with the center $e_{kl}+\frac{1}{2}e(T)-\frac{1}{2}h_{T,il,jk}=e_{jk}+e_{kl}+e_{il}$.

The facet $P(T)$ intersects the facets $P_{ij,kl}(Shr)$ and $P'_{ij,kl}(Shr)$ by 8-dimensional simplicial faces of type $P(Pe_9)$. Recall that these faces are also faces of facets of type $P(shr)$ and $P(Pe)$ of P_5.

6.3 Delaunay polytope P_3

Recall that a polytope of type P_3 is the convex hull of endpoints of vectors $\varepsilon(A)=\sum_{ij\in A}\varepsilon_{ij}e_{ij}$ for all $A\in\mathcal{D}_p$, where A is a three path of the coset $\mathcal{D}_p$ and $\varepsilon_i\in\{\pm1\}$. The 80 minimal vectors $\varepsilon(A)$ of the coset $M(0,\mathcal{D}_p)$ form a system $\mathcal{W}_3$ of vectors of norm 3 with $0,\pm1$ inner products. Call such a system a $(0,\pm1)$-*system of norm 3.*

Similarly to as it is done in Section 2, one can define a closure of a $(0,\pm1)$-system of norm 3. But, unfortunately, the closure can be not a $(0,\pm1)$-system, since it can contain a vector of norm 3 with inner products ± 2 with some vectors of the original system.

Shult and Yanushka study in [13] $(0, \pm 1)$-systems of norm 3 which are *tetrahedrally closed*. A $(0, \pm 1)$-system of norm 3 is tetrahedrally closed if, with any 3 vectors with inner products -1, it contains the sum of these vectors (which is a vector of norm 3). Clearly, a closed $(0, \pm 1)$-system is tetrahedrally closed. There are maximal $(0, \pm 1)$-systems of norm 3 which are not tetrahedral closed. (A $(0, \pm 1)$-system of norm 3 is called maximal if, without augmenting its dimension, one cannot add to this system a vector of norm 3 having inner products $0, \pm 1$ with all vectors of this system.)

In [13] one considers *regular* tetrahedrally closed $(0, \pm 1)$-systems $\mathcal{W}$ of norm 3 with the following strong property of a regularity. Let $G(\mathcal{W})$ be the graph vertices of which are lines spanned by vectors of $\mathcal{W}$. Two vertices of $G(\mathcal{W})$ are adjacent if the corresponding lines are skew (i.e. are not orthogonal). Then the regularity condition is equivalent to the graph $G(\mathcal{W})$ is strongly regular.

It is proved in [13] that there exist only 6 families of parameters of strongly regular graphs corresponding to regular $(0, \pm 1)$-systems of norm 3. In particular, it is constructed in [13] a regular $(0, \pm 1)$-system $\mathcal{W}_0$ whose graph $G(\mathcal{W}_0)$ is one of these strongly regular graphs. It has parameters $(v, k, \lambda, \mu) = (40, 27, 18, 18)$.

The graph $G(\mathcal{W}_3)$ related to the $(0, \pm 1)$-system of the Delaunay polytope P_3 is "almost" strongly regular: it has $v = 40$ vertices and it is co-edge-regular of valency $k = 27$ and $\mu = 18$, but it has $\lambda = 18$ for 480 edges and $\lambda = 26$ for other 60 edges.

We know two types of facets of P_3, the types $P(Pe)$ and $P(T)$. A facet of type $P(T)$ contains 9 pairs of matched vertices. There are two types of simplicial faces of the facet $P(T)$. Each type of faces defines a partition of $P(T)$ into two parallel faces of the same type.

A simplicial face of type $P(Pe_9)$ consists of vertices chosen by one from each pair of matched vertices of $P(T)$ such that vertices of the same group have the same form, i.e. all are c or all are $c - e_{\alpha\beta}$ $(c + e_{kl})$. Since there are 4 groups, the facet $P(T)$ has 2^4 faces of type $P(Pe_9)$ in P_3.

A simplicial face of the other type $P(3\Lambda)$ (the denotation is explained below) consists of 9 vertices chosen as follows. One chooses 3 vertices of the same form from the group of type $c, c + e_{kl}$, and by a pair of matched vertices from other groups. Then each of the first 3 vertices with the chosen pair of matched vertices below it induces a two-path Λ. All the chosen 9 vertices induce 3 disjoint two-paths, i.e. the graph 3Λ. It is easy to see that the facet $P(T)$ has also 2^4 faces of type $P(3\Lambda)$.

In general, a facet of type $P(T)$ in P_3 is defined by the vector of type $\varepsilon(T) = \sum_{ij \in T} \varepsilon_{ij} e_{ij}$. There is a one-to-one correspondence between sets T and three-paths $A \in \mathcal{D}_p$, where the coset $\mathcal{D}_p$ defines the Delaunay polytope P_3. If $A = (ijkl)$, then $T = E_5 - \{ik, il, jk, jl\}$.

We have the following analog of Proposition 10.

Proposition 12 *The facets of P_3 of type $P(Pe)$ are adjacent only to facets of type $P(T)$.*

Proof. At first we compute the number of the facets of type $P(Pe)$ in P_3. If we take the center of P_3 as the origin, then the vertices of P_3 are the vectors $\varepsilon(A)$ for all three-paths

$A \in \mathcal{D}_p$. The set of 10 vertices of a facet of type $P(Pe)$ consists of vectors $\varepsilon(A)$ for 10 distinct sets $A \in \mathcal{D}_p$ such that the signs ε_{ij} for each $ij \in E_5$ are the same in all $\varepsilon(A)$. Since there are 2^{10} possibilities of choosing signs, we obtain that there are 2^{10} facets of type $P(Pe)$ in P_3.

We saw that a facets of type $P(T)$ is uniquely determined by the vector $\varepsilon(T)$, where the set T is related to a three-path $A \in \mathcal{D}_p$. For a given T, there are 2^6 vectors $\varepsilon(T)$. Besides, there are $10 = |\mathcal{D}_p|$ sets T. Hence there are 10×2^6 facets of type $P(T)$ in P_3. Recall that $P(T)$ has 2^4 faces of type $P(Pe_9)$. Therefore we have $2^4 \times 10 \times 2^6 = 10 \times 2^{10}$ faces of type $P(Pe_9)$ in P_3. This is just the number of all 8-dimensional faces of the facets $P(Pe)$ in P_3. $\square$

The Delaunay polytope P_3 has a third type of facets defined by vectors of type $h_{[ijlk]} = e_{ij} + e_{jl} + e_{kl} + e_{ik}$. The polytope P_3 with the center $x(P_3) = e_{ij} + e_{jk} + e_{kl}$ is contiguous by this facet to a polytope of the same type P_3 with the center in $e_{jk} - e_{jl} - e_{ik}$. This Delaunay polytope belongs to the coset $M(0, \mathcal{D}_p)$, where the coset $\mathcal{D}_p$ contains the path $(ikjl)$. The facet defined by the vector $h_{[ijlk]}$ contains the following set of 12 vertices:

$$V(h_{[ijlk]}) = \left\{ \begin{array}{ll|ll|ll} c(\emptyset) & c^*(m; ij, kl) & c(k; kl) & c(k; kl, km) & c^*(im; kl) & c^*(im; kl, im) \\ 2e_{jk} & c^*(m; ij, kl, il) & c(j; ij) & c(j; ij, jm) & c^*(lm; ij) & c^*(lm; ij, lm) \end{array} \right\}$$

This array has 3 squares. The 4 vertices of each square induce a 4-circuit and span a 3-dimensional simplex with edges of norm 4 and 6. All the 3 simplices have the same squared radius 2 and the common center which is the center $e_{jk} + \frac{1}{2}(e_{ij} - e_{jl} + e_{kl} - e_{ik})$ of this facet. The spaces spanned by them are mutually orthogonal. This polytope is denoted in [1] by V_9^{333}. It belongs to a class of *repartitioning* polytopes. The squared radius of V_9^{333} is equal also to 2. Since there are 3 minimal dependencies, each containing two 4-circuits, all independent subsets of vectors of $V(h_{[ijlk]})$ have the same type. Namely, they consist of 9 vertices by 3 from each 4-circuit. The corresponding sets of vertices induce the graph 3Λ consisting of 3 disjoint two-paths.

Clearly V_9^{333} has $4^3 = 2^6$ faces of type $P(3\Lambda)$. In general, the facets of type V_9^{333} are defined by vectors $\varepsilon(E_5 - T)$, where $E_5 - T$ is the set of edges of the 4-circuit $[ijlk]$. Since there are 10 sets T, the polytope P_3 has 10×2^4 facets of type V_9^{333}. Therefore there are $2^6 \times 10 \times 2^4 = 10 \times 2^{10}$ faces of type $P(3\Lambda)$ in all facets V_9^{333} of P_3. Since this is the number of these faces in all facets $P(T)$, we have the following analog of the above proposition.

Proposition 13 *The facets of P_3 of type V_9^{333} are adjacent by faces of type $P(3\Lambda)$ only to facets of type $P(T)$. All 8-dimensional faces of V_9^{333} are of the type $P(3\Lambda)$.*

Concluding this section I give in Table 2 below the known me types of facets of the L-partition of the lattice Q_{10}. The type of the facet which is the common facet of adjacent Delaunay polytopes of types P and P' stays in the intersection of the row and the column corresponding to P and P'.

Table 2

	P_5	P_5'	P_3
P_5	$P(L_2(4))$	$P(Shr)$	$P(Pe)$
P_5'	$P(Shr)$	$P(L_2'(4))$	$P(T)$
P_3	$P(Pe)$	$P(T)$	V_9^{333}

Taking in attention the propositions of this section, one can conjecture that it is very plausible that the L-partition of the lattice Q_{10} has no other Delaunay polytope besides the polytopes of Table 2.

Acknowledgment

I am indebted to prof. Walter Deuber for his kind invitation me to work in SFB 343 Diskrete Strukturen in der Mathematik. The friendly atmosphere of this Abteilung helps me to do this work.

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