BOUNDS FOR THE ACCURACY OF POISSONIAN APPROXIMATIONS OF STABLE LAWS

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November 1995

Abstract. Stable laws G_α admit a well-known series representation of the type

$$G_\alpha = \mathcal{L}\Big(\sum\nolimits_{j=1}^{\infty} \Gamma_j^{-1/\alpha} X_j \Big), \qquad 0 < \alpha < 2,$$

where $\Gamma_1, \Gamma_2, \dots$ are the successive times of jumps of a standard Poisson process, and $X_1, X_2, \dots$ denote i.i.d. random variables, independent of $\Gamma_1, \Gamma_2, \dots$ We investigate the rate of approximation of G_α by distributions of partial sums $S_n = \sum_{j=1}^{n} \Gamma_j^{-1/\alpha} X_j$, and we get (asymptotically) optimal bounds for the variation of $G_\alpha - \mathcal{L}(S_n)$. The results obtained complement and improve results of A. Janicki and P. Kokoszka, and M. Ledoux and V. Paulauskas. Bound for the concentration function of S_n are also proved.

1. Introduction

The characteristic function g_α of any stable distribution function G_α allows the representation

$$g_\alpha(t) = \exp\Big\{ -c|t|^\alpha \big(1 + i\beta\varphi(\alpha, t)\big)\Big\}, \qquad 0 < \alpha < 2, \tag{1}$$

where $|\beta| \le 1$, and φ is a well-known function (see, for example Samorodnitsky and Taqqu (1994)).

Introduce a sequence $\lambda_1, \lambda_2, \dots$ of independent identically distributed (i.i.d.) random variables with exponential distribution, that is $\mathbf{P}\big\{\lambda_1 > x\big\} = e^{-x}$, for $x \ge 0$.

1991 *Mathematics Subject Classification.* Primary 62E20; secondary 60F05.

Key words and phrases. stable laws, Poissonian representation, convergence in variation, convergence rates, Berry–Esseen bounds, concentration functions.

[1]Research supported by the SFB 343 in Bielefeld

It is well-known that the sequence $\Gamma_n = \lambda_1 + \cdots + \lambda_n$ defines the successive times of the jumps of a standard Poisson process.

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables with common distribution function F such that

$$0 < \mathbf{E}\,|X|^\alpha < \infty \qquad \text{and} \qquad \mathbf{E}\,X = 0 \ \text{ if } \ \alpha > 1.$$

Assume that sequences $\lambda_1, \lambda_2, \dots$ and $X, X_1, X_2, \dots$ are independent. The representation

$$G_\alpha = \mathcal{L}\Big(\sum\nolimits_{j=1}^{\infty} \Gamma_j^{-1/\alpha} X_j \Big), \qquad \alpha \neq 1, \tag{2}$$

of stable random variables with distribution G_α by an almost surely convergent series goes back to P. Levy and was revived by R. LePage [LePage (1989, 1981), and LePage, Woodroofe and Zinn (1981)]. Recall, that the scale parameter c in (1) is connected to X by the following relation:

$$\lim_{x\to\infty} x^\alpha \big(1 - G_\alpha(x) + G_\alpha(-x)\big) = c\,\frac{1-\alpha}{\Gamma(2-\alpha)\,\cos \pi\alpha/2} \;=\; \mathbf{E}\,|X|^\alpha.$$

We shall investigate the convergence rates in (2), that is, we shall obtain bounds for $G_\alpha - \mathcal{L}(S_n)$, where S_n denotes the partial sum

$$S_n = \sum\nolimits_{j=1}^{n} \Gamma_j^{-1/\alpha} X_j.$$

General results on series representation of stable laws (including the cases $\alpha = 1$ and $\mathbf{E}\,X \neq 0$) and formulae connecting F with the parameters c and β in (1), can be found, for example, in Samorodnitsky and Taqqu (1994), Th. 1.4.5. Poissonian representations of stable laws or even infinitely divisible distributions [Ferguson and Klass (1972) and Rosinski (1990)] are valid in general Banach space setting. They are useful in treating structure problems related to stable laws [see, e.g., monographs Marcus and Pisier (1981), Ledoux and Talagrand (1991), Samorodnitsky and Taqqu (1994), Janicki and Weron (1994)]. Such representations have been used in resampling (bootstrap) problems in statistics [see, for example, LePage (1992), Kinateder (1992), LePage and Podgorski (1993), LePage, Podgorski and Ryznar (1994)], and in financial mathematics and option pricing models [Rachev and Samorodnitsky (1992), and LePage, Podgorski and Ryznar (1994)].

A more practical application of the representation theorem is the simulation of stable random variables and especially stable processes [Janicki and Weron (1994), Janicki and Kokoszka (1992a)].

Denote

$$\Delta_n = \Delta_n(\alpha, F) = \sup_x \Big| \,\mathbf{P}\left\{ S_n \leq x \right\} - G_\alpha(x) \Big|.$$

Write

$$\Lambda_r = \sup_{t>0} t^r \,\mathbf{P}\left\{ |X| > t \right\}$$

and assume that the number r satisfies

$$\alpha < r \le \min\left\{ 1 + \alpha,\, 2 \right\}.$$

We shall assume, in addition, that

$$\mathbf{E}\, X = 0, \qquad\qquad \text{if } r > 1.$$

By $C(\alpha, \beta, \dots)$ we shall denote generic constants which can depend only on $\alpha, \beta, \dots$ only.

Theorem 1. *We have*

$$\Delta_n = \sup_x \left| \mathbf{P}\left\{ S_n \le x \right\} - G_\alpha(x) \right| \le C(\alpha, r, F)\, \Lambda_r\, n^{-(r-\alpha)/\alpha}. \tag{3}$$

If $\mathbf{E}\,|X|^r < \infty$, *then*

$$\Delta_n = o(n^{-(r-\alpha)/\alpha}). \tag{4}$$

For explicit bounds for $C(\alpha, r, F)$ in (3), see Corollary 3 and Remark 4.

Theorem 1 is a consequence of Theorem 2 and its Corollary 3, where bounds for the variation distance between measures $\mathcal{L}(S_n)$ and $\mathcal{L}(S_m)$ are obtained. Bounds for Δ_n are similar to Berry–Esseen bounds in the classical Central Limit Theorem for i.i.d. summands in the case of Gaussian limiting law. The rate of convergence in the CLT is $o(n^{-\delta/2})$ (respectively, $O(n^{-\delta/2})$) provided that $\mathbf{E}\,|X|^{2+\delta} < \infty$ (respectively, $\Lambda_{2+\delta} < \infty$), $0 < \delta < 1$, [see, e.g., Ibragimov and Linnik (1971), Petrov (1995)]. Similarly to the case of the CLT, the bounds (3) and (4) are asymptotically optimal, as $n \to \infty$. More precisely, if the distribution of X is the stable distribution $F = G_r$, then we have

$$\Delta_n(\alpha, G_r) \ge C(\alpha, r)\, n^{-(r-\alpha)/\alpha} \tag{5}$$

[see Ledoux and Paulauskas (1995), Proposition 9]. Since Λ_r is finite in this case, the lower bound (5) establishes the optimality of (3).

Our main result—Theorem 2 will show that the convergence in the representation (2) is indeed rather strong. For $n < N \le \infty$, write

$$\delta_{n,N} = \sup_{A \in \mathcal{B}} \left| \mathbf{P}\left\{ S_n \in A \right\} - \mathbf{P}\left\{ S_N \in A \right\} \right|, \qquad \text{as well as } \ \delta_n = \delta_{n,n+1},$$

where $\mathcal{B}$ denotes the class of Borel sets. In order to measure the "degeneracy" of X, introduce

$$p = \mathbf{P}\left\{ c_1 \le |X| \le c_2 \right\}, \qquad\qquad \text{for } \ 0 < c_1 \le c_2 \le \infty.$$

Write $T = c_1 n^{1/\alpha}$, and introduce the sum of truncated moments

$$L_n = T^{-2}\, \mathbf{E}\, X^2\, \mathbf{I}\left\{ |X| \le T \right\} + T^{-1} \left| \mathbf{E}\, X\, \mathbf{I}\left\{ |X| \le T \right\} \right| + \mathbf{P}\left\{ |X| > T \right\}.$$

Let $[a]$ denote the integer part of a real number a.

Theorem 2. *We have*

$$\delta_n \le C(\alpha)\, p^{-1}\, (c_2/c_1)^{\alpha}\, L_m, \tag{6}$$

where $m=[np/2]$ *if* $p<1$, *and* $m=n$ *if* $p=1$. *In particular,*

$$\delta_n \le C(\alpha)\, c_1^{-r}\, (c_2/c_1)^{\alpha}\, p^{-1-r/\alpha}\, \Lambda_r\, n^{-r/\alpha}. \tag{7}$$

Furthermore, if $\mathbf{E}\,|X|^r<\infty$ *and* $p>0$, *then*

$$\delta_n = o(n^{-r/\alpha}).$$

The bound (6) trivially holds if its right-hand side is infinite, for example, when $p=0$. However, we can always choose $0<c_1\le c_2$ so that $p>0$, since $\mathbf{P}\,\{X=0\}<1$.

Notice that $\delta_{n,N}\le\sum_{j=n}^{N-1}\delta_j$. Thus we have the following corollary.

Corollary 3. *For all* $n<N\le\infty$, *we have*

$$\delta_{n,N} \le C(\alpha)\, c_1^{-r}\, (c_2/c_1)^{\alpha}\, p^{-1-r/\alpha}\, \Lambda_r\, n^{-(r-\alpha)/\alpha}.$$

Moreover,

$$\sup_{N>n} \delta_{n,N} = o\big(n^{-(r-\alpha)/\alpha}\big),$$

provided that $\mathbf{E}\,|X|^r<\infty$.

Corollary 3 implies Theorem 1, since $\Delta_n\le\delta_{n,\infty}$.

Remark 4. Using Esseen's type inequality and proving Theorem 1 directly, we can improve the bound for the constant. Namely, we shall prove that

$$\Delta_n \le C(\alpha)\, p^{-1}\, L_m.$$

Thus, the constant $C(\alpha,r,F)$ in (3) is bounded from above by $C(\alpha,r)\,c_1^{-r}\,p^{-1-r/\alpha}$.

Define the concentration function $Q(X;\lambda)$ of a random variable X by

$$Q(X;\lambda)=\sup_x \mathbf{P}\,\{x\le X\le x+\lambda\}, \qquad \text{for } \lambda\ge 0.$$

Theorem 5. *We have*

$$Q(S_n;\lambda)\le C(\alpha)\,\max\big\{\lambda\, c_1^{-1}\, p^{-1/\alpha};\, \exp\{-3np/4\}\big\}, \qquad \textit{for all } \lambda\ge 0.$$

If $p=\mathbf{P}\,\{|X|>c_1\}=1$, *then, for* $n\ge 2$,

$$Q(S_n;\lambda)\le C(\alpha)\, c_1^{-1}\,\lambda, \qquad \textit{for all } \lambda\ge 0,$$

and, in particular, the distribution function of S_n *has a density bounded from above by* $C(\alpha)\, c_1^{-1}$.

Using the obvious relation

$$\sigma_n^2 \stackrel{\text{def}}{=} \mathbf{E}\Big|\sum_{j=n+1}^{\infty} \Gamma_j^{-1/\alpha} X_j\Big|^2 = \mathbf{E}\, X^2 \sum_{j=n+1}^{\infty} \mathbf{E}\,\Gamma_j^{-2/\alpha},$$

A. Janicki and P. Kokoszka showed that $\sigma_n^2 = O\big(n^{-(2-\alpha)/\alpha}\big)$. This result yields rather week bounds for Δ_n, namely $\Delta_n = O\big(n^{-(2-\alpha)/3\alpha}\big)$. As stated, our results do not imply bounds for the rate of convergence of moments. However, we can obtain such bounds using estimates for the characteristic function (and its derivatives) of S_n.

Ledoux and Paulauskas (1995) obtained upper (and lower) bounds for Δ_n. The results of the present paper improve these bounds since the dependence of the bounds on n is asymptotically optimal, and neither moment nor smoothness conditions for X are required. The proof of the main Theorem 2 is based on conditioning and random selections techniques [Bentkus and Götze (1995, 1996)]. Using the techniques, it is possible as well to extend these results to the multidimensional case, to obtain asymptotic expansions of $\mathcal{L}(S_n)$, to estimate closeness of densities of $\mathcal{L}(S_n)$ and G_α, and so forth.

2. Main reduction

In this section we show that instead of the sum S_n, which has relatively complicated structure due to the dependence of random variables Γ_j's, we can study a simpler sum of specially chosen independent random variables (see Lemma 7).

Let $\xi_1, \xi_2, \dots$ denote i.i.d. random variables such that ξ_1 has a Pareto distribution, that is, $\mathbf{P}_2\{\xi_1 > x\} = x^{-\alpha}$, for $x \ge 1$. Assume that the sequences

$$\big\{X_i, i \ge 1\big\}, \qquad \big\{\xi_i, i \ge 1\big\}, \qquad \text{and} \qquad \big\{\lambda_i, i \ge 1\big\}$$

are independent.

Let $\mathcal{M}$ denote a class of subsets of $\mathbf{R}$ invariant with respect to shifts and multiplication by positive constants, that is

$$A \in \mathcal{M},\ \ \lambda > 0,\ \ a \in \mathbf{R} \implies \lambda A \in \mathcal{M},\ \ A + a \in \mathcal{M}.$$

For $n < N \le \infty$, write

$$\delta_{n,N}(\mathcal{M}) = \sup_{A \in \mathcal{M}} \Big| \mathbf{P}\big\{S_n \in A\big\} - \mathbf{P}\big\{S_N \in A\big\}\Big|, \qquad \text{and} \quad \delta_n(\mathcal{M}) = \delta_{n,n+1}(\mathcal{M}).$$

Notice that

$$\delta_{n,N} = \delta_{n,N}(\mathcal{B}) \qquad \text{and} \qquad \Delta_n = \delta_{n,\infty}(\mathcal{R}),$$

where $\mathcal{B}$ denotes the class of Borel sets, and $\mathcal{R} = \big\{(-\infty, x) : x \in \mathbf{R}\big\}$ is the class of half-intervals.

Lemma 6. *Write* $Z_n = \sum_{j=1}^{n} \xi_j X_j$. *Then*

$$\delta_n(\mathcal{M}) \le \sup_{A \in \mathcal{M}} \Big| \mathbf{P}\big\{ Z_n \in A \big\} - \mathbf{P}\big\{ Z_n + X \in A \big\} \Big|, \tag{8}$$

$$Q(S_n; \lambda) \le \mathbf{E} \sup_x \mathbf{P}\big\{ x \le Z_n \le x + \lambda \Gamma_{n+1}^{1/\alpha} \,\big|\, \Gamma_{n+1} \big\}. \tag{9}$$

Proof. We shall prove the estimate (8) only, since the proof of the inequality (9) is similar. Let $\xi_{n1} \ge \xi_{n2} \ge \cdots \ge \xi_{nn}$ be the ordered values of $\xi_1, \dots, \xi_n$. Then the random vectors

$$\mathcal{P} = (\xi_{n1}, \dots, \xi_{nn}) \qquad \text{and} \qquad \mathcal{G} = \Big(\Big(\frac{\Gamma_1}{\Gamma_{n+1}} \Big)^{-1/\alpha}, \dots, \Big(\frac{\Gamma_n}{\Gamma_{n+1}} \Big)^{-1/\alpha} \Big)$$

satisfy $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{G})$ [see, for example, LePage (1981)]. Observe [cf. Samorodnitsky and Taqqu (1994)] that the conditional distribution of $\mathcal{G}$, given Γ_{n+1}, equals the unconditional one, and thus equals the distribution of $\mathcal{P}$. The proof of this fact is similar to the proof that

$$\Big(\frac{\Gamma_1}{\Gamma_{n+1}}, \dots, \frac{\Gamma_n}{\Gamma_{n+1}} \Big)$$

has the same distribution as $(U_{n1}, \dots, U_{nn})$, where $U_{n1} \le \cdots \le U_{nn}$ are the ordered values of i.i.d. random variables $U_1, \dots, U_n$, uniformly distributed on [0,1] [Breiman (1968), p. 285]. Therefore we can write the following distributional equalities

$$S_n = \Gamma_{n+1}^{-1/\alpha} \sum_{j=1}^{n} \Big(\frac{\Gamma_j}{\Gamma_{n+1}} \Big)^{-1/\alpha} X_j \overset{\mathcal{D}}{=} \Gamma_{n+1}^{-1/\alpha} \sum_{j=1}^{n} \xi_{nj} X_j \overset{\mathcal{D}}{=} \Gamma_{n+1}^{-1/\alpha} \sum_{j=1}^{n} \xi_j X_j = \Gamma_{n+1}^{-1/\alpha} Z_n.$$

Proceeding similarly, using the independence assumption and $\mathcal{L}(X_{n+1}) = \mathcal{L}(X)$, we obtain

$$S_{n+1} = S_n + \Gamma_{n+1}^{-1/\alpha} X_{n+1} \overset{\mathcal{D}}{=} \Gamma_{n+1}^{-1/\alpha} \sum_{j=1}^{n} \xi_j X_j + \Gamma_{n+1}^{-1/\alpha} X_{n+1} = \Gamma_{n+1}^{-1/\alpha} (Z_n + X).$$

Finally, the definition of the class $\mathcal{M}$ combined with conditioning on $\Gamma_{n+1}^{-1/\alpha}$, implies the result. $\square$

In the next Lemma we shall get rid of the random variables X's, replacing them by constants. This is done by a method, introduced in Bentkus and Götze (1996).

For non-random

$$a_1, \dots, a_m \qquad \text{such that } c_1 \le a_j \le c_2, \qquad \text{for all } 1 \le j \le m, \tag{10}$$

introduce the sum

$$V_m = m^{-1/\alpha} \sum_{j=1}^{m} a_j \, \xi_j. \tag{11}$$

Write

$$\widetilde{\delta}_m(\mathcal{M}) = \sup_{A\in\mathcal{M}} \sup_{a} \Big| \, \mathbf{P}\left\{ V_m \in A \right\} - \mathbf{P}\left\{ V_m + m^{-1/\alpha} X \in A \right\} \Big|,$$

where $\sup_a$ denotes the supremum over all $a_1, \dots, a_m$ satisfying (10).

Lemma 7. *If $p = 1$, then $\delta_n(\mathcal{M}) \le \widetilde{\delta}_n(\mathcal{M})$ and*

$$Q(S_n; \lambda) \le \mathbf{E} \sup_{a} \sup_{x} \mathbf{P}\left\{ x \le V_n \le x + \lambda n^{-1/\alpha} \Gamma_{n+1}^{1/\alpha} \,\middle|\, \Gamma_{n+1} \right\}.$$

If $p < 1$, then we have

$$\delta_n(\mathcal{M}) \le \widetilde{\delta}_m(\mathcal{M}) + \exp\{-3pn/4\}, \tag{12}$$

$$Q(S_n; \lambda) \le \mathbf{E} \sup_{a} \sup_{x} \mathbf{P}\left\{ x \le V_m \le x + \lambda m^{-1/\alpha} \Gamma_{n+1}^{1/\alpha} \,\middle|\, \Gamma_{n+1} \right\} + \exp\{-3pn/4\}, \tag{13}$$

where $m = [pn/2]$.

Proof. If $p = 1$, we derive the result conditioning on $X_1, \dots, X_n$.

Thus, it remains to prove the bounds (12) and (13) assuming that $0 < p < 1$. We shall prove only (12) since the proof of (13) is similar.

Recall that $p = \mathbf{P}\left\{ c_1 \le |X| \le c_2 \right\}$. Introduce the i.i.d. Bernoulli random variables $\eta_j = \mathbf{I}\left\{ c_1 \le |X_j| \le c_2 \right\}$, for $j \ge 1$. Consider the event

$$B_n = \left\{ \eta_1 + \dots + \eta_n > pn/2 \right\}$$

and its complement B_n^c. Since $\mathbf{E}\,\eta_i = p$ and $|\eta_i - \mathbf{E}\,\eta_i| \le 1$, a well-known exponential inequality yields

$$\mathbf{P}\left\{ B_n^c \right\} \le \exp\{-3np/4\}. \tag{14}$$

In what follows $\mathbf{E}_{\{\dots\}}$ will denote the expectation taken with respect to those random variables, which are written in parenthesis as subscript to $\mathbf{E}$.

Combining (8) with (14) and conditioning on $X_1, \dots, X_n$, we have

$$\begin{aligned} \delta_n(\mathcal{M}) &\le \sup_{A\in\mathcal{M}} \Big| \, \mathbf{E}\left(\mathbf{I}\{Z_n \in A\} - \mathbf{I}\{Z_n + X \in A\} \right) \mathbf{I}\{B_n\} \Big| + \mathbf{P}\{B_n^c\} \\ &\le \mathbf{E}_{\{X_1,\dots,X_n\}} \, \delta_n'(\mathcal{M}) + \exp\{-3pn/4\}, \end{aligned}$$

where

$$\delta_n'(\mathcal{M}) = \mathbf{I}\{B_n\} \sup_{A\in\mathcal{M}} \Big| \, \mathbf{E}_{\{\xi_1,\dots,\xi_n,\, X\}} \left(\mathbf{I}\{Z_n \in A\} - \mathbf{I}\{Z_n + X \in A\} \right) \Big|.$$

Thus, in order to prove the Lemma, it is sufficient to verify that

$$\delta_n'(\mathcal{M}) \le \widetilde{\delta}_m(\mathcal{M}). \tag{15}$$

If $\mathbf{I}\{B_n\} = 0$, the inequality (15) is obviously fulfilled. Therefore we can assume that $\mathbf{I}\{B_n\} = 1$, that is, that the event B_n occurs. Consequently, there exist indices $l_1, \dots, l_k$ such that

$$k > np/2 \ge m \quad \text{and} \quad c_1 < |X_{l_j}| < c_2, \qquad \text{for } 1 \le j \le k.$$

Notice that k and $l_1, \dots, l_k$ are random (they depend on X's) but independent of $\xi_1, \dots, \xi_n$. Define the random sets

$$M_1 = \{l_1, \dots, l_m\} \qquad \text{and} \qquad M_2 = \{1, \dots, n\} \setminus M_1.$$

Introducing the notation

$$U = \sum_{j \in M_1} \xi_j X_j, \qquad V = \sum_{j \in M_2} \xi_j X_j \qquad \text{and} \qquad \mathbf{E}_M = \mathbf{E}_{\{\xi_i,\, i \in M\}},$$

we can write

$$\begin{aligned}
\delta_n'(\mathcal{M}) &= \sup_{A \in \mathcal{M}} \Big| \mathbf{E}_{M_1} \mathbf{E}_{M_2} \mathbf{E}_{\{X\}} \big(\mathbf{I}\{U \in A - V\} - \mathbf{I}\{U + X \in A - V\} \big) \Big| \\
&\le \mathbf{E}_{M_2} \sup_{A \in \mathcal{M}} \Big| \mathbf{E}_{M_1} \mathbf{E}_{\{X\}} \big(\mathbf{I}\{U \in A - V\} - \mathbf{I}\{U + X \in A - V\} \big) \Big| \\
&= \sup_{A \in \mathcal{M}} \Big| \mathbf{E}_{M_1} \mathbf{E}_{\{X\}} \big(\mathbf{I}\{U \in A\} - \mathbf{I}\{U + X \in A\} \big) \Big| \\
&\le \sup_{A \in \mathcal{M}} \sup_a \Big| \mathbf{E}_{\{M_1\}} \mathbf{E}_{\{X\}} \Big(\mathbf{I}\big\{ U \in A \big\} - \mathbf{I}\big\{ U + X \in A \big\} \Big) \Big| \\
&= \widetilde{\delta}_m(\mathcal{M}).
\end{aligned}$$

In the last step we replaced the sum U by $m^{-1/\alpha} U$ and used the obvious equality $\mathbf{E}_{\{M_1\}} f(\xi_j,\, j \in M_1) = \mathbf{E}\, f(\xi_1, \dots, \xi_m)$, which is valid for any measurable function f of m arguments since $\xi_1, \xi_2, \dots$ are identically distributed. □

3. Auxiliary lemmas

Write $p(x) = \alpha x^{-\alpha-1} \mathbf{I}\{x \ge 1\}$ for the Pareto distribution density, and $f(t)$ for its characteristic function.

Lemma 8. *Let $\delta > 0$. There exist positive constants $C_1(\alpha)$, $C_2(\alpha)$ and $C_3(\alpha, \delta) < 1$ such that*

$$
\begin{aligned}
&|f(t)| \le \exp\{-C_1(\alpha)\,|t|^\alpha\}, && \textit{for } |t| \le 1, && (16)\\
&|f'(t)| \le C_2(\alpha)\,|t|^{\alpha-1}, && \textit{for all } t \ne 0, && (17)\\
&|f(t)| \le C_2(\alpha)\,|t|^{-1}, && \textit{for all } t \ne 0, && (18)\\
&|f(t)| \le C_3(\alpha, \delta), && \textit{for } |t| > \delta. && (19)
\end{aligned}
$$

Proof. The Pareto distribution admits a density, thus (19) is fulfilled.

While proving (16)–(18) we shall assume that $t > 0$, since the case $t < 0$ can be treated similarly. Let us decompose the characteristic function f into a real and an imaginary part, $f(t) = f_1(t) + i\, f_2(t)$, where

$$f_1(t) = \alpha \int_1^\infty x^{-\alpha-1} \cos tx\, dx, \qquad f_2(t) = \alpha \int_1^\infty x^{-\alpha-1} \sin tx\, dx.$$

Notice that

$$f_1(t) = 1 - \alpha \int_1^\infty x^{-\alpha-1}\,(1 - \cos tx)\, dx = 1 - \alpha t^\alpha \int_t^\infty u^{-\alpha-1}\,(1 - \cos u)\, du.$$

A similar formula holds for f_2. Therefore, we easily obtain the estimates

$$\big|\, f_1(t) \,\big| \le 1 - C(\alpha)\,|t|^\alpha, \qquad \big|\, f_2(t) \,\big| \le C(\alpha)\,\big(|t|^\alpha + |t|\big), \qquad \text{for } |t| \le C_4(\alpha).$$

These estimates combined with $\big|\, f(t) \,\big| = \Big(\big|\, f_1(t) \,\big|^2 + \big|\, f_2(t) \,\big|^2\Big)^{1/2}$ imply that

$$\big|\, f(t) \,\big| \le \exp\big\{ -C_1(\alpha)\,|t|^\alpha \big\}, \qquad \text{for } |t| \le C_4(\alpha), \tag{20}$$

for some positive $C_4(\alpha)$.

The bound (16) follows from (20). Indeed, the estimate (20) implies (16) provided that $C_4(\alpha) \ge 1$. If $C_4(\alpha) < 1$, then we may choose a smaller constant $C_1(\alpha)$ (if necessary) and use (19) in order to obtain (16).

The proofs of (17) and (18) are similar to the proof of (16). Here some remarks on the proof of, say (18), will suffice. Integrating by parts we have

$$
\begin{aligned}
\big|\, f_1(t) \,\big| &= \alpha \Big| \int_1^\infty x^{-\alpha-1} \cos tx\, dx \Big| = \alpha t^{-1} \Big| \int_1^\infty x^{-\alpha-1}\, d \sin tx \Big| \\
&= \alpha t^{-1} \Big| \sin t + (\alpha + 1) \int_1^\infty x^{-\alpha-2} \sin tx\, dx \Big| \le C(\alpha)\, t^{-1}. \qquad \square
\end{aligned}
$$

Lemma 9. *We have $m^{-2/\alpha} \le p^{-1} L_m$.*

Proof. Indeed,

$$L_m \ge \left(c_1 m^{1/\alpha}\right)^{-2} c_1^2 \,\mathbf{P}\left\{c_1 \le |X| \le c_1 m^{1/\alpha}\right\} + \mathbf{P}\left\{|X| > c_1 m^{1/\alpha}\right\} \ge m^{-2/\alpha} p. \quad \square$$

We shall use the following version [Lemma 8.1 in Bentkus and Götze (1994)] of a smoothing lemma due to Prawitz (1972).

Lemma 10. *Let F and G be arbitrary distribution functions with the characteristic functions f and g. Then*

$$\sup_x \left| F(x) - G(x) \right| \le \frac{1}{2\pi} \int_{-H}^{H} \left| f(t) - g(t) \right| \frac{dt}{|t|} + R,$$

for any $H > 0$, where

$$|R| \le \frac{1}{H} \int_{-H}^{H} \left| f(t) \right| dt + \frac{1}{H} \int_{-H}^{H} \left| g(t) \right| dt.$$

4. Proof of theorems

To prove Theorem 2 we shall use the following lemma. Its proof will be given later.

Lemma 11. *The distribution function of V_m (as defined in (11)) has a differentiable density, say p_m, for $m \ge 10$. Furthermore, the quantities*

$$K_1 = \sup_{m \ge 5} \sup_a \int_{\mathbf{R}} \left| p'_m(x) \right| dx, \qquad K_2 = \sup_{m \ge 10} \sup_a \int_{\mathbf{R}} \left| p''_m(x) \right| dx$$

satisfy

$$K_2 \le C(\alpha)\, K_1^2, \qquad K_1^2 \le C(\alpha)\, c_1^{-2} (c_2/c_1)^{\alpha}.$$

Proof of Theorem 2. We shall only prove the bound (6) since the other statements of the Theorem follow from (6) by standard arguments. Write

$$J = \left| \mathbf{P}\left\{V_m \in A\right\} - \mathbf{P}\left\{V_m + m^{-1/\alpha} X \in A\right\} \right|.$$

It follows from Lemmas 7 and 9 that it suffices to prove that

$$\widetilde{\delta}_m(\mathcal{M}) = \sup_{A\in\mathcal{M}} \sup_{a} J \le C(\alpha)\,(c_2/c_1)^\alpha\, L_m, \tag{21}$$

where $m = [np/2]$, if $p < 1$, and $m = n$ if $p = 1$. In the proof of (21) we shall assume that $m \ge 10$ since otherwise (6) follows from $\delta_m(\mathcal{M}) \le 1$ and Lemma 9.

Introduce the truncated random variable $Y = X\,\mathbf{I}\big\{\,|X| \le c_1 m^{1/\alpha}\big\}$. Then

$$J \le J_t + \mathbf{P}\{|X| > c_1 m^{1/\alpha}\},$$

where

$$J_t = \Big|\,\mathbf{P}\,\big\{\,V_m \in A\big\} - \,\mathbf{P}\,\big\{\,V_m + m^{-1/\alpha}\,Y \in A\big\}\Big|.$$

Expanding $p_m(x - m^{-1/\alpha}\,Y)$ in powers of $m^{-1/\alpha}\,Y$, we have

$$\begin{aligned}
J_t &= \Big|\,\mathbf{E}\int_A \big(p_m(x) - p_m\big(x - m^{-1/\alpha}\,Y\big)\big)\,dx\Big| \\
&\le \Big|\,\mathbf{E}\,m^{-1/\alpha}\,Y \int_A p'_m(x)\,dx\Big| + \Big|\,\mathbf{E}\,m^{-2/\alpha}\,Y^2 \int_A p''_m(x + \theta\, m^{-1/\alpha}Y)\,dx\Big| \\
&\le m^{-1/\alpha}\,|\,\mathbf{E}\,Y|\, \sup_{A\in\mathcal{M}} \int_A \big|\,p'_m(x)\,\big|\,dx + m^{-2/\alpha}\,\mathbf{E}\,Y^2 \sup_{A\in\mathcal{M}} \int_{A+\theta\, m^{-1/\alpha}\,X^t} |p''_m(x)|\,dx \\
&\le K_1\, m^{-1/\alpha}\,|\,\mathbf{E}\,Y| + K_2\, m^{-2/\alpha}\,\mathbf{E}\,Y^2 \\
&\le C(\alpha)\,(c_2/c_1)^\alpha \Big(c_1^{-1}\, m^{-1/\alpha}\,|\,\mathbf{E}\,Y| + c_1^{-2}\, m^{-2/\alpha}\,\mathbf{E}\,Y^2\Big),
\end{aligned}$$

where $|\theta| \le 1$. Collecting the bounds, we obtain (21) since

$$\mathbf{P}\,\big\{\,|X| > c_1\, m^{1/\alpha}\,\big\} + c_1^{-1}\, m^{-1/\alpha}\Big|\,\mathbf{E}\,Y\Big| + c_1^{-2}\, m^{-2/\alpha}\,\mathbf{E}\,Y^2 = L_m. \qquad \square$$

Proof of Lemma 11. By $\widehat{p}$ we denote the Fourier transform of a function p.

Let us prove that $K_2 \le C(\alpha)\,K_1^2$. Let $k = [m/2]$. Then we can split

$$V_m = U_1 + U_2, \qquad \text{where} \qquad U_1 = m^{-1/\alpha} \sum_{j=1}^{k} a_j\,\xi_j, \qquad \text{and} \qquad U_2 = V_m - U_1.$$

Let u_1 and u_2 denote densities of U_1 and U_2, respectively. Then $p_m = u_1 * u_2$ is the convolution of u_1 and u_2 since random variables U_1 and U_2 are independent.

Notice that $|p''_m| = |u'_1 * u'_2| \le |u'_1| * |u'_2|$. Therefore, integrating and using Fubini's Theorem, we obtain $K_2 \le C(\alpha)\, K_1^2$.

Let f denote the characteristic function of the Pareto distribution. While estimating K_1^2, we shall use the inequality

$$\sup_{0\le k\le 4}\ \sup_{m\ge 5} \int_{\mathbf{R}} |t|^k \,|f(m^{-1/\alpha} t)|^{2m-4}\, dt \le C(\alpha). \tag{22}$$

In order to prove (22) split the real line $\mathbf{R}$ into three disjoint sets

$$\{\,|t| < m^{1/\alpha}\,\}, \qquad \{\,m^{1/\alpha} \le |t| < m^{2/\alpha}\,\}, \qquad \{\,|t| \ge m^{2/\alpha}\,\},$$

and on each of these sets apply estimates (16), (19) and (18), respectively.

Now we shall estimate K_1^2. Let $\varepsilon > 0$ denote an arbitrary fixed number. We shall prove that

$$K_1^2 \le C\varepsilon^{-1} \sup_{a,\, m\ge 3} I_0 + C\,\varepsilon \sup_{a,\, m\ge 3} I_2 + C\varepsilon^{-1} \sup_{a,\, m\ge 3} J, \tag{23}$$

where

$$I_k = \int_{\mathbf{R}} t^k \,\big|\hat{p}_m(t)\big|^2\, dt, \quad k = 0, 2, \qquad J = \int_{\mathbf{R}} t^2 \,\big|\hat{p}'_m(t)\big|^2\, dt.$$

Indeed, by Hölder's inequality, we have

$$K_1^2 \le C\varepsilon^{-1} \sup_{m\ge 3} \sup_{a} \int_{\mathbf{R}} (\varepsilon^2 + x^2)\, \big(p'_m(x)\big)^2\, dx.$$

The Parseval equality combined with the well-known properties of the Fourier transform imply

$$\int_{\mathbf{R}} \big(p'_m(x)\big)^2\, dx \le C \int_{\mathbf{R}} t^2 \big|\widehat{p}_m(t)\big|^2\, dx$$

and

$$\int_{\mathbf{R}} x^2\, \big(p'_m(x)\big)^2\, dx = \int_{\mathbf{R}} \big(x\, p'_m(x)\big)^2\, dx \le C \int_{\mathbf{R}} \big|\widehat{p}_m(t)\big|^2\, dt + C \int_{\mathbf{R}} t^2 \big|\hat{p}'_m(t)\big|^2\, dt.$$

Thus, (23) follows.

Let us estimate I_k. Write $f_j(t) = f(t\, a_j\, m^{-1/\alpha})$. Then $\widehat{p}_m = f_1 \dots f_m$. Applying the geometric-arithmetic mean inequality we have

$$|\widehat{p}_m|^2 \le m^{-1} \sum_{j=1}^{m} |f_j|^{2m}.$$

Therefore

$$
\begin{aligned}
I_k &\le m^{-1}\sum_{j=1}^{m}\sup_{m\ge 5}\sup_{a}\int_{\mathbf{R}} t^k\,|f_j(t)|^{2m}\,dt\\
&= \sup_{m\ge 5}\sup_{c_1\le a\le c_2}\int_{\mathbf{R}} t^k\,|f(a m^{-1/\alpha} t)|^{2m}\,dt\\
&\le c_1^{-k-1}\sup_{m\ge 5}\int_{\mathbf{R}} t^k\,|f(m^{-1/\alpha} t)|^{2m}\,dt,\\
&\le C(\alpha)\,c_1^{-k-1},
\end{aligned}
\tag{24}
$$

using the equality of all summands in the sum, using change of variables, and (22) as well.

Let us show that

$$
J \le Cm^2 \sup_{a,\,m\ge 3}\int_{\mathbf{R}} t^2\,|f_1'(t)|^2\,|f_2(t)|^{2m-4}\,dt. \tag{25}
$$

By the product formula

$$
\hat{p}_m' = \sum_{j=1}^{m} f_j' \prod_{1\le k\le m,\ k\ne j} f_k,
$$

whence

$$
(\hat{p}_m')^2 = \sum_{j=1}^{m}(f_j')^2 \prod_{1\le k\le m,\ k\ne j} f_k^2 + 2\sum_{1\le j<l\le m} f_j' f_l' \Big(\prod_{1\le k\le m,\ k\ne j} f_k^2\Big)\,\Big(\prod_{1\le k\le m,\ k\ne l} f_k^2\Big). \tag{26}
$$

Now we can proceed similarly to the proof of (24). Using $|f_j|\le 1$, we see that the integral over terms in the ordinary sum in (26) is bounded by the right hand side of (25). One can estimate the integrals over the remaining terms in (26) similarly. To this end it is necessary to apply first Hölder's inequality. For example, writing $A=|f_3(t)|^2\dots|f_m(t)|^2$, we have

$$
\int t^2\,|f_1'|\,|f_2'|\,A\,dt \le \Big(\sup_a \int t^2\,|f_1'|^2\,A\,dt\ \sup_a\int t^2\,|f_2'|^2\,A\,dt\Big)^{1/2} = \sup_a\int t^2\,|f_1'|^2\,A\,dt.
$$

By (17), we have $|f'(t)|\le C_2(\alpha)\,|t|^{\alpha-1}$, for all $t\in\mathbf{R}$. Therefore (25) yields

$$
J \le C(\alpha)\,c_2^{2\alpha}\sup_{m\ge 5}\sup_a\int_{\mathbf{R}} |t|^{2\,\alpha}\,|f_2(t)|^{2m-4}\,dt.
$$

Now by a change of the variable and (22), we obtain $J\le C(\alpha)\,c_2^{2\,\alpha}\,c_1^{-2\,\alpha-1}$.

Collecting the bounds for I_k and J and using (23) we have

$$K_1^2 \le C(\alpha)\, c_1^{-2}\Big(c_1\,\varepsilon^{-1}\big(1+(c_2/c_1)^{2\alpha}\big)+\varepsilon\, c_1^{-1}\Big) \le C(\alpha)\, c_1^{-2}\Big(c_1\,\varepsilon^{-1}\,(c_2/c_1)^{2\alpha}+\varepsilon\, c_1^{-1}\Big),$$

since $1 \le c_2/c_1$. The choice of $\varepsilon = c_1\,(c_2/c_1)^{\alpha}$ concludes the proof. □

Proof of Remark 4. We start as in the proof of Theorem 2. In view of Lemmas 7 and 9, we have to estimate $\sup_x \sup_a \big| F_m(x) - G_m(x) \big|$, where

$$F_m(x) = \mathbf{P}\,\big\{\, V_m < x \,\big\}, \qquad G_m(x) = \mathbf{P}\,\big\{\, V_m + m^{-1/\alpha}\, Y < x \,\big\},$$

and where the truncated random variable $Y = X\,\mathbf{I}\big\{\,|X| \le c_1\, m^{1/\alpha}\big\}$. Here we use the smoothing Lemma 10. Recall that $\widehat{p}_m$ denotes the characteristic function of V_m. Let g denote the characteristic function of $m^{-1/\alpha}\,Y$. Let us apply Lemma 10 with $H = \infty$ This is possible since Hölder's inequality and (24) together imply $\int_{\mathbf{R}} |\widehat{p}_m(t)|\,dt < \infty$. Using Taylor's formula for the difference $|1 - g(t)|$, we easily get the estimate

$$\sup_x \big| F_m(x) - G_m(x) \big| \le C(\alpha) \int_{\mathbf{R}} \big(m^{-1/\alpha}\, |\,\mathbf{E}\, Y| + |t|\, m^{-2/\alpha}\, \mathbf{E}\, Y^2 \big)\, |\widehat{p}_m(t)|\, dt \le C(\alpha)\, L_m$$

provided we again apply Hölder's inequality, and estimate the integrals as in (24). For example, writing $k = [m/2]$ and splitting $\widehat{p}_m(t) = A B$, where $A = f_1 \dots f_k$ and $B = f_{k+1} \dots f_m$, we obtain

$$\int_{\mathbf{R}} |t|\, |\widehat{p}_m(t)|\, dt \le \Big(\int_{\mathbf{R}} |t| A^2\, dt \int_{\mathbf{R}} |t| B^2\, dt \Big)^{1/2} \le C(\alpha)\, c_1^{-2}.$$

Proof of Theorem 5. By Lemma 7, we have to estimate

$$\mathbf{E} \sup_a \sup_x \mathbf{P}\,\big\{\, x \le V_m \le x + \lambda\, m^{-1/\alpha}\, \Gamma_{n+1}^{1/\alpha} \,\big|\, \Gamma_{n+1} \,\big\}.$$

The distribution of the random variable V_m has a density p_m, which is bounded from above by $A \overset{\text{def}}{=} \int_{\mathbf{R}} |\widehat{p}_m(t)|\, dt$. Using Hölder's inequality and (24), we get $A \le C(\alpha) c_1^{-1}$. Consequently, we obtain

$$\sup_x \mathbf{P}\,\big\{\, x \le V_m \le x + \lambda m^{-1/\alpha}\, \Gamma_{n+1}^{1/\alpha} \,\big|\, \Gamma_{n+1} \,\big\} \le C(\alpha)\, c_1^{-1}\, \lambda m^{-1/\alpha}\, \Gamma_{n+1}^{1/\alpha}.$$

Taking expectations and using $\mathbf{E}\,\Gamma_n^{1/\alpha} \le C(\alpha)\, n^{1/\alpha}$, we conclude the proof for the case $p < 1$.

In the case $p = 1$ the proof becomes simpler since Lemma 7 can be replaced by Lemma 6, and we can repeat the previous arguments with $m = n$. □

References

Bentkus, V. and Götze, F., *On the Lattice Point Problem for Ellipsoids*, Preprint 94-111 SFB 343, Universität Bielefeld (1994).

Bentkus, V. and Götze, F., *Optimal rates of convergence in functional limit theorems for quadratic forms*, Preprint 95-091 SFB 343, Universität Bielefeld (1995).

Bentkus, V. and Götze, F., *Optimal rates of convergence in the CLT for quadratic forms*, Ann. Prob. (1996), no. 1.

Breiman, L., *Probability*, Addison–Wesley, Reading, Massachusetts, 1968.

Ferguson, T. and Klass, M., *A representation theorem of independent increment processes without Gaussian component*, Ann. Math. Stat. **43** (1972), 1634–1643.

Ibragimov, I.A. and Linnik, Yu.V., *Independent and stationary sequences of random variables*, Wolters-Nordhoff publishing, Groningen, 1971.

Janicki, A. and Kokoszka, P., *Computer investigation of the rate of convergence of Lepage type series to α-stable random variables*, Statistics **23** (1992a), 365–373.

Janicki, A. and Kokoszka, P., *On the rate of convergence of LePage type series to finite dimensional distributions of Levy motion*, preprint (1992b).

Janicki, A. and Weron, A., *Simulation and Chaotic Behaviour of α-Stable Stochastic Processes*, Marcel Dekker, New York, 1994.

Kinateder, J., *An invariance principle applicable to the bootstrap*, in Exploring the limits of bootstrap (R. LePage and L. Billard, eds.), 1992, pp. 177–181.

Ledoux, M. and Paulauskas, V., *A rate of convergence in the Poissonian representation of stable distributions*, Vilnius university preprint 95-20 (1995).

Ledoux, M. and Talagrand, M., *Probability in Banach Spaces*, Springer, Berlin, 1991.

LePage, R., *Multidimensional infinitely divisible variables and processes. Part I: Stable case*, Technical Report No 292, Statistics Department, Stanford, (1980) reprinted in Lecture Notes in Math., **1391** (1989).

LePage, R., *Multidimensional infinitely divisible variables and processes. Part II*, Lecture Notes in Mathematics **860** (1981), 279–284.

LePage, R. (1992), *Bootstraping signs*, in Exploring the limits of bootstrap (R. LePage and L. Billard, eds.), 1992, pp. 215–224.

LePage, R. and Podgorski, K. 1993, *Resampling permutations in regresion without second moment*, submitted to J. Mult. Analysis (1993).

LePage, R., Podgorski, K. and Ryznar, M., *Strong and conditional invariance principles for samples attracted to stable laws*, submitted to Probab. Theory and Related Fields (1994).

LePage, R., Woodroofe, M. and Zinn, J., *Convergence to a stable distribution via order statistics*, Ann. Probab. **9** (1981), 624–632.

Marcus, M.B. and Pisier, G., *Random Fourier Series with Applications to Harmonic Analysis. Ann. Math. Stud., 101*, Princeton Univ. Press, 1981.

Petrov, V.V., *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*, Oxford University Press, 1995.

Prawitz, H. (1972), *Limits for a distribution, if the characteristic function is given in a finite domain*, Skand. Aktuar. Tidskr. (1972), 138–154.

Rachev, S.T. and Samorodnitsky, G., *Option pricing formulae for speculative prices modelled by subordinated stochastic process*, preprint (1992).

Rosinski, J., *On series representation of infinitely divisible random vectors*, Ann. Probab. **18** (1990), 405–430.

Samorodnitsky, G. and Taqqu, M.S., *Stable non-Gaussian Random Processes. Stochastic Models with Infinite Variance*, Chapman and Hall, New York, London, 1994.

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