WHEN IS THE STUDENT t–STATISTIC ASYMPTOTICALLY STANDARD NORMAL?

BY EVARIST GINÉ*, FRIEDRICH GÖTZE† AND DAVID M. MASON‡

University of Connecticut, Universität Bielefeld and University of Delaware

Abstract

Let X, X_i, $i \in \mathbf{N}$, be independent identically distributed random variables. It is shown that the Student t–statistic based upon the sample $\{X_i\}_{i=1}^n$ is asymptotically $N(0,1)$ if and only if X is in the domain of attraction of the normal law. It is also shown that, for any X, if the self–normalized sums $U_n := \sum_{i=1}^n X_i \Big/ \left(\sum_{i=1}^n X_i^2\right)^{1/2}$, $n \in \mathbf{N}$, are stochastically bounded then they are uniformly subgaussian that is, $\sup_n \mathbf{E} \exp(\lambda U_n^2) < \infty$ for some $\lambda > 0$.

February 1996

AMS 1991 *subject classifications.* Primary: 60F05, 62E20.

Key words and phrases: Student t–statistic, self–normalized sums, domains of attraction, convergence of moments.

* Research partially supported by the SFB 343 at Bielefeld and by NSF Grant No. DMS-93–00725.

† Research supported by the SFB 343 at Bielefeld.

‡ Research partially supported by NSF Grant No. DMS–95–04758.

1. Introduction. Let $X, X_i,\ i \in \mathbf{N}$, be independent, identically distributed random variables and let

$$S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2, \quad n \in \mathbf{N}. \tag{1.1}$$

Using this notation we can write the classical Student t–statistic as

$$T_n \; = \; \frac{S_n/V_n}{\sqrt{(n-(S_n/V_n)^2)/(n-1)}}. \tag{1.2}$$

The main object of this article is to prove that T_n or, equivalently, the self–normalized sum S_n/V_n, is asymptotically standard normal if and only if X is in the domain of attraction of the normal law, written $X \in DAN$, and $\mathbf{E}X = 0$.

A basic requirement in the proof of this result is that the moments of S_n/V_n converge to those of the standard normal whenever S_n/V_n is asymptotically $N(0,1)$. This is a trivial consequence of our second main result, which states that, if the sequence $\{S_n/V_n\}$ is stochastically bounded, then it is uniformly subgaussian in the sense that $\sup_n \mathbf{E}\exp\{\lambda(S_n/V_n)^2\} < \infty$ for some $\lambda > 0$ or, equivalently, $\sup_n \mathbf{E}\exp\{tS_n/V_n\} \le 2\exp\{ct^2\}$ for all $t \in \mathbf{R}$ and some $c < \infty$. To compare this with previous results, we recall that Logan et *al.* (1973) show, among other things, that if X is in the domain of attraction of an α–stable law, $0 < \alpha \le 2$, centered if $\alpha > 1$ and symmetric if $\alpha = 1$, then the sequence of self–normalized sums converges in distribution to a limit which is subgaussian, and if moreover X is symmetric, then the moments of S_n/V_n also converge to the corresponding moments of the limit. As a consequence of our second result we obtain that if the sequence $\{S_n/V_n\}$ is stochastically bounded then all its subsequential limits in law are (uniformly) subgaussian and there is convergence of the moment generating function of S_n/V_n along each convergent (in distribution) subsequence to the moment generating function of the limit, for all $t \in \mathbf{R}$ (there is even convergence of some square exponential moments).

If X is symmetric then the self–normalized sums are always stochastically bounded as a consequence of Khinchin's inequality. But if X is not symmetric, there are even centered random variables with good moment conditions for which S_n/V_n fails to be shift tight. We present examples of this type and give as well a criterion for stochastic boundednesss of self–normalized sums.

The line of research leading to our results starts perhaps with Efron (1969), who studied the limiting behavior of the Student t–statistic and, equivalently, of self–normalized sums in some nonstandard cases. In a more strict sense, it actually begins with the conjecture of Logan et *al.* (1973) stating that 'S_n/V_n is asymptotically normal if [and perhaps only if] X is in the domain of attraction of the normal law' (and X is centered).

The 'if' part of this conjecture follows rather easily from basic principles (Raikov's theorem), as was noticed, among others, by Maller (1981), but the parenthetical 'only if' part has remained open until now for the general case of not necessarily symmetric random variables. For X symmetric, Griffin and Mason (1991) attribute to Roy Erickson a beautiful proof of the fact that convergence in distribution of the self–normalized sums S_n/V_n to the standard normal law does indeed imply that $X \in DAN$. This result and its method of proof directly inspired ours.

As in the symmetric case (Griffin and Mason, *loc. cit.*), proving that $X \in DAN$ under the assumption that the self–normalized sums are asymptotically standard normal, ultimately reduces –via O'Brien's (1980) observation that $X \in DAN$ is equivalent to $\max_{i \le n} |X_i|/V_n \to 0$ in probability– to the analysis of the terms in the development of $\mathbf{E}(S_n/V_n)^4$. Most of these terms vanish in the symmetric case. The main difficulty in the absence of symmetry consists in showing that the terms which are zero under symmetry are indeed asymptotically negligible in the general case. Control of these terms is achieved by means of certain estimates (Lemma 3.1) of the moments

$$\mathbf{E}\big(X_1^{m_1} \cdots X_r^{m_r}/V_n^k\big), \quad m_1 + \ldots m_r = k, \tag{1.3}$$

in combination with the result on uniform subgaussianness of S_n/V_n (only the consequence on boundedness of moments is used here). The key to the proof of this last theorem, is another estimate (Lemma 2.1) of the same moments (1.3) which allow us to derive a sort of converse Hölder inequality for S_n/V_n. The (two types of) estimates of (1.3) just alluded to, which are probably of independent interest, are the main new technical tools used in the present work.

Section 2 contains the result on boundedness of exponential moments of stochastically bounded self–normalized sums and some of its consequences, as well as examples of variables for which the self–normalized sums are not shift tight. Section 3 is devoted to the asymptotic normality of self–normalized sums and the Student t–statistic which, as indicated above, is our main goal.

Letting T_n be the Student t–statistic as in (1.2) based on n samples from X, Bentkus and Götze (1994) obtained sharp bounds for $\sup_{x \in \mathbf{R}} |\Pr\{T_n < x\} - \Phi(x)|$ that depend rather explicitly on the law of X and tend to zero whenever $X \in DAN$. Because of our main result, remarkably, these estimates tend to zero whenever there is convergence. For other important aspects of self–normalized sums we refer to LePage et *al.* (1981), S. Csörgő (1989) and the references therein.

Finally here are some notational conventions. Most statements of results in this article refer to the self–normalized sums S_n/V_n, with S_n and V_n as in (1.1) from a sequence of i.i.d. random variables X_i distributed like X. We implicitly assume this setup throughout and avoid describing it in every statement. $\mathcal{L}(Y)$ will denote the probability distribution of the random variable Y. Limits of probability laws are always to be understood in the sense of weak convrgence, that is, in the sense of limits in distribution of the variables involved. If $V_n(\omega) = 0$ (in which case, also $S_n(\omega) = 0$), with some abuse of notation we define $S_n(\omega)/V_n(\omega) := 0$, that is, we adhere to the convention $0/0 = 0$.

2. Uniform subgaussianness of stochastically bounded self–normalized sums and some consequences. The proof of the main result of this section (Theorem 2.5) is ultimately based on the observation that the components in the development of $\mathbf{E}(S_n/V_n)^k$, $k > 1$, are dominated, up to precise multiplicative constants, by powers of moments of order one, $\big[\mathbf{E}|S_\ell/V_\ell|\big]^s$, for some $\ell \le n$ and $s \le k$. We isolate this fact as the following lemma:

2.1. Lemma. *Let $r, k, n, m_1, \ldots, m_r$ be natural numbers such that $1 \le r \le k \le n$, $m_i \ge 1$ for all i and $m_1 + \ldots + m_r = k$. Define $n_r = [n/r]$ and $s = \#\{i \le r : m_i = 1\}$. Then the* *following inequality holds for any set of i.i.d. random variables $X_i,\ i \le n$:*

$$n_r^r \binom{k}{m_1, \ldots, m_r}^{1/2} \left| \mathbf{E} \frac{X_1^{m_1} \cdots X_r^{m_r}}{V_n^k} \right| \le \left(\mathbf{E} \left| \frac{S_{n_r}}{V_{n_r}} \right| \right)^s . \tag{2.1}$$

Proof. Setting

$$S(i) := \sum_{j=1}^{n_r} X_{(j-1)n_r+i}^{m_i}, \quad V(i) := \Big[\sum_{j=1}^{n_r} X_{(j-1)n_r+i}^2 \Big]^{1/2}, \quad \text{for } i = 1, \ldots, r,$$

we have

$$n_r^r \left| \mathbf{E} \frac{X_1^{m_1} \cdots X_r^{m_r}}{V_n^k} \right| = \left| \mathbf{E} \frac{S(1) \cdots S(r)}{V_n^k} \right| .$$

Also, by the multinomial theorem,

$$V_n^k = \left(\Big[\sum_{j=1}^{r} \big(V(j)\big)^2 \Big]^k \right)^{1/2} \ge \binom{k}{m_1, \ldots, m_r}^{1/2} \prod_{j=1}^{r} \big(V(j)\big)^{m_j},$$

so that

$$\binom{k}{m_1, \ldots, m_r}^{1/2} \left| \frac{S(1) \cdots S(r)}{V_n^k} \right| \le \prod_{i=1}^{r} \left| \frac{S(i)}{\big(V(i)\big)^{m_i}} \right| .$$

(Note that, provided we adhere to our convention $0/0 = 0$, this inequality holds even if some of the $V(i)$ are zero as, in this case, both sides are zero.) Since the random variables $S(i)/\big(V(i)\big)^{m_i}$ are independent, we obtain, from the above, that

$$n_r^r \binom{k}{m_1, \ldots, m_r}^{1/2} \left| \mathbf{E} \frac{X_1^{m_1} \cdots X_r^{m_r}}{V_n^k} \right| \le \prod_{i=1}^{r} \mathbf{E} \left| \frac{S(i)}{\big(V(i)\big)^{m_i}} \right| .$$

Now, if $m_i > 1$ then $S(i) \le \big(V(i)\big)^{m_i}$, and therefore the last product is dominated by the product of the terms for which $m_i = 1$,

$$\prod_{1 \le i \le r : m_i = 1} \mathbf{E} \left| \frac{S(i)}{\big(V(i)\big)^{m_i}} \right| = \left(\mathbf{E} \left| \frac{S_{n_r}}{V_{n_r}} \right| \right)^s ,$$

giving inequality (2.1).

□

Inequality (2.1) produces a kind of reverse Hölder (or hypercontractivity) inequality for the variables $|S_n/V_n|$, as follows:

2.2. Lemma. *The following inequality holds for all $n, k \in \mathbf{N}$:*

$$\left|\mathbf{E}\Big(\frac{S_n}{V_n}\Big)^k\right| \le c(k)\max\Big[1, \max_{\ell\le n}\mathbf{E}\Big|\frac{S_\ell}{V_\ell}\Big|\Big]^k \tag{2.2}$$

where $c(k)$ can be taken to be

$$c(k) = \Big(\frac{4e}{3}+1\Big)^k (k!)^{1/2} \le 5^k (k!)^{1/2}. \tag{2.3}$$

Proof. For $n \le 4k$ we have

$$|S_n/V_n|^k \le n^{k/2} \le 2^k k^{k/2} < c(k)$$

since $k! \ge 2k^k/e^k$ for all k (we will make repeated use of this elementary inequality). So, we can assume $n > 4k$, in which case, $1/n_r < \frac{4}{3}r/n$ for all $r \le k$. Let us set

$$M_n = \max\Big[1, \max_{\ell\le n}\mathbf{E}\Big|\frac{S_\ell}{V_\ell}\Big|\Big]. \tag{2.4}$$

Then, inequality (2.1) gives

$$\begin{aligned}
\left|\mathbf{E}\Big(\frac{S_n}{V_n}\Big)^k\right| &= \left|\sum_{r=1}^k \sum_{\substack{m_1+\ldots+m_r=k\\ m_i\ge 1}} \binom{n}{r}\binom{k}{m_1,\ldots,m_r}\mathbf{E}\frac{X_1^{m_1}\cdots X_r^{m_r}}{V_n^k}\right| \\
&\le M_n^k \sum_{r=1}^k \binom{n}{r} n_r^{-r} \sum_{\substack{m_1+\ldots+m_r=k\\ m_i\ge 1}} \binom{k}{m_1,\ldots,m_r}^{1/2} \\
&\le M_n^k (k!)^{1/2} \sum_{r=1}^k \Big(\frac{4}{3}\Big)^r \frac{r^r}{r!}\binom{k-1}{r-1} \\
&\le \frac{1}{2} M_n^k (k!)^{1/2} \sum_{r=1}^k \Big(\frac{4e}{3}\Big)^r \binom{k-1}{r-1} \\
&\le \frac{2e}{3} M_n^k (k!)^{1/2} (1+4e/3)^{k-1} < c(k) M_n^k,
\end{aligned}$$

where we use that $\#\{(m_1,\ldots,m_r) : m_1+\ldots+m_r = k, m_i \ge 1, m_i \in \mathbf{N}\} = \binom{k-1}{r-1}$ and bound the multinomial coefficient by $k!$.

$\square$

Lemma 2.2 has the following immediate consequence:

2.3. Lemma. *With M_n defined by (2.4), we have that, for all $n \in \mathbf{N}$,*
i)

$$\mathbf{E}\exp\Big\{\frac{(S_n/V_n)^2}{4(1+4e/3)^2 M_n^2}\Big\} \le 2 \tag{2.5}$$

and
ii)

$$\mathbf{E}\exp\Big\{t|S_n/V_n|\Big\} \le 2\exp\Big\{(1+4e/3)^2M_n^2t^2\Big\} \tag{2.6}$$

for all $t>0$.

PROOF. Let us fix n and set

$$\lambda_0 = \frac{1}{4(1+4e/3)^2M_n^2}.$$

Then, inequality (2.2) and the definition of $c(k)$ readily give

$$\begin{aligned}\sup_n \mathbf{E}\exp\{\lambda_0(S_n/V_n)^2\} &\le \sum_{k=0}^{\infty}\frac{\lambda_0^k\big((2k)!\big)^{1/2}(1+4e/3)^{2k}M_n^{2k}}{k!}\\ &< \sum_{k=0}^{\infty}\lambda_0^k2^k(1+4e/3)^{2k}M_n^{2k}=2,\end{aligned}$$

since $(2k)!\le 2^kk!$, proving (2.5). Using the inequality $|ab|\le a^2/2+b^2/2$ we have, for all $t>0$,

$$t|S_n/V_n| \le \frac{1}{2}\big(2^{1/2}\lambda_0^{1/2}|S_n/V_n|\big)^2+\frac{1}{2}\big(t/(2\lambda_0)^{1/2}\big)^2=\lambda_0\Big(\frac{S_n}{V_n}\Big)^2+\frac{t^2}{4\lambda_0}.$$

Then, (2.5) gives

$$\mathbf{E}\exp\Big\{t|S_n/V_n|\Big\}\le 2\exp\Big\{\frac{t^2}{4\lambda_0}\Big\},$$

that is, (2.6).

□

Inequality (2.2) also allows us to apply Paley–Zygmund's argument (e.g., Kahane, 1968, page 6, or Kwapień and Woyczyński, 1992, Lemma 0.2.1) along a subsequence and conclude that if the sequence $\{S_n/V_n\}$ is stochastically bounded then it is L_1–bounded, as follows:

2.4. LEMMA. *If the sequence* $\{S_n/V_n\}$ *is stochastically bounded then*

$$\sup_n \mathbf{E}\Big|\frac{S_n}{V_n}\Big|<\infty. \tag{2.7}$$

PROOF. Suppose, to the contrary, that

$$\sup_n \mathbf{E}\big|S_n/V_n\big|=\infty,$$

and, for each $\ell\in\mathbf{N}$, define

$$n_\ell=\min\Big\{n:\mathbf{E}\Big|\frac{S_n}{V_n}\Big|>\ell^{1/2}\Big\}.$$

Then,

$$n_\ell < \infty \quad \text{for all } \ell \in \mathbf{N} \quad \text{and} \quad n_\ell > \ell \to \infty.$$

(n_ℓ is finite by the assumption, and $n_\ell > \ell$ because, as observed above, $\left|S_n/V_n\right| \le n^{1/2}$ for all n.) Moreover, the definition of n_ℓ implies

$$\max_{n \le n_\ell} \mathbf{E}\left|\frac{S_n}{V_n}\right| = \mathbf{E}\left|\frac{S_{n_\ell}}{V_{n_\ell}}\right| > \ell^{1/2}.$$

Therefore, inequality (2.2) for $n = n_\ell$ and $k = 2$ becomes

$$\mathbf{E}\left(\frac{S_{n_\ell}}{V_{n_\ell}}\right)^2 \le c(2)\left(\mathbf{E}\left|\frac{S_{n_\ell}}{V_{n_\ell}}\right|\right)^2. \tag{2.8}$$

Now we observe that, by Hölder, for all $a > 0$,

$$\mathbf{E}\left|\frac{S_n}{V_n}\right| \le a + \mathbf{E}\left(\left|\frac{S_n}{V_n}\right| I\left(|S_n/V_n| > a\right)\right) \le a + \left[\mathbf{E}\left(\frac{S_n}{V_n}\right)^2\right]^{1/2}\left[\Pr\left\{\left|\frac{S_n}{V_n}\right| > a)\right\}\right]^{1/2}.$$

Then, this last inequality for $a = a_\ell = \mathbf{E}\left|S_{n_\ell}/V_{n_\ell}\right|/2$, in combination with inequality (2.8), gives

$$\left(\Pr\left\{\left|\frac{S_{n_\ell}}{V_{n_\ell}}\right| > a_\ell\right\}\right)^{\frac{1}{2}} \ge \frac{\frac{1}{2}\mathbf{E}\left|S_{n_\ell}/V_{n_\ell}\right|}{\left[\mathbf{E}\left(S_{n_\ell}/V_{n_\ell}\right)^2\right]^{1/2}} \ge \frac{1}{2\sqrt{c(2)}} > 0$$

for all $\ell \in \mathbf{N}$ (this is Paley–Zygmund's argument). Since $n_\ell > \ell \to \infty$ and $a_\ell > \ell^{1/2}/2 \to \infty$ as $\ell \to \infty$, it follows that the sequence $\left\{S_n/V_n\right\}_{n=1}^\infty$ is not stochastically bounded, contradiction.

□

Combining Lemmas 2.3 and 2.4 we obtain the main result of this section:

2.5. Theorem. *Let the sequence $\{S_n/V_n\}$ be stochastically bounded. Then*

$$M := \max\left[1, \sup_n \mathbf{E}\left|\frac{S_n}{V_n}\right|\right] < \infty \tag{2.9}$$

and, moreover,
i)

$$\sup_n \mathbf{E}\exp\left\{\frac{(S_n/V_n)^2}{4(1+4e/3)^2 M^2}\right\} \le 2 \tag{2.10}$$

and
ii)

$$\sup_n \mathbf{E}\exp\left\{t\left|S_n/V_n\right|\right\} \le 2\exp\left\{(1+4e/3)^2 M^2 t^2\right\} \tag{2.11}$$

for all $t > 0$.

PROOF. $M < \infty$ by Lemma 2.4, and inequalities (2.10) and (2.11) follow from (2.5) and (2.6) in Lemma 2.3 since $M \ge M_n$ for all n.

□

A random variable Z, or its probability law, is *subgaussian* if $\Pr\{|Z| > t\} = O(e^{-ct^2})$ as $t \to \infty$, for some $c < \infty$ or, equivalently, if $\mathbf{E}e^{tZ} \le e^{\tau t^2}$ for all $t \in \mathbf{R}$ and some $\tau < \infty$ (see Kahane, 1968, page 62). Theorem 2.5 then shows that if the self–normalized sums are uniformly tight they are automatically uniformly subgaussian.

2.6. COROLLARY. *Assume the sequence $\{S_n/V_n\}$ is stochastically bounded, let Z be any of its subsequential limits in law, and let $\{n_k\}$ be a subsequence such that*

$$\lim_{k\to\infty} \mathcal{L}\big(S_{n_k}/V_{n_k}\big) = \mathcal{L}(Z).$$

Then Z is subgaussian and

$$\lim_{k\to\infty} \mathbf{E}\exp\Big\{\frac{(S_{n_k}/V_{n_k})^2}{4(1+4e/3)^2M^2}\Big\} = \mathbf{E}\exp\Big\{\frac{Z^2}{4(1+4e/3)^2M^2}\Big\} \le 2. \tag{2.12}$$

Hence,

$$\lim_{k\to\infty} \mathbf{E}\exp\big\{t(S_{n_k}/V_{n_k})\big\} = \mathbf{E}\exp\big\{tZ\big\} \le 2\exp\Big\{(1+4e/3)^2M^2t^2\Big\} \tag{2.13}$$

for all $t \in \mathbf{R}$ and, in particular, the moments of S_{n_k}/V_{n_k} also converge to the corresponding moments of Z.

PROOF. Letting $\lambda = (4(1+4e/3)^2M^2)^{-1}$, which is positive by Theorem 2.5, the proof of Lemma 2.3 shows that e.g. $\sup_n \mathbf{E}\exp\Big\{\frac{4\lambda}{3}(S_n/V_n)^2\Big\} < \infty$, thus providing enough uniform integrability to interchange limits and expectations in the relevant sequences.

□

Since subgaussian variables are determined by their moments, by e.g. Proposition 8.49, page 182 in Breiman (1968), this is true in particular for any subsequential limits of a sequence of stochastically bounded self–normalized sums. (This is relevant to the fact that, when we prove below that if the self–normalized sums are asymptotically $N(0,1)$ then $X \in DAN$, the only use we make of the particular form of the limit is that the second and fourth moments of $N(0,1)$ are respectively 1 and 3.)

Theorem 2.5 and Corollary 2.6 are best possible except perhaps for the constants in the exponents: if X is in the domain of attraction of the normal law and X is centered then S_n/V_n converges in law to $N(0,1)$, as mentioned in the Introduction and proved in Section 3.

The symmetric case of Theorem 2.5 deserves special consideration.

2.7. REMARK. *The symmetric case.* If X is symmetric Theorem 2.5 has a stronger version with a much simpler proof as a consequence of Khinchin's inequality (e.g. Kwapień and Woyczyński, 1990, inequality (3.4.3)). Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a Rademacher sequence independent of the sequence $\{X_i\}_{i=1}^{\infty}$. (The variables ε_i are i.i.d. and $\Pr\{\varepsilon_i = 1\} = \Pr\{\varepsilon_i = 1\} = 1/2$.) Then, by symmetry, the sequences $\{X_i\}$ and $\{\varepsilon_i X_i\}$ have the same joint distributions. This and Khinchin's inequality readily give

$$
\begin{aligned}
\sup_n \mathbf{E}\left|\frac{S_n}{V_n}\right|^k &= \sup_n \mathbf{E}\left|\frac{\sum_{i=1}^n \varepsilon_i X_i}{V_n}\right|^k \\
&= \sup_n \mathbf{E}\mathbf{E}_\varepsilon\left|\frac{\sum_{i=1}^n \varepsilon_i X_i}{V_n}\right|^k \\
&\le (k-1)^{k/2} \sup_n \mathbf{E}\left[\frac{\sum_{i=1}^n X_i^2}{V_n^2}\right]^{\frac{k}{2}} \\
&= (k-1)^{k/2} \sup_n \Pr\{V_n \neq 0\} \\
&= (k-1)^{k/2}, \qquad (2.14)
\end{aligned}
$$

where $\mathbf{E}_\varepsilon$ denotes conditional expectation given X_i, $i \in \mathbf{N}$. As a consequence of these inequalities, it follows that $\{S_n/V_n\}$ *is always stochastically bounded when X is symmetric.* Moreover, proceeding as in the proof of Theorem 2.5 but using the estimate (2.14) instead of inequality (2.2), we obtain

$$
\sup_n \mathbf{E} \exp\{\lambda (S_n/V_n)^2\} < \infty
$$

for $\lambda < (2e)^{-1}$, which is an improvement on inequality (2.9).

In view of the previous remark, we may ask whether the sequence of self–normalized sums is stochastically bounded for all random variables X, symmetric or not. A (related) question that one may also ask is whether such a result, as well as Theorem 2.5, follows in fact from the inequalities (2.14) by some sort of symmetrization argument. Both questions have a negative answer since, as we see immediately below, there exist centered random variables X for which the sequence $\{S_n/V_n - c_n\}_{n=1}^{\infty}$ is not stochastically bounded for any choices of centering constants c_n (that is, it is not shift tight) but, on the other hand, by Remark 2.6, the sequence of randomized self–normalized sums $\sum_{i=1}^n \varepsilon_i X_i/V_n$, $n \in \mathbf{N}$, is stochastically bounded, which is in contrast with the case of numerical normalizations.

2.8. Example. Let X be a random variable such that $\mathbf{E}X = 0$ and, for some $1 < \alpha < 2$,

$$
\Pr\{X = 2^{2^k}\} = c_1 2^{-\alpha 2^k}, \; k = 0, 1, 2, \ldots, \; \Pr\{X = -c_2\} = c_3,
$$

with $c_1 \neq 0$. We will show that if $\{X_i\}$ is a sequence of i.i.d. random variables with the law of X, then there is no sequence of constants $\{c_n\}$ such that the sequence $\{S_n/V_n - c_n\}$ of shifted self–normalized sums is stochastically bounded. This will be a direct consequence of the following fact: the row sums of the triangular array of two dimensional random vectors

$$
\mathbf{Y}_{k,i} = \left(\frac{X_i}{2^{2^k}}, \frac{X_i^2}{2^{2^{k+1}}}\right), \quad 1 \le i \le n_k := \left[2^{\alpha 2^k}\right],
$$

satisfy

$$\lim_{k\to\infty} \mathcal{L}\Big(\sum_{i=1}^{n_k} \mathbf{Y}_{k,i}\Big) = \mathcal{L}\big(N_{c_1} - c_1, N_{c_1}\big), \tag{2.15}$$

where N_{c_1} is a Poisson random variable with parameter c_1. (2.15) follows in a standard way from the general central limit theorem in $\mathbf{R}^2$ (the CLT in $\mathbf{R}^d$ is usually stated for the Euclidean norm, but the details in the derivation of (2.15), which we omit, work better if one takes instead the norm $|(x_1, x_2)| := \max(|x_1|, |x_2|)$; for a precise statement in any norm, see for example Theorem 7.7, Chapter 3, in Araujo and Giné, 1980, taking into account that condition (3) there is automatically satisfied for finite dimensional Banach spaces). As a consequence of (2.15), letting $0 < \tau < c_1 \wedge 1$, $0 < \delta < 1$, we obtain

$$\begin{aligned}
\lim_{M\to\infty} \limsup_n \Pr\Big\{\Big|\frac{S_n}{V_n}\Big| > M\Big\} & \\
&\geq \lim_{\delta\to 0} \limsup_k \Pr\Big\{\frac{\sum_{i=1}^{n_k} X_i}{2^{2^k}} \in (-c_1 - \tau, -c_1 + \tau), \frac{\sum_{i=1}^{n_k} X_i^2}{2^{2^{k+1}}} < \delta\Big\} \\
&= \lim_{\delta\to 0} \Pr\big\{N_{c_1} \in (-\tau, \tau), N_{c_1} < \delta\big\} \\
&= \lim_{\delta\to 0} \Pr\big\{N_{c_1} = 0\big\} = e^{-c_1} > 0,
\end{aligned}$$

showing that the sequence $\{S_n/V_n\}$ is not stochastically bounded. Finally, we show that this sequence is not shift tight either, that is, that there is no sequence of numerical constants c_n for which the shifted sequence $\big\{\frac{S_n}{V_n} - c_n\big\}$ is stochastically bounded. By the previous computation, such a sequence necessarily satisfies that $|c_{n_k}| \to \infty$. On the other hand, if $r \in \mathbf{N}$ is such that $r > c_1 + 1$ and $0 < \tau < 1$, and we set $J := \big((r - \tau - c_1)/\sqrt{r + \tau}, (r + \tau - c_1)/\sqrt{r - \tau}\big)$ then, obviously, $|c_{n_k}| \to \infty$ implies that, for all $M > 0$, the shifted interval $J - c_{n_k}$ is eventually contained in $\{|x| > M\}$, and therefore we have, again by (2.15),

$$\begin{aligned}
\liminf_k \Pr\Big\{\Big|\frac{S_{n_k}}{V_{n_k}} - c_{n_k}\Big| > M\Big\} &\geq \liminf_k \Pr\Big\{\frac{S_{n_k}}{V_{n_k}} \in J\Big\} \\
&\geq \lim_k \Pr\Big\{r - \tau - c_1 < S_{n_k}/2^{2^k} < r + \tau - c_1, \\
&\qquad\qquad r - \tau < V_{n_k}^2/2^{2^{k+1}} < r + \tau\Big\} \\
&= \Pr\big\{N_{c_1} = r\big\} > 0,
\end{aligned}$$

contradicting the stochastic boundedness of the sequence $\big\{\frac{S_{n_k}}{V_{n_k}} - c_n\big\}$.

Example 2.8 shows that for every $1 < \tau < 2$ there are centered random variables X with finite absolute moment of order τ for which the sequence $\{S_n/V_n\}$ is not shift tight.

2.9. Example. Let X be a random variable in the domain of attraction of a stable law of index $\alpha = 1$, written $X \in D(1)$. It is easy to see from the general central limit theorem in $\mathbf{R}^2$ and from the properties of $D(1)$ that there exist constants $a_n \nearrow \infty$ and b_n such that

$$\lim_{n\to\infty} \mathcal{L}\Big(\frac{S_n - b_n}{a_n}, \frac{V_n^2}{a_n^2}\Big) = \mathcal{L}\big(\theta_1, \theta_{1/2}\big),$$

where θ_1 is 1–stable and $\theta_{1/2}$ is positive and $\frac{1}{2}$–stable (see Csörgő, Haeusler and Mason, 1988, and Csörgő, 1989, for an approach to domains of attraction based on quantiles and Feller, 1971, Section XVII.5, or Araujo and Giné, 1980, Section 2.6, for the classical approach). Clearly then, whenever $|b_n/a_n| \to \infty$ it is impossible for $\{S_n/V_n\}_{n=1}^{\infty}$ to be shift tight. The following example shows that this can even happen when $\mathbf{E}X = 0$. Let X be any centered random variable whose tail probabilities are of the order

$$\Pr\{X > t\} \sim \frac{c_1}{t(\log t)(\log\log t)^2}, \quad \Pr\{X < -t\} \sim \frac{c_2}{t(\log t)(\log\log t)^2}$$

as $t \to \infty$, with $c_1 \neq c_2$ and at least one of the constants c_1 and c_2 different from zero. Concretely we can take X to be such that its quantile function, defined as $Q(s) = \inf\{x : \Pr\{X \leq x\} \geq s\}$, $0 < s < 1$, is precisely

$$Q(s) = -h(s) + \int_0^1 h(s)ds$$

with $h(s) = s^{-1}\big(L(s^{-1})\big)^{-1}\big(LL(s^{-1})\big)^{-2}$ and $L(s) := \log(s \vee e)$. Then, by choosing a_n and b_n as in Corollary 3 of Csörgő, Haeusler and Mason (1988), it is readily checked that

$$b_n \sim \frac{-n}{\log n \log\log n} \quad \text{and} \quad a_n \sim \frac{n}{\log n(\log\log n)^2},$$

so that $|b_n/a_n| \to \infty$.

This example is interesting in that it shows that $\{S_n/V_n\}$ may fail to be shift tight even if X is centered and there are numerical normalizers a_n for which the sequence $\{S_n/a_n\}$ is shift tight.

In view of the previous examples, it makes sense to ask for analytic conditions ensuring stochastic boundedness of self–normalized sums. Although this subject is tangential to the main object of this article, we present one such necessary and sufficient condition as another corollary of Theorem 2.5.

2.10. Corollary. *A necessary and sufficient condition for the sequence of self–normalized sums $\{S_n/V_n\}_{n=1}^{\infty}$ to be stochastically bounded is that for some $\delta > 0$,*

$$\limsup_n n^2 \int_0^\delta \Big[\mathbf{E}X\exp(-\lambda X^2)\Big]^2\Big[\mathbf{E}\exp(-\lambda X^2)\Big]^{n-2} d\lambda < \infty. \tag{2.16}$$

Proof. If X is degenerate, the result is trivial, so we can assume X takes more than one value. By Corollary 2.6 the sequence $\{S_n/V_n\}_{n=1}^{\infty}$ is stochastically bounded if and only if the sequence of second moments is bounded. Since $\mathbf{E}\big(S_n/V_n\big)^2 = 1+\mathbf{E}\big(\sum_{1\leq i\neq j\leq n} X_iX_j/V_n^2\big)$, stochastic boundedness of the self–normalized sums is equivalent to the condition

$$\limsup_{n\to\infty} n^2\mathbf{E}\frac{X_1X_2}{V_n^2} < \infty. \tag{2.17}$$

For any $\delta > 0$ we can write

$$\frac{X_1X_2}{V_n^2} = \int_0^\delta X_1X_2e^{-\lambda V_n^2}\,d\lambda + \int_\delta^\infty X_1X_2e^{-\lambda V_n^2}\,d\lambda \tag{2.18}$$

(this identity is obvious for $V_n \neq 0$, and it holds also for $V_n = 0$ because of the convention in force for 0/0.) Now, since $\left|\sum_{1\le i\neq j\le n} X_iX_j\right| \le \sum_{i,j\le n}(X_i^2+X_j^2)/2 = nV_n^2$, we have

$$\begin{aligned}
\mathbf{E}\int_\delta^\infty X_1X_2e^{-\lambda V_n^2}\,d\lambda &= \mathbf{E}\Big[\frac{X_1X_2}{V_n^2}e^{-\delta V_n^2}\Big] \\
&= \frac{1}{n(n-1)}\mathbf{E}\Bigg[\sum_{1\le i\neq j\le n}\frac{X_iX_j}{V_n^2}e^{-\delta V_n^2}\Bigg] \\
&\le \frac{1}{n}\mathbf{E}e^{-\delta V_n^2} \\
&= \frac{1}{n}\Big[\mathbf{E}e^{-\delta X^2}\Big]^n.
\end{aligned}$$

X being non–degenerate, there exists $\tau > 0$ such that $\Pr\{|X| \le \tau\} < 1$ and therefore,

$$\mathbf{E}e^{-\frac{\delta}{2}X^2} \le \Pr\{|X|\le\tau\} + e^{-\frac{\delta\tau^2}{2}}\Pr\{|X|>\tau\} := \sigma_\delta < 1.$$

Hence,

$$n^2\mathbf{E}\int_\delta^\infty X_1X_2e^{-\lambda V_n^2}\,d\lambda \le n\sigma_\delta^n \to 0 \quad \text{as} \quad n\to\infty \tag{2.19}$$

for all $\delta > 0$. Since

$$\mathbf{E}\int_0^\delta X_1X_2e^{-\lambda V_n^2}\,d\lambda = \int_0^\delta \Big[\mathbf{E}X\exp(-\lambda X^2)\Big]^2\Big[\mathbf{E}\exp(-\lambda X^2)\Big]^{n-2}d\lambda,$$

the proposition follows from (2.18) and (2.19).

□

In connection with (2.16), it is interesting to note that if $\mathbf{E}X = 0$ then $\mathbf{E}\big[X\exp(-\lambda X^2)\big] = -\lambda\mathbf{E}\big[X^3\exp(-\lambda\theta X^2)\big]$, where θ is uniform on $[0,1]$, independent of X. For example, using Corollary 2.10 and this observation it can be proved without much effort that if X is centered and there exist $C > 0$ and $\alpha > 1$ such that $\limsup_{n\to\infty}\mathbf{E}X^2I(|X|\le tx)/\mathbf{E}X^2I(|X|\le t) \le Cx^{2-\alpha}$ for all $x > 1$ then the sequence $\{S_n/V_n\}$ is stochastically bounded (this condition means that X is in the Feller class, 'above $\alpha = 1$'; see Feller, 1966, Section 9).

3. The Student t–statistic and the domain of attraction of the normal law. If we only assume stochastic boundedness of the self–normalized sums, then Lemma 2.1 and Theorem 2.5 give that $\mathbf{E}\big(X_1^{m_1}\cdots X_r^{m_r}/V_n^k\big) = O\big(n^{-r}\big)$, a bound that cannot in general be improved. However, the proof of our main theorem requires improving this bound to $o\big(n^{-r}\big)$ when $m_i = 1$ for at least one $1 \le i \le r$, under the assumption that the self–normalized sums are asymptotically $N(0,1)$. This is what the following Hölder type inequality will do for us.

3.1. Lemma. *Let $r, k, n, m_1, \ldots, m_r$ be natural numbers such that k is even, $1 \le r \le k$, $m_i \ge 1$ for all i, $m_1 + \ldots + m_r = k$ and $n \ge \max\bigl(2m_1, 2(k-m_1)\bigr)$. Then, the following inequality holds for any set of i.i.d. random variables X_i, $i \le n$:*

$$\Bigl|\mathbf{E}\frac{X_1^{m_1}\cdots X_r^{m_r}}{V_n^k}\Bigr| \tag{3.1}$$
$$\le \frac{\bigl[(m_1-1)!(k-m_1-1)!\bigr]^{1/2}}{\bigl(\frac{k}{2}-1\bigr)!}\Bigl[\mathbf{E}\frac{X_1^{m_1}X_2^{m_1}}{V_n^{2m_1}}\Bigr]^{1/2}\Bigl[\mathbf{E}\frac{X_3^{m_2}X_4^{m_2}\cdots X_{2r-1}^{m_r}X_{2r}^{m_r}}{V_n^{2(k-m_1)}}\Bigr]^{1/2},$$

where the expected values at the right side are non–negative.

Proof. We apply the elementary identity

$$\int_0^\infty \lambda^{m-1}\exp(-\lambda a)d\lambda = \frac{(m-1)!}{a^m},$$

which is valid for all positive integers m and positive real numbers a, with $a = V_n^2$ and $m = k/2$, to obtain

$$\mathbf{E}\frac{X_1^{m_1}\cdots X_r^{m_r}}{V_n^k} = \frac{1}{\bigl(\frac{k}{2}-1\bigr)!}\mathbf{E}\int_0^\infty \lambda^{\frac{k}{2}-1}X_1^{m_1}\cdots X_r^{m_r}\exp(-\lambda V_n^2)d\lambda.$$

(Because of the $0/0 = 0$ convention, this identity is true even if $V_n(\omega)$ is not different from zero for all ω.) The order of the integrals can be reversed since $\frac{|X_1^{m_1}\cdots X_r^{m_r}|}{V_n^k}I(V_n \ne 0) \le 1$. Hence, setting

$$\varphi(\lambda) = \mathbf{E}\exp(-\lambda X^2),$$

we have

$$\begin{aligned}
\mathbf{E}\frac{X_1^{m_1}\cdots X_r^{m_r}}{V_n^k} &= \frac{1}{\bigl(\frac{k}{2}-1\bigr)!}\int_0^\infty \lambda^{\frac{k}{2}-1}\mathbf{E}\bigl[X_1^{m_1}\cdots X_r^{m_r}\exp(-\lambda V_n^2)\bigr]d\lambda \\
&= \frac{1}{\bigl(\frac{k}{2}-1\bigr)!}\int_0^\infty \lambda^{\frac{k}{2}-1}\Bigl[\prod_{j=1}^r\bigl(\mathbf{E}X^{m_j}\exp(-\lambda X^2)\bigr)\Bigr]\Bigl[\mathbf{E}\exp(-\lambda X^2)\Bigr]^{n-r}d\lambda \\
&= \frac{1}{\bigl(\frac{k}{2}-1\bigr)!}\int_0^\infty \Bigl[\prod_{j=1}^r\frac{\lambda^{m_j/2}\mathbf{E}X^{m_j}\exp(-\lambda X^2)}{\varphi(\lambda)}\Bigr]\frac{\varphi^n(\lambda)}{\lambda}d\lambda. \qquad (3.2)
\end{aligned}$$

Now we can apply Hölder's inequality with respect to the measure $d\mu(\lambda) = \lambda^{-1}\varphi^n(\lambda)d\lambda$, and obtain from (3.2) that

$$\begin{aligned}
\Bigl|\mathbf{E}\frac{X_1^{m_1}\cdots X_r^{m_r}}{V_n^k}\Bigr| &\le \frac{1}{\bigl(\frac{k}{2}-1\bigr)!}\Bigl[\int_0^\infty\Bigl(\frac{\lambda^{m_1/2}\mathbf{E}X^{m_1}\exp(-\lambda X^2)}{\varphi(\lambda)}\Bigr)^2\frac{\varphi^n(\lambda)}{\lambda}d\lambda\Bigr]^{1/2} \\
&\qquad\times\Bigl[\int_0^\infty\Bigl(\prod_{j=2}^r\frac{\lambda^{m_j/2}\mathbf{E}X^{m_j}\exp(-\lambda X^2)}{\varphi(\lambda)}\Bigr)^2\frac{\varphi^n(\lambda)}{\lambda}d\lambda\Bigr]^{1/2} \\
&= \frac{\bigl[(m_1-1)!(k-m_1-1)!\bigr]^{1/2}}{\bigl(\frac{k}{2}-1\bigr)!}\Bigl[\mathbf{E}\frac{X_1^{m_1}X_2^{m_1}}{V_n^{2m_1}}\Bigr]^{1/2}\Bigl[\mathbf{E}\frac{X_3^{m_2}X_4^{m_2}\cdots X_{2r-1}^{m_r}X_{2r}^{m_r}}{V_n^{2(k-m_1)}}\Bigr]^{1/2},
\end{aligned}$$

and it is clear from these identities that the last two expected values are non–negative.

□

Proving that if $X \in DAN$ and $\mathbf{E}X = 0$ then the self–normalized sums are asymptotically normal is a simple consequence of Raikov's theorem as observed e.g. by Maller (1981). For this, as well as for proving that X is centered if S_n/V_n is asymptotically normal, we require a small variation on Raikov's theorem, and we include its very simple proof for the reader's convenience.

3.2. Lemma. *Let $X, X_i,\ i \in \mathbf{N}$, be i.i.d. random variables in the domain of attraction of the normal law and such that $\mathbf{E}X^2 = \infty$. Let $b_n \nearrow \infty$ be a set of constants for which*

$$\mathcal{L}\Big(\frac{1}{b_n}\sum_{i=1}^{n}(X_i - \mathbf{E}X)\Big) \to_w N(0,1). \tag{3.3}$$

Then,

$$\frac{1}{b_n^2}\sum_{i=1}^{n} X_i^2 \to 1 \ \text{ in probability.} \tag{3.4}$$

Proof. We should recall that if $X \in DAN$ then $\mathbf{E}|X|^{2-\tau} < \infty$ for all $0 < \tau < 2$ and that constants b_n for which (3.3) holds always exist and have the form $b_n = n^{1/2}\ell(n)$ for some function ℓ slowly varying at infinity. If $\mathbf{E}X^2 = \infty$, truncated variances can be replaced by truncated second moments in the general criterion for normal convergence and therefore, if (3.3) holds, then

$$\lim_{n\to\infty}\frac{n}{b_n^2}\mathbf{E}X^2 I(|X| \le \delta b_n) = 1 \ \text{ and } \ \lim_{n\to\infty} n\Pr\big\{|X| > \delta b_n\big\} = 0$$

for all $\delta > 0$. (For these and other facts on domains of attraction, see Csörgő, Haeusler and Mason, 1988, Feller, 1971, or Araujo and Giné, 1980). The last two limits imply that for all $\varepsilon > 0$ and $\delta > 0$ there is an $N < \infty$ such that, for $n \ge N$,

$$\begin{aligned}
\Pr\Big\{\Big|\frac{\sum_{i=1}^n X_i^2}{b_n^2} - 1\Big| > \varepsilon\Big\} &\le n\Pr\Big\{|X| > \delta b_n\Big\} \\
&\quad + \Pr\Big\{\Big|\sum_{i=1}^{n} X_i^2 I(|X_i| \le \delta b_n) - n\mathbf{E}X^2 I(|X_i| \le \delta b_n)\Big| > b_n^2\varepsilon/2\Big\} \\
&\le n\Pr\Big\{|X| > \delta b_n\Big\} + \frac{4n}{\varepsilon^2 b_n^4}\mathbf{E}X^4 I(|X_i| \le \delta b_n) \\
&\le n\Pr\Big\{|X| > \delta b_n\Big\} + \frac{4\delta^2}{\varepsilon^2}\frac{n}{b_n^2}\mathbf{E}X^2 I(|X_i| \le \delta b_n),
\end{aligned}$$

and (3.4) follows by letting first $n \to \infty$ and then $\delta \to 0$.

□

We are now ready to prove our main result characterizing asymptotic normality of self–normalized sums.

3.3. THEOREM. *The following two statements are equivalent:*

a) *X is in the domain of attraction of the normal law and $EX = 0$.*

b)

$$\lim_{n\to\infty} \mathcal{L}\Big(\frac{S_n}{V_n}\Big) = N(0,1). \tag{3.5}$$

PROOF. Assume condition a) holds. Then, if $\mathbf{E}X^2 < \infty$, the self–normalized sums converge in distribution to the standard normal law by the central limit theorem and the law of large numbers. If $\mathbf{E}X^2 = \infty$, then (3.5) is a direct consequence of (3.3) and (3.4) in Lemma 3.2. So, b) is proved.

If $X \in DAN$ then the sequence $\big\{(S_n - n\mathbf{E}X)/V_n\big\}_{n=1}^{\infty}$ converges in law (as above, by the CLT and the LLN if $\mathbf{E}X^2 < \infty$, and by Lemma 3.2 if $\mathbf{E}X^2 = \infty$). Suppose for the moment that the sequence $\{S_n/V_n\}_{n=1}^{\infty}$ also converges in law. Then the sequence of centerings $n(\mathbf{E}X)/V_n$, $n \in \mathbf{N}$, must be stochastically bounded. But if b_n is as in Lemma 3.2 for $\mathbf{E}X^2 = \infty$, and $b_n = n^{1/2}$ for $\mathbf{E}X^2 < \infty$, then b_n/V_n converges in probability to a finite number whereas $n/b_n \to \infty$ (recall $b_n = n^{1/2}\ell(n)$ with $\ell(x)$ slowly varying at infinity). Hence, we must have $\mathbf{E}X = 0$.

The proof of the theorem is thus reduced to showing that if b) holds then $X \in DAN$. One of the several necessary and sufficient conditions for $X \in DAN$ is that

$$\lim_{n\to\infty} \frac{\max_{i\le n}|X_i|}{V_n} = 0 \ \text{ in probability,} \tag{3.6}$$

(O'Brien, 1980), as mentioned above. Then, since

$$\Pr\Big\{\max_{i\le n}\frac{|X_i|}{V_n} > \varepsilon\Big\} \le n\Pr\Big\{\frac{|X_1|}{V_n} > \varepsilon\Big\} \le \frac{n}{\varepsilon^4}\mathbf{E}\Big(\frac{X_1}{V_n}\Big)^4,$$

it suffices to prove

$$\mathbf{E}\Big(\frac{X_1}{V_n}\Big)^4 = o\big(n^{-1}\big) \tag{3.7}$$

under the assumption that S_n/V_n is asymptotically $N(0,1)$. Let us then assume (3.5). The left side of (3.7) is one of the summands in the development of $\mathbf{E}(S_n/V_n)^4$, $n \ge 4$,

$$\mathbf{E}(S_n/V_n)^4 = \sum_{r=1}^{4}\sum_{\substack{m_1+\cdots+m_r=4\\ m_i\ge 1}}\binom{n}{r}\binom{4}{m_1,\ldots,m_r}\mathbf{E}\frac{X_1^{m_1}\cdots X_r^{m_r}}{V_n^4}$$

$$= n\mathbf{E}\Big(\frac{X_1}{V_n}\Big)^4 + 6\binom{n}{2}\mathbf{E}\frac{X_1^2X_2^2}{V_n^4} + 8\binom{n}{2}\mathbf{E}\frac{X_1X_2^3}{V_n^4} + 36\binom{n}{3}\mathbf{E}\frac{X_1X_2X_3^2}{V_n^4} + 24\binom{n}{4}\mathbf{E}\frac{X_1\cdots X_4}{V_n^4}.$$

We will obtain (3.7) by taking limits in this identity using the fact that the moments of S_n/V_n converge together with Lemmas 2.1 and 3.1. Since $\sum_{1\le i\ne j\le n} X_i^2X_j^2/V_n^4 = 1 - \sum_{i=1}^n X_i^4/V_n^4$, we have

$$2\binom{n}{2}\mathbf{E}\frac{X_1^2X_2^2}{V_n^4} = 1 - n\mathbf{E}\Big(\frac{X_1}{V_n}\Big)^4$$

and the previous development simplifies to

$$\mathbf{E}(S_n/V_n)^4 = 3 - 2n\mathbf{E}\Big(\frac{X_1}{V_n}\Big)^4 + 8\binom{n}{2}\mathbf{E}\frac{X_1X_2^3}{V_n^4} + 36\binom{n}{3}\mathbf{E}\frac{X_1X_2X_3^2}{V_n^4} + 24\binom{n}{4}\mathbf{E}\frac{X_1\cdots X_4}{V_n^4}. \tag{3.8}$$

Now, by Corollary 2.6, there is also convergence of moments in the limit (3.5). Hence,

$$\lim_{n\to\infty} \mathbf{E}(S_n/V_n)^4 = 3, \tag{3.9}$$

and therefore (3.7) will be proved if we show that the last three summands at the right of equation (3.8) tend to zero. So, we must obtain the bound

$$\left|\mathbf{E}\frac{X_1X_2^{m_2}\cdots X_r^{m_r}}{V_n^4}\right| = o\big(n^{-r}\big) \tag{3.10}$$

for all $1 \le r \le 4$ and $1 + m_2 + \ldots + m_r = 4$. For this we use Lemma 3.1, which gives that, for $n \ge 6$,

$$\left|\mathbf{E}\frac{X_1X_2^{m_2}\cdots X_r^{m_r}}{V_n^4}\right| \le 2^{1/2}\left[\mathbf{E}\frac{X_1X_2}{V_n^2}\right]^{1/2}\left[\mathbf{E}\frac{X_2^{m_2}X_3^{m_3}\cdots X_{2r-1}^{m_r}X_{2r}^{m_r}}{V_n^6}\right]^{1/2}, \tag{3.11}$$

To bound the first factor at the right of (3.11) we observe that by Corollary 2.6 and the limit (3.5),

$$\lim_{n\to\infty} \mathbf{E}(S_n/V_n)^2 = 1$$

which, since $(S_n/V_n)^2 = 1 - \sum_{1\le i\ne j\le n} X_iX_j/V_n^2$, implies

$$\mathbf{E}\frac{X_1X_2}{V_n^2} = o\big(n^{-2}\big). \tag{3.12}$$

Theorem 2.5 implies, by (3.5), that $\sup_n \mathbf{E}|S_n/V_n| < \infty$, and this, combined with Lemma 2.1, yields

$$\mathbf{E}\frac{X_2^{m_2}X_3^{m_2}\cdots X_{2r-1}^{m_r}X_{2r}^{m_r}}{V_n^6} = O\big(n^{-2(r-1)}\big). \tag{3.13}$$

(3.11)–(3.13) give (3.10), proving the theorem.

□

As mentioned in the Introduction, the classical Student t_n–statistic,

$$T_n(X) = \frac{\frac{1}{n^{1/2}}\sum_{i=1}^n X_i}{\Big[\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2\Big]^{1/2}},$$

where $\bar{X}_n := S_n/n$, satisfies equation (1.2). Therefore, if $T_n(X)$ or S_n/V_n has a limiting distribution, so does the other and both coincide (this is well known). Hence, replacing X by $X - a$ in this observation and in Theorem 3.3 (note that $X \in DAN$ iff $X - a \in DAN$), we obtain:

3.4. Theorem. *The following two statements are equivalent:*

i) X is in the domain of attraction of the normal law.

ii) There exists a finite constant a such that

$$\lim_{n\to\infty} \mathcal{L}\big(T_n(X-a)\big) = N(0,1).$$

Moreover, if either i) or ii) holds, then $a = \mathbf{E}X$.

Theorem 3.4 provides a full answer to this article's title question.

References

Araujo, A. and Giné, E (1980). *The Central Limit Theorem for Real and Banach valued random variables.* Wiley, New York.

Bentkus, V. and Götze, F. (1994). The Berry-Esseen bound for Student's statistic. Universität Bielefeld, SFB 343, Preprint No 94–033. To appear in *Ann. Probab.* **24**, No. 1.

Breiman, L. (1968). *Probability.* Addison–Wesley, Reading, Massachusetts.

Csörgő, S. (1989). Notes on extreme and self–normalised sums from the domain of attraction of a stable law. *J. London Math. Soc.* **39** 369–384.

Csörgő, S., Haeusler, E. and Mason, D. M. (1988). A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables. *Advances in Applied Mathematics* **9** 259–333.

Efron, B. (1969). Student's t–test under symmetry conditions. *J. Amer. Statist. Assoc.* **64** 1278–1302.

Feller, W. (1966). On regular variation and local limit theorems. In *Proc. Fifth Berkeley Symp.*, Vol. 2, Part 1, pp. 373–388.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications.* Vol. II. Wiley, New York.

Griffin, P. S. and Mason, D. M. (1991). On the asymptotic normality of self–normalized sums. *Proc. Cambridge Phil. Soc.* **109** 597–610.

Kahane, J.–P. (1968). *Some random series of functions.* D. C. Heath, Lexington, Massachusetts.

Kwapień, S. and Woyczyński, W. (1992). *Random series and stochastic integrals.* Birkhäuser, Boston.

LePage, R, Woodroofe, M. and Zinn, J. (1981). Convergence to a stable distribution via order statistics. *Ann. Probab.* **9** 624–632.

Logan, B. F., Mallows, C. L., Rice, S. O., and Shepp, L. A. (1973) Limit distributions of self–normalized sums. *Ann. Probab.* **1** 788–809.

Maller, R. A. (1981). A theorem on products of random variables, with application to regression. *Austral. J. Statist.* **23** 177–185.

O'Brien, G. L. (1980). A limit theorem for sample maxima and heavy branches in Galton–Watson trees. *J. Appl. Probab.* **17** 539–545.

Evarist Giné
Department of Mathematics
Department of Statistics
University of Connecticut
Storrs, CT 06269
USA
gine@uconnvm.uconn.edu

Friedrich Götze
Fakultät für Mathematik
Universität Bielefeld
Postfach 8640
4800 Bielefeld 1
Germany
goetze@mathematik.uni-bielefeld.de

David M. Mason
Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716
davidm@math.udel.edu