Stochastic quantization of the two-dimensional polymer measure

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Abstract

We prove that there exists a diffusion process whose invariant measure is the two-dimensional polymer measure ν_g. The diffusion is constructed by means of the theory of Dirichlet forms on infinite-dimensional state spaces. We prove the closability of the appropriate pre-Dirichlet form which is of gradient type, using a general closability result by two of the authors. This result does not require an integration by parts formula (which does not hold for the two-dimensional polymer measure ν_g) but requires the quasi-invariance of ν_g along a basis of vectors in the classical Cameron-Martin space such that the Radon-Nikodym derivatives (have versions which) form a continuous process. We also show the Dirichlet form to be irreducible or equivalently that the diffusion process is ergodic under time translations.

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1 Introduction and main results

Let ν be a given probability measure on a space X. The stochastic quantization of ν means the construction of a Markov process whose invariant measure is just ν. Parisi and Wu [PW81] first studied the stochastic quantization for certain gauge fields in quantum field theory. After that, many works (see e.g. [J-LM85], [AR89a], [AR89b], [AR91]) on the stochastic quantization of Euclidean quantum fields such as the $P(\phi)_2$ - field appeared. There is a close connection between the Edward's model (or say polymer measure) and the ϕ^4 - field (see e.g. [Sy69, AFH-KL86]). This naturally directs the interest to the stochastic quantization of a polymer measure. The main aim of the present paper is to study the stochastic quantization of the two dimensional polymer measure.

Let us first introduce the (rigorous) definition of the two dimensional polymer measure ν_g. Let $X := C([0,1], \mathbb{R}^2)_0$ be the set of all continuous paths in $\mathbb{R}^2$ indexed by $[0,1]$ and starting at zero. Let $\mathcal{B}$ be the σ-algebra generated by all maps $\omega \mapsto \omega_t,\ t \geq 0$, from X to $\mathbb{R}^2$ and let ν_0 denote the Wiener measure on $(X, \mathcal{B})$. Let $\alpha(x, A)$ be the self-intersection local time at $x \in \mathbb{R}^2$ of $\omega \in X$ on the set $A \subset [0,1] \times [0,1]$, i.e.

$$\alpha(x, A) = \int_A \delta_x(\omega_s - \omega_t) ds dt.$$

(For its precise definition, the reader is referred to [Ro86] and the references therein.) It is well-known that $\alpha(0, \{(s,t) : 0 \leq s < t \leq 1\}) = \infty,\ \nu_0 - a.e.$ Therefore, one has to use renormalization. To this end we set

$$\alpha_{i,k} = \alpha\Big(0, \Big[\frac{2(i-1)}{2^k}, \frac{2i-1}{2^k}\Big) \times \Big[\frac{2i-1}{2^k}, \frac{2i}{2^k}\Big)\Big), \ \ i = 1, \ldots, 2^{k-1},\ k = 1, 2, \ldots.$$

Let E be the expectation with respect to ν_0, and

$$\xi_n = \sum_{k=1}^{n} \sum_{i=1}^{2^{k-1}} (\alpha_{i,k} - E\alpha_{i,k}), \quad n \geq 1.$$

Then one can prove (see [LG85], [V69]) that $(\xi_n)_{n \geq 0}$ is almost surely convergent to a random variable $\xi \in L^2(\Omega; \nu_0)$ and $\lim_{n \to \infty} E|\xi_n - \xi|^2 = 0$. The random variable ξ is usually called the normalized self-intersection local time of planar Brownian motion. It is proved in [AHZ95] that the random variable ξ is not differentiable in the sense of Meyer-Watanabe. One can prove that there is $g_0 \in (0, \infty)$ (see e.g. [LG94], [PY96]) such that

$$E \exp(-g\xi) \begin{cases} < \infty, & \forall g \in (-g_0, \infty), \\ = \infty, & \forall g \in (-\infty, -g_0). \end{cases}$$

The two dimensional polymer measure ν_g is defined by

$$\nu_g := (E \exp(-g\xi))^{-1} \exp(-g\xi)\ \nu_0\,, \quad g \in (-g_0, \infty).$$

In this paper we always assume that $g \in (-g_0, \infty)$.

For the construction of the stochastic quantization of ν_g we shall use the theory of Dirichlet forms. We want to emphasize that this method only gives a way to "*reconstruct*" ν_g (cf. Theorems 1.1 and 1.5 below), since it uses ν_g and is based on some of its specific properties. However, also in all approaches to the stochastic quantization of measures in Euclidean field theory (such as e.g. $P(\phi)_2$-fields) the respective measures have been used in an essential way. A truly constructive approach is not known. It seems that the situation for measures in Euclidean field theory as well as for polymer measures is just too singular. (E.g. all attempts using strong solutions for appropriate stochastic differential equations to obtain the required Markov processes have failed so far.) To describe the Dirichlet form in question precisely, we need some preparations.

X equipped with the supremum-norm is a separable real Banach space. Let X' denote its dual. Let $H \subset X$ be the classical Cameron-Martin space, i.e., $H := \{h \in X : \ h \text{ is absolutely continuous and } |h|_H^2 = \int_0^1 |h_t'|^2 dt < \infty\}$. Furthermore, let $K := \{h \in H : \ \sup_{0 \le t \le 1} |h'(t)| < \infty\}$. $(H, \langle\ ,\ \rangle_H)$ is a real separable Hilbert space which is densely and continuously embedded into X, and K is a dense linear subspace of $(H, \langle\ ,\ \rangle_H)$. Identifying H with its dual we obtain that X' is densely embedded into H. Hence $X' \subset H \subset X$, and $\langle\ ,\ \rangle_H$ restricted to $X' \times H$ coincides with the dualization between X' and X. Define the set of smooth bounded cylinder functions by

$$\mathcal{F}C_b^\infty := \{f(l_1, \dots, l_m) \ : \ m \in \mathbb{N}, \ f \in C_b^\infty(\mathbb{R}^m), \ l_1, \dots, l_m \in X'\},$$

where $C_b^\infty(\mathbb{R}^m)$ denotes the set of all bounded infinitely differentiable functions on $\mathbb{R}^m$ with all partial derivatives bounded. Let $k \in X$. For $u \in \mathcal{F}C_b^\infty$, $\omega \in X$, define

$$\frac{\partial u}{\partial k}(\omega) := \frac{d}{ds} u(\omega + sk)|_{s=0}$$

and let $\nabla u(\omega)$ denote the unique element in H such that

$$\langle \nabla u(\omega), k \rangle_H = \frac{\partial u}{\partial k}(\omega) \quad \text{for all } k \in H.$$

Define for $u, v \in \mathcal{F}C_b^\infty$

$$\mathcal{E}_{\nu_g}(u, v) := \int \langle \nabla u(\omega), \nabla v(\omega) \rangle_H \ \nu_g(d\omega)$$

and $\mathcal{E}_{\nu_g,1} := \mathcal{E}_{\nu_g} + (\ ,\ )_{L^2(X;\nu_g)}$. Note that since $\nu_g \ll \nu_0$ and since $\nu_0(U) > 0$ for every $\emptyset \neq U \subset \Omega$, U open, we can identify each $u \in \mathcal{F}C_b^\infty$ with the corresponding class in $L^2(X; \nu_g)$. In particular, $(\mathcal{E}_{\nu_g}, \mathcal{F}C_b^\infty)$ is a symmetric positive definite bilinear form on $L^2(X; \nu_g)$. Now we can formulate our first main result.

Theorem 1.1 *$(\mathcal{E}_{\nu_g}, \mathcal{F}C_b^\infty)$ is closable on $L^2(X;\nu_g)$ and the closure $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$ is a symmetric* Dirichlet form *(i.e., a closed positive definite symmetric bilinear form such that $u^\# := \min(\max(u,0),1) \in D(\mathcal{E}_{\nu_g})$ and $\mathcal{E}_{\nu_g}(u^\#, u^\#) \leq \mathcal{E}_{\nu_g}(u,u)$ for all $u \in D(\mathcal{E}_{\nu_g})$).*

For more details on Dirichlet forms the reader is referred e.g. to [MR92], [FOT94], [BH91], [Si74], and for the special type of Dirichlet form appearing here, namely so-called *classical Dirichlet forms* to [AR90]. For the convenience of the reader, however, we recall all notions necessary here in Section 3 below.

Let $(L, D(L))$ be the generator of $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$, i.e., the unique negative definite self-adjoint operator on $L^2(X;\nu_g)$ such that

$$D(\sqrt{-L}) = D(\mathcal{E}_{\nu_g}) \qquad \text{and} \qquad (\sqrt{-L}u, \sqrt{-L}v) = \mathcal{E}_{\nu_g}(u,v) \quad \text{for all } u,v \in D(\mathcal{E}_{\nu_g}). \tag{1.1}$$

Let $T_t := e^{tL},\ t \geq 0$.

Theorem 1.2 *There exists a diffusion process $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_\omega)_{\omega\in X})$ which is* associated *with $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$, i.e., for all ($\nu_g$-versions of) $f \in L^2(X;\nu_g)$ and all $t \geq 0$ the function*

$$\omega \mapsto p_t f(\omega) := \int_\Omega f(X_t)\, dP_\omega\ , \quad \omega \in X,$$

is a ν_g-version of $T_t f$. $\mathbb{M}$ is conservative and ν_g-symmetric. In particular, ν_g is an invariant measure for $\mathbb{M}$.

Remark 1.3 (i) $\mathbb{M}$ is in fact even *properly associated* to $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$ in the sense of [MR92, Chap. IV, Definition 2.5].

(ii) Ω in Theorem 1.2 can, of course, always be taken to be $C([0,\infty), X)$, since $\mathbb{M}$ has continuous sample paths and is conservative.

The proofs of Theorems 1.1 and 1.2 well be given in Section 3 below. The most difficult part is to prove the closability of $(\mathcal{E}_{\nu_g}, \mathcal{F}C_b^\infty)$ on $L^2(X;\nu_g)$. The rest follows from known results. The closability is implied by a general closability criterion (cf. [AR89b, Corollary 2.5]) and the following result on ν_g which is certainly of its own interest. For $k \in X$ define $\tau_k(\omega) := \omega + k$, $\omega \in X$.

Theorem 1.4 *Let $k \in K$. Then ν_g is k-*quasi-invariant*, i.e., $\nu_g \circ \tau_{sk}^{-1}$ is absolutely continuous w.r.t. ν_g for all $s \in \mathbb{R}$. If $a_{sk} := \frac{d(\nu_g \circ \tau_{sk}^{-1})}{d\nu_g}$, $s \in \mathbb{R}$, then the process $(a_{sk})_{s\in\mathbb{R}}$ has a version which has $\nu_0 - a.e.$ continuous sample paths.*

The proof of Theorem 1.4 is given (independently of the rest of this paper) in Section 2 below.

We want to emphasize that [AR89b, Corollary 2.5] (which just states that whenever $(a_{sk})_{s\in\mathbb{R}}$ as above has a continuous version for k in an orthonormal basis of H, then the corresponding classical Dirichlet form is closable) is a closability criterion of a totally different kind than the ones proved in older papers like [AH-K77a], [AH-K77b], [Ku82]. There, always an integration by parts formula (cf. e.g. [AKuR90]) was assumed to hold. In the case of the polymer measure ν_g such an integration by parts formula does not hold (cf. [AHZ95]), but nevertheless we have closability. [AR89b, Corollary 2.5] is a consequence of the main result in [AR90, Sect. 3], which "characterizes" closability.

Our final result is the following

Theorem 1.5 *If $g < g_0$, then the Dirichlet form $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$ is* irreducible *(i.e., $u \in D(\mathcal{E}_{\nu_g})$ with $\mathcal{E}_{\nu_g}(u,u) = 0$ implies that u is a constant) or equivalently the associated diffusion $\mathbb{M}$ when started with ν_g is ergodic under time translations. In particular, $\lim_{t\to\infty} p_t f = \int f d\nu_g \quad \nu_g - a.e.$ for all bounded $\mathcal{B}$-measurable functions $f : X \to \mathbb{R}$.*

Remark 1.6 The last convergence in Theorem 1.5 in fact holds even quasi-everywhere. This follows from Fukushima's quasi-everywhere ergodic theorem (c.f. [F83], which by the "transfer method" in [MR92, Chap. VI] generalizes to this "non-locally compact case").

The proof of Theorem 1.5 will be given at the end of Section 3.

Finally, we would like to mention that Theorems 1.1, 1.2 and 1.4 also hold for the polymer measure in three dimensions. The proof of Theorem 1.4 is, however, much more complicated since ν_g is no longer absolutely continuous w.r.t. ν_0 in this case. All this is the contents of the forthcoming paper [ARZ95].

2 Quasi-invariance of the two-dimensional polymer measure

In this section we prove Theorem 1.4. To this end we will use a new approximation for the random variable ξ. Some ideas given below are also suitable for the three-dimensional case. In a forthcoming paper [ARZ95] we shall use a similar approach to prove that the three dimensional polymer measure is also quasi-invariant.

We set

$$g_n(x) := cI_{[0,n]}(|x|), \quad n \geq 1, \ x \in \mathbb{R}^2,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$, and the constant $c \in (0,\infty)$ will be specified below. It is easy to show that there is a constant $c \in (0,\infty)$ such that the following convergence holds in the vague sense

$$\int_{\mathbb{R}^2} e^{i\langle x,y\rangle} g_n(y)dy \to \delta_0(x), \quad n \to \infty,$$

where δ_0 is the Dirac function at point $0 \in \mathbb{R}^2$, and $\langle x,y\rangle = x_1y_1 + x_2y_2$ if $x = (x_1,x_2)$ and $y = (y_1,y_2)$. Let

$$\eta_{n,\varepsilon}(\omega) := \int_{T_\varepsilon} \int_{\mathbb{R}^2} e^{i\langle y,\, \omega_t - \omega_s\rangle} g_n(y)dy\ ds\ dt,$$

where $T_\varepsilon := \{(s,t):\ t-s \geq \varepsilon,\ 0 \leq s < t \leq 1\}$. Let

$$\tilde{\eta}_{n,\varepsilon}(u,k) := \eta_{n,\varepsilon}(\omega + uk) - \eta_{n,\varepsilon}(\omega).$$

Let us first prove two lemmas.

Lemma 2.1 *Let $\gamma \in (0,1)$ and $k \in K$ be given. Then there is a constant $C \in (0,\infty)$ such that*

$$E|\tilde{\eta}_{n,\varepsilon}(u_1,k) - \tilde{\eta}_{n,\varepsilon}(u_2,k)|^2 \leq C|u_1 - u_2|^{1+\gamma}$$

for all $u_1, u_2 \in \mathbb{R}$, $\varepsilon \in (0,1)$ and $n \geq 1$, where E denotes expectation w.r.t. ν_0.

Proof. By definition we know that $\tilde{\eta}_{n,\varepsilon}(u_1,k) - \tilde{\eta}_{n,\varepsilon}(u_2,k)$ is equal to

$$\int_{T_\varepsilon} \int_{\mathbb{R}^2} e^{i\langle y,\omega_t-\omega_s\rangle} (e^{i\langle y,u_1(k_t-k_s)\rangle} - e^{i\langle y,u_2(k_t-k_s)\rangle}) g_n(y)dydsdt.$$

Let

$$\begin{aligned}
T_1(\varepsilon) &:= \{0 \leq s_1 < t_1 < s_2 < t_2 \leq 1\} \cap \cap_{i=1}^2 \{t_i - s_i \geq \varepsilon\},\\
T_2(\varepsilon) &:= \{0 \leq s_1 < s_2 < t_1 < t_2 \leq 1\} \cap \cap_{i=1}^2 \{t_i - s_i \geq \varepsilon\},\\
T_3(\varepsilon) &:= \{0 \leq s_1 < s_2 < t_2 < t_1 \leq 1\} \cap \cap_{i=1}^2 \{t_i - s_i \geq \varepsilon\},
\end{aligned}$$

and let $I_{l,\varepsilon}$ be equal to

$$\int_{T_l(\varepsilon)} \int_{\mathbb{R}^4} Ee^{i\langle y_1,\omega_{t_1}-\omega_{s_1}\rangle + i\langle y_2,\omega_{t_2}-\omega_{s_2}\rangle} (e^{iu_1\langle y_1,k_{t_1}-k_{s_1}\rangle} - e^{iu_2\langle y_1,k_{t_1}-k_{s_1}\rangle})$$
$$\times (e^{iu_1\langle y_2,k_{t_2}-k_{s_2}\rangle} - e^{iu_2\langle y_2,k_{t_2}-k_{s_2}\rangle}) g_n(y_1)g_n(y_2) \prod_{j=1}^2 dy_j ds_j dt_j, \quad l = 1,2,3.$$

Then it is easy to see that

$$E|\tilde{\eta}_{n,\varepsilon}(u_1,k)-\tilde{\eta}_{m,\varepsilon}(u_2,k)|^2 = 2\sum_{l=1}^{3} I_{l,\varepsilon}.$$

If $(s_1,s_2,t_1,t_2)\in T_1(\varepsilon)$, then

$$Ee^{i\langle y_1,\omega_{t_1}-\omega_{s_1}\rangle+i\langle y_2,\omega_{t_2}-\omega_{s_2}\rangle} = e^{-\frac{1}{2}|y_1|^2(t_1-s_1)-\frac{1}{2}|y_2|^2(t_2-s_2)}. \tag{2.1}$$

Since $\sin(-x)=-\sin x$, $I_{1,\varepsilon}$ is equal to

$$\begin{aligned}&\int_{T_1(\varepsilon)}\int_{\mathbb{R}^4} e^{-\frac{1}{2}|y_1|^2(t_1-s_1)-\frac{1}{2}|y_2|^2(t_2-s_2)}g_n(y_1)g_n(y_2)\\ &\qquad\times\prod_{i=1}^{2}(\cos(u_1\langle y_i,k_{t_i}-k_{s_i}\rangle)-\cos(u_2\langle y_i,k_{t_i}-k_{s_i}\rangle))\ dy_i\ ds_i\ dt_i.\end{aligned}$$

We remark that for any given $\alpha\in(0,1]$ there is a constant $C\in(0,\infty)$ (from now on, $C\in(0,\infty)$ will denote a constant which might be different from line to line) such that

$$|\cos y_1-\cos y_2|\le C|y_1-y_2|^{\alpha}\wedge 1,\quad y_1,y_2\in\mathbb{R}. \tag{2.2}$$

By (2.1) and (2.2) it follows that $I_{1,\varepsilon}$ is less than

$$\begin{aligned}&C|u_1-u_2|^{1+\gamma}\int_{T_1(0)}\int_{\mathbb{R}^4} e^{-\frac{1}{2}|y_1|^2-\frac{1}{2}|y_2|^2}g_n(y_1/\sqrt{t_1-s_1})g_n(y_2/\sqrt{t_2-s_2})\\ &\quad\times\prod_{i=1}^{2}(t_i-s_i)^{-1}\langle y_i,(k_{t_i}-k_{s_i})/\sqrt{t_i-s_i}\rangle^{(1+\gamma)/2}dy_ids_idt_i\\ &\le C|u_1-u_2|^{1+\gamma}\int_{T_1(0)}\int_{\mathbb{R}^4} e^{-\frac{1}{2}|y_1|^2-\frac{1}{2}|y_2|^2}g_n(y_1/\sqrt{t_1-s_1})g_n(y_2/\sqrt{t_2-s_2})\\ &\quad\times|y_1|^{(1+\gamma)/2}|y_2|^{(1+\gamma)/2}\prod_{i=1}^{2}(t_i-s_i)^{-(3-\gamma)/4}\ dy_i\ ds_i\ dt_i\\ &\le C|u_1-u_2|^{1+\gamma}.\end{aligned}$$

We now consider $I_{2,\varepsilon}$. Let $\sigma=(t_2-s_2)^{-1}((t_2-t_1)(t_1-s_1)+(t_1-s_2)(s_2-s_1))$. If $(s_1,s_2,t_1,t_2)\in T_2(\varepsilon)$, then the left hand side of (2.1) is equal to

$$\exp\Big(-\frac{1}{2}|y_1|^2\sigma-\frac{1}{2}\Big|y_2+y_1\frac{t_1-s_2}{t_2-s_2}\Big|^2(t_2-s_2)\Big),$$

which is denoted by $f_1(s,t,y)$. We also remark that for any given $\alpha\in(0,1]$ there is a constant $C\in(0,\infty)$ such that

$$|\sin y_1-\sin y_2|\le C|y_1-y_2|^{\alpha}\wedge 1,\quad y_1,y_2\in\mathbb{R}. \tag{2.3}$$

Using (2.2) and (2.3) we readily see that $I_{2,\varepsilon}$ is equal to

$$\begin{aligned}
&\int_{T_2(\varepsilon)}\int_{\mathbb{R}^4} g_n(y_1)g_n(y_2)\Big(-\prod_{i=1}^{2}(\sin(u_1\langle y_i,k_{t_i}-k_{s_i}\rangle-\sin(u_2\langle y_i,k_{t_i}-k_{s_i}\rangle))\\
&\qquad+\prod_{i=1}^{2}(\cos(u_1\langle y_i,k_{t_i}-k_{s_i}\rangle)-\cos(u_2\langle y_i,k_{t_i}-k_{s_i}\rangle))\Big)f_1(s,t,y)\prod_{i=1}^{2}dy_i ds_i dt_i\\
&\le C|u_2-u_1|^{1+\gamma}\int_{T_2(0)}\int_{\mathbb{R}^4}\prod_{i=1}^{2}dy_i ds_i dt_i e^{-\frac12|y_1|^2\sigma-\frac12|y_2|^2(t_2-s_2)}g_n(y_1)g_n\Big(y_2-y_1\frac{t_1-s_2}{t_2-s_2}\Big)\\
&\qquad\times|y_1|^{(1+\gamma)/2}\Big|y_2-y_1\frac{t_1-s_2}{t_2-s_2}\Big|^{(1+\gamma)/2}\prod_{i=1}^{2}(t_i-s_i)^{(1+\gamma)/2}\\
&\le C|u_1-u_2|^{1+\gamma}\int_{T_2(0)}\int_{\mathbb{R}^4}\prod_{i=1}^{2}dy_i ds_i dt_i e^{-\frac12|y_1|^2\sigma-\frac12|y_2|^2(t_2-s_2)}\\
&\qquad\times\Big(|y_1y_2|^{(1+\gamma)/2}+\Big||y_1|^2\frac{t_1-s_2}{t_2-s_2}\Big|^{(1+\gamma)/2}\Big)\prod_{i=1}^{2}(t_i-s_i)^{(1+\gamma)/2}\\
&\le C|u_1-u_2|^{1+\gamma}\int_{T_2(0)}\prod_{i=1}^{2}ds_i dt_i(\sigma^{-1-(1+\gamma)/4}(t_1-s_1)^{(1+\gamma)/2}(t_2-s_2)^{-1+(1+\gamma)/4}\\
&\qquad\quad+\sigma^{-1-(1+\gamma)/2}(t_1-s_2)^{(1+\gamma)/2}(t_1-s_1)^{(1+\gamma)/2})\\
&\le C|u_1-u_2|^{1+\gamma}.
\end{aligned}$$

We now consider $I_{3,\varepsilon}$. If $(s_1,s_2,t_1,t_2)\in T_3(\varepsilon)$, then the left hand side of (2.1) is equal to

$$e^{-\frac12|y_1|^2(t_1-s_1-t_2+s_2)-\frac12|y_1+y_2|^2(t_2-s_2)},$$

which is denoted by $f_2(s,t,y)$. As for $I_{2,\varepsilon}$ we see that $I_{3,\varepsilon}$ is equal to

$$\begin{aligned}
&\int_{T_3(\varepsilon)}\int_{\mathbb{R}^4} g_n(y_1)g_n(y_2)\Big(-\prod_{i=1}^{2}(\sin(u_1\langle y_i,k_{t_i}-k_{s_i}\rangle-\sin(u_2\langle y_i,k_{t_i}-k_{s_i}\rangle))\\
&\qquad+\prod_{i=1}^{2}(\cos(u_1\langle y_i,k_{t_i}-k_{s_i}\rangle)-\cos(u_2\langle y_i,k_{t_i}-k_{s_i}\rangle))\Big)f_2(s,t,y)\prod_{i=1}^{2}dy_i\ ds_i\ dt_i\\
&\le C|u_2-u_1|^{1+\gamma}\int_{T_3(0)}\int_{\mathbb{R}^4} e^{-\frac12|y_1|^2(t_1-s_1-t_2+s_2)-\frac12|y_2|^2(t_2-s_2)}g_n(y_1)g_n(y_2-y_1)\\
&\qquad\times(|y_1|^{(1+\gamma)/2}|y_2-y_1|^{(1+\gamma)/2})\prod_{i=1}^{2}(t_i-s_i)^{(1+\gamma)/2}\prod_{i=1}^{2}\ dy_i\ ds_i\ dt_i\\
&\le C|u_1-u_2|^{1+\gamma}\int_{T_3(0)}\int_{\mathbb{R}^4}\prod_{i=1}^{2}\ dy_i\ ds_i\ dt_i e^{-\frac12|y_1|^2(t_1-s_1-t_2+s_2)-\frac12|y_2|^2(t_2-s_2)}\\
&\qquad\times(|y_1y_2|^{(1+\gamma)/2}+|y_1|^{1+\gamma}\prod_{i=1}^{2}(t_i-s_i)^{(1+\gamma)/2}\\
&\le C|u_1-u_2|^{1+\gamma}\int_{T_3(0)}\Big((t_1-s_1-t_2+s_2)^{-1-(1+\gamma)/4}(t_1-s_1)^{(1+\gamma)/2}(t_2-s_2)^{-1+(1+\gamma)/4}
\end{aligned}$$

$$+(t_1-s_1-t_2+s_2)^{-1-(1+\gamma)/2}(t_1-s_1)^{(1+\gamma)/2}(t_2-s_2)^{(1+\gamma)/2}\Big)\prod_{i=1}^{2}\ ds_i\ dt_i$$
$$\le C|u_1-u_2|^{1+\gamma}.$$

Combining the estimates for $I_{1,\varepsilon}$, $I_{2,\varepsilon}$ and $I_{3,\varepsilon}$ given before, we conclude that for any given $\gamma \in (0,1)$ there is a constant $C \in (0,\infty)$ such that

$$I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon} \le C|u_1-u_2|^{1+\gamma}, \quad \forall u_1,u_2 \in \mathbb{R}\ \varepsilon \in (0,1),\ n \ge 1,$$

which implies the conclusion given in Lemma 2.1. □

Lemma 2.2 *Let $\tilde{\xi}(u,k)=\xi(\omega+uk)-\xi(\omega)$ for $u \in \mathbb{R}$ and $k \in K$. Then, for any given $\gamma \in (0,1)$ and $k \in K$ there is a constant $C \in (0,\infty)$ such that*

$$E|\tilde{\xi}(u_1,k)-\tilde{\xi}(u_2,k)|^2 \le C|u_1-u_2|^{1+\gamma}, \quad u_1,u_2 \in \mathbb{R}.$$

Proof. From the choice of $(g_n)_{n\ge1}$ we see that $\eta_{n,\varepsilon}$ is convergent in probability (with respect to ν_0) to $\int_{T_\varepsilon}\delta(\omega_t-\omega_s)dsdt$ as $n\to\infty$, if $\varepsilon \in (0,1)$ is fixed. Moreover, we have that

$$\lim_{n\to\infty} E\eta_{n,\varepsilon} = E\int_{T_\varepsilon}\delta(\omega_t-\omega_s)dsdt, \quad \varepsilon \in (0,1).$$

Therefore, $\eta(n,\varepsilon)-E\eta(n,\varepsilon)$ is convergent in probability with respect to ν_0 to

$$\int_{T_\varepsilon}\delta(\omega_t-\omega_s)dsdt - E\int_{T_\varepsilon}\delta(\omega_t-\omega_s)dsdt$$

as $n\to\infty$, if $\varepsilon \in (0,1)$ is fixed. However, it is easy to check that

$$\lim_{\varepsilon\to0+}\left(\int_{T_\varepsilon}\delta(\omega_s-\omega_t)dsdt - E\int_{T_\varepsilon}\delta(\omega_s-\omega_t)dsdt\right)=\xi, \quad \text{in } L^2(\nu_0).$$

Then there are sequences $(m_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} m_n=\infty$ and $\lim_{n\to\infty}\varepsilon_n=0$ such that $\eta_{m_n,\varepsilon_n}-E\eta_{m_n,\varepsilon_n}$ is convergent to ξ in probability with respect to ν_0, as $n\to\infty$. Hence, $\eta_{m_n,\varepsilon_n}(\cdot+uk)-E\eta_{m_n,\varepsilon_n}$ is also convergent to $\xi(\cdot+uk)$ in probability with respect to ν_0, as $n\to\infty$. This proves that $\tilde{\eta}_{m_n,\varepsilon_n}(u,k)$ is convergent to $\tilde{\xi}(u,k)$ in probability with respect to ν_0, as $n\to\infty$. Thus we get the desired result immediately from Lemma 2.1 □

Having Lemma 2.2 we can easily prove Theorem 1.4

Proof of Theorem 1.4 By Lemma 2.2 we know that for any given $k \in K$ there is $\{\tilde{\xi}'(u,k),\ u \in \mathbb{R}\}$ such that

$$\nu_0(\tilde{\xi}(u,k) = \tilde{\xi}'(u,k)) = 1, \quad u \in \mathbb{R}, \tag{2.4}$$
$$\nu_0(\tilde{\xi}'(u,k) \text{ is continuous with respect to } u \in \mathbb{R}) = 1. \tag{2.5}$$

By the definition of the two dimensional polymer measure ν_g for $g \in (-g_0, \infty)$, i.e.,

$$\nu_g := (E\exp(-g\xi))^{-1}\exp(-g\xi)\ \nu_0,$$

we can easily see that $\nu_g \circ \tau_{uk}^{-1}$ is indeed absolutely continuous with respect to ν_g for all $u \in \mathbb{R}$ and $k \in K$. Let

$$a_{uk}(\omega) := \frac{\nu_g \circ \tau_{uk}^{-1}}{d\nu_g}(\omega).$$

By definition of ν_g we know that

$$a_{uk}(\omega) = e^{-g\tilde{\xi}(-u,k)} \cdot e^{u\int_0^1 k_t' d\omega_t - \frac{1}{2}u^2 \int_0^1 (k_t')^2 dt},$$

with $\int_0^1 k_t' d\omega_t$ understood as an Itô stochastic integral. Let

$$a'_{uk}(\omega) := e^{-g\tilde{\xi}'(-u,k)} \cdot e^{u\int_0^1 k_t' d\omega_t - \frac{1}{2}u^2 \int_0^1 (k_t')^2 dt}.$$

By (2.4) and (2.5) we know that

$$\nu_0(a_{uk} = a'_{uk}) = 1, \quad u \in \mathbb{R}, k \in K, \tag{2.6}$$
$$\nu_0(a'_{uk} \text{ is continuous with respect to } u \in \mathbb{R}) = 1, \quad k \in K. \tag{2.7}$$

Since ν_g is absolutely continuous with respect to ν_0, both (2.6) and (2.7) hold with ν_g replacing ν_0. Hence the proof is complete.

□

3 Proofs of Theorems 1.1, 1.2 and 1.5

Before we prove Theorem 1.1 we recall the necessary definitions. $(\mathcal{E}_{\nu_g}, \mathcal{F}C_b^\infty)$ is *closable* on $L^2(X;\nu_g)$ means that the unique continuous extension of the continuous embedding $\mathcal{F}C_b^\infty \subset L^2(X;\nu_g)$ (where $\mathcal{F}C_b^\infty$ is equipped with norm $(\mathcal{E}_{\nu_g,1})^{1/2}$ and $L^2(X;\nu_g)$ with $(\ ,\ )_{L^2(X;\nu_g)}^{1/2}$) to the completion of $\mathcal{F}C_b^\infty$ (w.r.t. $(\mathcal{E}_{\nu_g,1})^{1/2}$) is still one-to-one. The *closure* $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$ is then the smallest closed extension of $(\mathcal{E}_{\nu_g}, \mathcal{F}C_b^\infty)$ on $L^2(X;\nu_g)$.

Proof of Theorem 1.1. Since K is dense in H we can find an orthonormal basis $\{k_n\ :\ n \in \mathbb{N}\}$ of H in K. Obviously, we have for $u, v \in \mathcal{F}C_b^\infty$

$$\mathcal{E}_{\nu_g}(u,v) = \sum_{n=1}^\infty \int \frac{\partial u}{\partial k_n}\frac{\partial v}{\partial k_n}\ d\nu_g\ . \tag{3.1}$$

Hence e.g. by [MR92, Chap.I, Proposition 3.7(i)] it suffices to prove the closability of each $(\mathcal{E}_{k_n}, \mathcal{F}C_b^\infty)$ on $L^2(X;\nu_g)$ where

$$\mathcal{E}_{k_n}(u,v) := \int \frac{\partial u}{\partial k_n}\frac{\partial v}{\partial k_n}\, d\nu_g \ , \ u,v \in \mathcal{F}C_b^\infty \ .$$

But this is a direct consequence of Theorem 1.4 and [AR89b, Corollary 2.5]. The fact that the closure $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$ of $(\mathcal{E}_{\nu_g}, \mathcal{F}C_b^\infty)$ on $L^2(X;\nu_g)$ is then a Dirichlet form, follows by [AR90, Theorem 3.8] or [MR92, Chap.I, Proposition 3.5].

□

Proof of Theorem 1.2 By virtue of the representation (3.1) the existence of a ν_g-symmetric $I\!M$ is a consequence of [MR92, Chap.IV, Subsection 4b and Theorem 3.5] or [Sch90]. The continuity of the sample paths follows from [MR92, Chap.V, Example 1.12(ii) and Theorem 1.11]. The conservativity of $I\!M$ is obvious since $1 \in D(L)$ and $L1 = 0$, hence $T_t 1 = 1$ for all $t \geq 0$. The fact that ν_g is an invariant measure for $I\!M$ follows then immediately.

□

Proof of Theorem 1.5. Suppose $u \in D(\mathcal{E}_{\nu_g})$ such that

$$\mathcal{E}_{\nu_g}(u,u) = 0.$$

We may assume that u is bounded (cf. [AKR95, Proof of Proposition 2.3(ii)⇒(i)]). Since $g \in (-g_0, g_0)$ we can find $\lambda \in (\frac{1}{2}, 1)$ such that

$$E\ \exp\left(\frac{\lambda}{1-\lambda} g\xi\right) < \infty \quad \text{(i.e., } -\frac{\lambda}{1-\lambda} g \in (-g_0, \infty)\text{)}.$$

By Hölder's inequality we have that for all $f \in L^2(X;\nu_g)$

$$\begin{aligned} \int |f|^{2\lambda} d\nu_0 &= \int |f|^{2\lambda} \exp(-\lambda g\xi)\exp(\lambda g\xi) d\nu_0 \\ &\leq \left(\int |f|^2 \exp(-g\xi) d\nu_0\right)^{\lambda} \\ &\quad \left(\int \exp\left(\frac{\lambda}{1-\lambda} g\xi\right) d\nu_0\right)^{1-\lambda} . \end{aligned}$$

Hence

$$\left(\int |f|^{2\lambda} d\nu_0\right)^{\frac{1}{2\lambda}} \leq c\ |f|_{L^2(X;\nu_g)} \tag{3.2}$$

for some constant $c \in (0,\infty)$ independent of f. Applying (3.2) to $|\nabla v|_H$ for $v \in \mathcal{F}C_b^\infty$ we obtain that

$$\left(\int |\nabla v|_H^{2\lambda} d\nu_0\right)^{\frac{1}{2\lambda}} \leq c\ \mathcal{E}_{\nu_g}(v,v)^{1/2} \quad \text{for all } v \in \mathcal{F}C_b^\infty \ . \tag{3.3}$$

By definition of the closure $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$, we can find $v_n \in \mathcal{F}C_b^\infty$, $n \in \mathbb{N}$, such that $v_n \underset{n\to\infty}{\longrightarrow} u$ and $|\nabla v_n|_H \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(X;\nu_g)$. Applying (3.2), (3.3) we see that the same convergence holds in $L^{2\lambda}(X;\nu_0)$. Hence $u \in D^{1,2\lambda}$ and $\nabla u = 0$ (cf. [N95, Chap. 1, Sect. 1.5] for the notation). Since $2\lambda > 1$ it follows by [N95, Proposition 1.5.5 and Remark 1 on p. 35] that u is constant ν_0-a.e., hence ν_g-a.e. The statement that $P_{\nu_g} := \int_X P_\omega \nu_g d(\omega)$ is ergodic unter time shifts on Ω $(= C([0,\infty), X)$ (cf. Remark 1.3(ii)) is well-known to be equivalent with the irreducibility of $(\mathcal{E}_{\nu_g}, D(\mathcal{E}_{\nu_g}))$ (cf. [F82] which by the "transfer method" in [MR92, Chap.VI] generalizes immediately to this "non-locally compact case"). The last statement follows similarly from [F82, F83]. $\square$

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$$\xi = -\int_0^1 \langle d\omega_u, \int_0^u ds \frac{\omega_u - \omega_s}{|\omega_u - \omega_s|^2}\rangle + \int_0^1 ds \log|\omega_1 - \omega_s|$$

to prove Theorem 1.4. We thank him very much for the above suggestion and other interesting comments. The financial support of SFB 343 Bielefeld and SFB 237 Bochum-Düsseldorf-Essen is gratefully acknowledged.

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