Stochastic quantization of the three-dimensional polymer measure

Sergio Albeverio[1)] Michael Röckner[2)] Xian Yin Zhou[1), 3)]

Abstract

We prove that there exists a diffusion process whose invariant measure is the three-dimensional polymer measure ν_λ. The diffusion is constructed by means of the theory of Dirichlet forms on infinite-dimensional state spaces. We prove the closability of the appropriate pre-Dirichlet form which is of gradient type, using a general closability result by two of the authors. This result does not require an integration by parts formula (which does not even hold for the two-dimensional polymer measure ν_λ) but requires the quasi-invariance of ν_λ along a basis of vectors in the classical Cameron-Martin space such that the Radon-Nikodym derivatives (have versions which) form a continuous process.

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1) Institut für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany.
2) Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
3) Institute of Mathematics, Beijing Normal University, Beijing 100875, China

1 Introduction

In [AHRZ95] the stochastic quantization of the two-dimensional polymer measure was constructed using the theory of Dirichlet forms on infinite-dimensional state spaces. In this paper the same is done in the case of the three-dimensional polymer measure which is technically much more difficult since in three dimensions the polymer measure is no longer absolutely continuous w.r.t. Wiener measure. For the motivation and the background literature on stochastic quantization and the polymer measure (sometimes also called Edward's model) we refer to [AHRZ95, Section 1] and also to [W80, W82].

In order to describe our results more precisely we start with recalling some facts about the three-dimensional polymer measure. The polymer measure is informally defined by

$$\nu_\lambda(d\omega) = N_\lambda^{-1} \exp\Big(-\lambda \int_0^1 \int_s^1 \delta(\omega_t - \omega_t) ds dt \Big) \nu_0(d\omega), \tag{1.1}$$

where $\lambda \in [0,\infty)$ is the coupling constant, N_λ is a renormalization constant to assure that ν_λ is a probability measure, ν_0 is Wiener measure on $X := C_0([0,1] \to \mathbb{R}^3)$ and $\omega_t \mapsto \omega$, $t \geq 0$, the evaluation maps from X to $\mathbb{R}^3$. The three dimensional polymer measure was first constructed by J. Westwater (see [W80] and [W82]). A simpler construction for small values of the coupling constant was recently given by E. Bolthausen [B93]. In fact, the latter approach is still suitable for all positive coupling constants, and the polymer measure given by Bolthausen is actually the same as the one given by Westwater (see the recent work by Albeverio and Zhou [AZ95]). It was shown in [W82] that the three-dimensional polymer measure is singular with respect to Wiener measure. A rigorous definition of ν_λ can be given as follows (cf. [B93], [AZ95]). For the rest of this paper we fix $\lambda \in [0,\infty)$. For $\varepsilon \in (0,1)$ we set $\kappa_1(\varepsilon) := 2(2\pi)^{-3/2}(\varepsilon^{-1/2} - 1)$, and

$$\kappa_2(\varepsilon) := \int_0^1 \int_{s_1 \vee \varepsilon}^1 \int_{s_2 \vee (s_1+\varepsilon)}^1 (2\pi)^{-3}(s_1(s_2-s_1)+s_1(s_3-s_2)+(s_2-s_1)(s_3-s_2))^{-3/2} ds_1 ds_2 ds_3.$$

It is easy to check that $\lim_{\varepsilon\to 0+}(2\pi)^2\kappa_2(\varepsilon)|\log\varepsilon|^{-1} = 1$. Let

$$\begin{aligned} J^\varepsilon_{s,t;u,v} &:= \int_s^t \int_{(u\vee(\sigma+\varepsilon))\wedge v}^v \delta(\omega_\sigma - \omega_\tau) d\sigma d\tau, \\ \bar{J}^{\varepsilon,\lambda}_{s,t} &:= \lambda J^\varepsilon_{s,t} - \lambda(t-s)\kappa_1(\varepsilon) + \lambda^2(t-s)\kappa_2(\varepsilon), \end{aligned}$$

where $J^\varepsilon_{s,t} := J^\varepsilon_{s,t;s,t}$ (the functional $J^\varepsilon_{s,t;u,v}$ was already discussed in [Ro83] where the reader also finds the precise definition of the double integral). Let

$$\mu_{\varepsilon,\lambda} := (E\exp(-\bar{J}^{\varepsilon,\lambda}_{0,1}))^{-1}\exp(-\bar{J}^{\varepsilon,\lambda}_{0,1})\ \nu_0,$$

where E is the expectation with respect to ν_0. It was shown in [B93] and [AZ95] that the limit $\lim_{\varepsilon\to 0+}\mu_{\varepsilon,\lambda}$ exists in the weak sense. This limit is denoted by ν_λ. ν_λ is obviously a rigorous version of (1.1).

As already emphasized in [AHRZ95] the construction of the stochastic quantization of ν_λ by the theory of Dirichlet forms uses ν_λ and is based on some of its specific properties.

It thus does not lead to an independent *construction of* ν_λ itself. However, also in all approaches to the stochastic quantization of measures in Euclidean field theory (such as e.g. $P(\phi)_2$-fields, see for example [J-LM85], [AR89a, 89b, 91]) the respective measures have been used in an essential way. A truly constructive approach is not known. It seems that the situation for measures in Euclidean field theory as well as for polymer measures is just too singular. (E.g. all attempts using strong solutions for appropriate stochastic differential equations to obtain the required Markov processes have failed so far.) To describe the Dirichlet form in question precisely, we need some preparations.

X equipped with the supremum-norm is a separable real Banach space. Let X' denote its dual. Let $H \subset X$ be the classical Cameron-Martin space, i.e., $H := \{h \in X : h$ is absolutely continuous and $|h|_H^2 = \int_0^1 |h_t'|^2 dt < \infty\}$. Furthermore, let $K := \{h \in H : \sup_{0\le t\le 1} |h'(t)| < \infty\}$. $(H, \langle\ ,\ \rangle_H)$ is a real separable Hilbert space which is densely and continuously embedded into X, and K is a dense linear subspace of $(H, \langle\ ,\ \rangle_H)$. Identifying H with its dual we obtain that X' is densely embedded into H. Hence $X' \subset H \subset X$, and $\langle\ ,\ \rangle_H$ restricted to $X' \times H$ coincides with the dualization between X' and X. Define the set of smooth bounded cylinder functions by

$$\mathcal{F}C_b^\infty := \{f(l_1, \ldots, l_m) \ :\ m \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^m),\ l_1, \ldots, l_m \in X'\},$$

where $C_b^\infty(\mathbb{R}^m)$ denotes the set of all bounded infinitely differentiable functions on $\mathbb{R}^m$ with all partial derivatives bounded. Let $k \in X$. For $u \in \mathcal{F}C_b^\infty$, $\omega \in X$, define

$$\frac{\partial u}{\partial k}(\omega) := \frac{d}{ds} u(\omega + sk)|_{s=0}$$

and let $\nabla u(\omega)$ denote the unique element in H such that

$$\langle \nabla u(\omega), k\rangle_H = \frac{\partial u}{\partial k}(\omega) \quad \text{for all } k \in H.$$

Define for $u, v \in \mathcal{F}C_b^\infty$

$$\mathcal{E}_{\nu_\lambda}(u, v) := \int \langle \nabla u(\omega), \nabla v(\omega)\rangle_H \ \nu_\lambda(d\omega)$$

and $\mathcal{E}_{\nu_\lambda,1} := \mathcal{E}_{\nu_\lambda} + (\ ,\ )_{L^2(X;\nu_\lambda)}$.

Now we can formulate our first main result.

Theorem 1.1 *(i)* $\operatorname{supp} \nu_\lambda = X$ *(i.e.,* $\nu_\lambda(U) > 0$ *for all non-empty open* $U \subset X$*). In particular, we can identify each* $u \in \mathcal{F}C_b^\infty$ *with the corresponding class in* $L^2(X; \nu_\lambda)$*, and thus* $(\mathcal{E}_{\nu_\lambda}, \mathcal{F}C_b^\infty)$ *is a symmetric positive definite bilinear form on* $L^2(X; \nu_\lambda)$*.*
(ii) $(\mathcal{E}_{\nu_\lambda}, \mathcal{F}C_b^\infty)$ *is closable on* $L^2(X; \nu_\lambda)$ *and the closure* $(\mathcal{E}_{\nu_\lambda}, D(\mathcal{E}_{\nu_\lambda}))$ *is a symmetric* Dirichlet form *(i.e., a closed positive definite symmetric bilinear form such that* $u^\# := \min(\max(u, 0), 1) \in D(\mathcal{E}_{\nu_\lambda})$ *and* $\mathcal{E}_{\nu_\lambda}(u^\#, u^\#) \le \mathcal{E}_{\nu_\lambda}(u, u)$ *for all* $u \in D(\mathcal{E}_{\nu_\lambda})$*).*

$(\mathcal{E}_{\nu_\lambda}, \mathcal{F}C_b^\infty)$ is *closable* on $L^2(X; \nu_\lambda)$ means that the unique continuous extension of the continuous embedding $\mathcal{F}C_b^\infty \subset L^2(X; \nu_\lambda)$ (where $\mathcal{F}C_b^\infty$ is equipped with norm $(\mathcal{E}_{\nu_\lambda,1})^{1/2}$

and $L^2(X;\nu_\lambda)$ with $(\ ,\ )^{1/2}_{L^2(X;\nu_\lambda)}$) to the completion of $\mathcal{F}C_b^\infty$ (w.r.t. $(\mathcal{E}_{\nu_\lambda,1})^{1/2}$) is still one-to-one. The *closure* $(\mathcal{E}_{\nu_\lambda}, D(\mathcal{E}_{\nu_\lambda}))$ is then the smallest closed extension of $(\mathcal{E}_{\nu_\lambda}, \mathcal{F}C_b^\infty)$ on $L^2(X;\nu_\lambda)$. For more details on Dirichlet forms the reader is referred e.g. to [MR92], [FOT94], [BH91], [Si74], and for the special type of Dirichlet form appearing here, namely so-called *classical Dirichlet forms* to [AR90a].

Let $(L, D(L))$ be the generator of $(\mathcal{E}_{\nu_\lambda}, D(\mathcal{E}_{\nu_\lambda}))$, i.e., the unique negative definite self-adjoint operator on $L^2(X;\nu_\lambda)$ such that

$$D(\sqrt{-L}) = D(\mathcal{E}_{\nu_\lambda}) \qquad \text{and} \qquad (\sqrt{-L}u, \sqrt{-L}v) = \mathcal{E}_{\nu_\lambda}(u,v) \quad \text{for all } u, v \in D(\mathcal{E}_{\nu_\lambda}). \tag{1.2}$$

Let $T_t := e^{tL}$, $t \geq 0$.

Theorem 1.2 *There exists a diffusion process* $I\!\!M = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_\omega)_{\omega\in X})$ *which is* associated *with* $(\mathcal{E}_{\nu_\lambda}, D(\mathcal{E}_{\nu_\lambda}))$*, i.e., for all ($\nu_\lambda$-versions of) $f \in L^2(X;\nu_\lambda)$ and all $t \geq 0$ the function*

$$\omega \mapsto p_t f(\omega) := \int_\Omega f(X_t)\, dP_\omega\ , \quad \omega \in X,$$

is a ν_λ-version of $T_t f$. $I\!\!M$ is conservative and ν_λ-symmetric. In particular, ν_λ is an invariant measure for $I\!\!M$.

Remark 1.3 (i) $I\!\!M$ is in fact even *properly associated* to $(\mathcal{E}_{\nu_\lambda}, D(\mathcal{E}_{\nu_\lambda}))$ in the sense of [MR92, Chap. IV, Definition 2.5].

(ii) Ω in Theorem 1.2 can, of course, always be taken to be $C([0,\infty), X)$, since $I\!\!M$ has continuous sample paths and is conservative.

For the proof of Theorem 1.1 we need the following result which is of its own interest and is in fact the heart of this paper. For $k \in X$ define $\tau_k(\omega) := \omega + k$, $\omega \in X$, and let

$$K_0 := \{k \in K | \sup_{0\leq t\leq 1} |k''(t)| < \infty\}.$$

Theorem 1.4 *Let $k \in K_0$. Then ν_λ is k-*quasi-invariant*, i.e., $\nu_\lambda \circ \tau_{sk}^{-1}$ is absolutely continuous w.r.t. ν_λ for all $s \in I\!\!R$. If $a_{sk} := \frac{d(\nu_\lambda\circ\tau_{sk}^{-1})}{d\nu_\lambda}$, $s \in I\!\!R$, then the process $(a_{sk})_{s\in I\!\!R}$ has a version which has $\nu_\lambda - a.e.$ continuous sample paths.*

The proof of Theorem 1.4 is given in Section 5 below after the necessary preparations in Sections 2 to 4 have been made. The proof of Theorem 1.1 follows from Theorem 1.4, [AR90b, Proposition 2.7], and the general closability criterion [AR89b, Corollary 2.5]. Details are given in Section 6. The proof of Theorem 1.2 then follows by the standard machinery of Dirichlet forms on infinite-dimensional state spaces and is exactly the same as that of [AHRZ95, Theorem 1.2] and therefore omitted.

In contrast to the two-dimensional case we have not yet succeeded in proving that $(\mathcal{E}_{\nu_\lambda}, D(\mathcal{E}_{\nu_\lambda}))$ in Theorem 1.1 is irreducible.

2 Some estimates in Wiener space

As a preparation, in this section we first prove an estimate which is valid in the Wiener space. For this purpose, we set

$$g_n(x) := c\prod_{i=1}^{3} I_{\{|x_i|\le n\}}, \quad k\ge 1, x\in \mathbb{R}^3,$$

where $x=(x_1,x_2,x_3)$ and $c\in(0,\infty)$ is a constant.

It is easy to show that there is a constant $c\in(0,\infty)$ such that the following holds in the vague sense

$$\int_{\mathbb{R}^3} e^{i\langle x,y\rangle} g_n(y)dy \to \delta_0(x), \quad n\to\infty,$$

where δ_0 is the Dirac function at point $0\in\mathbb{R}^3$. From now on we always choose the constant $c\in(0,\infty)$ such that the above convergence holds. Let

$$\eta_{n,\varepsilon}(\omega) := \int_{T_\varepsilon}\int_{\mathbb{R}^3} e^{i\langle y,\omega_t-\omega_s\rangle} g_n(y)dydsdt,$$
$$\tilde\eta_{n,\varepsilon}(u,k) := \eta_{n,\varepsilon}(\omega+uk)-\eta_{n,\varepsilon}(\omega),$$

where $T_\varepsilon := \{(s,t):\ t-s\ge\varepsilon,\ 0\le s<t\le 1\}$, and $\langle x,y\rangle=\sum_{i=1}^3 x_iy_i$ if $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$. The main result in this section is as follows.

Proposition 2.1 *Let $\gamma\in(0,1/2)$ and $k\in K$ be given. Let $\alpha\in(0,(1/2-\gamma)/2)$ and $\varepsilon_n\in(0,1)$, $n\in\mathbb{N}$. For any given $p\ge 1$ there is a constant $C\in(0,\infty)$ such that*
(i) $E|\tilde\eta_{n,\varepsilon_n}(u,k)-\tilde\eta_{m,\varepsilon_m}(u,k)|^p\le C|u|^{\gamma p}(\varepsilon_m^{\alpha p}+m^{-(1/2-\gamma)p/2})$ for all $u\in\mathbb{R}$, $n\ge m\ge 1$.
(ii) $E|\tilde\eta_{n,\varepsilon_n}(u_1,k)-\tilde\eta_{n,\varepsilon_n}(u_2,k)|^p\le C|u_1-u_2|^{\gamma p}$ for all $u_1,u_2\in\mathbb{R}$, $n\ge 1$.

Proof. We first prove that the following estimate holds for any given integer $p\ge 1$.

$$E|\tilde\eta_{n,\varepsilon_m}(u,k)-\tilde\eta_{m,\varepsilon_m}(u,k)|^p\le C|u|^{\gamma p}m^{-(1/2-\gamma)[p/2]} \tag{2.1}$$

for all $u\in\mathbb{R}$, $n>m\ge 1$, where $\gamma\in(0,1/2)$. The constant $C\in(0,\infty)$ given in this lemma is independent of n,m,u,u_1 and u_2, which might be different from line to line.

We remark that for any given $\varepsilon\in(0,1]$ there is a constant $C\in(0,\infty)$ such that

$$|\sin x|\le C|x|^\varepsilon\wedge 1;\quad |1-\cos x|\le C|x|^\varepsilon\wedge 1,\quad \forall x\in\mathbb{R}. \tag{2.2}$$

Let

$$T^p_{\varepsilon_m} := \underbrace{T_{\varepsilon_m}\times\cdots\times T_{\varepsilon_m}}_{p-\text{times}}.$$

Then (2.2) implies that

$$E|\tilde\eta_{n,\varepsilon_m}(u,k)-\tilde\eta_{m,\varepsilon_m}(u,k)|^p \tag{2.3}$$

$$
\begin{aligned}
&= (-1)^p \int_{T^p_{\varepsilon m}} \int_{\mathbb{R}^{3p}} e^{-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^p \langle y_j, \omega_{t_j}-\omega_{s_j}\rangle\right)} \\
&\qquad \times \prod_{j=1}^p (e^{i\langle y_j, u(k_{t_j}-k_{s_j})\rangle} - 1)(g_m(y_j) - g_n(y_j)) dy_j ds_j dt_j \\
&\le C|u|^{\gamma p} \int_{T^p_{\varepsilon m}} \int_{\mathbb{R}^{3p}} e^{-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^p \langle y_j, \omega_{t_j}-\omega_{s_j}\rangle\right)} \\
&\qquad \times \prod_{j=1}^p |y_j|^\gamma |t_j - s_j|^\gamma (g_m(y_j) - g_n(y_j)) dy_j ds_j dt_j.
\end{aligned}
$$

As in the proof of [Ro83, Lemma 2] we denote the pair (s_j, t_j) by S^j and set $(z_1, z_2, \cdots, z_{2p}) := (s_1, t_1, s_2, t_2, \cdots, s_p, t_p)$. We may assume that $T^p_{\varepsilon m}$ is replaced by $T^p_{\varepsilon m} \cap \Delta_p(\pi)$ for some permutation π of $\{1, \cdots, 2p\}$, where

$$\Delta_p(\pi) := \{(z_1, \cdots, z_{2p}) | z_{\pi(1)} < \cdots < z_{\pi(2p)}\},$$

since the right hand side of (2.3) is a sum of integrals over such regions. Let us define disjoint intervals

$$R_i := [z_{\pi(i)}, z_{\pi(i+1)}], \quad 1 \le i \le 2p-1,$$

and let $W(R_i) := \omega_{z_{\pi(i+1)}} - \omega_{z_{\pi(i)}}$. Then

$$\omega_{t_j} - \omega_{s_j} = \sum_{R_i \subset [s_j, t_j]} W(R_i).$$

Since $W(R_i)$, $i = 1, \cdots, 2p-1$, are independent, we have

$$
\begin{aligned}
\mathrm{Var}\Big(\sum_{j=1}^p \langle y_j, \omega_{t_j} - \omega_{s_j}\rangle\Big) &= \mathrm{Var}\Big(\sum_{j=1}^p \Big\langle y_j, \sum_{R_i \subset [s_j, t_j]} W(R_i)\Big\rangle\Big) \\
&= \sum_i |\bar{y}^i|^2 |R_i|,
\end{aligned}
$$

where $\bar{y}^i := \sum_{j: R_i \subset [s_j, t_j]} y_j$. For convenience we let R_0 be the interval on the left of R_1 and R_{2p} the interval on the right of R_{2p-1}. Let $\bar{y}^0 := 0$, $\bar{y}^{2p} := 0$. Thus, to prove (2.1) we only need to prove that

$$
\begin{aligned}
&\int_{T^p_{\varepsilon m} \cap \Delta_p(\pi)} \int_{\mathbb{R}^{3p}} \exp\Big(-\frac{1}{2}\sum_{i=1}^{2p-1} |\bar{y}^i|^2 |R_i|\Big) \prod_{j=1}^p |y_j|^\gamma |t_j - s_j|^\gamma (g_m(y_j) - g_n(y_j)) dy_j ds_j dt_j \\
&\le C m^{-(1/2-\gamma)[p/2]}. \qquad (2.4)
\end{aligned}
$$

For this purpose we may also assume that $\cup_j [s_j, t_j]$ is connected, working with each component separately. We define $f(l)$ and $r(l)$ by requiring that $R_{f(l)}$ is the first and $R_{r(l)-1}$ is the last of the intervals R_i in $[s_l, t_l]$. Then

$$[s_l, t_l] = R_{f(l)} \cup R_{f(l)+1} \cup \cdots \cup R_{r(l)-1},$$

and so

$$y_l = \bar{y}^{f(l)} - \bar{y}^{f(l)-1}, \quad -y_l = \bar{y}^{r(l)} - \bar{y}^{r(l)-1}.$$

Using this we get

$$|y_l| \le (|\bar{y}^{f(l)}| + |\bar{y}^{f(l)-1}|)^{1/2}(|\bar{y}^{r(l)}| + |\bar{y}^{r(l)-1}|)^{1/2}.$$

Since each i is either an $f(l)$ or an $r(l)$ for a unique l, and either an $f(q)-1$ or an $r(q)-1$ for a unique q, we have

$$\begin{aligned}\prod_{i=1}^{p} |y_l| &\le \prod_{l=1}^{p}(|\bar{y}^{f(l)}| + |\bar{y}^{f(l)-1}|)^{1/2}(|\bar{y}^{r(l)}| + |\bar{y}^{r(l)-1}|)^{1/2} \\ &\le 2^{p/2}\prod_{i=1}^{2p-1}(1+|\bar{y}^i|^2)^{1/2},\end{aligned}$$

by using $(a+b)^{1/2} \le 2^{1/4}(1+a^2)^{1/4}(1+b^2)^{1/4}$.

We now change the variable: $(S^1, \cdots, S^p) \to (z_{\pi(1)}, |R_1|, \cdots, |R_{2p-1}|)$. For each j, $1 \le j \le p$, we can choose some $R_{d(j)} \subset [s_j, t_j]$ such that $|R_{d(j)}| \ge (t_j - s_j)/2p$. For the $d|R_{d(j)}|$ integral we have

$$\int_{(t_j-s_j)/2p}^{1} e^{-|\bar{y}^{d(j)}|^2 t} dt \le \begin{cases} C(1+|\bar{y}^{d(j)}|^2)^{-1}, & |\bar{y}^{d(j)}|^2 \le (t_j-s_j)^{-1}, \\ C|t_j - s_j|^{-\gamma q}(1+|\bar{y}^{d(j)}|^2)^{-1-\gamma q}, & |\bar{y}^{d(j)}|^2 \ge (t_j-s_j)^{-1}, \end{cases}$$

where $q \ge 1$ is any fixed constant. For later use, we always choose $q \ge 1$ such that $(q-1)\gamma < 1$ and $2q\gamma > 3/2$. For the other integral we have

$$\int_0^1 e^{-|y|^2 t} dt \le C(1+|y|^2)^{-1}, \quad \forall y \in \mathbb{R}^3. \tag{2.5}$$

From these we can see that the left hand side of (2.4) can be bounded from above by a sum of terms of the following form:

$$\begin{aligned} &C\int_{T^p_{\varepsilon_m}\cap\Delta(\pi)} \prod_{j=1}^{p} |t_j - s_j|^{-(q-1)\gamma} ds_j dt_j \\ &\quad \times \int_{\mathbb{R}^{3p}} \prod_{j=1}^{p}(1+|\bar{y}^{f(j)}|^2)^{-1+\gamma/2}(1+|\bar{y}^{r(j)}|^2)^{-1+\gamma/2}(1+|\bar{y}^{d(j)}|^2)^{-\gamma q}(g_m(y_j)-g_n(y_j))dy_j \\ &\le C\int_{\mathbb{R}^{3p}} \prod_{j=1}^{p}(1+|\bar{y}^{f(j)}|^2)^{-1+\gamma/2}(1+|\bar{y}^{r(j)}|^2)^{-1+\gamma/2}(1+|\bar{y}^{d(j)}|^2)^{-\gamma q}(g_m(y_j)-g_n(y_j))dy_j, \end{aligned}$$

which is by the Cauchy-Schwarz inequality less than

$$\begin{aligned} &C\Big[\int_{\mathbb{R}^{3p}} \prod_{j=1}^{p}(1+|\bar{y}^{f(j)}|^2)^{-2+\gamma}(g_m(y_j)-g_n(y_j))dy_j\Big]^{1/2} \\ &\quad \times \Big[\int_{\mathbb{R}^{3p}} \prod_{j=1}^{p}(1+|\bar{y}^{r(j)}|^2)^{-2+\gamma}(1+|\bar{y}^{d(j)}|^2)^{-2q\gamma}dy_j\Big]^{1/2}. \end{aligned} \tag{2.6}$$

We remark that $\{\bar{y}^{f(1)}, \cdots, \bar{y}^{f(p)}\}$ is a (nonsingular) set of coordinates for $\mathbb{R}^{3p}$ (see [Ro83, Lemma 3]) and that we can also choose a (nonsingular) set of coordinates for $\mathbb{R}^{3p}$ from the set

$$\bar{y}^{r(1)}, \bar{y}^{r(2)}, \cdots, \bar{y}^{r(p)}, \bar{y}^{d(1)}, \cdots, \bar{y}^{d(p)}$$

(see [Ro83, Lemma 4]). Moreover, if $|y_j| \geq m$ for $j = 1, \cdots, p$, then there are $1 \leq i_1 < \cdots < i_{[p/2]} \leq p$ such that $|\bar{y}^{f(i_l)}| \geq m/2$ for $l = 1, \cdots, [p/2]$. To see this, let us recall the definitions of $\bar{y}^i$ and $f(1), \cdots, f(p)$. Without loss of generality, we may assume $s_1 < \cdots < s_p$. In this case we have $\bar{y}^{f(1)} = y_1$, $\bar{y}^{f(2)} = y_1 + y_2$, $\cdots, \bar{y}^{f(p)} = y_1 + \cdots + y_p$. From these expressions we can easily see that if $|y_j| \geq m$ for $j = 1, \cdots, p$, then there must be $1 \leq i_1 < \cdots, i_{[p/2]} \leq p$ such that $|\bar{y}^{f(i_l)}| \geq m/2$ for $l = 1, \cdots, [p/2]$.

From these observations we can see that (2.6) is less than $Cm^{-(1/2-\gamma)[p/2]}$, if $\gamma \in (0, 1/2)$ and $2q\gamma > 3/2$. This proves that the right hand side of (2.3) is less than $C|u|^{\gamma p} m^{-(1/2-\gamma)[p/2]}$ for all $u \in \mathbb{R}$ and $n > m \geq 1$, which implies (2.1).

We now use the approach for proving (2.1) given before to prove the following estimate

$$E|\tilde{\eta}_{n,\varepsilon_n} - \tilde{\eta}_{n,\varepsilon_m}|^p \leq C|u|^{\gamma p} \varepsilon_m^{\alpha p}, \quad u \in \mathbb{R}, n \geq m \geq 1, \tag{2.7}$$

where $\gamma \in (0, 1/2)$, $\alpha \in (0, (1/2 - \gamma)/2)$. To get this estimate we only need to use the following estimate instead of (2.5):

$$\int_0^{\varepsilon_m} e^{-|y|^2 t} dt \leq C\varepsilon_m^{\alpha} (1 + |y|^2)^{-1+\alpha},$$

where $\alpha \in (0, (1/2 - \gamma)/2)$. As in the derivation of (2.1), we can use the above estimate to prove that (2.7) is indeed true (here we have to choose $\alpha \in (0, (1/2 - \gamma)/2)$).

Since the integer $p \geq 1$ is arbitrary, we get from (2.7) and (2.1) that

$$E|\tilde{\eta}_{n,\varepsilon_n}(u,k) - \tilde{\eta}_{m,\varepsilon_m}(u,k)|^{2p} \leq C|u|^{2\gamma p} (\varepsilon_m^{2\alpha p} + m^{-(1/2-\gamma)p}),$$

from which we get the desired result (i).

To prove (ii) we first remark that

$$\begin{aligned}
&E|\tilde{\eta}_{n,\varepsilon}(u_1,k) - \tilde{\eta}_{n,\varepsilon}(u_2,k)|^p \\
&= \int_{T^p_{\varepsilon_m}} \int_{\mathbb{R}^{3p}} e^{-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^p \langle y_j, \omega_{t_j} - \omega_{s_j}\rangle\right)} \\
&\quad \times \prod_{j=1}^p (e^{i\langle y_j, u_1(k_{t_j} - k_{s_j})\rangle} - e^{i\langle y_j, u_2(k_{t_j} - k_{s_j})\rangle}) g_n(y_j) dy_j ds_j dt_j.
\end{aligned} \tag{2.8}$$

We also remark that for any given $\varepsilon \in (0, 1]$ there is a constant $C \in (0, \infty)$ such that

$$|\sin x_1 - \sin x_2| + |\cos x_1 - \cos x_2| \leq C|x_1 - x_2|^{\varepsilon}, \quad x_1, x_2 \in R.$$

Using this fact we can show that the right hand side of (2.8) is less than

$$\begin{aligned}
&C|u_1 - u_2|^{\gamma p} \int_{T^p_{\varepsilon_m}} \int_{\mathbb{R}^{3p}} e^{-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^p \langle y_j, \omega_{t_j} - \omega_{s_j}\rangle\right)} \\
&\quad \times \prod_{j=1}^p |y_j|^{\gamma} |t_j - s_j|^{\gamma} g_n(y_j) dy_j ds_j dt_j.
\end{aligned} \tag{2.9}$$

Repeating the argument given before one can show that the right hand side of (2.9) is less than $C|u_1 - u_2|^{\gamma p}$. The proof of Proposition 2.1 is then complete. $\square$

3 A key lemma

We first remark that if for $x \in \mathbb{R}^3$ we set $p_t(x) := (2\pi t)^{-3/2} \exp(-\frac{|x|^2}{2t})$, and

$$g_t^{\varepsilon,\lambda}(x) := E(\exp(-\bar{J}_{0,t}^{\varepsilon,\lambda}) I_{\{\omega_t = x\}}).$$

In [B93] and [AZ95] it was shown that for any given $\lambda \in [0, \infty)$ there is a constant $C \in (0, \infty)$ such that

$$|g_t^{\varepsilon,\lambda}(x) - p_t(x)| \leq C t^{1/2} p_{2t}(x), \quad \forall t \in [0,1], x \in \mathbb{R}^3. \tag{3.1}$$

Let $\psi : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be bounded and infinitely often differentiable with bounded derivatives of all orders, and for $0 \leq s < t \leq 1$, let $\Psi_{s,t}$ be defined by

$$\Psi_{s,t} := \exp\Big(\int_s^t \psi(u, \omega_u) du\Big). \tag{3.2}$$

As was shown in [B93, §4], (3.1) implies that for any such ψ and $\lambda \in [0, \infty)$ there is a constant $C \in (0, \infty)$ such that

$$|E(\exp(-\bar{J}_{0,1}^{\varepsilon_1,\lambda})\Psi_{0,1}) - E(\exp(-\bar{J}_{0,1}^{\varepsilon_2,\lambda})\Psi_{0,1})| \leq C(\varepsilon_1^\delta + \varepsilon_2^\delta) \tag{3.3}$$

for all $\varepsilon_1, \varepsilon_2 \in (0,1)$, where $\delta \in (0,1)$.

Let $f_i : \mathbb{R}^3 \to \mathbb{R}$ be a bounded measurable function for $i = 1, 2$. Let $\Delta_{u,v;s,t} := \{(\sigma, \tau) : u \leq \sigma \leq v, s \leq \tau \leq t\}$, and for any fixed $a_1, a_2 \in (0,1)$

$$\Phi_{u,v;s,t} := \int_{T_{a_1} \cap \Delta_{u,v;s,t}} \int_{\mathbb{R}^3} f_1(\omega_\sigma - \omega_\tau) d\sigma d\tau - \int_{T_{a_2} \cap \Delta_{u,v;s,t}} \int_{\mathbb{R}^3} f_2(\omega_\sigma - \omega_\tau) d\sigma d\tau.$$

For short we set $\Phi := \Phi_{0,1;0,1}$ and $\Phi_{s,t} := \Phi_{s,t;s,t}$. We set

$$\rho(\varepsilon) := E(\exp(-\bar{J}_{0,1}^{\varepsilon,\lambda})|\Phi|).$$

In this section we fix $k \in K$. The main aim of this section is to prove the following lemma.

Lemma 3.1 *There are constants $C \in (0, \infty)$ and $\delta_0 \in (0,1)$ such that*

$$|\rho(\varepsilon_1) - \rho(\varepsilon_2)| \leq C \max_{x \in R^3}(|f_1(x)| + |f_2(x)|)(\varepsilon_1^{\delta_0} + \varepsilon_2^{\delta_0}), \quad \varepsilon_1, \varepsilon_2 \in (0,1). \tag{3.4}$$

A typical example of Φ to which Lemma 3.1 applies is:

$$\Phi := \tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k) - \tilde{\eta}_{2^{n-1},2^{-\varepsilon_0(n-1)}}(u,k).$$

The estimate (3.4) for the above special Φ will be used in the proof of Theorem 4.1 below. A general form of (3.4) will be given in Remark 3.2 below. The estimate (3.4), with $|\Phi|$ replaced by $\Psi_{0,1}$ in (3.2), was already proved by Bolthausen in [B93, §4]. By modifying his argument one can see that the discussion given in [B93, §4] is also suitable for the present case. Since $|\Phi_{0,t+s}|$ is not equal to $|\Phi_{0,s}||\Phi_{s,s+t}|$, some details given in the proof of Lemma 3.1 are different from those given in [B93, §4].

Proof of Lemma 3.1. For short we will denote $\sup_{x\in\mathbb{R}^3}(|f_1(x)|+|f_2(x)|)$ by $\max|f|$. As in [B93, (4.2)] we have the following relation:

$$\frac{d}{d\varepsilon}\rho(\varepsilon) = \lambda\int_0^{1-\varepsilon} dsE(e^{-\bar{J}_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})|\Phi|) + (\lambda\frac{d}{d\varepsilon}\kappa_1(\varepsilon) - \lambda^2\frac{d}{d\varepsilon}\kappa_2(\varepsilon))\rho(\varepsilon),$$
$$\bar{J}_{0,1} = \tilde{J}^s_{0,1} + \lambda J_{0,s;s,s+\varepsilon} + \lambda J_{s,s+\varepsilon;s+\varepsilon,1} - \lambda\varepsilon\kappa_1(\varepsilon) + O(\varepsilon^{3/4}),$$

where $\tilde{J}^s_{0,1} := \bar{J}_{0,s} + \bar{J}_{s+\varepsilon,1} + \lambda J_{0,s;s+\varepsilon,1}$, and $J_{s,t;u,v} : := J^\varepsilon_{s,t;u,v}$ was defined in Section 1. Using the fact that $1-x \le e^{-x} \le 1-x+O(1)x^2,\ x>0$, one shows that

$$\begin{aligned} E(e^{-\bar{J}_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})|\Phi|) &= E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})|\Phi|)e^{\lambda\varepsilon\kappa_1(\varepsilon)-O(\varepsilon^{3/4})} \\ &-\lambda E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})(J_{0,s;s,s+\varepsilon}+J_{s,s+\varepsilon;s+\varepsilon,1})|\Phi|)(1+O(\varepsilon^{1/2})) \\ &+O(1)\lambda^2E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})(J_{0,s;s,s+\varepsilon}+J_{s,s+\varepsilon;s+\varepsilon,1})^2|\Phi|)(1+O(\varepsilon)). \end{aligned} \tag{3.5}$$

By (3.1) we know that $Ee^{-\bar{J}^{\varepsilon,\lambda}_{s,t}}$ is bounded from above by a constant which is independent of $\varepsilon\in(0,1)$ and $s,t\in[0,1]$. By definition we also know that

$$|J_{0,s;s,s+\varepsilon}|^2 = \int_{([0,s]\times[s,s+\varepsilon])^2}\prod_{i=1}^2 I_{\{\tau_i-\sigma_i\ge\varepsilon\}}\delta(\omega_{\sigma_i}-\omega_{\tau_i})d\sigma_i d\tau_i.$$

Using (3.1) again we obtain that

$$Ee^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})\prod_{i=1}^2\delta(\omega_{\sigma_i}-\omega_{\tau_i}) \le CE\delta(\omega_{2s}-\omega_{2s+2\varepsilon})\prod_{i=1}^2\delta(\omega_{2\sigma_i}-\omega_{2\tau_i}).$$

Therefore,

$$\begin{aligned} &Ee^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})|J_{0,s;s,s+\varepsilon}|^2 \\ &\le C\int_{([0,s]\times[s,s+\varepsilon])^2}E\delta(\omega_{2s}-\omega_{2s+2\varepsilon})\prod_{i=1}^2\delta(\omega_{2\sigma_i}-\omega_{2\tau_i})I_{\{\tau_i-\sigma_i\ge\varepsilon\}}d\sigma_i d\tau_i \\ &\le C\varepsilon^{-3/2}\varepsilon = C\varepsilon^{-1/2}. \end{aligned}$$

Similarly, we have that

$$Ee^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})|J_{s,s+\varepsilon,s+\varepsilon,1}|^2 \le C\varepsilon^{-1/2}.$$

From these estimates we obtain that the last term on the right hand side of (3.5) is less than

$$\begin{aligned} & C\max|f|\lambda^2 E e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})(J_{0,s;s,s+\varepsilon}+J_{s,s+\varepsilon;s+\varepsilon,1})^2 \\ & \le C\max|f|\lambda^2\varepsilon^{-1/2}. \end{aligned} \tag{3.6}$$

We now consider the first term on the right hand side of (3.5). By definition we know that

$$|\Phi_{s,s+\varepsilon;0,1}|+|\Phi_{0,1;s,s+\varepsilon}|\le C\max|f|\varepsilon,$$

which implies that

$$\begin{aligned} & E e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})(|\Phi_{s,s+\varepsilon;0,1}|+|\Phi_{0,1;s,s+\varepsilon}|) \\ & \le C\max|f|\varepsilon p_\varepsilon(0)Ee^{-\bar{J}_{0,s}-\bar{J}_{s+\varepsilon,1}}\le C\max|f|\varepsilon^{-1/2}. \end{aligned} \tag{3.7}$$

This shows that

$$\begin{aligned} & E e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})|\Phi| \\ & = O(1)\max|f|\varepsilon^{-1/2}+Ee^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})|\Phi-\Phi_{s,s+\varepsilon;0,1}-\Phi_{0,1;s,s+\varepsilon}| \\ & = O(1)\max|f|\varepsilon^{-1/2}+p_\varepsilon(0)Ee^{-\bar{J}_{0,1-\varepsilon}-\lambda Y_s}|\Phi_{0,s}+\Phi_{s+\varepsilon}\circ\theta_\varepsilon+\Phi_{0,s;s,1-\varepsilon}| \\ & = O(1)\max|f|\varepsilon^{-1/2}+p_\varepsilon(0)[Ee^{-\bar{J}_{0,1-\varepsilon}}|\Phi_{0,1-\varepsilon}|-\lambda EY_s e^{-\bar{J}_{0,1-\varepsilon}}|\Phi_{0,1-\varepsilon}|] \\ & \qquad +O(1)p_\varepsilon(0)\lambda^2 EY_s^2 e^{-\bar{J}_{0,1-\varepsilon}}|\Phi|, \end{aligned} \tag{3.8}$$

where $\theta_\varepsilon(\omega)(t):=\omega_{t-\varepsilon}$ and $Y_s:=\int\int dudv I_{\{0\le u\le s\le v\le 1-\varepsilon, v-u\le\varepsilon\}}\delta(\omega_u-\omega_v)$. It is easy to prove (see e.g. [Ro83]) that $E|Y_s|^2\le C\varepsilon$. Thus, as in the derivation of (3.6) we can show that the last term on the right hand side of (3.8) is less than $C\max|f|\varepsilon^{-1/2}$.

To handle the second term on the right hand side of (3.8), let us first remark that

$$Ee^{-\bar{J}_{0,1-\varepsilon}}|\Phi_{0,1-\varepsilon}-\Phi|+EY_s e^{-\bar{J}_{0,1-\varepsilon}}|\Phi_{0,1-\varepsilon}-\Phi|\le C\max|f|\varepsilon. \tag{3.9}$$

Therefore, the second term on the right hand side of (3.8) is equal to

$$p_\varepsilon(0)[Ee^{-\bar{J}_{0,1-\varepsilon}}|\Phi|-\lambda EY_s e^{-\bar{J}_{0,1-\varepsilon}}|\Phi|]+O(\varepsilon^{-1/2}\max|f|).$$

If $u_2-u_1\le\varepsilon$, then

$$\begin{aligned} & E(\delta(\omega_{u_2}-\omega_{u_1})e^{-\bar{J}_{0,1-\varepsilon}}|\Phi|) \\ & = E[\delta(\omega_{u_2}-\omega_{u_1})e^{-\bar{J}_{0,u_1}-\bar{J}_{u_2,1-\varepsilon}-\lambda J_{0,u_1;u_2,1-\varepsilon}}|\Phi|(1+O(\varepsilon^{1/2})+O(1)(J_{0,u_1;u_1,u_2}+J_{u_1,u_2;u_2,1-\varepsilon}))]. \end{aligned}$$

As in the derivation of (3.6) one proves that

$$\begin{aligned} & E[\delta(\omega_{u_2}-\omega_{u_1})e^{-\bar{J}_{0,u_1}-\bar{J}_{u_2,1-\varepsilon}-\lambda J_{0,u_1;u_2,1-\varepsilon}}|\Phi|(J_{0,u_1;u_1,u_2}+J_{u_1,u_2;u_2,1-\varepsilon})] \\ & \le C\max|f|(u_2-u_1)^{-1}, \quad u_2-u_1\le\varepsilon. \end{aligned}$$

Using similar estimates as for (3.7) and (3.9) one also proves that

$$
\begin{aligned}
&E(\delta(\omega_{u_2}-\omega_{u_1})e^{-\bar{J}_{0,u_1}-\bar{J}_{u_2,1-\varepsilon}-\lambda J_{0,u_1;u_2,1-\varepsilon}}|\Phi|)\\
&=p_{u_2-u_1}(0)E(e^{-\bar{J}_{0,1-\varepsilon-(u_2-u_1)}}|\Phi|)+O(1)(u_2-u_1)^{-1/2}\max|f|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&E(\delta(\omega_{u_2}-\omega_{u_1})e^{-\bar{J}_{0,1-\varepsilon}}|\Phi|)\\
&=p_{u_2-u_1}(0)E(e^{-\bar{J}_{0,1-\varepsilon-(u_2-u_1)}}|\Phi|)+O(1)(u_2-u_1)^{-1/2-\delta_1}\max|f|,
\end{aligned}
$$

where $\delta_1\in(0,1)$. Suppose we have proven that

$$
|E(e^{-\bar{J}_{0,1-r}}|\Phi|)-E(e^{-\bar{J}_{0,1}}|\Phi|)|\le C\max|f|\varepsilon^{1/2+\delta_2},\quad \forall r\le 2\varepsilon \tag{3.10}
$$

for some constant $\delta_2>0$. We remark that

$$
\int\int dudvI_{\{u\le 0\le v,v-u\le\varepsilon\}}p_{v-u}(0)=\varepsilon\kappa_1(\varepsilon).
$$

Then, the second term on the right hand side of (3.8) is equal to

$$
p_\varepsilon(0)\rho(\varepsilon)(1-O(\max|f|\varepsilon^{1/2+\delta_2}))-\lambda p_\varepsilon(0)\rho(\varepsilon)\varepsilon\kappa_1(\varepsilon)(1+O(\varepsilon^{1-\delta_1}\max|f|)).
$$

By (3.8) we then know that the first term on the right hand side of (3.5) is equal to

$$
\begin{aligned}
&O(1)\max|f|\varepsilon^{-\delta_1}+p_\varepsilon(0)\rho(\varepsilon)(1-\lambda\varepsilon\kappa_1(\varepsilon))e^{\lambda\varepsilon\kappa_1(\varepsilon)}(1+O(\max|f|\varepsilon^{3/2-\delta_1}))\\
&=p_\varepsilon(0)\rho(\varepsilon)+O(1)\max|f|\varepsilon^{-\delta_1}.
\end{aligned}
$$

Thus, by (3.5) and (3.6) we know that

$$
\begin{aligned}
\frac{d}{d\varepsilon}\rho(\varepsilon)=&-\lambda^2\int_0^{1-\varepsilon}dsE(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})(J_{0,s;s,s+\varepsilon}+J_{s,s+\varepsilon;s+\varepsilon,1})|\Phi|)(1+O(\varepsilon^{1/2}))\\
&-\lambda^2\frac{d}{d\varepsilon}\kappa_2(\varepsilon)\rho(\varepsilon)+O(1)\max|f|\varepsilon^{-\delta_1},
\end{aligned}\tag{3.11}
$$

provided that (3.10) has been proven.

We now prove that

$$
\int_0^{1-\varepsilon}dsE(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})J_{0,s;s,s+\varepsilon}|\Phi|)=-\frac{1}{2}\frac{d}{d\varepsilon}\kappa_2(\varepsilon)\rho(\varepsilon)+O(1)\max|f|\varepsilon^{-\delta_1}. \tag{3.12}
$$

As in [B93, §4] we choose $\beta>\frac{1}{2}>\alpha$ and $\gamma>\frac{1}{2}$ with $\gamma<\beta$. Then the left hand side of (3.12) is equal to

$$
\int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha}ds\int_{s-\varepsilon^\beta}^{s}du\int_s^{s+\varepsilon}dvE(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s-\omega_{s+\varepsilon})\delta(\omega_u-\omega_v)|\Phi|+\max|f|\varepsilon^{-\delta_1}.
$$

For $s > \varepsilon^\gamma$ and $s + \varepsilon^\gamma \leq 1$ we set

$$\check{J}^s_{0,1} := \bar{J}_{0,s-\varepsilon^\gamma} + \bar{J}_{s+\varepsilon^\gamma,1} + \lambda J_{0,s-\varepsilon^\gamma;s+\varepsilon^\gamma,1}.$$

Then it follows that

$$\begin{aligned}
&\int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv E(e^{-\bar{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\delta(\omega_u - \omega_v)|\Phi| \\
&= \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv E(e^{-\check{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\delta(\omega_u - \omega_v)|\Phi| + \max|f|\varepsilon^{-\delta_1} \\
&= \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv E(e^{-\check{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\delta(\omega_u - \omega_v) \\
&\qquad \times|\Phi_{0,s-\varepsilon^\gamma} + \Phi_{s+\varepsilon^\gamma,1} + \Phi_{0,s-\varepsilon^\gamma;s+\varepsilon^\gamma,1}| + \max|f|\varepsilon^{-\delta_1} \\
&= \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv \int dx \int dy E(e^{-\check{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})\delta(\omega_u - \omega_v) \\
&\quad \times\delta(x - \omega_{s-\varepsilon^\gamma})\delta(y - \omega_{s+\varepsilon^\gamma})|\Phi_{0,s-\varepsilon^\gamma} + \Phi_{s+\varepsilon^\gamma,1} + \Phi_{0,s-\varepsilon^\gamma;s+\varepsilon^\gamma,1}| + \max|f|\varepsilon^{-\delta_1} \\
&= \int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv \int dx \int dy E(e^{-\check{J}^s_{0,1}}\delta(x - \omega_{s-\varepsilon^\gamma})\delta(y - \omega_{s+\varepsilon^\gamma}) \\
&\qquad \times|\Phi_{0,s-\varepsilon^\gamma} + \Phi_{s+\varepsilon^\gamma,1} + \Phi_{0,s-\varepsilon^\gamma;s+\varepsilon^\gamma,1}|q_{u,s,v}(y-x) + \max|f|\varepsilon^{-\delta_1},
\end{aligned} \tag{3.13}$$

where $q_{u,s,v} := p_{2\varepsilon^\gamma - \varepsilon-(s-u)} * (p_{s-u}p_{v-s}p_{s+\varepsilon-v})$. Thus, using the argument given in [B93] (see the top of page 96) we get that the right hand side of (3.13) is equal to

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{d\varepsilon}k_2(\varepsilon)\int_{\varepsilon^\alpha}^{1-\varepsilon^\alpha} ds \int_{s-\varepsilon^\beta}^{s} du \int_s^{s+\varepsilon} dv \int dx \int dy E(e^{-\check{J}^s_{0,1}}|\Phi| + \max|f|\varepsilon^{-\delta_1} \\
&= -\frac{1}{2}\frac{d}{d\varepsilon}k_2(\varepsilon)\int_0^{1-\varepsilon} ds \int_0^{s} du \int_s^{s+\varepsilon} dv E e^{-\bar{J}_{0,1}}|\Phi| + \max|f|\varepsilon^{-\delta_1},
\end{aligned}$$

which proves (3.12). Similarly, we prove that

$$\begin{aligned}
&\int_0^{1-\varepsilon} ds E(e^{-\tilde{J}^s_{0,1}}\delta(\omega_s - \omega_{s+\varepsilon})J_{s,s+\varepsilon;s+\varepsilon,1}|\Phi|) \\
&= -\frac{1}{2}\frac{d}{d\varepsilon}\kappa_2(\varepsilon)\rho(\varepsilon) + O(1)\max|f|\varepsilon^{-\delta_1}.
\end{aligned}$$

This then complets the proof of (3.11), provided that (3.10) has been proven.

We now turn to the proof of (3.10). Note that

$$\begin{aligned}
\frac{d}{dv}E(\exp(-\bar{J}_{0,v})|\Phi|) &= \lambda\int_0^{v-\varepsilon} ds E(\delta(\omega_s - \omega_v)\exp(-\bar{J}_{0,v})|\Phi|) \\
&\quad +(-\lambda k_1(\varepsilon) + \lambda^2 k_2(\varepsilon))E\exp(-\bar{J}_{0,v})|\Phi|.
\end{aligned} \tag{3.14}$$

As in [B93, (4.13)] we easily show that

$$|\frac{d}{dv}E\exp(-\bar{J}_{0,v})|\Phi|| \leq \max|f|\varepsilon^{-1/2}. \tag{3.15}$$

We now derive a slightly better bound, which implies the estimate (3.10). It is easy to show that

$$\int_0^{v-\varepsilon^{3/4}} dsE(\delta(\omega_s-\omega_v)e^{-\bar{J}_{0,v}}|\Phi|) \le \max|f|\varepsilon^{-3/8}.$$

Moreover, by (3.15) it follows that

$$\begin{aligned}
&\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} dsE(\delta(\omega_s-\omega_v)e^{-\bar{J}_{0,v}}|\Phi|)\\
&=\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} dsE(e^{-\bar{J}_{0,v}}|\Phi_{0,s}+\Phi_{v,1}|\delta(\omega_s-\omega_v)+\max|f|\varepsilon^{-1/4}\\
&=\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} dsEe^{-\bar{J}_{0,s}}|\Phi_{0,s}+\Phi_{v,1}|\delta(\omega_s-\omega_v)+\max|f|\varepsilon^{-1/4}\\
&\qquad+O(1)\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} dsE(e^{-\bar{J}_{0,s}}|\Phi_{0,s}+\Phi_{v,1}|\delta(\omega_s-\omega_v))|\bar{J}_{0,v}-\bar{J}_{0,s}|ds\\
&=\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} dsp_{v-s}(0)Ee^{-\bar{J}_{0,s}}|\Phi_{0,1-(v-s)}|+\max|f|\varepsilon^{-1/4}\\
&\qquad+O(1)\max|f|\int_{v-\varepsilon^{3/4}}^{v-\varepsilon} dsEe^{-\bar{J}_{0,s}}\delta(\omega_s-\omega_v)|\bar{J}_{0,v}-\bar{J}_{0,s}|ds\\
&=\int_0^{v-s} dsp_{v-s}(0)Ee^{-\bar{J}_{0,v}}|\Phi|+\max|f|\varepsilon^{-1/4}\\
&=k_1(\varepsilon)E\exp(-\bar{J}_{0,v})|\Phi|+\max|f|\varepsilon^{-1/4},
\end{aligned}$$

where we have used the results given in [B93, page 97]. Thus, by (3.14) we know that (3.10) holds. This then completes the proof of Lemma 3.1. □

In fact, using the above approach we are also able to prove a more general result. For later use we state this general result in the following remark. Since there is no essential difference between the proofs of Lemma 3.1 and Remark 3.2, we omit the proof of the latter.

Remark 3.2 For $i=1,\cdots,m$ let $f_i:\mathbb{R}^3\to\mathbb{R}$ be a bounded measurable function. For $a_i\in(0,1)$ and $i=1,\cdots,m$ set

$$\Psi_i := \int_{T_{a_i}\cap\Delta_{0,1;0,1}} f_i(\omega_\tau-\omega_\sigma)d\sigma d\tau.$$

Let k be a given function in K_0 and $U(k)=k_1'\omega_1-\int_0^1\omega_t k_t''dt$, where K_0 is as defined in Section 1. Then, using the same approach as in the proof of Lemma 3.1 we prove that for any given $l\ge 1$ there are constants $C\in(0,\infty)$ and $\delta_0\in(0,1)$ such that

$$\begin{aligned}
&\left|E(\exp(-\bar{J}_{0,1}^{\varepsilon_1,\lambda})U(k)^l\prod_{i=1}^m\Psi_i - E(\exp(-\bar{J}_{0,1}^{\varepsilon_2,\lambda})U(k)^l\prod_{i=1}^m\Psi_i\right|\\
&\le C\max_{x\in R^3}\prod_{i=1}^m|f_i(x)|(\varepsilon_1^{-\delta_0}+\varepsilon_2^{\delta_0}),\quad \varepsilon_1,\varepsilon_2\in(0,1).
\end{aligned}$$

Let $p \geq 1$ be a given constant. We can also show that there are constants $C \in (0,\infty)$ and $\delta_0 \in (0,1)$ such that

$$
\begin{aligned}
&|E\exp(-\bar{J}_{0,1}^{\varepsilon_1,\lambda})||\Psi_1|^p - |\Psi_2|^p| - E\exp(-\bar{J}_{0,1}^{\varepsilon_2,\lambda})||\Psi_1|^p - |\Psi_2|^p|| \\
&\leq C\max_{x\in R^3}(|f_1(x)| + |f_2(x)|)(\varepsilon_1^{\delta_0} + \varepsilon_2^{\delta_0}), \quad \varepsilon_1,\varepsilon_2 \in (0,1).
\end{aligned}
$$

4 Some estimates with respect to ν_λ

In Section 2 we actually proved that the sequence $(\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k))_{n\geq 1}$ is Cauchy in $L^p(\nu_0)$ for $p \in (0,\infty)$. The aim of this section is to use Lemma 3.1 above to prove that this sequence is also Cauchy in $L^1(\nu_\lambda)$ for $\lambda \in (0,\infty)$. The main result in this section, which will play a very important role in the proof that ν_λ is k-quasi-invariant for $k \in K_0$, is as follows.

Theorem 4.1 *(i) For any given $u \in \mathbb{R}$, $\varepsilon_0, \lambda \in (0,\infty)$ and $k \in K$, there is a random variable $\rho(u,k)$ in $L^1(\nu_\lambda)$ such that*

$$\lim_{n\to\infty} E_{\nu_\lambda}|\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k) - \rho(u,k)| = 0,$$

where E_{ν_λ} is the expectation with respect to ν_λ.
(ii) For any given $\lambda \in [0,\infty)$, $M \in (0,\infty)$, $k \in K$, $p > 1$ and $\gamma \in (0,1/2)$ there is a constant $C \in (0,\infty)$ such that

$$E_{\nu_\lambda}|\rho(u_1,k) - \rho(u_2,k)|^p \leq C|u_1 - u_2|^{\gamma(p-1)}, \quad |u_1|,|u_2| \leq M.$$

Proof. (i): For $\varepsilon_0 \in (0,\infty)$, $u \in \mathbb{R}$ and $k \in K$ we let

$$\Phi_n := \tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k) - \tilde{\eta}_{2^{n-1},2^{-\varepsilon_0(n-1)}}(u,k)\,,\ n \in \mathbb{N}.$$

It is easy to see that

$$
\begin{aligned}
\Phi_n \;=\; &\int_{\Delta_{0,1;0,1}} d\sigma d\tau \Big(I_{(\sigma,\tau)\in T_{2^{-\varepsilon_0 n}}} \int_{\mathbb{R}^3} e^{i\langle y,\omega_\sigma-\omega_\tau\rangle}(e^{i\langle y,u(k_\sigma-k_\tau)\rangle} - 1)g_{2^n}(y)dy \\
&\qquad - I_{(\sigma,\tau)\in T_{2^{-\varepsilon_0(n-1)}}} \int_{\mathbb{R}^3} e^{i\langle y,\omega_\sigma-\omega_\tau\rangle}(e^{i\langle y,u(k_\sigma-k_\tau)\rangle} - 1)g_{2^{n-1}}(y)dy\Big).
\end{aligned}
$$

By Lemma 3.1 we know that there are constants $\delta_0 \in (0,1)$ and $C \in (0,\infty)$ such that

$$
\begin{aligned}
&|E\exp(-\bar{J}_{0,1}^{\varepsilon_1,\lambda})|\Phi_n| - E\exp(-\bar{J}_{0,1}^{\varepsilon_2,\lambda})|\Phi_n|| \\
&\leq C2^{3n}|\varepsilon_1^{\delta_0} - \varepsilon_2^{\delta_0}|, \quad \varepsilon_1,\varepsilon_2 \in (0,1).
\end{aligned}
$$

Let $N(\lambda) := \lim_{\varepsilon\to 0^+} E\exp(-\bar{J}_{0,1}^{\varepsilon,\lambda})$ (by (3.3) we know that the latter limit exists). Moreover, it is proved in [AZ95] that $N(\lambda) > 0$ for all $\lambda \in [0,\infty)$. Therefore,

$$|N(\lambda)E_{\nu_\lambda}|\Phi_n| - E\exp(-\bar{J}_{0,1}^{\varepsilon,\lambda})|\Phi_n|| \leq C2^{3n}\varepsilon^{\delta_0}.$$

We choose $\varepsilon_n := 2^{-4\delta_0^{-1}n}$. Then

$$|N(\lambda)E_{\nu_\lambda}|\Phi_n| - E\exp(-\bar{J}_{0,1}^{\varepsilon_n,\lambda})|\Phi_n|| \le C2^{-n}. \tag{4.1}$$

By Proposition 2.1 we know that there is a constant $\delta_1 \in (0,1)$ such that

$$E|\Phi_n|^p \le C_p 2^{-\delta_1 pn}, \quad p \ge 1.$$

From the definition of $\bar{J}_{0,1}^{\varepsilon,\lambda}$ given in Section 1, we see that for $p > 1$

$$\begin{aligned}
&E\exp(-p/(p-1)\bar{J}_{0,1}^{\varepsilon_n,\lambda}) \\
&= E\exp(-\bar{J}_{0,1}^{\varepsilon,p\lambda/(p-1)} + (p^2/(p-1)^2 - p/(p-1))\lambda^2 k_2(\varepsilon_n)) \\
&= E\exp(-\bar{J}_{0,1}^{\varepsilon,p\lambda/(p-1)})\exp(p(p-1)^{-2}\lambda^2 k_2(\varepsilon_n)),
\end{aligned}$$

which implies that

$$\begin{aligned}
&E\exp(-\bar{J}_{0,1}^{\varepsilon_n,\lambda})|\Phi_n| \\
&\le E^{1-1/p}\exp(-p(p-1)^{-1}\bar{J}_{0,1}^{\varepsilon_n,\lambda}) \cdot E^{1/p}|\Phi_n|^p \\
&\le C2^{-\delta_1 n}\exp(\lambda^2(p-1)^{-1}k_2(\varepsilon_n))E\exp(-\bar{J}_{0,1}^{\varepsilon_n,p\lambda/(p-1)}) \\
&\le C2^{-\delta_1 n}\exp(\lambda^2(p-1)^{-1}k_2(\varepsilon_n)) \\
&\le C2^{-\delta_1 n/2},
\end{aligned}$$

if $p > 1$ is large enough such that $\lambda^2(p-1)^{-1}k_2(\varepsilon_n) \le \delta_1 n/2$. By (4.1) we then know that

$$N(\lambda)E_{\nu_\lambda}|\Phi_n| \le C(2^{-n} + 2^{-\delta_1 n/2}),$$

which proves that the sequence $(\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k))_{n\ge 1}$ is Cauchy in $L^1(\nu_\lambda)$. Thus, there is $\rho(u,k)$ such that

$$\lim_{n\to\infty} E_{\nu_\lambda}|\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k) - \rho(u,k)| = 0$$

and assertion (i) is proved.

(ii): Let

$$\xi_n(u_1,u_2) := \tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u_1,k) - \tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u_2,k),$$

and

$$\Phi_n' := |\xi_n(u_1,u_2)|^p - |\xi_{n-1}(u_1,u_2)|^p.$$

It is easy to see that there exists $f_n^{(u_1,u_2)}$ such that

$$\xi_n(u_1,u_2) = \int_{T_{2^{-\varepsilon_0 n}}\cap\Delta_{0,1;0,1}} f_n^{(u_1,u_2)}(\omega_\sigma - \omega_\tau)d\sigma d\tau,$$

and for any given $\gamma \in (0,1/2)$ there is a constant $C \in (0,\infty)$ such that

$$\max|f_n^{(u_1,u_2)}| \le C2^{3n(1+\gamma)}|u_1-u_2|^\gamma.$$

Thus, by a similar argument as in the proof of Lemma 3.1 (see Remark 3.2) one can show that there is $\delta_0 \in (0,1)$ such that

$$
\begin{aligned}
&|E \exp(-\bar{J}_{0,1}^{\varepsilon_1,\lambda})|\Phi'_n| - E \exp(-\bar{J}_{0,1}^{\varepsilon_2,\lambda})|\Phi'_n|| \\
&\le C 2^{3n(1+\gamma)p} |u_1 - u_2|^{\gamma p} (\varepsilon_1^{\delta_0} + \varepsilon_2^{\delta_0}), \quad \varepsilon_1, \varepsilon_2 \in (0,1).
\end{aligned}
$$

Let $\varepsilon'_n := 2^{-4n(1+\gamma)p\delta_0^{-1}}$. Then

$$
|N(\lambda) E_{\nu_\lambda} |\Phi'_n| - E \exp(-\bar{J}_{0,1}^{\varepsilon'_n,\lambda})|\Phi'_n|| \le C 2^{-(1+\gamma)pn} |u_1 - u_2|^{\gamma p}.
$$

By Proposition 2.1 we have that there is $\alpha \in (0,1)$ such that

$$
\begin{aligned}
E|\Phi'_n|^q &\le E^{1/2} |\xi_n(u_1,u_2) - \xi_{n-1}(u_1,u_2)|^{2q} \\
&\quad \cdot E^{1/2} \Big(\sum_{i=0}^{p-1} |\xi_n(u_1,u_2)|^i |\xi_{n-1}(u_1,u_2)|^{p-1-i} \Big)^{2q} \\
&\le C 2^{-\alpha q n} |u_1 - u_2|^{(p-1)\gamma q}, \quad |u_1|, |u_2| \le M,
\end{aligned}
$$

where $\gamma \in (0, 1/2)$, $M \ge 1$ is a given constant and the constant $C \in (0,\infty)$ is independent of u_1, u_2 and $n \ge 1$. Thus, as in (i) we obtain that if $q > 1$ is large enough,

$$
\begin{aligned}
&E \exp(-\bar{J}_{0,1}^{\varepsilon_n,\lambda})|\Phi'_n| \\
&\le E^{1-1/q} \exp(-q(q-1)^{-1} \bar{J}_{0,1}^{\varepsilon_n,\lambda}) E^{1/q} |\Phi'_n|^q \\
&\le C 2^{-\alpha n/2} |u_1 - u_2|^{(p-1)\gamma}, \quad |u_1|, |u_2| \le M.
\end{aligned}
$$

This proves that

$$
E_{\nu_\lambda} |\Phi'_n| \le C 2^{-\alpha n/2} |u_1 - u_2|^{(p-1)\gamma}, \quad |u_1|, |u_2| \le M.
$$

Therefore,

$$
E_{\nu_\lambda} |\xi_n(u_1,u_2)|^p \le C |u_1 - u_2|^{(p-1)\gamma}, \quad |u_1|, |u_2| \le M.
$$

By (i) we conclude that

$$
E|\rho(u_1,k) - \rho(u_2,k)|^p \le C |u_1 - u_2|^{(p-1)\gamma}, \quad |u_1|, |u_2| \le M,
$$

which proves assertion (ii) in Theorem 4.1. □

5 Proofs of Theorems 1.4 and 1.1

Let

$$
\begin{aligned}
\bar{\eta}_{n,\lambda} &:= \lambda \eta_{2^n, 2^{-\varepsilon_0 n}} - \lambda \kappa_1(2^{-\varepsilon_0 n}) + \lambda^2 \kappa_2(2^{-\varepsilon_0 n}), \\
\nu_{n,\lambda} &:= (E \exp(-\bar{\eta}'_{n,\lambda}))^{-1} \exp(-\bar{\eta}'_{n,\lambda})\, \nu_0,
\end{aligned}
$$

where $\bar{\eta}'_{n,\lambda} := \bar{\eta}_{n,\lambda} I_{\{|\eta_{2^n,2^{-\varepsilon_0 n}} - J_{0,1}^{2^{-\varepsilon_0 n}}| \le 2^{-n/8}\}}$ and $J_{s,t}^{\varepsilon}$ was defined in Section 1. Let us first prove two lemmas.

Lemma 5.1 *For any given* $\lambda \in [0, \infty)$

$$\nu_{n,\lambda} \underset{n\to\infty}{\longrightarrow} \nu_\lambda \quad \textit{in the weak sense,}$$

if $\varepsilon_0 \in (0,1)$ *is sufficiently small.*

Proof. Let

$$\begin{aligned}
I_1 &:= \int_{T_1(2^{-\varepsilon_0 n})} \int_{\mathbb{R}^6} e^{-\frac{1}{2}|y_1|^2(t_1-s_1)-\frac{1}{2}|y_2|^2(t_2-s_2)} g_{2^{m_1},2^{m_2}}(y_1,y_2) \prod_{i=1}^{2} dy_i ds_i dt_i \\
I_2 &:= \int_{T_2(2^{-\varepsilon_0 n})} \int_{\mathbb{R}^6} f_1(s,t,y) g_{2^{m_1},2^{m_2}}(y_1,y_2) \prod_{i=1}^{2} dy_i ds_i dt_i \\
I_3 &:= \int_{T_3(2^{-\varepsilon_0 n})} \int_{\mathbb{R}^6} f_2(s,t,y) g_{2^{m_1},2^{m_2}}(y_1,y_2) \prod_{i=1}^{2} dy_i ds_i dt_i.
\end{aligned}$$

Then we have the following relation:

$$E|\eta_{2^{m_1},2^{-\varepsilon_0 n}} - \eta_{2^{m_2},2^{-\varepsilon_0 n}}|^2 = I_1 + I_2 + I_3.$$

Without loss of generality we may assume $m_1 < m_2$. It is easy to show that

$$\begin{aligned}
I_1 &\leq \int_{T_1(2^{-\varepsilon_0 n})} (t_1-s_1)^{-3/2}(t_2-s_2)^{-3/2} e^{-\frac{1}{4}(t_1-s_1+t_2-s_2)2^{2m_1}} \prod_{i=1}^{2} dt_i ds_i \\
&\leq 2^{\varepsilon_0 n} e^{-\frac{1}{4}2^{\varepsilon_0 n}2^{2m_1}} \leq C 2^{2\varepsilon_0 n} 2^{-2m_1}, \quad m_1 \geq n,
\end{aligned}$$

if $\varepsilon_0 \in (0,1)$. Moreover, we obtain that

$$\begin{aligned}
I_2 &\leq \int_{T_2(2^{-\varepsilon_0 n})} \prod_{i=1}^{2} \sigma^{-3/2}(t_2-s_2)^{-3/2} \int_{\mathbb{R}^6} e^{-\frac{1}{2}|y_1|^2-\frac{1}{2}|y_2|^2} \\
&\qquad \times g_{2^m,2^{m_1}}\left(y_1\sigma^{-1/2}, y_2(t_2-s_2)^{-1/2} - y_1\sigma^{-1/2}\frac{t_1-s_2}{t_2-s_2}\right) dy_1 dy_2 \\
&\leq C 2^{2\varepsilon_0 n} 2^{-m_1}, \quad m_1 \geq n,
\end{aligned}$$

if $\varepsilon_0 \in (0,1)$ is sufficiently small, where

$$\sigma := (t_2-s_2)^{-1}((t_2-t_1)(t_1-s_1) + (t_1-s_2)(s_2-s_1)).$$

By similar methods it follows that

$$I_3 \leq C 2^{2\varepsilon_0 n - m_1}, \quad m_1 \geq n,$$

if $\varepsilon_0 \in (0,1)$ is sufficiently small. Therefore,

$$E|\eta_{2^{m_1},2^{-\varepsilon_0 n}} - \eta_{2^{m_2},2^{-\varepsilon_0 n}}|^2 \leq C 2^{-m_1/2}, \quad m_2 > m_1 \geq n,$$

if $\varepsilon_0 \in (0,1)$ is sufficiently small. Hence

$$E|\eta_{2^n,2^{-\varepsilon_0 n}} - J_{0,1}^{2^{-\varepsilon_0 n}}|^2 \le \limsup_{m\to\infty} E|\eta_{2^m,2^{-\varepsilon_0 n}} - \eta_{2^n,2^{-\varepsilon_0 n}}|^2 \le C2^{-n/2}, \tag{5.1}$$

if $\varepsilon_0 \in (0,1)$ is sufficiently small. Let $\Psi_{0,1}$ be defined as in (3.2) and let $\lambda \in [0,\infty)$ be given. We note that there is a constant $c \in (0,\infty)$ which is independent of λ such that

$$\begin{aligned} E^{1/2}\exp(-2\bar{J}_{0,1}^{2^{-\varepsilon_0 n},\lambda}) &\le E^{1/2}\exp(-\bar{J}_{0,1}^{2^{-\varepsilon_0 n},2\lambda} + 2\lambda^2\kappa_2(2^{-\varepsilon_0 n})) \\ &\le Ce^{\lambda^2\kappa_2(2^{-\varepsilon_0 n})} \le C2^{c\lambda^2\varepsilon_0 n}. \end{aligned}$$

By (5.1) we then know that

$$\begin{aligned} &|E(\Psi_{0,1}(\exp(-\bar{\eta}'_{n,\lambda}) - \exp(-\bar{J}_{0,1}^{2^{-\varepsilon_0 n},\lambda})))| \\ &\le \lambda 2^{-n/8}E(\Psi_{0,1}\exp(-\bar{J}_{0,1}^{2^{-\varepsilon_0 n},\lambda})) + E(\Psi_{0,1}(1+\exp(-\bar{J}_{0,1}^{2^{-\varepsilon_0 n},\lambda}))I_{\{|\eta_{2^n,2^{-\varepsilon_0 n}} - J_{0,1}^{2^{-\varepsilon_0 n}}|\ge 2^{-n/8}\}}) \\ &\le C2^{-n/8} + C2^{c\lambda^2\varepsilon_0 n}2^{-n/16} \le C2^{-n/32}, \end{aligned} \tag{5.2}$$

if $\varepsilon_0 \in (0,1)$ is sufficiently small. By (3.3) we know that

$$\lim_{\varepsilon\to 0+}(E\exp(-\bar{J}_{0,1}^{\varepsilon,\lambda}))^{-1}E\exp(-\bar{J}_{0,1}^{\varepsilon,\lambda})\Psi_{0,1} = E_{\nu_\lambda}\Psi_{0,1}.$$

It follows from (5.2) that

$$\lim_{n\to\infty}(E\exp(-\bar{\eta}'_{n,\lambda}))^{-1}E\exp(-\bar{\eta}'_{n,\lambda})\Psi_{0,1} = E_{\nu_\lambda}\Psi_{0,1},$$

if $\varepsilon_0 \in (0,1)$ is sufficiently small. This completes the proof of Lemma 5.1. □

From now on we assume that $\varepsilon_0 \in (0,1)$ is sufficiently small. Let

$$A_n(\omega) := \{|\eta_{2^n,2^{-\varepsilon_0 n}}(\omega) - \bar{J}_{0,1}^{2^{-\varepsilon_0 n}}(\omega)| \le 2^{-\delta n}\}$$

and let $\delta \in (0, 1/16)$ be a small constant which will be specified in the proof of Lemma 5.2 below. Set

$$D_{n,\lambda}(\omega) := e^{-\bar{\eta}'_{n,\lambda}(\omega-uk)I_{A_n(\omega-uk)} + \bar{\eta}'_{n,\lambda}(\omega)I_{A_n(\omega)}} \cdot e^{u\int_0^1 k'_t d\omega_t - \frac{1}{2}u^2\int_0^1 (k'_t)^2 dt},$$

with $\int_0^1 k'_t d\omega_t$ understood as an Itô stochastic integral. It is clear that $D_{n,\lambda} = \frac{d\tau_{uk}(\nu_{n,\lambda})}{d\nu_{n,\lambda}}$. Since ν_λ is singular with respect to ν_0 and we do not know at present whether $(\omega_t)_{t\in[0,1]}$, as the coordinate process under ν_λ, is a semimartingale or not, the quantity $\int_0^1 k'_t d\omega_t$ is not well-defined under the polymer measure ν_λ. If $k \in K_0$, however, by Itô's formula we have that

$$\int_0^1 k'_t d\omega_t = k'_1\omega_1 - \int_0^1 \omega_t dk'_t.$$

The right hand side of the above equality is well-defined under the measure ν_λ. From now on we assume that $k \in K_0$, and the left hand side (i.e. $\int_0^1 k_t' d\omega_t$) of the above equality is always understood as its right hand side under the measure ν_λ. Let

$$a_{uk} := e^{-\lambda\rho(-u,k)+V(u,k)},$$

where $V(u,k) := uk_1'\omega_1 - u\int_0^1 \omega_t k_t'' dt - \frac{1}{2}u^2 \int_0^1 (k_t')^2 dt$, and $\rho(u,k)$ is as defined in Theorem 4.1.

Lemma 5.2 *For any given $u \in \mathbb{R}$, $k \in K_0$ and $\lambda \in [0,\infty)$ the following holds*

$$\frac{d\tau_{uk}(\nu_\lambda)}{d\nu_\lambda} = a_{uk}, \quad \nu_\lambda\text{-}a.e.$$

Proof. For $M > 1$ we set

$$B_n(\omega) := \{|\lambda\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(-u,k)| \geq \log M/4\} \cup A_n(\omega - uk) \cup A_n(\omega) \cup \{|V(u,k)| \geq \log M/4\}.$$

By (3.1) and (5.2) we obtain that

$$\sup_{n\geq 1} E_{\nu_{n,\lambda}}\left(|\omega_1| + \left(\int_0^1 |\omega_t|^2 dt\right)^{1/2}\right) \leq C < \infty.$$

From this it follows that

$$\sup_{n\geq 1} \nu_{n,\lambda}\left(\left|uk_1'\omega_1 - u\int_0^1 \omega_t k_t'' dt\right| + \left|\frac{1}{2}u^2\int_0^1 (k_t')^2 dt\right| \geq \log M/4\right) \leq C(\log M)^{-1}. \quad (5.3)$$

From the proof of Theorem 4.1 (i) we see that

$$E\exp(-\bar{J}_{0,1}^{\varepsilon,\lambda})|\Phi_n| \leq C2^{-\delta_0 n}, \quad n \geq 1,$$

where $\delta_0 \in (0,1)$ and $\Phi_n := \tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k) - \tilde{\eta}_{2^{n-1},2^{-\varepsilon_0(n-1)}}(u,k)$. Thus, as in the derivation of (6.2) we prove that

$$E_{\nu_{n,\lambda}}|\Phi_m| \leq C2^{-\delta_1 m}, \quad \forall m,n \geq 1,$$

with a constant $\delta_1 \in (0,1)$. By (3.1) it follows that $\sup_{n\geq 1} E_{\nu_{n,\lambda}}|\tilde{\eta}_{2,2^{-\varepsilon_0}}(u,k)|$ is finite. Hence,

$$\begin{aligned} &\sup_{n\geq 1} E_{\nu_{n,\lambda}}|\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k)| \\ \leq\ &\sup_{n\geq 1} E_{\nu_{n,\lambda}}|\tilde{\eta}_{2,2^{-\varepsilon_0}}(u,k)| + \sup_{n\geq 1}\sum_{m=2}^{\infty} E_{\nu_{n,\lambda}}|\Phi_m| \leq C, \end{aligned}$$

which implies that

$$\nu_{n,\lambda}(|\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k)| \geq \log M/4) \leq C(\log M)^{-1}, \quad \forall n \geq 1. \quad (5.4)$$

From (5.2) we see that

$$\nu_{n,\lambda}(A_n^c) \le C2^{-n/16}, \quad n \ge 1. \tag{5.5}$$

To estimate $\nu_{n,\lambda}(A_n^c(\cdot + uk))$ let us first remark that

$$\eta_{2^m,2^{-\varepsilon_0 n}}(\omega + uk) = \tilde{\eta}_{2^m,2^{-\varepsilon_0 n}}(u,k) + \eta_{2^m,2^{-\varepsilon_0 n}}(\omega).$$

By Proposition 2.1 and (5.1) we then know that

$$\begin{aligned}
&E|\eta_{2^n,2^{-\varepsilon_0 n}}(\cdot + uk) - J_{0,1}^{2^{-\varepsilon_0 n}}(\cdot + uk)|^2 \\
&\le \limsup_{m\to\infty} E|\eta_{2^m,2^{-\varepsilon_0 n}}(\cdot + uk) - \eta_{2^n,2^{-\varepsilon_0 n}}(\cdot + uk)|^2 \\
&\le 2\limsup_{m\to\infty} E|\tilde{\eta}_{2^m,2^{-\varepsilon_0 n}}(u,k) - \tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k)|^2 + 2\limsup_{m\to\infty} E|\eta_{2^m,2^{-\varepsilon_0 n}}(\cdot) - \eta_{2^n,2^{-\varepsilon_0 n}}(\cdot)|^2 \\
&\le C2^{-\delta_2 n}, \quad \forall n \ge 1,
\end{aligned}$$

with a constant $\delta_2 \in (0,1)$. We now choose $\delta = \delta_2/4 \wedge 1/16$ in the definition of the set A_n. Thus, as in the derivation of (5.2) we prove that

$$E_{\nu_{n,\lambda}}|\eta_{2^n,2^{-\varepsilon_0 n}}(\cdot + uk) - J_{0,1}^{2^{-\varepsilon_0 n}}(\cdot + uk)|^2 \le C2^{-\delta_2 n/4},$$

which implies that

$$\nu_{n,\lambda}(A_n^c(\cdot + uk)) \le C2^{-\delta_2 n/8}, \quad n \ge 1. \tag{5.6}$$

Combining the estimates (5.3) - (5.6) we conclude that

$$\begin{aligned}
\nu_{n,\lambda}(B_n(\cdot + uk)) \;\le\;& \nu_{n,\lambda}(|\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k)| \ge \log M/4) + \nu_{n,\lambda}(A_n^c) + \nu_{n,\lambda}(A_n^c(\cdot + uk)) \\
&+\nu_{n,\lambda}\Big(\Big|uk_1'\omega_1 - u\int_0^1 \omega_t k_t'' dt\Big| + \Big|\frac{1}{2}u^2\int_0^1 (k_t')^2 dt\Big| \ge \log M/4\Big) \\
\le\;& C((\log M)^{-1} + 2^{-\delta_2 n/4} + 2^{-n/16}), \quad n \ge 1.
\end{aligned}$$

Using this estimate we obtain that for $A \in \mathcal{F}$

$$\begin{aligned}
&\int_{A\cap B_n} \frac{d\tau_{uk}(\nu_{n,\lambda})}{d\nu_{n,\lambda}} d\nu_{n,\lambda} = \int_{A\cap B_n} d\tau_{uk}(\nu_{n,\lambda}) \\
&= \int_{(A-uk)\cap B_n(\cdot+uk)} d\nu_{n,\lambda} \le C((\log M)^{-1} + 2^{-\delta_2 n/4} + 2^{-n/16}), \quad n \ge 1,
\end{aligned} \tag{5.7}$$

where $A - uk := \{\omega - uk : \ \omega \in A\}$.

Now we set

$$\begin{aligned}
\zeta_n &:= -\lambda\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(-u,k) I_{A_n(\omega)\cap A_n(\omega+uk)} + V(u,k), \\
\bar{\zeta}_n &:= -\lambda\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(-u,k) + V(u,k).
\end{aligned}$$

The random variable $|\zeta_n|$ is bounded on the set B_n^c. By Taylor's expansion we can find a constant $L = L(M) \geq 1$ such that

$$\Big|\int_{A\cap B_n^c} e^{\zeta_n} d\nu - \int_{A\cap B_n^c} \sum_{i=0}^{L} \frac{1}{i!}\zeta_n^i d\nu\Big| \leq C(\log M)^{-1}, \quad n \geq 1,$$

where $\nu := \nu_{n,\lambda}$ or $= \nu_\lambda$. By (5.5), (5.6) and Hölder's inequality it easily follows that

$$\lim_{n\to\infty}\Big|\int_{A\cap B_n^c} (\zeta_n^i - \bar{\zeta}_n^i) d\nu_{n,\lambda}\Big| = 0, \quad A \in \mathcal{F}, i = 0, 1, \cdots, L. \tag{5.8}$$

Using a similar argument as for proving (3,3) (see Remark 3.2 above or [B93, §4]) one can easily show that there are constants $\delta_3 \in (0,1)$ and $C \in (0,\infty)$ such that

$$|E((e^{-\bar{J}_{0,1}^{\varepsilon_1,\lambda}} - e^{-\bar{J}_{0,1}^{\varepsilon_2,\lambda}})V(u,k))| \leq C(\varepsilon_1^{\delta_3} + \varepsilon_2^{\delta_3})$$

for all $\varepsilon_1, \varepsilon_2 \in (0,1)$. Thus, Lemma 3.1 implies that

$$E|\exp(-\bar{J}_{0,1}^{2^{-\varepsilon_0 n},\lambda})\bar{\zeta}_n - E\exp(-\bar{J}_{0,1}^{\varepsilon,\lambda})\bar{\zeta}_n| \leq C2^{-\delta_4 n}, \varepsilon \in (0, 2^{-\varepsilon_0 n}]$$

for some constants $\delta_4 \in (0,1)$ and $C \in (0,\infty)$. Let $\mathcal{F}_0$ be the collection of all sets of the form

$$\{\omega:\ \omega_{t_1} \in O_1, \cdots, \omega_{t_n} \in O_n\}$$

for $0 \leq t_1 < \cdots < t_n \leq 1$, and open sets $O_1, \cdots, O_n$ in $\mathbb{R}^3$ and all $n \geq 1$. Then it follows that for any given $A \in \mathcal{F}_0$

$$\Big|\int_{A\cap B_n^c} \bar{\zeta}_n d\nu_{n,\lambda} - \int_{A\cap B_n^c} \bar{\zeta}_n d\nu_\lambda\Big| \leq C2^{-\delta_5 n}, \quad n \geq 1$$

for some constants $\delta_5 \in (0,1)$ and $C \in (0,\infty)$. By Remark 3.2 we know that

$$|E((e^{-\bar{J}_{0,1}^{2^{-\varepsilon_0 n},\lambda}} - e^{-\bar{J}_{0,1}^{\varepsilon,\lambda}})V^j(u,k)(\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k))^{i-j})| \leq C2^{-\delta_4 n}, \quad \varepsilon \in (0, 2^{-\varepsilon_0 n}]$$

for $0 \leq j \leq i \leq L$. This implies that the following holds for any given $A \in \mathcal{F}_0$

$$\Big|\int_{A\cap B_n^c} V^j(u,k)(\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k))^{i-j} d\nu_{n,\lambda} - \int_{A\cap B_n^c} V^j(u,k)(\tilde{\eta}_{2^n,2^{-\varepsilon_0 n}}(u,k))^{i-j} d\nu_\lambda\Big| \leq C2^{-\delta_5 n}, \quad n \geq 1,\ 0 \leq j \leq i \leq L.$$

From this we obtain that for any given $A \in \mathcal{F}_0$

$$\Big|\int_{A\cap B_n^c} \bar{\zeta}_n^i d\nu_{n,\lambda} - \int_{A\cap B_n^c} \bar{\zeta}_n^i d\nu_\lambda\Big| \leq C2^{-\delta_5 n}, \quad n \geq 1$$

for $i \in [1, L]$. By (5.8) we know that for $A \in \mathcal{F}_0$

$$\lim_{n\to\infty}\Big|\int_{A\cap B_n^c} \sum_{i=1}^{L} \frac{1}{i!}\zeta_n^i d\nu_{n,\lambda} - \int_{A\cap B_n^c} \sum_{i=0}^{L} \frac{1}{i!}\zeta_n^i d\nu_\lambda\Big| = 0.$$

Hence, if $n \geq 1$ is large enough and $A \in \mathcal{F}_0$ is fixed

$$\left| \int_{A \cap B_n^c} e^{\zeta_n} d\nu_{n,\lambda} - \int_{A \cap B_n^c} e^{\zeta_n} d\nu_\lambda \right| \leq C(\log M)^{-1}. \tag{5.9}$$

Since $D_{n,\lambda}$ is bounded on the set B_n^c, by the dominated convergence theorem and Theorem 4.1, it follows that for any given $A \in \mathcal{F}_0$

$$\begin{aligned} \int_{A \cap B^c} a_{uk} d\nu_\lambda &\leq \liminf_{n \to \infty} \int_{A \cap B_n^c} D_{n,\lambda} d\nu_\lambda \\ &\leq \limsup_{n \to \infty} \int_{A \cap B_n^c} D_{n,\lambda} d\nu_\lambda \leq \int_{A \cap B'^c} a_{uk} d\nu_\lambda, \quad A \in \mathcal{F}_0, \end{aligned}$$

where

$$\begin{aligned} B &:= \{|\rho(-u,k)| \geq \log M/4\} \cup \{|V(u,k)| \geq \log M/4\}, \\ B' &:= \{|\rho(-u,k)| > \log M/4\} \cup \{|V(u,k)| > \log M/4\}. \end{aligned}$$

Since $M > 1$ is arbitrary, by (5.7) and (5.9) we then obtain that

$$\lim_{n \to \infty} \int_A D_{n,\lambda} d\nu_{n,\lambda} = \int_A a_{uk} d\nu_\lambda, \quad A \in \mathcal{F}_0. \tag{5.10}$$

By Lemma 5.1 we also know that for any given $u \in \mathbb{R}$ and $k \in K$

$$\tau_{uk}(\nu_{n,\lambda}) \underset{n \to \infty}{\longrightarrow} \tau_{uk}(\nu_\lambda) \text{ weakly.}$$

Thus, we obtain that for any given $A \in \mathcal{F}_0$

$$\int_A d\tau_{uk}(\nu_{n,\lambda}) \to \int_A d\tau_{uk}(\nu_\lambda), \quad n \to \infty.$$

Since $\int_A d\tau_{uk}(\nu_{n,\lambda})$ is by definition equal to $\int_A D_{n,\lambda} d\nu_{n,\lambda}$, by (5.10) we have that

$$\int_A \frac{d\tau_{uk}(\nu_\lambda)}{d\nu_\lambda} d\nu_\lambda = \int_A a_{uk} d\nu_\lambda, \quad \forall A \in \mathcal{F}_0.$$

Using this we easily deduce the desired result. $\square$

Proof of Theorem 1.4 We have to show that for each $k \in K_0$ there is $\{a'_{uk}, u \in \mathbb{R}\}$ such that

$$\nu_\lambda(a'_{uk} = a_{uk}) = 1, \quad \forall u \in \mathbb{R}, \tag{5.11}$$

$$\nu_\lambda(a'_{uk} \text{ is continuous with respect to } u \in \mathbb{R}) = 1. \tag{5.12}$$

We choose a sufficiently small $\varepsilon_0 \in (0,1)$ such that the assertion in Lemma 5.1 holds. By Lemma 5.2 we know that $a_{uk} = \frac{d\tau_{uk}(\nu_\lambda)}{d\nu_\lambda}$, ν_λ-a.e. By Theorem 4.1 we know that there is $\rho'(u,k)$ such that

$$\begin{aligned} &\nu_\lambda(\rho'(u,k) = \rho(u,k)) = 1, \quad u \in \mathbb{R}, \\ &\nu_\lambda(\rho'(u,k) \text{ is continuous with respect to } u \in \mathbb{R}) = 1. \end{aligned}$$

For $k \in K_0$ we set

$$a'_{uk} = e^{-\lambda\rho'(-u,k)+V(u,k)}.$$

Then $\{a'_{uk}, u \in \mathbb{R}\}$ satisfies (5.11) and (5.12). $\square$

Proof of Theorem 1.1 Since obviously K_0 is dense in H, hence in X, it follows by [AR90b, Proposition 2.7] that $\mathrm{supp}_{\nu_\lambda} = X$. This proves assertion (i). Since we can make use of Theorem 1.4 and hence can apply the general closability criterion [AR89b, Corollary 2.5] as in the two-dimensional case, the proof of assertion (ii) is word by word the same as that of [AHRZ95, Theorem 1.1] except that K has to be replaced by K_0.

$\square$

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