Uniqueness of Gibbs states for quantum lattice systems

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Abstract

We prove uniqueness of Gibbs states for certain quantum lattice systems with unbounded spins. We use Dobrushin's uniqueness criterion. The necessary estimates for the Vasershtein distance between the corresponding one-point conditional distributios with boundary conditions differing only at one side, are obtained by proving a Log–Sobolev inequality on the infinite dimensional single spin (= loop) spaces. Some important classes of concrete examples to which all this applies are discussed.

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Contents

1 Introduction.

The purpose of this paper is to prove uniqueness of Gibbs states for certain quantum lattice systems with unbounded spins. It was proposed in [AH-K] to construct and study such states through a *Euclidean approach.* This approach transforms the problem of constructing such Gibbs states on the algebra of local observables into the problem of studying Gibbs measures for classical lattice systems, but with complicated infinite dimensional single spin spaces (as e.g. loop spaces). We shall describe this approach and the particular types of Hamiltonians considered here in more detail in Section 2 below. Section 3 is devoted to describe the corresponding Gibbsian formalism, i.e., the underlying local characteristics given by the heuristic Hamiltonian are defined rigorously through specifications determining the associated Gibbs states via the DLR (Dobrushin - Lanford - Ruelle) - equations.

Our main result, i.e., the uniqueness of the Gibbs states, is formulated and proved in Section 4 (Theorem 4.1). The proof is based on Dobrushin's fundamental uniqueness criterion. The necessary estimates for the Vasershtein distance between the corresponding one point conditional distributions with boundary conditions differing only at one site, are reduced to estimates of variances of Lipschitz continuous functions on the loop
(= single spin) space w.r.t. the one point conditional distributions. The technique to obtain these latter estimates is the new feature of this work. We derive them by proving that L. Gross' *Log–Sobolev inequality* holds for the
(of course, non-Gaussian) one point conditional distributions. By the Rothaus - Simon theorem this yields a mass gap, and thus immediately the desired variance estimates.

The proof of the Log–Sobolev inequality on the (infinite dimensional) single spin space is presented in detail in Section 5. Section 6 is devoted to some important classes of concrete examples. In Section 7 we give some concluding remarks and describe, in particular, some generalizations concerning the interaction part of our Hamiltonians.

In this paper we present this new technique (which yields a standard method to check Dobrushin's uniqueness criterion), for particular, though complicated models (i.e, models with infinite dimensional single spin spaces). We would like to emphasize, however, that this method is quite universal for lattice models (at least with finite range interactions) and gives some structural insights. The main point is that the metric chosen for the Vasershtein distance is the intrinsic metric of some Dirichlet form on the single spin space satisfying a Log–Sobolev inequality. This will be elaborated in more detail in a forthcoming paper.

Finally, we would like to give some comments on the relevance of our main result (Theorem 4.1) for quantum physics. Condition (4.1) yielding uniquenes of the Gibbs space can be achieved by choosing various physical parameters of the system sufficiently small. We discuss in the examples of Sections 6 and 7 the influence of physical parameters as the inverse temperature and the intensity of the interaction. We also analyze the connection between the shape of the one-particle potentials and the uniqueness of the corresponding Gibbs states. The results we have obtained here can be considered as rigorous mathematical proofs of some facts from the theory of phase transitions in quantum statistical physics, see e.g. [BC].

2 Temperature loop space (TLS) representation for quantum Gibbs states.

Let $\mathbb{Z}^d$, $d \in \mathbb{N}$, be the integer lattice with the Euclidean distance $|k-j|$, $k,j \in \mathbb{Z}^d \subset \mathbb{R}^d$. We consider a translation invariant system of interacting quantum anharmonic oscillators with the heuristic Hamiltonian

$$H \;=\; -\frac{1}{2m}\sum_{k\in\mathbb{Z}^d}\triangle_k + \frac{a^2}{2}\sum_{k\in\mathbb{Z}^d}x_k^2 + \frac{J}{2}\sum_{<k,j>}(x_k-x_j)^2 + \sum_{k\in\mathbb{Z}^d}V(x_k). \quad (2.1)$$

We specify the assumptions on the system (2.1) as follows. The parameter $m>0$ has the meaning of a physical mass. The two–particle harmonic interactions $(x_k-x_j)^2$ have an intensity $J>0$ and are taken over all nearest–neighbour pairs $<k,j>$ in $\mathbb{Z}^d$ ($\mid k-j\mid = 1$). The harmonic self–interaction is given by $a^2x_k^2/2$ with a constant $a^2>0$. The anharmonic self–interaction potential has the form

$$V(q) \;=\; V_0(q) + W(q)\ . \quad (2.2)$$

Here V_0 is a convex function with polynomially bounded derivatives up to the second order, that means there exist $b \in \mathbb{R}\setminus\{0\}$, $M \in \mathbb{N}$, and $C>0$ such that

$$V_0 \in C^2(\mathbb{R}^1),\ V_0''(q) \geq b^2,$$

for all $q \in \mathbb{R}^1$,

$$\mid V_0^{(l)}(q)\mid \leq\ C(1+\mid q\mid)^M, l=0,1,2, \quad (2.3)$$

for all $q \in \mathbb{R}^1$. The perturbation W is given by a bounded function

$$W \in C_b(\mathbb{R}^1). \quad (2.4)$$

We set

$$\delta(W) \;:=\; \sup_{\mathbb{R}^1} W - \inf_{\mathbb{R}^1} W.$$

A basic problem of equilibrium quantum statistical physics is the study of temperature (i.e., Gibbs) states. We will follow the Euclidean approach based on a path space representation for Gibbs states (see e.g. the early work of S. Albeverio and R. Høegh–Krohn [AH-K] and its further developments in [KL], [GK]). Under this approach the problem of giving a proper meaning to a Gibbs state G_β of the lattice system (2.1) at inverse temperature $\beta > 0$ can be reduced to the problem of constructing some Euclidean Gibbs measure ν_β on the space Ω_β of periodic trajectories (loops) on $[0, \beta]$ with values in $\mathbb{R}^{\mathbb{Z}^d} := \times_{k\in\mathbb{Z}^d}\mathbb{R}^1$.

Namely, let S_β be a circle having length β (it is convenient to interpret S_β as the interval $[0, \beta]$ with the ends identified). The spaces $C(S_\beta)$ and $L^2(S_\beta)$ consist of functions $\omega(\cdot) : S_\beta \to \mathbb{R}^1$ which are continuous resp. square integrable with respect to Lebesgue measure on S_β. If $C(S_\beta)$ is equipped with the sup-norm $\|\cdot\|_{C(S_\beta)}$ and $L^2(S_\beta)$ with its natural norm, for the respective Borel σ - algebras the following relation holds

$$\mathcal{B}(C(S_\beta)) \ = \ \mathcal{B}(L^2(S_\beta)) \cap C(S_\beta).$$

We define as the configuration space the temperature loop space (TLS)

$$\begin{aligned} \Omega_\beta \ &:= \ C(S_\beta)^{\mathbb{Z}^d} \\ &= \ \{\omega(\cdot) = (\omega_k(\cdot))_{k\in\mathbb{Z}^d} \mid \omega : S_\beta \to \mathbb{R}^{\mathbb{Z}^d}, \omega_k(\cdot) \in C(S_\beta)\} \qquad (2.5) \end{aligned}$$

equipped with the product topology and with the σ- algebra $\sum_{cyl} = \mathcal{B}(\Omega_\beta)$ generated by the cylinder sets

$$\{\omega \in \Omega_\beta \mid (\omega_k)_{k\in\Lambda} \in B_\Lambda \in \mathcal{B}(C(S_\beta)^\Lambda), \Lambda \subset \mathbb{Z}^d, \mid \Lambda \mid < \infty\}.$$

The Euclidean measure ν_β corresponding to a Gibbs state G_β of the system (2.1) has the heuristic representation

$$d\nu_\beta(\omega(\cdot)) \ = \ \frac{1}{Z_\beta} \exp\{-E_\beta(\omega(\cdot))\} \times_{k\in\mathbb{Z}^d, \tau\in S_\beta} d\omega_k(\tau) \qquad (2.6)$$

where

$$\begin{aligned} E_\beta(\omega(\cdot)) \ &= \ \int_{S_\beta} [\frac{m}{2} \sum_{k\in\mathbb{Z}^d} \dot{\omega}_k^2(\tau) + \frac{a^2}{2} \sum_{k\in\mathbb{Z}^d} \omega_k^2(\tau) \\ &+ \ \frac{J}{2} \sum_{<k,j>} (\omega_k(\tau) - \omega_j(\tau))^2 + \sum_{k\in\mathbb{Z}^d} V(\omega_k(\tau))] d\tau \qquad (2.7) \end{aligned}$$

is the Euclidean action functional of the system and Z_β is a normalizing factor. In full analogy with classical statistical mechanics, a rigorous meaning can be given to the measure ν_β by the Dobrushin–Lanford–Ruelle formalism as an infinite volume Gibbs distribution with spin space $C(S_\beta)$. But besides the definition, the further study of Euclidean Gibbs distributions, including existence and uniqueness problems, is rather complicated, since the topological features of the functional spin spaces must be taken into account properly.

3 Gibbsian formalism for TLS measures.

So we start from the definition of the corresponding family of local specifications (π_Λ), $\Lambda \subset \mathbb{Z}^d, \mid \Lambda \mid < \infty$, satisfying the consistency condition. Below let a, m, J and V be as specified at the beginning of Section 2.

To construct a reference measure $\times_{k\in\mathbb{Z}^d} d\sigma_\beta(\omega_k(\cdot))$ one should specify the following one–particle distribution on $C(S_\beta)$ which heuristically looks like

$$d\sigma_\beta(\omega(\cdot)) \;=\; \frac{1}{Z_\beta}\exp\{-\int_{S_\beta}[\frac{m}{2}\dot{\omega}^2(\tau) + \frac{a^2}{2}\omega^2(\tau) + V(\omega(\tau))]d\tau\} \times_{\tau\in S_\beta} d\omega(\tau). \tag{3.1}$$

Let us consider the Hilbert space $\mathcal{H}_\beta := L^2(S_\beta)$ (with scalar product $<\cdot,\cdot>_\beta$ and norm $||\cdot||_\beta)$ and a positive operator $A_\beta = -m\triangle_\beta + a^2 1\!\!1$ in $\mathcal{H}_\beta$ (here $\triangle_\beta$ is the Laplace–Beltrami operator on the circle S_β). Let γ_β be the Gaussian measure on $(\mathcal{H}_\beta, \mathcal{B}(\mathcal{H}_\beta))$ with zero mean value and correlation operator $C_\beta = A_\beta^{-1}$. Such a measure is given uniquely by its Fourier transform

$$\int_{\mathcal{H}_\beta} e^{i<\varphi,\omega>_\beta} d\gamma_\beta(\omega) \;=\; e^{-\frac{1}{2}<C_\beta\varphi,\varphi>_\beta},\ \varphi \in \mathcal{H}_\beta. \tag{3.2}$$

Actually, the set $C(S_\beta)$ of continuous loops has full external measure $(\gamma_\beta^*(C(S_\beta)) = 1)$, and the measure γ_β on the space $(C(S_\beta), \mathcal{B}(C(S_\beta)))$ is canonically generated by the well-known oscillator bridge process of length β [Si 2]. We then define

$$d\sigma_\beta(\omega(\cdot)) \;=\; \frac{1}{Z_\beta}e^{-\int_{S_\beta} V(\omega(\tau))d\tau} d\gamma_\beta(\omega(\cdot)) \tag{3.3}$$

as a probability measure on $(C(S_\beta), \mathcal{B}(C(S_\beta)))$. Obviously, this definition is meaningful for the class of potentials $V \in C(\mathbb{R}^1)$, $\inf_{\mathbb{R}^1} V > -\infty$, under consideration. The measure σ_β has all moments of the form

$$E_{\sigma_\beta}\{||\omega||^M_{C(S_\beta)}\} \; < \; \infty, \; M \in I\!N, \tag{3.4}$$

since the Gaussian measure γ_β has all such moments. The heuristic presentation (3.1) of the measure σ_β originates from a standard approximation used in constructive field theory ([GRS], [Si 1]).

Next, corresponding to each $\xi \in \Omega_\beta$ and each finite $\Lambda \subset Z\!\!\!Z^d$, we define the Gibbs distribution in the volume Λ with boundary condition ξ as the probability measure

$$\begin{aligned} &\nu_{\beta,\Lambda}(d\omega_\Lambda(\cdot)|\xi(\cdot)) \\ = \; &\frac{1}{Z_{\beta,\Lambda}(\xi(\cdot))}\exp\{-I_{\beta,\Lambda}(\omega_\Lambda(\cdot)|\xi(\cdot))\} \times_{k\in\Lambda} d\sigma_\beta(\omega_k(\cdot)) \end{aligned} \tag{3.5}$$

on the loop space

$$\Omega_{\beta,\Lambda} \; := \; C(S_\beta)^\Lambda.$$

Here

$$\omega_\Lambda(\cdot) := (\omega_k(\cdot))_{k\in\Lambda} \in \Omega_{\beta,\Lambda}$$

and

$$Z_{\beta,\Lambda}(\xi(\cdot)) \; := \; \int_{\Omega_{\beta,\Lambda}} \exp\{-I_{\beta,\Lambda}(\omega(\cdot)|\xi(\cdot))\} \times_{k\in\Lambda} d\sigma^\Lambda_\beta(\omega_k(\cdot)) \tag{3.6}$$

is the finite volume partition function. Furtheremore,

$$\begin{aligned} &I_{\beta,\Lambda}(\omega_\Lambda(\cdot)|\xi(\cdot)) : \\ = \; &\frac{J}{2}\int_{S_\beta}\left[\sum_{<k,j>,\; k,j\in\Lambda}(\omega_k(\tau)-\omega_j(\tau))^2 + \sum_{<k,j>,\; k\in\Lambda, j\in\Lambda^c}(\omega_k(\tau)-\xi_j(\tau))^2\right]d\tau \end{aligned} \tag{3.7}$$

is the pair interaction in the volume Λ under the external boundary condition

$$\xi_{\Lambda^c} \; := \; (\xi_j(\cdot))_{j\in\Lambda^c} \in C(S_\beta)^{\Lambda^c}, \; \Lambda^c := Z\!\!\!Z^d \setminus \Lambda.$$

Obviously, $I_{\beta,\Lambda}(\omega_\Lambda(\cdot)|\xi(\cdot))$ is a continuos function with respect to $\xi(\cdot) \in C(S_\beta)^{Z\!\!\!Z^d}$ (or even $\xi(\cdot) \in L^2(S_\beta)^{Z\!\!\!Z^d}$). Below we shall refer to this as the regularity property of $I_{\beta,\Lambda}$ (resp. $\pi_{\beta,\Lambda}$, where the latter is defined in (3.10) below). Heuristically, in (2.6), (2.7),

$$\nu_{\beta,\Lambda}(d\omega_\Lambda(\cdot)|\xi(\cdot)) \;=\; \frac{1}{Z_{\beta,\Lambda}}\exp\{-E_{\beta,\Lambda}(\omega_\Lambda(\cdot)|\xi(\cdot))\}\times_{k\in\Lambda,\tau\in S_\beta} d\omega_k(\tau) \quad (3.8)$$

with the finite volume Euclidean action functional

$$E_{\beta,\Lambda}(\omega_\Lambda(\cdot)|\xi(\cdot)) \;=\; I_{\beta,\Lambda}(\omega_\Lambda(\cdot)|\xi(\cdot))$$
$$+\;\int_{S_\beta}[\frac{m}{2}\sum_{k\in\Lambda}\dot{\omega}_k^2(\tau)+\frac{a^2}{2}\sum_{k\in\Lambda}\omega_k^2(\tau)+\sum_{k\in\Lambda}V(\omega_k(\tau))]d\tau. \qquad (3.9)$$

On the space $(\Omega_\beta,\mathcal{B}(\Omega_\beta))$ it is reasonable to consider the associated family of probability measures

$$\pi_{\beta,\Lambda}(B|\xi(\cdot)) \;:=\; \int_{\Omega_{\beta,\Lambda}} 1\!\!1_B(\omega_\Lambda(\cdot)\times\xi_{\Lambda^c}(\cdot))\nu_{\beta,\Lambda}(d\omega_\Lambda(\cdot)|\xi(\cdot))\;,$$
$$B\in\mathcal{B}(\Omega_\beta),\;\xi(\cdot)\in\Omega_\beta. \qquad (3.10)$$

In fact, each $\pi_{\beta,\Lambda}(d\omega(\cdot)|\xi(\cdot))$ is concentrated on configurations of the form: $\omega(\cdot)=(\omega_\Lambda(\cdot),\xi_{\Lambda^c}(\cdot))$. An essential point is that the stochastic kernels (3.10) set up a family of local Gibbs specifications $\pi_{\beta,\Lambda},\;\Lambda\subset\mathbb{Z}^d,|\Lambda|<\infty$ [Ge], such that the consistency conditions hold. Namely, let $\Lambda\subset\Lambda'$ and $B\in\mathcal{B}(\Omega_\beta)$, then

$$\pi_{\beta,\Lambda'}\pi_{\beta,\Lambda}(B|\xi(\cdot)) \;:=\; \int_{\Omega_\beta}\pi_{\beta,\Lambda'}(d\omega(\cdot)|\xi(\cdot))\pi_{\beta,\Lambda}(B|\omega(\cdot))$$
$$=\;\pi_{\beta,\Lambda'}(B|\xi(\cdot))\;\text{ for all }\;\xi(\cdot)\in\Omega_\beta. \qquad (3.11)$$

Definition 3.1 *A probability measure ν_β on $(\Omega_\beta,\mathcal{B}(\Omega_\beta))$ is called Euclidean Gibbs state in the TLS representation (corresponding to the lattice system (2.1) at inverse temperature β) if and only if it satisfies the Dobrushin–Lanford–Ruelle (DLR) equilibrium equations:*

for all $\Lambda\subset\mathbb{Z}^d$, $|\Lambda|<\infty$, and $B\in\mathcal{B}(\Omega_\beta)$

$$\nu_\beta\pi_{\beta,\Lambda}(B) \;:=\; \int_{\Omega_\beta} d\nu_\beta(\omega(\cdot))\pi_{\beta,\Lambda}(B|\omega(\cdot)) \;=\; \nu_\beta(B),$$
$$i.e.,\quad \nu_\beta\pi_{\beta,\Lambda}=\nu_\beta. \qquad (3.12)$$

In other words, ν_β is a measure on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ such that its conditional expectations with respect to the σ-algebras generated by $\omega \mapsto \omega_k(\cdot)$, $k \in \Lambda^c$, coincide ν_β – a.e. with the prescribed local specifications $\pi_{\beta,\Lambda}$ for all finite $\Lambda \subset \mathbb{Z}^d$.

We denote by $\mathcal{G}_\beta$ the set of all Gibbs measures for our system at a fixed inverse temperature $\beta > 0$. But the definition, as it was given, is too extensive, and the usual way to avoid Gibbs states without physical relevance is to impose some growth conditions at infinity [GRS]. Let $S'(\mathbb{Z}^d)$ be the space of slowly increasing sequences over $\mathbb{Z}^d$, i.e.,

$$S'(\mathbb{Z}^d) \ := \ \text{ind} \lim_{p \in \mathbb{Z}^d_+} l_{2,-p}(\mathbb{Z}^d)$$

where

$$l_{2,-p}(\mathbb{Z}^d) \ := \ \{x \in \mathbb{R}^{\mathbb{Z}^d} | \ ||x||^2_{l_{2,-p}} = \sum_{k \in \mathbb{Z}^d} (1+|k|)^{-p} x_k^2 < \infty\}. \qquad (3.13)$$

We will restrict our considerations to the class $\mathcal{G}^t_\beta$ of tempered Gibbs measures specified by the following assumption on their moment sequence:

$$(m_k)_{k \in \mathbb{Z}^d} \ := \ (< ||\omega_k||_\beta >_{\nu_\beta})_{k \in \mathbb{Z}^d} \in S'(\mathbb{Z}^d). \qquad (3.14)$$

Here and further on we write

$$< f >_{\nu_\beta} = \int_{\Omega_\beta} f(\omega) d\nu_\beta(\omega) \qquad (3.15)$$

for any $f: \ \Omega_\beta \to \mathbb{R}^1$ as soon as the integral makes sense.

The first step in the study of Gibbs measures is to check whether the set $\mathcal{G}^t_\beta$ is nonempty, which is not evident for unbounded spin systems in general. In many cases it can be verified by looking into the relationship between Gibbs measures and thermodynamical limits of local Gibbs specifications π_Λ when Λ tends to $\mathbb{Z}^d$. It is worth noting that, depending on the specific class of quantum lattice models that one considers, the convergence

$$\pi_{\Lambda_n} \to \nu_\beta \in \mathcal{G}^t_\beta, \ \Lambda_n \nearrow \mathbb{Z}^d \ ,$$

can be proved by one or more of the following methods:

1) General criteria for existence of limiting Gibbs distributions (quantum analogues of the existence theorem of R. Dobrushin [Do]) [BaK 1].

2) Superstability estimates (extension of results of D. Ruelle [Ru], J. L. Lebowitz and E. Presutti [LP] on classical spin systems to the quantum case) [PY 1].

3) Method of cluster expansions [AH-K], [KR], [Pa], [PY 2].

4) Method of correlation inequalities [GK], [KS].

5) Reflection positivity method for Gibbs states with periodic boundary conditions [BaK 2,5], [Ko 1,2].

6) Stochastic quantization method [AKT].

Regarding the quantum lattice system with Hamiltonian (2.1), the simplest way to show that $\mathcal{G}^t_\beta \neq \emptyset$ is to take advantage of the fact that the interaction is superstable for all values of the parameters $\beta > 0, a^2 > 0, b^2 \geq 0$ under the assumptions (2.2) – (2.4). As it was shown in [PY 1], for a wide class of boundary conditions $\xi(\cdot) \in \Omega_\beta$ (for instance, such that $\sup_k ||\xi_k||_\beta < \infty$) the associated family of distributions $\pi_\Lambda(d\omega|\xi)$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, is tight in the sense of local weak convergence on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ and has at least one accumulating point $\nu_\beta \in \mathcal{G}^t_\beta$. An analogous result can also be proved using the more involved technique of stochastic quantization [AKT]. Let us mention that also the method of reflection positivity is applicable to the systems with Hamiltonian (2.1). This method gives not only existence of so-called periodic Gibbs states but also produces a-priori estimates for corresponding functional integrals [BaK, 2,5]. Moreover, the reflection positivity method can be used in the study of phase transitions in the models discussed, see [BaK 3,4], [DLP], [PK].

The next open problem is to give a topological description of the set of quantum Gibbs measures as it was done for the classical case in [COPP], [BH-K]. (So far, some information is available only about a subclass $\mathcal{G}^0_\beta \subset \mathcal{G}^t_\beta$ with an additional regularity assumption on the support of ν_β [PY 1].) We do not concentrate on this problem since the purpose of our paper is to provide a sufficient condition for $|\mathcal{G}^t_\beta| = 1$.

4 Uniqueness theorem.

In this section we present the central result of this paper.

Theorem 4.1 *Suppose that the parameters of the system (2.1) (resp. its rigorous implementation through (3.5), (3.10) and Definition 3.1) satisfy the relation*

$$\frac{e^{\beta\delta(W)}}{2d + J^{-1}(a^2 + b^2)} \; < \; \frac{1}{2d} \; , \tag{4.1}$$

then the set $\mathcal{G}^t_\beta$ of tempered Gibbs measures coinsists of exactly one point.

The proof of the above theorem will be based on a well-known general uniqueness criterion by R.L. Dobrushin. In order to formulate it, we first need to introduce the notion of Vasershtein distance between probability distributions.

Let $(\mathcal{X}, \rho)$ be a complete separable metric space, $\mathcal{M}$ the space of all probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and

$$\mathcal{M}_1 = \{\mu \in \mathcal{M} | \int_{\mathcal{X}} \rho(x, x_0)\mu(dx) < \infty\}$$

for some (or equivalently all) $x_0 \in \mathcal{X}$.

The Vasershtein distance R is defined for two measures $\mu_1, \; \mu_2 \; \in \; \mathcal{M}_1$ by

$$R(\mu_1, \mu_2) \; := \; \inf_{\varphi_1, \varphi_2} E\rho(\varphi_1, \varphi_2) \; , \tag{4.2}$$

where the infimum is taken over all pairs of $\mathcal{X}$ - valued random variables φ_1, φ_2 with distributions μ_1 and μ_2 respectively. A definition for the Vasershtein distance R equivalent to (4.2) is given by the Kantorovich-Rubinstein dual relation. Namely, for any $\mu_1, \mu_2 \in \mathcal{M}_1$

$$R(\mu_1, \mu_2) \; = \; \sup_{f \in Lip_1(\mathcal{X})} \left| \int_{\mathcal{X}} f d\mu_1 - \int_{\mathcal{X}} f d\mu_2 \right| \tag{4.3}$$

where

$$Lip_1(\mathcal{X}) \; := \; \{f : \mathcal{X} \to I\!R^1 | [f]_{Lip} \; := \; \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)} \; \leq 1\} \; . \tag{4.4}$$

Moreover, $(\mathcal{M}_1, R)$ is a complete metric space and the following characterization of convergence in the Vasershtein metric holds: $R(\mu_n, \mu) \to 0, \; n \to \infty$, if and only if $\mu_n \to \mu$ in the topology of weak convergence and

$$\int_{\mathcal{X}} \rho(x, x_0)(\mu_n - \mu)(dx) \to 0, \; n \to \infty \; .$$

For more details on the Vasershtein and other probability metrics see e.g. the reviews [Du], [Ra].

Let us recall now the definition of the Dobrushin matrix. Let π_Λ, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, be a family of local Gibbs specifications with spin space $\mathcal{X}$ that satisfies the standard regularity and consistency assumptions [Do], [Ge]. In fact, we should only look at the one-dimensional Gibbs distributions $\nu_k(dx_k|\xi)$ in volumes $\Lambda_k = \{k\}$, $k \in \mathbb{Z}^d$, with boundary conditions $\xi = (\xi_j)_{j\in\mathbb{Z}^d} \in \mathcal{X}^{\mathbb{Z}^d}$, which are given by

$$\nu_k(B|\xi) \ := \ \pi_{\{k\}}(\{x \in \mathcal{X}^{\mathbb{Z}^d}, x_k \in B\}|\xi), \ B \in \mathcal{B}(\mathcal{X}) \ . \tag{4.5}$$

For every $k, j \in \mathbb{Z}^d$, $k \neq j$, we are interested in the dependence of the measure $\nu_k(dx_k|\xi)$ on the values of the boundary condition ξ at site j. This can be described by the Dobrushin matrix $C = (C_{kj})_{k,j\in\mathbb{Z}^d}$, where

$$C_{kj} \ := \ \sup_{\substack{\xi,\eta \,\in\, \mathcal{X}^{\mathbb{Z}^d} \\ \forall i \neq j \,:\, \xi_i = \eta_i}} \left\{ \frac{R(\nu_k(\cdot|\xi), \nu_k(\cdot|\eta))}{\rho(\xi_j, \eta_j)} \right\} \ . \tag{4.6}$$

Proposition 4.1 *(Dobrushin's uniqueness criterion [Do], [DS], [DSK], [Fö], [Kü]).*

Suppose that the following condition of weak dependence holds

$$\sup_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d,\, j\neq k} C_{kj} \ < \ 1 \ , \tag{4.7}$$

then there is at most one tempered Gibbs measure ν (where tempered means that $(m_k)_{k\in\mathbb{Z}^d} = (<\rho(x_k,0)>_\nu)_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d)$) for the prescribed specifications $\pi_{\{k\}}, k \in \mathbb{Z}^d$.

Now we are in a position to turn to

Proof of Theorem 4.1. The existence problem has been discussed in Section 2. As for uniqueness, we will prove even the more general fact that there exists not more than one Gibbs measure ν_β on the configuration space $\Omega'_\beta \ := \ (L^2(S_\beta))^{\mathbb{Z}^d} \supset \Omega_\beta$, which satisfies the growth condition (3.14). Note that the regularity property of the specifications allows us to define the extension of the family (3.10) of local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d, |\Lambda| < \infty$ to the new spin space $\mathcal{X} = \mathcal{H}_\beta$.

Due to the translation invariance of the system and the nearest–neighbours interaction, it will be sufficient to obtain the following estimate on the Dobrushin coefficients C_{kj}, $|k-j|=1$:

$$C_{kj} \ := \sup_{\substack{\xi,\eta\,\in\,\Omega'_\beta \\ \forall i \neq j\,:\ \xi_i = \eta_i}} \frac{R(\nu_{\beta,k}(\cdot|\xi), \nu_{\beta,k}(\cdot|\eta))}{||\xi_j-\eta_j||_\beta} \ < \ \frac{1}{2d}\,. \tag{4.8}$$

Returning to the notations (3.5) – (3.7), we observe that

$$\nu_{\beta,k}(d\omega_k|\xi) \ = \ \frac{1}{Z_{\beta,k}(\xi)}\exp\{-Jd||\omega_k||_\beta^2 + J\sum_{|i-k|=1} < \omega_k, \xi_i >_\beta\}d\sigma_\beta(\omega_k), \tag{4.9}$$

where as in (3.6) $Z_{\beta,k}(\xi)$ is a normalizing factor. This measure depends on the boundary conditions only through the value $x_k = \sum_{|i-k|=1}\xi_i \in \mathcal{H}_\beta$. Hence we have an equivalent representation for the Dobrushin coefficients

$$C_{kj} \ = \sup_{\substack{x,\,y\,\in\,\mathcal{H}_\beta \\ x\neq y}} \frac{R(\nu^x(\cdot),\nu^y(\cdot))}{||x-y||_\beta} \tag{4.10}$$

where $\nu^x(d\omega)$ is a family of probability measures on $(\mathcal{H}_\beta, \mathcal{B}(\mathcal{H}_\beta))$ parametrized by $x \in \mathcal{H}_\beta$ and given by

$$d\nu^x(\omega) \ = \ \frac{1}{Z_x}\exp\{-Jd||\omega||_\beta^2 + J < \omega, x >_\beta\}d\sigma_\beta(\omega)\ ,$$

$$Z_x \ = \ \int_{\mathcal{H}_\beta}\exp\{-Jd||\omega||_\beta^2 + J < \omega, x >_\beta\}d\sigma_\beta(\omega)\ . \tag{4.11}$$

Consider the mapping

$$\mathcal{H}_\beta \ni x \mapsto < f >_{\nu^x} := \int_{\mathcal{H}_\beta} f(\omega)d\nu^x(\omega) \in I\!R^1 \tag{4.12}$$

with fixed $f \in Lip_1(\mathcal{H}_\beta)$. Because of (3.4) this mapping is Fréchet differentiable and the corresponding derivative in direction $\varphi \in \mathcal{H}_\beta$ can be written as

$$< \nabla_x < f >_{\nu^x}, \varphi >_\beta =$$

$$J(< f < \cdot, \varphi >_\beta >_{\nu^x} - < f >_{\nu^x} << \cdot, \varphi >_\beta >_{\nu^x}) \tag{4.13}$$
$$= J\, Cov_{\nu^x}(f, < \cdot, \varphi >_\beta) \; . \tag{4.14}$$

Thus,

$$| < \nabla_x < f >_{\nu^x}, \varphi >_\beta | \;\leq\; J(Var_{\nu^x} f)^{1/2} (Var_{\nu^x} < \cdot, \varphi >_\beta)^{1/2} \tag{4.15}$$

where as usual

$$Var_{\nu^x} f \;:=\; \int_{\mathcal{H}_\beta} (f - < f >_{\nu^x})^2 d\nu^x \tag{4.16}$$

denotes the variance of the function f with respect to the measure ν^x.

Now we are at the crucial point in the proof. As a direct consequence of the logarithmic Sobolev (LS) inequality for the measures ν^x (to be proved in Section 5; cf. Theorem 5.1) we know by Corrolary 5.2 below that

$$Var_{\nu^x} f \leq C \;:=\; \frac{e^{\beta\delta(W)}}{2Jd + a^2 + b^2} \; . \tag{4.17}$$

Hence (4.15) implies that

$$| < \nabla_x < f >_{\nu^x}, \varphi >_\beta | \;\leq\; JC ||\varphi||_\beta, \; \varphi \in \mathcal{H}_\beta \; ,$$

which by the mean-value theorem implies

$$\left| \int_{\mathcal{H}_\beta} f d\nu^x - \int_{\mathcal{H}_\beta} f d\nu^y \right| \;\leq\; JC ||x - y||_\beta, \; f \in Lip_1(\mathcal{H}_\beta) \; , \tag{4.18}$$

or, in terms of the Vasershtein distance,

$$R(\nu^x(\cdot), \nu^y(\cdot)) \;\leq\; JC ||x - y||_\beta \; . \tag{4.19}$$

By (4.10) and the assumption (4.1) we obtain the desired estimate (4.8) which, as explained above, implies

$$|\mathcal{G}_\beta^t| \;=\; 1 \; .$$

and the proof is complete. □

5 Log–Sobolev inequality for one–particle temperature measures.

Let us first refine some notations of Sections 2 – 4. The function V_0 (as a convex function on $\mathbb{R}^1$, see (2.2), (2.3)) has the following decomposition

$$V_0(q) = \frac{b^2 q^2}{2} + \lambda q + \tilde{V}_0(q), \quad q \in \mathbb{R}^1, \tag{5.1}$$

where $\tilde{V}_0$ is a convex function: $\tilde{V}''_0(q) \geq 0, \ q \in \mathbb{R}^1$, semibounded from below. Having in mind this decomposition we can extract the quadratic potential from V_0 (that produces an obvious corresponding change of the measure γ_β). Analogously, the linear term in V_0 can be included into the boundary condition in (4.11). At last, all our considerations will be uniform with respect to the mass parameter $m > 0$. So, we put for a simplicity the notations $m = 1$. As a result, the subject of our study will be the one–particle measure ν_β on the spin space $C(S_\beta) \subset L^2(S_\beta) := \mathcal{H}_\beta$, which has a heuristic representation

$$\begin{aligned} d\nu_\beta(\omega(\cdot)) \ &= \ \frac{1}{Z_\beta} \exp\{-\int_{S_\beta} [\frac{1}{2}\dot{\omega}^2(\tau) + \frac{a^2}{2}\omega^2(\tau) - \xi(\tau)\omega(\tau) \\ &+ \ V_0(\omega(\tau)) + W(\omega(\tau))] d\tau\} \times_{\tau \in S_\beta} d\omega(\tau) \end{aligned} \tag{5.2}$$

or the rigorous one

$$d\nu_\beta(\omega(\cdot)) \ = \ \frac{e^{<\xi,\omega>_\beta}}{Z_\beta} \exp\{-\int_{S_\beta} (V_0 + W)(\omega(\tau)) d\tau\} d\gamma_\beta(\omega(\cdot)) \tag{5.3}$$

where γ_β is a normal disribution in $\mathcal{H}_\beta$ with covariance operator $C_\beta = (-\triangle_\beta + a^2 1\!\!1)^{-1}$ and $V_0 \in C^2(\mathbb{R}^1), \ V_0'' \geq 0$, is polynomially bounded up to its second derivative, and semibounded from below (since we have extracted the quadratic and linear terms from V_0). Here $\xi \in \mathcal{H}_\beta$ is any fixed element and we do not stress the dependence on ξ in our notations. Note that ν^x defined in (4.11) obviously can be rewritten as (5.3). Let us mention that the parameter a^2 in (5.2) should be expressed in terms of the parameters of the measure (4.11) as $a^2 + b^2 + 2Jd$.

Next we introduce the spaces of smooth functions on $\mathcal{H}_\beta$ which will be needed below. We define the space of k times, $k \in \mathbb{N} \cup \{0, \infty\}$, continuously differentiable cylinder functions on $\mathcal{H}_\beta$

$$\mathcal{F}C^k(\mathcal{H}_\beta) := \{u : \mathcal{H}_\beta \to I\!R^1 | u(\omega) = u_n(<\omega, l_1>_\beta, \ldots, <\omega, l_n>_\beta),$$
$$u_n \in C^k(I\!R^n), \;\; l_1, \ldots, l_n \in \mathcal{H}_\beta, \;\; n \in I\!N\} \tag{5.4}$$

and its subspace

$$\mathcal{F}C^k_b(\mathcal{H}_\beta) := \{u : \mathcal{H}_\beta \to I\!R^1 | u(\omega) = u_n(<\omega, l_1>_\beta, \ldots, <\omega, l_n>_\beta),$$
$$u_n \in C^k_b(I\!R^n), l_1, \ldots, l_n \in \mathcal{H}_\beta, n \in I\!N\}. \tag{5.5}$$

We shall use the symbol $\nabla u(\omega) \in \mathcal{H}_\beta$ for the gradient realization of the derivative $u'(\omega) \in \mathcal{L}(\mathcal{H}_\beta, I\!R^1)$.

On the domain $\mathcal{F}C^\infty_b(\mathcal{H}_\beta)$ we define the classical pre-Dirichlet form associated with the measure ν_β:

$$\mathcal{E}_{\nu_\beta}(u, v) := \int_{\mathcal{H}_\beta} < \nabla u(\omega), \nabla v(\omega) >_\beta d\nu_\beta(\omega) \; . \tag{5.6}$$

Clearly, $(\mathcal{E}_{\nu_\beta}, \mathcal{F}C^\infty_\beta(\mathcal{H}_\beta))$ is closable on $L^2(\nu_\beta) := L^2(\mathcal{H}_\beta, \nu_\beta)$ [AR 1]. We denote its closure by $(\mathcal{E}_{\nu_\beta}, \mathcal{D}(\mathcal{E}_{\nu_\beta}))$. The extension of ∇ to $\mathcal{D}(\mathcal{E}_{\nu_\beta})$ will again be denoted by ∇. For a more detailed discussion of infinite–dimensional Dirichlet forms and additional references see e.g. [AKR].

Theorem 5.1 *(Log–Sobolev inequality).*
For all $u \in \mathcal{F}C^\infty_b(\mathcal{H}_\beta)$ the following inequality is true

$$\int_{\mathcal{H}_\beta} |u|^2 \log |u| d\nu_\beta \; \le \; C_{LS} \int_{\mathcal{H}_\beta} ||\nabla u||^2_\beta d\nu_\beta + ||u||^2_{L^2(\nu_\beta)} \log ||u||_{L^2(\nu_\beta)} \tag{5.7}$$

with the Sobolev coefficient

$$C_{LS} \; = \; \frac{1}{a^2} e^{\beta \delta(W)} \; . \tag{5.8}$$

Corollary 5.1 *(Rothaus–Simon mass gap theorem [Ro], [Si 3], [Gr 2]).*
For all $u \in \mathcal{F}C^\infty_b(\mathcal{H}_\beta)$, such that $u \perp 1$ in $L^2(\nu_\beta)$, we have

$$\mathcal{E}_{\nu_\beta}(u, u) \; \ge \; \frac{1}{C_{LS}} ||u||^2_{L^2(\nu_\beta)} \; . \tag{5.9}$$

Corollary 5.2 *(Variation estimate).*

For all $f \in Lip(\mathcal{H}_\beta)$ the following inequality holds:

$$Var_{\nu_\beta} f \;:=\; \int (f - < f >_{\nu_\beta})^2 d\nu_\beta \;\leq\; C_{LS}[f]^2_{Lip}\,. \tag{5.10}$$

Proof of Corollary 5.2. We give two different proofs that (5.9) implies (5.10). We first note that clearly $Lip(\mathcal{H}_\beta) \subset L^2(\nu_\beta)$.

Proof 1. It is clear that (5.9) extends to all $\mathcal{D}(\mathcal{E}_{\nu_\beta})$. So, it suffices to show that for every $f \in Lip(\mathcal{H}_\beta)$ we have that $f \in \mathcal{D}(\mathcal{E}_{\nu_\beta})$ with

$$\int \|\nabla f\|_\beta d\nu_\beta \leq [f]^2_{Lip}. \tag{5.11}$$

But this is exactly the assertion in [Ku, Lemma 1.3] stated, however, there for a Dirichlet form $(\mathcal{E}^+_{\nu_\beta}, \mathcal{D}(\mathcal{E}^+_{\nu_\beta}))$ (and a corresponding gradient D) for which by definition a priori $\mathcal{D}(\mathcal{E}^+_{\nu_\beta}) \supset \mathcal{D}(\mathcal{E}_{\nu_\beta})$. To see that in fact $\mathcal{D}(\mathcal{E}_{\nu_\beta}) = \mathcal{D}(\mathcal{E}^+_{\nu_\beta})$ in our situation, we may obviously assume that $W \equiv 0$. Denoting the density of ν_β w.r.t. γ_β in (5.3) by φ^2 in this case, we see that condition (2.12) in [RZ, Proposition 2.8] is fulfilled. (Note that $H^{1,2}(E;K;\varphi^2\mu)$ in [RZ, Section 2] with $E = H = \mathcal{H}_\beta$ and $\mu = \gamma_\beta$ really coincides with $D(\mathcal{E}^+_{\nu_\beta})$ by [AR 2, Proposition 3.2]). Hence $D(\mathcal{E}_{\nu_\beta}) = D(\mathcal{E}^+_{\nu_\beta})$ by [RZ, Theorem 2.3].

□

Proof 2. We start with the spectral gap inequality (5.9) which can be written (after an obvious approximation) as

$$\begin{aligned}\int_{\mathcal{H}_\beta} (u - < u >_{\nu_\beta})^2 d\nu_\beta \;&=\; \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} (u(\omega_1) - u(\omega_2))^2 d\nu_\beta(\omega_1) d\nu_\beta(\omega_2) \\ &\leq\; C_{LS}\int_{\mathcal{H}_\beta} \|\nabla u\|^2_\beta d\nu_\beta\end{aligned} \tag{5.12}$$

for any $u \in C^1_b(\mathcal{H}_\beta)$. Let further $v \in C^1(\mathcal{H}_\beta) \cap Lip(\mathcal{H}_\beta)$. We introduce a sequence of scalar functions

$$[0, +\infty) \ni s \mapsto \Theta_N(s) \in [0,1],\;\; \Theta_N \in C^\infty_b(\mathbb{R}^1),\;\; N \in \mathbb{N},$$

having the properties

$$\Theta_N(s) = 1,\; s \in [0,N];\; \Theta_N(s) = 0,\; s \geq N+1; \tag{5.13}$$

$$\Theta_{N+1}(s+1) = \Theta_N(s),\; s \geq 0.$$

Then $u_N(\cdot) := v(\cdot)\Theta_N(\|\cdot\|_\beta) \in C_b^1(\mathcal{H}_\beta),\ N \in I\!N$. Using the estimates

$$\begin{aligned} |v(\omega)| &\le K(1+\|\omega\|_\beta), \\ \int \|\omega\|_\beta^2 d\nu_\beta &< \infty, \end{aligned} \tag{5.14}$$

it is easy to see that

$$||v-u_N||_{L^2(\nu_\beta)} \to 0,\ \int_{\mathcal{H}_\beta} \|\nabla(v-u_N)\|_\beta^2 d\nu_\beta \to 0,\ N\to\infty\ ,$$

and hence (5.12) holds for any $v \in C^1(\mathcal{H}_\beta)\cap Lip(\mathcal{H}_\beta)$. Thus

$$Var_{\nu_\beta} v \le C_{LS} \sup_{\omega\in\mathcal{H}_\beta} ||\nabla v(\omega)||_\beta^2 = C_{LS}[v]^2_{Lip(\mathcal{H}_\beta)}\ . \tag{5.15}$$

The latter equality in (5.15) is well-known.

The idea how to prove (5.10) for a general $f \in Lip(\mathcal{H}_\beta)$ is to construct a proper approximation by $(v_M)_{M\in I\!N} \subset \mathcal{F}C^\infty(\mathcal{H}_\beta)\cap Lip(\mathcal{H}_\beta)$ and to pass to the limit in (5.15) using Lebesgue's dominated convergence theorem.

Let $(l_j)_{j=1}^\infty$ be an orthogonal basis in $\mathcal{H}_\beta$ and $(P_n)_{n\in I\!N}$ be the corresponding sequence of projectors on the finite-dimensional subspaces generated by the vectors $(l_j)_{j=1}^n$. We define

$$f_n(<\omega,l_1>_\beta,\ldots,<\omega,l_n>_\beta) := f(P_n\omega),\ n\in I\!N\ ;$$

It is obvious that

$$[f_n]_{Lip(I\!R^n)} = [f(P_n(\cdot))]_{Lip(\mathcal{H}_\beta)} \le [f]_{Lip(\mathcal{H}_\beta)}$$

and

$$||f(P_n(\cdot))-f||_{L^2(\nu_\beta)} \to 0,\ \ n\to\infty. \tag{5.16}$$

So, the problem is reduced to the proof of (5.10) for cylinder Lipschitz functions on $\mathcal{H}_\beta$. But for every Lipschitz function $g(x_1,\ldots,x_n)$ on $I\!R^n$ there exists a sequence $(g_M)_{M\in I\!N} \subset C^\infty(I\!R^n)\cap Lip(I\!R^n)$ such that

$$[g_M]_{Lip(I\!R^n)} \le [g]_{Lip(I\!R^n)}$$

and

$$g_M \to g,\ M\to\infty\ , \tag{5.17}$$

pointwise on $I\!R^n$. (Here we can e.g. use general properties of approximation by convolutions $g_M = \rho_M * g$ in Hölder spaces, see for instance [Tr]). And finally, based on (5.16) and (5.17), the sequence

$$v_M(\omega) \ = \ g_M(<\omega, l_1>_\beta, \ldots, <\omega, l_n>_\beta), \ M \in I\!N \ ,$$

gives us the desired approximation of

$$f(P_n\omega) \ = \ g(<\omega, l_1>_\beta, \ldots, <\omega, l_n>_\beta).$$

$\square$

Proof of Theorem 5.1. Let us consider first of all the case $W \equiv 0$. We introduce a new Dirichlet form $\mathcal{E}_{\nu_\beta, C_\beta}$ on $\mathcal{F}C_b^\infty(\mathcal{H}_\beta)$ with the coefficient operator $C_\beta = A_\beta^{-1}$ by the formula

$$\mathcal{E}_{\nu_\beta, C_\beta}(u,v) \ = \ \int_{C(S_\beta)} < C_\beta \nabla u, \nabla v >_\beta \, d\nu_\beta \ \ (u, v \in \mathcal{F}C_b^\infty(\mathcal{H}_\beta)).$$

Due to the estimate $A_\beta \geq a^2 1\!\!1$ we have

$$\mathcal{E}_{\nu_\beta, C_\beta}(u,v) \ \leq \ \frac{1}{a^2}\mathcal{E}_{\nu_\beta}(u,v) \ \ (u, v \in \mathcal{F}C_b^\infty(\mathcal{H}_\beta)). \tag{5.18}$$

Let us denote by $\mathcal{H}_\beta^\pm$ the Hilbert spaces defined as the closure of $C^\infty(S_\beta)$ in the scalar products $<\varphi, \psi>_{\pm\beta} = <A_\beta^{\pm 1}\varphi, \psi>_\beta$ (and norms $\| \cdot \|_\pm$) respectively. Obviously, $\mathcal{H}_\beta^\pm$ are isomorphic to the Sobolev spaces $W_2^{\pm 1}(S_\beta)$. So, as a result we have the following rigging

$$\mathcal{H}_\beta^- \ \supset \ \mathcal{H}_\beta \ \supset \ \mathcal{H}_\beta^+ \ , \tag{5.19}$$

where both embeddings belong to the Hilbert–Schmidt class because the operator $C_\beta = (-\triangle_\beta + a^2)^{-1}$ is a trace class operator in $\mathcal{H}_\beta$. Note that the operators $A_\beta^{\pm 1/2}$ generate natural isometric mappings

$$A_\beta^{\pm 1/2} : \mathcal{H}_\beta^\pm \to \mathcal{H}_\beta, \quad A_\beta^{\pm 1/2} : \mathcal{H}_\beta \to \mathcal{H}_\beta^\mp.$$

Let γ_β^- be the canonical Gaussian measure corresponding to the space $\mathcal{H}_\beta$, i.e., a measure on $\mathcal{B}(\mathcal{H}_\beta^-)$ with the Fourier transform

$$\int_{\mathcal{H}_\beta^-} e^{i<\varphi,x>_\beta} d\gamma_\beta^-(x) \;=\; e^{-\frac{1}{2}<\varphi,\varphi>_\beta}, \;\; \varphi \in \mathcal{H}_\beta^+ \;.$$

Due to Minlos theorem the measure γ_β^- is uniquely given by this formula.

Using (5.3) we write

$$\mathcal{E}_{\nu_\beta,C_\beta}(u,v) \;=\; \int_{C(S_\beta)} < A_\beta^{-1}\nabla u(\omega), \nabla v(\omega) >_\beta \qquad (5.20)$$

$$\times \; \frac{1}{Z_\beta} e^{-\int_0^\beta V_0(\omega(\tau))d\tau + <\xi,\omega>_\beta} d\gamma_\beta(\omega) \;.$$

Let us consider the mapping

$$\mathcal{H}_\beta \supset C(S_\beta) \ni \omega \;\mapsto\; A_\beta^{1/2}\omega = x \in \mathcal{H}_\beta^- \qquad (5.21)$$

which transforms the measure γ_β in γ_β^-, see e.g. [BK, Ch.2]. Then for any $l \in \mathcal{H}_\beta$ the linear functional $< l, \omega >_\beta, \omega \in \mathcal{H}_\beta$, turns into

$$< l,\omega >_\beta = < l, A^{-1/2}x >_\beta = < A^{-1/2}l, x >_\beta = < \hat{l}, x >_\beta, \;\; x \in \mathcal{H}_\beta^- \;,$$

with $\hat{l} = A^{-1/2}l \in \mathcal{H}_\beta^+$. This means that the change of variables (5.21) transforms the spaces of cylinder functions

$$\mathcal{F}C_b^\infty(\mathcal{H}_\beta) \ni u(\omega) \;\mapsto\; \hat{u}(x) = u(A_\beta^{-1/2}x) \in \mathcal{F}C_b^\infty(\mathcal{H}_\beta^+)$$

where

$$\mathcal{F}C_b^\infty(\mathcal{H}_\beta^+) \;:=\; \{\hat{u} : \mathcal{H}_\beta^- \to I\!R^1 | \hat{u}(x) = u_n(< x, \hat{l}_1 >_\beta, \ldots, < x, \hat{l}_n >_\beta) \;,$$

$$u_n \in C_b^\infty(I\!R^n), \; \hat{l}_1, \ldots, \hat{l}_n \in \mathcal{H}_\beta^+, \; n \in I\!N\} \;.$$

Doing so in (5.20) we obtain

$$\mathcal{E}_{\nu_\beta,C_\beta}(u,v) = \int_{\mathcal{H}_\beta^-} < \nabla\hat{u}(x), \nabla\hat{v}(x) >_\beta$$

$$\frac{1}{Z_\beta}\exp\{-\int_0^\beta V_0((A_\beta^{-1/2}x)(\tau))d\tau + < \xi, A_\beta^{-1/2}x >_\beta\} d\gamma_\beta^-$$

$$:= \int_{\mathcal{H}_\beta^-} < \nabla\hat{u}(x), \nabla\hat{v}(x) >_\beta \, d\nu_\beta^-(x)$$

$$= \mathcal{E}_{\nu_\beta^-}(\hat{u},\hat{v}), \quad \hat{u},\hat{v} \in \mathcal{F}C_b^\infty(\mathcal{H}_\beta^+) \; .$$

For the Dirichlet form $\mathcal{E}_{\nu_\beta^-}$ we can prove an LS inequality by using an approximative approach (similar to [AKR, Theorem 4]. But we shall not use the latter below). To this end we write

$$d\nu_\beta^-(x) \;=\; \frac{1}{Z_\beta} e^{-U_0(x)} d\gamma_\beta^-(x) \; ,$$

where $U_0(x) = \int_0^\beta V_0((A_\beta^{-1/2}x)(\tau))d\tau \, - < \xi, A_\beta^{-1/2}x >_\beta$. This function is defined on the set $\{x \in \mathcal{H}_\beta^- | x = A_\beta^{1/2}\omega, \omega \in C(S_\beta)\}$ which has full γ_β^- – measure. The approximation of U_0 by smooth cylinder functions on $\mathcal{H}_\beta^-$ may be constructed in the following way.

We introduce a standard orthonormal basis $\{e_k^\beta \mid k \in \mathbb{Z}\} \subset C^\infty(S_\beta)$ in $\mathcal{H}_\beta$ as follows

$$e_k^\beta(\tau) := \begin{cases} \beta^{-1/2}, & k = 0 \\ (\frac{2}{\beta})^{\frac{1}{2}} \sin \frac{2\pi k\tau}{\beta}, & k > 0 \\ (\frac{2}{\beta})^{\frac{1}{2}} \cos \frac{2\pi k\tau}{\beta}, & k < 0 \end{cases} \qquad , \tau \in S_\beta.$$

For any $N \in \mathbb{N}$ the operator

$$\mathcal{H}_\beta \ni \omega(\cdot) \mapsto (S_N\omega)(\cdot) = \sum_{|k|\le N} \omega_k e_k^\beta(\cdot) \in C^\infty(S_\beta),$$

$$\omega_k :=< \omega, e_k^\beta >_\beta, \;\; k \in \mathbb{Z},$$

is an orthonormal projector in $\mathcal{H}_\beta$. (Note that $S_N\omega$ is nothing but the partial sum for the Fourier series of ω). Since $C^\infty(S_\beta) \subset \mathcal{H}_\beta^+$, this projector has an obvious extension to $\mathcal{H}_\beta^-$:

$$\mathcal{H}_\beta^- \ni x \mapsto S_N x \in \mathcal{H}_N := S_N \mathcal{H}_\beta$$

and it will be useful to have the following isomorphysm in mind:

$$\mathcal{H}_N \ni (S_N x)(\cdot) = \sum_{|k|\le N} x_k e_k^\beta(\cdot) \mapsto x_{(N)} := (x_k)_{k=-N}^{k=N} \in \mathbb{R}^{2N+1}.$$

For any $N \in \mathbb{N}$ we put

$$U_N(x) := U_0(S_N x) = \int_0^\beta V_0(\sum_{|k|\le N} \lambda_k^{-\frac{1}{2}} x_k e_k^\beta(\tau))d\tau - < A_\beta^{-1/2} S_N x, \xi >_\beta, \tag{5.22}$$

where $\lambda_k = a^2 + (2\pi k)^2 \beta^{-2}$ (such that $A_\beta e_k^\beta = \lambda_k e_k^\beta,\ k \in \mathbb{Z}$). We introduce a sequence of probability measures on $\mathcal{H}_\beta^-$ in the following way:

$$d\mu_N(x) = \frac{1}{Z_N} e^{-U_N(x)} d\gamma_\beta^-(x), \quad N \in \mathbb{N}.$$

First of all we need to show weak convergence $\mu_N \to \nu_\beta^-, N \to \infty$. To this end we consider the function $U_N(x),\ x \in \mathcal{H}_\beta^-$, for $x = A_\beta^{-1/2}\omega$ with $\omega \in H^{1/4}(S_\beta)$, where $H^{1/4}(S_\beta)$ denotes the space of all Hölder continuous functions on S_β with Hölder-exponent $1/4$. It is well-known that $\gamma_\beta(H^{1/4}(S_\beta)) = 1$ (as well as for any Hölder-exponent less than $1/2$) and, therefore, the points x under consideration form a set of full measure for γ_β^-, see e.g. [Si 2]. For any such x we have

$$U_N(x) = \int_0^\beta V_0((S_N\omega)(\tau))d\tau - \int_0^\beta \xi(\tau)(S_N\omega)(\tau)d\tau \to U_0(x),$$

as $N \to \infty$ because the sequence $S_N\omega$ converges uniformly on S_β to ω as $N \to \infty$ (due to the Dini test). This implies the convergence

$$\exp(-U_n(x)) \ \to \exp(-U_0(x)), \quad N \to \infty,$$

for γ_β^--a.a. $x \in \mathcal{H}_\beta^-$. To be able to apply Lebesgue's dominated convergence theorem to this sequence, we construct an integrable majorant as follows.

Using the semiboundedness from below of V_0 we have

$$0 \le e^{-U_N(x)} \le C_{V_0} \exp(|< A_\beta^{-1/2} S_N x, \xi >_\beta|) \le C_{V_0} \exp(\|S_N x\|_{-\beta} \|\xi\|_\beta)$$
$$\le C_{V_0} \exp(\|\xi\|_\beta \|x\|_{-\beta}), \quad x \in \mathcal{H}_\beta^-, \quad N \in \mathbb{N}.$$

The latter function is integrable with respect to γ_β^- for any $\xi \in \mathcal{H}_\beta$ due to the trace class property of the operator A_β^{-1} in $\mathcal{H}_\beta$, see e.g. [BK,Ch.2]. Then the required convergence of the measures $\mu_N, N \in \mathbb{N}$ follows.

Having chosen our orthonormal basis in $\mathcal{H}_\beta$ we obtain the isomorphism

$$\mathcal{H}_\beta^- \ni x(\cdot) \mapsto \mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in l_2(\mathbb{Z}, \lambda^{-1}) := \{\mathbf{x} | \sum_k x_k^2 \lambda_k^{-1} < \infty\}.$$

The measure γ_β^- is transformed under this isomorphism into a Gaussian product-measure on $l_2(\mathbb{Z}, \lambda^{-1})$ of the form

$$d\gamma(\mathbf{x}) = \times_{k\in\mathbb{Z}} d\gamma(x_k), \ d\gamma(x_k) = \frac{1}{2\pi} e^{-\frac{x_k^2}{2}} dx_k,$$

see e.g. [BeK, Ch.2]. We shall also use the space $\mathcal{F}C_b^\infty := \mathcal{F}C_b^\infty(span\{e_k^\beta | k \in \mathbb{Z}\})$ which is defined as before (replacing $\mathcal{H}_\beta^+$ by $span\{e_k^\beta | k \in \mathbb{Z}\})$). Any function from $\mathcal{F}C_b^\infty$ depends on a finite number of coordinates $x_k, k \in \mathbb{Z}$, and this space is dense in $L^2(\mu)$ for any probability measure μ on $\mathcal{B}(\mathcal{H}_\beta^-)$, see e.g. [BeK, Ch.2].

The measures μ_N in coordinate representation can be written as

$$d\mu_N(\mathbf{x}) = d\chi_N(x_{(N)})\ d\gamma(x^{(N)}),$$

where $x_{(N)} = (x_k)_{|k|\le N}$, $x^{(N)} = (x_k)_{|k|>N}$,

$$d\gamma(x^{(N)}) = \times_{|k|>N} d\gamma(x_k),$$

and

$$d\chi_N(x_{(N)}) = \frac{1}{Z_N}\exp(-U_N(x_{(N)}))d\gamma(x_{(N)}) \tag{5.23}$$

with $U_{(N)}(x_{(N)}) \in C^2(\mathbb{R}^{2N+1})$ given by (5.22). This function is convex on $\mathbb{R}^{2N+1}$ which implies via the Bakry-Emery criterion the following LS inequality for the measure $d\chi_N$: for all $f \in C_b^1(\mathbb{R}^{2N+1})$

$$\int |f(x_{(N)}|^2 \log |f(x_{(N)}| d\chi_N(x_{(N)}) \ \le \tag{5.24}$$

$$\int < \nabla f(x_{(N)}), \nabla f(x_{(N)}) >_{\mathbb{R}^{2N+1}}\ d\chi_N(x_{(N)}) + \|f\|^2_{L^2(\chi_N)} \log \|f\|_{L^2(\chi_N)},$$

see [DeS, Th. 6.2.42]. The same LS inequality we have for the Gaussian measure $d\gamma(x^{(N)})$ on all cylinder functions from $\mathcal{F}C_b^\infty$ which depend on $x^{(N)}$ (so-called Gaussian LS inequality, see [Gr 2]). Using the additivity property of the LS inequalities [Gr 2] we obtain (coming back to the functional integral representation) the following LS inequality

$$\int_{\mathcal{H}_\beta^-} |\hat{u}(x)|^2 \log |\hat{u}(x)| d\mu_N(x)$$

$$\le \int < \nabla \hat{u}(x), \nabla \hat{u}(x) >_\beta\ d\mu_N(x) + ||\hat{u}||^2_{L^2(\mu_N)} \log ||\hat{u}||_{L^2(\mu_N)}$$

$$(\hat{u} \in \mathcal{F}C_b^\infty)\ .$$

Now as in [AKR] we can use the weak convergence of the measures μ_N, $N \in \mathbb{N}$ to ν_β^- in the latter inequality. As a result we have the LS inequality for the Dirichlet form $\mathcal{E}_{\nu_\beta^-}$ with LS constant equal to 1. Namely,

$$\int_{\mathcal{H}_\beta^-} |\hat{u}(x)|^2 \log |\hat{u}(x)| d\nu_\beta^-(x)$$

$$\leq \int < \nabla \hat{u}(x), \nabla \hat{u}(x) >_\beta d\nu_\beta^-(x) + ||\hat{u}||^2_{L^2(\nu_\beta^-)} \log ||\hat{u}||_{L^2(\nu_\beta^-)}$$

$$(\hat{u} \in \mathcal{F}C_b^\infty) \ .$$

Producing the inverse change of variables (that preserves $\mathcal{F}C_b^\infty$) and using a standard approximation by cylinder functions we come to

$$\int_{C(S_\beta)} |u(\omega)|^2 \log |u(\omega)| d\nu_\beta(\omega)$$

$$\leq \mathcal{E}_{\nu_\beta, C_\beta}(u, u) + ||u||^2_{L^2(\nu_\beta)} \log ||u||_{L^2(\nu_\beta)}$$

$$(u \in C_b^1(\mathcal{H}_\beta))$$

which together with (5.18) proves the theorem in the case $W = 0$.

The statement of the theorem for general potentials W from the considered class follows from the trivial estimate

$$\beta \min_{q \in \mathbb{R}^1} W(q) \ \leq \ \int_0^\beta W(\omega(\tau)) d\tau \ \leq \ \beta \max_{q \in \mathbb{R}^1} W(q), \ (\omega \in C(S_\beta))$$

and a simple perturbation theorem for the LS constant, see [St].

□

Remark 5.1 *(i) We note that the essential self-adjointness on $L^2(\nu_\beta)$ of the restriction of the generators of $(\mathcal{E}_{\nu_\beta}, D(\mathcal{E}_{\nu_\beta}))$ to $\mathcal{F}C_\beta^\infty(\mathcal{H}_\beta)$ (see [AKR]) follows (at least if $W \equiv 0$) directly from [LiSe].*
(ii) Both the Log-Sobolev inequality for $(\mathcal{E}_{\nu_\beta}, D(\mathcal{E}_{\nu_\beta}))$ and the essential self-adjointness mentioned in (i) are presently also investigated by Y. M. Park and H. J. Yoo. Their approach , however, is based on the methods developed in [AKR] (more precisely, Theorem 3 resp. Theorem 1 therein).

6 Examples.

Example 1. Let us fix parameters $a^2, J, \beta > 0$ and suppose that the anharmonic self–interaction is given by the double–well potential

$$V(q) \;=\; q^4 - \kappa^2 q^2,\; q \in I\!R^1,$$

which is standard in the theory of phase transitions both in classical and quantum statistical physics (see e.g. [BaK 3,4], [DLP], [PK], [VZ]). We shall show that for all $|\kappa| < \kappa_0$ the system (2.1) satisfies Dobrushin's uniqueness theorem and the critical value $\kappa_0 = \kappa(a^2, J, \beta) > 0$ can be estimated from below. For this purpose let us decompose V as follows:

$$V(q) \;=\; V_0(q) + W(q) \tag{6.1}$$

with functions

$$V_0(q) \;=\; V(q)\,, \qquad |q| \geq |\kappa|;$$

$$V_0(q) \;=\; \tfrac{2}{3\kappa^2}(q^6 - \kappa^6) - q^2(q^2 - \kappa^2),\; |q| \leq |\kappa|,$$

$$W(q) \;=\; 0\,, \qquad |q| \geq |\kappa|;$$

$$W(q) \;=\; -\tfrac{2}{3\kappa^2}(q^6 - \kappa^6) + 2q^2(q^2 - \kappa^2),\; |q| \leq |\kappa|,$$

such that

$$V_0^{(l)}(\pm\kappa) \;=\; V^{(l)}(\pm\kappa),\; W^{(l)}(\pm\kappa) \;=\; 0,\;\; l = 0, 1, 2.$$

It is easy to calculate that

$$\inf_{q \in I\!R^1} V_0''(q) \;\geq\; \frac{1}{5}\kappa^2,$$

$$\delta(W) \;\leq\; 2 \sup_{q \in I\!R^1} |W(q)| \;\leq\; \frac{7}{3}\kappa^4.$$

Correspondingly, the uniqueness condition (4.1) can be written as

$$e^{\beta\delta(W)} \;<\; 1 + (2Jd)^{-1}(a^2 + \inf_{I\!R^1} V_0''). \tag{6.2}$$

But for $|\kappa| > 0$ so small that $\beta\delta(W) \;\leq \frac{7}{3}\beta\kappa^4 < 1$, we have the estimate

$$e^{\beta\delta(W)} \;\le\; 1+\frac{7}{3}\beta\kappa^4+\left(\frac{7}{3}\beta\kappa^4\right)^2.$$

This means that (6.2) holds as soon as

$$\frac{7}{3}\beta\kappa^4+\left(\frac{7}{3}\beta\kappa^4\right)^2 \;<\; (2Jd)^{-1}\left(a^2+\frac{1}{5}\kappa^2\right) \;,$$

$$\frac{7}{3}\beta\kappa^4 \;\le\; 1,$$

and the latter is true for all $|\kappa|<\kappa_0(a^2,J,\beta)$.

Example 2. Analogously we can analyze the case of general φ^4-type potentials of the form

$$V(q)=\sigma q^4-\lambda q^2,\; q\in I\!R^1,\;\; \sigma,\lambda>0. \tag{6.3}$$

This potentials has a two-well shape with minima at the points $q^{\pm}_{min} = \pm\sqrt{\frac{\lambda}{2\sigma}}$. The depth of the wells is equal to

$$|\min_{q\in I\!R^1} V(q)| = \frac{\lambda^2}{4\sigma}.$$

Doing decomposition (6.1) for the potential (6.3) (which is a simple transformation of the decomposition of Example 1) we obtain a sufficient condition for uniqueness of the corresponding Euclidean Gibbs measure $\nu_\beta \in \mathcal{G}^t_\beta$ in the form

$$\exp\left(\frac{7\beta}{3}\frac{\lambda^2}{\sigma}\right) \;<\; 1+(2Jd)^{-1}\left(a^2+\frac{\lambda}{5}\right). \tag{6.4}$$

As a result we see that the uniqueness can be obtained in case of a sufficiently small depth of the wells (which is proportional to λ^2/σ).

Example 3. Let us consider the space-scaling of the potential (6.3), that is

$$V^{\varepsilon}(q):=V(\varepsilon^{-1}q)=\frac{\sigma}{\varepsilon^4}q^4-\frac{\lambda}{\varepsilon^2}q^2,\;\; q\in I\!R^1,\;\; \varepsilon>0. \tag{6.5}$$

The uniqueness condition (6.4) has now the form

$$\exp\left(\frac{7\beta}{3}\frac{\lambda^2}{\sigma}\right) \;<\; 1+(2Jd)^{-1}\left(a^2+\frac{\lambda}{5\varepsilon^2}\right). \tag{6.6}$$

This condition is always valid for a small enough $\varepsilon>0$. Making ε small we draw together the wells of the potential (6.5) while keeping the depth of the wells fixed. This corresponds to applying pressure to the quantum system considered. So, as a result we have a rigorous proof of the well-known physical phenomenon that the pressure can remove the critical behaviour of the system, see e.g. [BC] for a physical description and discussions of experimental results.

Example 4. For given $M \geq 2$, $\lambda > 0$, let us consider a one-particle potential of the form

$$V(q) = q^{2M} - \lambda^{2M-2} q^2, \quad q \in \mathbb{R}^1. \tag{6.7}$$

The decomposition (6.1) can be taken as follows:

$$V_0(q) = \begin{cases} V(q), \quad |q| \geq \lambda, \\ \frac{2(M-1)}{\lambda^2(M+1)}(q^{2M+2} - \lambda^{2M+2}) - q^{2M} + q^2\lambda^{2M-2}, \ |q| \leq \lambda, \end{cases}$$

$$W(q) = \begin{cases} 0, \quad |q| \geq \lambda, \\ -\frac{2(M-1)}{\lambda^2(M+1)}(q^{2M+2} - \lambda^{2M+2}) + 2q^{2M} - 2q^2\lambda^{2M-2}. \end{cases}$$

From these formulas it is not hard to see that we have estimates for the convextity of V_0 and the oscillation of W of the form

$$\text{for all } \ M \geq 2 \quad \min_{q \in \mathbb{R}^1} V_0''(q) \geq C_1 \lambda^{2M-2}, \ q \in \mathbb{R}^1, \ \delta(W) \leq C_2 \lambda^{2M}$$

with some absolute constants $C_1, C_2 > 0$. Then the uniqueness condition (6.2) for the potential (6.7) can be written as

$$e^{C_2 \beta \lambda^{2M}} < 1 + (2Jd)^{-1}(a^2 + C_1 \lambda^{2M-2}).$$

For $\lambda < 1$ we obtain the uniqueness of $\nu_\beta \in \mathcal{G}_\beta^t$ by taking big enough M. Let us mention that these cases include the two-well potentials with fixed zeros $(0, \pm\lambda)$ and minima at the points $\pm M^{-\frac{1}{2M-2}}\lambda \approx \pm\lambda, M \to \infty$. Choosing a big M in the case $\lambda < 1$ we make the depth of the wells (which is proportional to λ^{2M}) small. The latter produces the uniqueness that corresponds to the phenomenological description in [BC].

Actually, we can generalize these examples and consider all potentials of the form

$$V(q) \ := \ P_\kappa(q) \ = \ P_0(q) + \kappa R(q), \ \ q \in \mathbb{R}^1, \tag{6.8}$$

where P_0 is a convex polynomial and R is any polynomial with degree less or equal to the degree of P_0. The previous assertion is still valid: there exists $\kappa_0 > 0$ such that for all $|\kappa| < \kappa_0$ the system (2.1) satisfies the Dobrushin uniqueness theorem. The analogous result for classical lattice systems with the polynomial self–interaction (6.8) has been proved in [Roy]. The Dobrushin matrix has been also estimated in [BP] using a technique different from ours for classical systems with potentials $V(q) = V_0(q) + W(q)$ such that $\inf_{I\!R^1} V_0'' \geq 0$ and, in addition, $\sup_{I\!R^1} |W'| < \infty$. It should be noted that our methods work both in quantum and classical cases and a forthcoming paper of the authors will be especially devoted to classical lattice systems.

7 Concluding remarks and discussions.

1. The assertion of Theorem 4.1 clearly reflects the fact that the uniqueness of ν_β can be achieved, keeping fixed the other parameters of the system (2.1), but choosing a sufficiently high temperature, i.e., a small $\beta > 0$, or a small intensity of the pair interaction $J > 0$, or a W with a small oscillation. These effects have an obvious physical interpretation.

2. We can generalize the model (2.1) in the following way. Instead of $\frac{1}{2}J\sum_{<k,j>}(x_k - x_j)^2$ let us consider the anharmonic two–particle interaction given by $J\sum_{<k,j>} U(x_k - x_j)$ where $U : I\!R^1 \to I\!R^1$ is a convex function such that

$$\inf_{q\in I\!R^1} U''(q) \; > \; 0, \quad \sup_{q\in I\!R^1} U''(q) \; < \; +\infty.$$

Then all the previous arguments hold and the uniqueness condition (4.1) can now be stated as follows:

$$\frac{e^{\beta\lambda\delta(W)}\sup_{I\!R^1} U''}{2d\inf_{I\!R^1} U'' + J^{-1}(a^2+b^2)} \; < \; \frac{1}{2d}\,. \tag{7.1}$$

3. One of the standard applications of the Dobrushin contraction technique (see e.g. [Gr 1], [Fö], [Kü]) is a result about the exponential decay of correlations for the Gibbs measure ν_β. For instance, under the assumptions of Theorem 4.1, we obtain that

$$Cov_{\nu_\beta}(<\omega_k, l_1>_\beta, <\omega_j, l_2>_\beta) \;\leq\; \mathcal{K}e^{-\varepsilon|k-j|}||l_1||_\beta||l_2||_\beta \tag{7.2}$$

with some $\mathcal{K}, \varepsilon > 0$ uniformly for all $k, j \in Z\!\!Z^d$, $\varphi_1, \varphi_2 \in L^2(S_\beta)$.

It should be also noted that, due to the uniqueness of $\nu_\beta \in \mathcal{G}^t_\beta$ and the superstability of the interactions in our system (2.1), the measure ν_β

must necessarily be of a sub-Gaussian growth on $(L^2(S_\beta))^{\mathbb{Z}^d}$ (see [PY 1, Proposition 2.5]). As a consequence, the following estimate holds

$$\sup_k \int_{\Omega_\beta} ||\omega_k||_\beta^M d\nu_\beta \ < \ \infty, \quad M \in \mathbb{N} \ . \tag{7.3}$$

The latter estimate also follows from so-called chess-board estimates for our system which hold for periodic Gibbs states and due to the uniqueness, thus for ν_β) (see [BaK, 2,5]).

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