Geometric aspects of finite and infinite dimensional Fleming-Viot processes

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Abstract

Some geometric aspects of Fleming-Viot processes are studied . In the finite-dimensional case we give an analytic proof that the Fleming-Viot process with parent-independent mutation is the only one that has a symmetrizing measure. This gives rise to a new proof that there is a unique symmetrizing measure for the infinite-dimensional Fleming-Viot process with parent-independent mutation. We show that the finite dimensional Fleming-Viot process with parent-independent mutation is a time changed Brownian motion with respect to a proper metric, and in some special cases is even a Brownian motion itself. A sequence of these finite-dimensional Fleming-Viot processes converges in probability to the infinite dimensional Fleming-Viot process. The main result for the infinite-dimensional Fleming-Viot process is on its intrinsic metric. We prove that the intrinsic metric is uniformly equivalent to the metric defined by the total variation norm.

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Contents

0 Introduction

This paper is on the geometry of reversible Fleming–Viot processes. This type of measure-valued diffusions and its corresponding analytic quantities have been studied intensively in recent years. For excellent survey articles we refer to [EK93] and [D93] and the references therein. For a more analytic approach based on the theory of Dirichlet forms we refer to recent joint work by the authors and B. Schmuland (see [ORS95]).

The purpose of this paper is threefold: first, we give a complete analytic proof for the characterization of reversible Fleming–Viot processes in finite dimensions (leading to an alternative proof of reversibility in infinite dimensions); secondly, we show that the finite-dimensional Fleming–Viot is the time change of a Brownian motion (provided that the underlying simplex is not two–dimensional) and approximate the infinite-dimensional Fleming-Viot processes by these time–changed Brownian motions; thirdly, we identify the associated intrinsic metric in both the finite and infinite-dimensional case.

The contents of this paper's five sections of which the first three (and the Appendix) are solely on the finite dimensional case, is as follows.

After giving some neccesary definitions in Sect. 1 we introduce a convenient coordinate system.

In Sect. 2 we treat the "folklore result" that the only reversible finite dimensional Fleming–Viot process is the one with *parent–independent mutation* (abbreviated PIM). Since we could not find a reference for the "necessity–part", we include a detailed proof in the Appendix. We also study the corresponding symmetric Dirichlet forms with both *absorbing* (i.e., *Dirichlet*) and *reflecting* (i.e., *Neumann*) boundary conditions.

In Sect. 3 we prove that if the underlying simplex $\Delta(\subset \mathbb{R}^d, \text{cf.}(1.1))$ is not two–dimensional, the Fleming–Viot with PIM is a time–change of a Brownian motion with respect to a properly chosen metric on Δ. Moreover, if the mutation is uniform (with $q_i = \frac{1}{2}$, $1 \leq i \leq d$, see Subsect. 3.2), it is itself a Brownian motion w.r.t. that metric. To prove this result we apply Dirichlet form techniques, in particular [FOT94, §6.2].

As an application of the results in Sect. 2, in Sect. 4.1 we give a new proof for the known fact (cf. [EK93]) that there exists a unique symmetrizing measure for the *infinite-dimensional* Fleming–Viot process with PIM, i.e., with state space equal to the probability measures $M_1(S)$ on a standard measurable space $(S, \mathcal{S})$. In Sect. 4.2 we construct an approximation of more general infinite-dimensional Fleming-Viot processes, namely those with a *symmetric interactive selection* as in [ORS95], by finite-dimensional processes. In the case without selection the convergence is even in probability. By this result one can thus consider the infinite-dimensional Fleming-Viot processes with PIM, as a kind of (time-changed) Brownian motion on $M_1(S)$.

In Sect. 5 we study the corresponding intrinsic metric d (defined via the intrisic gradient of a natural bundle structure over $M_1(S)$) (cf. Subsect 5.1). The main result (Theorem 5.1) states that d is on $M_1(S)$ uniformly equivalent to the metric defined by the variation norm (giving a geometric interpretation for the variation norm). In particular, $(M_1(S), d)$ is always complete, and, if S is countable, $(M_1(S), d)$ is separable. In the finite-dimensional case (i.e., $S = \{1, \ldots, d\}$, hence $M_1(S) = \bar{\Delta}$) this means that the intrinsic metric is uniformly equivalent to the Euclidean metric.

Future research of the authors (which is in progress) concerns the generalization of Section 3 and the results on the above intrinsic metric to general measure–valued diffusions. Furthermore, recent results on the geometry induced by Dirichlet forms on the state space (e.g. in terms of the intrinsic metric) in [St93] will be adapted in order to apply them to our situation (cf. [G95]).

1 Definitions and Preliminaries

In this section we recall the definition of the generator associated with a finite-dimensional Fleming-Viot process. Let $d \geq 2$ be fixed throughout this paper.

The state space of the process is the $(d-1)$-dimensional simplex.

$$\bar{\Delta} := \{(x_1, \ldots, x_d) \,|\, x_i \geq 0,\ i = 1, \ldots, d,\ \sum_{i=1}^{d} x_i = 1\} \tag{1.1}$$

Set $\Delta := \mathring{\bar{\Delta}}$. For a closed set $A \subset \mathbb{R}^d$ let $C^2(A)$ denote the set of functions on A which are restrictions of twice continuously differentiable functions on $\mathbb{R}^d$. On $C^2(\bar{\Delta})$ we define an operator L which is defined as the restriction to $\bar{\Delta}$ of the operator

$$L_1 f(x) := \sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\, \frac{\partial^2 f}{\partial x_i\, \partial x_j}(x) + \sum_{i=1}^{d} \left(\sum_{j=1}^{d} q_{ji} x_j \right) \frac{\partial f}{\partial x_i}(x)\ ,\quad f \in C^2(\mathbb{R}^d)\ . \tag{1.2}$$

Here $Q = (q_{ij})_{\substack{1 \leq i \leq d \\ 1 \leq j \leq d}}$ is a matrix and $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else .} \end{cases}$

Remark 1.1 The operator L and the Markov process associated with $(-L)$ is well known and appears in mathematical genetics. It describes a genetic evolution of a population with d types. Roughly speaking the diffusion matrix $(x_i(\delta_{ij} - x_j))_{1 \leq i,j \leq d}$ represents the genetic drift caused by random mating, whereas the matrix Q reflects the effect of mutation, i.e., q_{ji} is the intensity that a type j mutates to a type i. Because of the absence of selection the process is called 'neutral mutation model'. For further information about the applications in genetics we refer to to the survey article [EK93].

1.1 Convenient coordinates

In order to investigate the operator L on the $d-1$-dimensional manifold Δ we introduce the following coordinate transformation.

$$\Phi :\ \mathbb{R}^d \to \mathbb{R}^d\ ,\quad \Phi(x) := y \ \text{ with }\ y_i := x_i\ ,\ \ i = 1, \ldots, d-1,\ y_d := \sum_{i=1}^{d} x_i\ . \tag{1.3}$$

The inverse transformation $x = \Phi^{-1}(y)$ can be written by $x_i = y_i$, $i = 1, \ldots, d-1$, $x_d = y_d - \sum_{i=1}^{d-1} y_i$.

If we define for every $u \in C^2(\mathbb{R}^d)$ $\tilde{u}$ on $\Phi(\mathbb{R}^d)$ $(= \mathbb{R}^d)$ by $\tilde{u} := u \circ \Phi^{-1}$ then we get for $y = \Phi(x) \in \Phi(\mathbb{R}^d)$ that $Lu(x)$ $(= L(u \circ \Phi^{-1} \circ \Phi)(x))$ $= \tilde{L}\tilde{u}(y)$, where

$$\begin{aligned}\tilde{L}\tilde{u}(y) := \sum_{i,j=1}^{d} y_i(\delta_{ij} - y_j) \frac{\partial^2 \tilde{u}}{\partial y_i \, \partial y_j}(y) + 2 \sum_{i=1}^{d-1} y_i \frac{\partial^2 \tilde{u}}{\partial y_i \, \partial y_d}(y) \\ - \sum_{i=1}^{d-1} \left[q_{id} \sum_{l=1}^{d-1} y_l - (Q^{\mathrm{T}} y)^i \right] \frac{\partial \tilde{u}}{\partial y_i}(y) \ .\end{aligned}$$

Because every $y \in \Phi(\Delta)$ satisfies $y_d = 1$ we have for $\tilde{u} \in C^2(\Phi(\bar{\Delta}))$ by expanding the above sums appropriately, that

$$\begin{aligned}\tilde{L}\tilde{u}(y) \quad = \quad & \sum_{i=1}^{d-1} y_i \frac{\partial^2 \tilde{u}}{\partial y_i^2}(y) - \sum_{i,j=1}^{d-1} y_i y_j \frac{\partial^2 \tilde{u}}{\partial y_i \, \partial y_j}(y) \\ & - \sum_{i=1}^{d-1} \left[q_{di} \sum_{l=1}^{d-1} y_l - \sum_{j=1}^{d-1} q_{ji} y_j - q_{di} \right] \frac{\partial \tilde{u}}{\partial y_i}(y) \ .\end{aligned} \tag{1.4}$$

2 Divergence form and reversibility

2.1 Divergence form

By the product rule we can rewrite $\tilde{L}$ in divergence form:

Lemma 2.1 *On $C^2(\Phi(\bar{\Delta}))$ we have*

$$\begin{aligned}\tilde{L}\tilde{u} \quad = \quad & \sum_{i=1}^{d-1} \frac{\partial}{\partial y_i} \left(y_i \frac{\partial \tilde{u}}{\partial y_i} \right) - \sum_{i,j=1}^{d-1} \frac{\partial}{\partial y_i} \left(y_i y_j \frac{\partial \tilde{u}}{\partial y_j} \right) \\ & - \sum_{i=1}^{d-1} (1 - d y_i) \frac{\partial \tilde{u}}{\partial y_i} \\ & - \sum_{i=1}^{d-1} \left[q_{di} \sum_{l=1}^{d-1} y_l - \sum_{j=1}^{d-1} q_{ji} y_j - q_{di} \right] \frac{\partial \tilde{u}}{\partial y_i} \ .\end{aligned} \tag{2.1}$$

2.2 Bilinear forms

Using Φ and the $(d-1)$-dimensional Lebesgue measure dy on $\{y \in \mathbb{R}^d \mid y_d = 1\}$ $(\equiv \mathbb{R}^{d-1})$ we can induce measures on $\bar{\Delta}$ (hence Δ) and bilinear forms associated with L. Define $C_0^2(\Delta) := \{u \circ \Phi^{-1} \mid u \in C_0^2(\Phi(\Delta))\}$. Note that $C_0^2(\Delta) \subset C^2(\bar{\Delta})$. Let $\tilde{\varrho}$ be a measure on $\Phi(\Delta)$ and for simplicity of notation we denote the image measure of dy under Φ^{-1} on Δ again by dy. Define $\hat{\varrho}$ on Δ to be s.t. $\hat{\varrho} \circ \Phi^{-1} = \tilde{\varrho}$ and

$$\mathcal{E}^{\hat{\varrho}}(u,v) := -\int Lu\, v\, d\hat{\varrho} \quad ; \quad u, v \in C_0^2(\Delta)\ . \tag{2.2}$$

Clearly, we then have

$$\begin{aligned} \mathcal{E}^{\tilde{\varrho}}(\tilde{u},\tilde{v}) \ &:= \ -\int_{\Phi(\Delta)} \tilde{L}\tilde{u}\, \tilde{v}\, d\tilde{\varrho} = -\int_{\Delta} \tilde{L}\tilde{u}\circ\Phi\ \tilde{v}\circ\Phi\, d\hat{\varrho} \\ &= \ -\int_{\Delta} L(\tilde{u}\circ\Phi)\, \tilde{v}\circ\Phi\, d\hat{\varrho} = \mathcal{E}^{\hat{\varrho}}(\tilde{u}\circ\Phi, \tilde{v}\circ\Phi) \end{aligned}$$

$$\text{for all} \quad \tilde{u}, \tilde{v} \in C_0^2(\Phi(\Delta))\ . \tag{2.3}$$

From now on we want to restrict ourselves to the case which is important for the genetic interpretation. There, in fact, Q is the generator of a continuous-time Markov process with state-space $\{1, \ldots, d\}$. Such a generator is called "Q-matrix" and has (in the conservative case, which is the only one considered here) the defining properties that $q_{ij} \geq 0$, if $i \neq j$, and $\sum_{j=1}^d q_{ij} = 0$.

The following result characterizes when $(\mathcal{E}^{\hat{\varrho}}, C_0^2(\Delta))$ is symmetric, a result which is kind of "folklore" knowledge in the area of Fleming-Viot processes. However, we could not find an explicit reference for the necessity part. Therefore, we include a complete proof in the appendix.

Theorem 2.2 *Assume that Q is a Q-matrix. Then*

$$\mathcal{E}^{\hat{\varrho}}(u,v) = \mathcal{E}^{\hat{\varrho}}(v,u) \quad \textit{for all} \quad u, v \in C_0^2(\Delta) \tag{2.4}$$

if and only if

$$\begin{aligned} q_{ij} = q_{dj} =: q_j \quad &, \quad i, j = 1, \ldots, d-1 \ \textit{and}\ i \neq j \\ q_{id} = q_{1d} =: q_d \quad &, \quad i = 1, \ldots, d-1, \end{aligned} \tag{2.5}$$

and the measure $\hat{\varrho}$ has a density ϱ w.r.t. dy of the form

$$\varrho(y) = \frac{\Gamma(q_1 + \ldots + q_d)}{\Gamma(q_1)\ldots\Gamma(q_d)} y_1^{q_1 - 1} \cdots y_{d-1}^{q_{d-1}-1} \left(1 - \sum_{l=1}^{d-1} y_l\right)^{q_d - 1} \quad . \tag{2.6}$$

In this case

$$\mathcal{E}^{\hat{\varrho}}(u,v)=\mathcal{E}^{\hat{\varrho}}(v,u) \qquad \textit{for all} \quad u,v\in C^2(\bar{\Delta}) \tag{2.7}$$

provided $q_i>0,\ i=1,\dots,d$ *(where* $\mathcal{E}^{\hat{\varrho}}(u,v)$ *for* $u,v\in C^2(\bar{\Delta})$ *is defined again by (2.7)).*

From now on we consider the operator L with the special Q-matrix specified in (2.5) and the special density ϱ resp. $\varrho(\Phi^{-1})$ given in (2.6).

Hence for all $\tilde{u},\tilde{v}\in C_0^2(\Phi(\Delta))$ resp. $C^2(\Phi(\bar{\Delta}))$ provided $q_i>0,\ 1\le i\le d$,

$$\begin{aligned}\tilde{L}\tilde{u}(y) &= \sum_{i,j=1}^{d-1}\frac{\partial}{\partial y_i}\left((\delta_{ij}y_i-y_iy_j)\frac{\partial\tilde{u}}{\partial y_j}\right)\\ &\quad -\sum_{i=1}^{d-1}\left[1-q_i+y_i\left(\sum_{l=1}^{d}q_l-d\right)\right]\frac{\partial\tilde{u}}{\partial y_i}\end{aligned} \tag{2.8}$$

and (as calculated in the appendix)

$$\begin{aligned}\mathcal{E}(u,v) &:= \mathcal{E}^{\varrho}(u,v)=\mathcal{E}^{\varrho(\Phi^{-1})}(\tilde{u},\tilde{v})\\ &= \sum_{i,j=1}^{d-1}\int_{\Phi(\Delta)}y_i(\delta_{ij}-y_j)\frac{\partial\tilde{u}}{\partial y_i}(y)\frac{\partial\tilde{v}}{\partial y_j}(y)\,\varrho(\Phi^{-1})(y)\,dy\ .\end{aligned} \tag{2.9}$$

In particular, $\mathcal{E}$ is positive definite (since $\sum_{i=1}^{d-1}y_i\le 1$ for all $y=(y_1,\dots,y_{d-1})\in\Phi(\Delta)$).

2.3 Closability

Recall that (depending on $q_i,\ 1\le i\le d$) we have two symmetric positive definite bilinear forms (shortly called *symmetric forms* below) on $L^2(\Delta;\varrho\,dy)$, namely $(\mathcal{E},C_0^2(\Delta))$ and, provided $q_i>0$ for all $1\le i\le d$, $(\mathcal{E},C^2(\bar{\Delta}))$. Since both are defined in terms of the symmetric operators $(L,C_0^2(\Delta))$ resp. $(L,C^2(\bar{\Delta}))$ on $L^2(\Delta;\varrho\,dy)$, both forms are closable on $L^2(\Delta;\varrho\,dy)$ (cf. [F80, Problem 1.1.2]). Let us denote their closures by $(\mathcal{E},D_0(\mathcal{E}))$, $(\mathcal{E},D(\mathcal{E}))$ respectively. Clearly, in case $q_i>0,\ 1\le i\le d$, $D_0(\mathcal{E})\subset D(\mathcal{E})$ and the two forms coincide on $D_0(\mathcal{E})$. The question, when they coincide, has been answered in [Shi76].

Lemma 2.3 *Suppose* $q_i>0$ *for all* $1\le i\le d$. *Then* $D(\mathcal{E})=D_0(\mathcal{E})$ *if and only if* $q_i\ge 1$ *for all* $1\le i\le d$.

Proof [Shi76, Lemma 1.1]. □

Note that since $\partial\Delta$ has zero dy-measure, $C^2(\bar{\Delta})\subset L^2(\Delta;\varrho\,dy)$.

Remark 2.4

(i) Clearly, both $(\mathcal{E}, D_0(\mathcal{E}))$ and $(\mathcal{E}, D(\mathcal{E})$ are *Dirichlet forms* on $L^2(\Delta; \varrho\, dy)$ (cf. [F80, MR92]); i.e., they are closed symmetric forms such that

$$u^\sharp := u^+ \wedge 1 \in D(\mathcal{E}) \text{ and } \mathcal{E}(u^\sharp, u^\sharp) \leq \mathcal{E}(u,u) \text{ for all } u \in D_{(0)}(\mathcal{E}) \ . \tag{2.10}$$

Moreover, they are both local and regular and their state spaces Δ resp. $\bar{\Delta}$ are separable metric spaces. So, by the fundamental theorem of Fukushima (cf. [F80, Sect. 6]) for any of the above Dirichlet forms there exists a unique associated diffusion process $\mathbf{M}$ on Δ resp. $\bar{\Delta}$, and these are in fact Fleming-Viot processes with (in the case of $(\mathcal{E}, D_0(\mathcal{E}))$) absorbing resp. (in the case of $(\mathcal{E}, D(\mathcal{E}))$) reflecting boundary conditions.

(ii) According to [ORS95, Proposition 7.6] the boundary of $\bar{\Delta}$ is a capacity zero set for $D(\mathcal{E})$ if $q_i \geq 1$ for all $1 \leq i \leq d$. Hence Lemma 2.3 has a nice intuitive, probabilistic interpretation: In the case $q_i \geq 1,\ 1 \leq i \leq n$ the process associated with $(\mathcal{E}, D(\mathcal{E}))$ does not hit the boundary and can therefore not decide wether there are Dirchlet- (i.e., reflecting) or Neumann- (i.e., absorbing) boundary conditions. In fact [ORS95, Proposition 7.6] (for finite dimensions) can be considered as the probabilistic counterpart of Lemma 2.3 above.

3 The Fleming-Viot process as a time-changed Brownian motion

In this section we shall identify a metric on the manifold Δ w.r.t. which the Fleming-Viot processes (of the previous section) are time-changed Brownian motions. First we recall the definition of a Brownian motion on a manifold which can be done most easily in terms of Dirichlet forms (cf. [F84]).

3.1 Brownian motion on a manifold

Let (M, g) be a connected oriented Riemannian manifold (possibly with boundary) with volume element $d_g\mathrm{vol}$. Consider the canonical pre-Dirichlet form on $L^2(M; d_g\mathrm{vol})$ with domain $D := C_0^\infty(M)$ resp. $C^\infty(M) \cap L^2(M; d_g\mathrm{vol})$

$$\mathcal{E}(u,v) := \int_M g(\nabla u, \nabla v) d_g\mathrm{vol}\ ,\ \ u, v \in D\ .$$

We suppose from now on that $(\mathcal{E}, D)$ is closable on $L^2(M; d_g\mathrm{vol})$ (which is always the case if $D = C_0^\infty(M)$ or ∂M is nice). Then the closure $(\mathcal{E}, \bar{D})$ is a regular local Dirichlet form on $L^2(M; d_g\mathrm{vol})$. Hence by [F80, Sect. 6] there exists a unique diffusion $\mathbf{W}$ (possibly with explosion) on M resp. $\bar{M}$ associated with $(\mathcal{E}, \bar{D})$. Its boundary behaviour depends on (and is entirely determined by) the choice of D. $\mathbf{W}$ is called *Brownian motion on M* resp. $\bar{M}$ (with "absorbing" or "reflecting" boundary conditions). The corresponding generator evaluated on $C_0^\infty(M)$ is just the Laplace-Beltrami operator and it is well-known that (up to equivalence, see [F80]) this definition of Brownian motion coincides with usual ones (as e.g. in [IW84]).

We have the following elementary lemma whose proof is omitted.

Lemma 3.1 *Let A be a real $n \times n$-matrix, $n \neq 2$, and $\alpha \in]\,0, \infty\,[$. Define $B := (\alpha^2 \det A)^{-1/(n-2)} A$. Then $\alpha A = (\det B)^{-1/2} B$.*

As a consequence we obtain the following result more or less well-known in Dirichlet space theory. Let g, as usual, denote the determinant of the metric $g(\cdot, \cdot)$.

Proposition 3.2 *Let $n := \dim M \neq 2$. Let $\sigma : M \to]0, \infty[$ be a C^∞-function, and consider the closure $(\mathcal{E}^\sigma, \bar{D})$ of*

$$\mathcal{E}^\sigma(u, v) := \int_M g(\nabla u, \nabla v)\, \sigma\, d_g\mathrm{vol}\ ,\quad u, v \in D\ , \tag{3.1}$$

on $L^2(M; d_g\mathrm{vol})$, where D is as above. Then the diffusion process $\mathbf{X}$ on M resp. $\bar{M}$ (which exists by [F80, Sect. 6]) associated with the Dirichlet form $(\mathcal{E}^\sigma, \bar{D})$ on $L^2(M; d_g\mathrm{vol})$ is a time change of the Brownian motion W associated with $(\mathcal{E}, \bar{D})$ on M resp. $\bar{M}$ w.r.t. the metric $\tilde{g}(\cdot, \cdot) := \sigma^{2/(n-2)} g(\cdot, \cdot)$. More precisely, we have for the corresponding paths

$$X_t = W_{\tau_t},\ t \geq 0,\ \textit{where } \tau_t := \inf\left\{ s > 0 \,\Big|\, \int_0^s \sigma^{-n/(n-2)}(W_r)\, dr > t \right\}\ .$$

Proof. In local coordinates with $g(\cdot, \cdot) = (g_{ij})_{i,j}$ and $(g^{ij})_{i,j} = (g_{ij})^{-1}_{i,j}$, by Lemma 3.1 we have that for $\tilde{g}(\cdot, \cdot) = (\tilde{g}_{ij})_{i,j}$, $\tilde{g} := \det \tilde{g}(\cdot, \cdot)$,

$$\begin{aligned} \mathcal{E}^\sigma(u, v) &= \int g^{ij} \partial_i u\, \partial_j v\, \sigma \sqrt{g}\, dx \\ &= \int \tilde{g}^{ij}\, \partial_i u\, \partial_j v \sqrt{\tilde{g}}\, dx \\ &= \int \tilde{g}(u, v)\, d_{\tilde{g}}\mathrm{vol} \end{aligned}$$

on $L^2(M; \sigma_1\, d_{\tilde{g}}\mathrm{vol})$ with $\sigma_1 := \sqrt{g}/\sqrt{\tilde{g}} = \sigma^{-n/(n-2)}$. Now the assertion follows by [FOT94, Theorem 6.2.1]. □

3.2 Application to the Fleming-Viot process

The results in Subsection 3.1 apply to the Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ and $(\mathcal{E}, D_0(\mathcal{E}))$ defined in Subsection 2.3. Both are of type (3.1) with $M = \Delta$, $n = d-1$, and g, $d_g\mathrm{vol}$ in the coordinates introduced in Subsection 1.1 are given as follows (since $g^{ij} = y_i(\delta_{ij} - y_j)$, $1 \leq i, j \leq d-1$, cf. (2.9))

$$
\begin{aligned}
g_{ij}(y) &= \delta_{ij} y_i^{-1} + \left(1 - \sum_{l=1}^{d-1} y_l\right)^{-1}, \quad 1 \leq i, j \leq d-1 \\
g(y) &= \det g_{ij}(y) = y_1^{-1} \cdots y_{d-1}^{-1} \left(1 - \sum_{l=1}^{d-1} y_l\right)^{-1} \\
d\mathrm{vol}_g &= y_1^{-1/2} \cdots y_{d-1}^{-1/2} \left(1 - \sum_{l=1}^{d-1} y_l\right)^{-1/2} dy_1 \cdots dy_{d-1} \,.
\end{aligned}
$$

Consequently if $d \neq 3$, Proposition 3.2 applies where the function σ appearing in the time change is then given by (cf. (2.6))

$$
\sigma = \frac{\varrho}{\sqrt{g}} = y_1^{q_1 - 1/2} \cdots y_{d-1}^{q_{d-1} - 1/2} \left(1 - \sum_{l=1}^{d-1} y_l\right)^{q_d - 1/2} .
$$

The metric for the underlying Brownian motion is

$$
\tilde{g}_{ij} = y_1^{\frac{2q_1 - 1}{d-3}} \cdots y_{d-1}^{\frac{2q_{d-1}-1}{d-3}} \left(1 - \sum_{l=1}^{d-1} y_l\right)^{\frac{2q_d - 1}{d-3}} g_{ij} \,, \quad 1 \leq i, j \leq d-1 \,.
$$

Remark 3.3 We note that if $q_i = \frac{1}{2}$ for all $1 \leq i \leq d-1$, the Fleming-Viot is itself already the underlying Brownian motion and no time change is necessary. It is in fact the Brownian motion on the sphere, since in this case the sectional curvature can be computed explicitly, i.e.:

$$
R\langle a, b \rangle b \cdot ga = \frac{1}{4}[(ga \cdot a)(gb \cdot b) - (ga \cdot b)^2] \,, \tag{3.2}
$$

hence our manifold (Δ, g) is an n-sphere of radius 2. This was first noted by B. Driver (cf. [Dr94]).

4 The infinite-dimensional Fleming-Viot process with parent-independent mutation. Reversibility and Approximation

The infinite-dimensional Fleming-Viot process with PIM is the infinite-dimensional analogue of the process considered in Section 2 with $q_{ij} = q_i$, $i = 1, \dots, d$ and is defined as the unique solution of a martingale problem. On the basis of our results in Section 2 we will reprove in Subsection 4.1 the well-known fact (cf. [EK86]) that the only symmetrizing probability measure of this process is the infinite Poisson-Dirichlet distribution m_{θ,ν_0}, cf. (4.5) and the Fleming-Viot process is in fact reversible w.r.t. m_{θ,ν_0}. However, our proof is different from the one in [EK86]. In Subsection 4.2 we shall show that a sequence of finite-dimensional Fleming-Viot processes as in Subsection 2.3 converges to the infinite-dimensional Fleming-Viot process. In the case of symmetric interactive selection (cf. [ORS95, Subsection 5.2]) the convergence is in law. If there is no interaction, the convergence is even in probability.

As we know by Section 3 the finite-dimensional Fleming-Viot process can be viewed as time-changed Brownian motions. Hence we may regard the infinite-dimensional Fleming-Viot process with parent-independent mutation (i.e., with $\varphi = 1$) as a good substitute for a time-changed Brownian motion on the space of probability measures $E = \mathcal{M}_1(S)$ viewed as a "manifold", where the tangent space at the point μ is $L^2(S, \mu)$.

Definition 4.1 Let $\theta > 0$ and $\nu_0 \in E := \mathcal{M}_1(S)$, the space of probability measures over a standard measurable space $(S, \mathcal{S})$ (i.e., there exists a Polish topology on S whose Borel σ-algebra coincides with $\mathcal{S}$). Assume that ν_0 has full support w.r.t. some Polish topology on S compatible with $\mathcal{S}$. Let $\mathcal{S}_b$ denote the set of all bounded $\mathcal{S}$-measurable functions on $\mathcal{S}$.

(i) An E-valued process X on a filtered probability measure space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ is a Fleming-Viot process with PIM-parameter (θ, ν_0) if for every finitely based function $u \in \mathcal{F}C^\infty(\mathcal{S}_b)$, i.e.,

$$u(\mu) = \Phi\left(\int f_1\, d\mu, \dots, \int f_l\, d\mu\right),\ \Phi \in C^\infty(\mathbb{R}^l),\ f_i \in \mathcal{S}_b,\quad i = 1, \dots l,\ \mu \in E, \quad (4.1)$$

the process

$$u(X_t) - u(X_0) - \int_0^t \mathcal{A}u(X_s)\, ds \quad (4.2)$$

is a martingale, where

$$\begin{aligned}\mathcal{A}u(\mu) &= \theta \int_E \mu(dy) \int \nu_0(dx)\, (\nabla_x u(\mu) - \nabla_y u(\mu)) \\ &\quad + \int_E \mu(dy) \int_E (\delta_y(dx) - \mu(dx)) \nabla_x \nabla_y u(\mu) \ .\end{aligned} \tag{4.3}$$

The gradient $\nabla_x u(\mu)$ is defined by $\lim_{s\to 0} \frac{u(\mu+s\varepsilon_x)-u(\mu)}{s}$, where ε_x is Dirac measure in $x \in S$. For u as above we have

$$\nabla_x u(\mu) = \sum_{i=1}^{l} \frac{\partial \Phi}{\partial x_i} \left(\int f_1\, \mu, \dots, \int f_l \, d\mu \right) f_i(x). \tag{4.4}$$

(ii) Let $\mathcal{B}$ denote the smallest σ-algebra on E such that all functions

$$\mu \mapsto \int f\, d\mu \ , \quad f \in \mathcal{S}_b \ ,$$

are $\mathcal{B}$-measurable. Define a probability measure m_{θ,ν_0} on $\mathcal{B}$ by

$$m_{\theta,\nu_0}(A) := P\left[\sum_{i=1}^{\infty} \varrho_i \delta_{\xi_i} \in A \right] \ , \ A \in \mathcal{B} \ . \tag{4.5}$$

The sequence $(\varrho_i)_{i\in\mathbb{N}}$ is defined by

$$\varrho_i := \frac{s_{[i]}}{\sum_{j=1}^{\infty} s_j}$$

where $(s_i)_{i\in\mathbb{N}}$ is a Poisson point process on $[0,\infty)$ with intensity $\theta x^{-1} e^{-\theta x}$ and $s_{[i]}$ is the i-th order statistic. The sequence $(\xi_i)_{i\in\mathbb{N}}$ is $\nu_0^{\otimes\mathbb{N}}$distributed and independent of $(\varrho_i)_{i\in\mathbb{N}}$.

4.1 Reversibility

In [EK86] Ethier and Kurtz show that the measure m_{θ,ν_0} is the unique stationary distribution and then Ethier [E90] proved that m_{θ,ν_0} is reversible. The main tool is the calculation of the moments of the measure m_{θ,ν_0}, i.e., expressions of type

$$\int_{\mathcal{M}_1(E)} \left(\int f_1 \, d\mu \right) \dots \left(\int f_l\, d\mu \right) m_{\theta,\nu_0}(d\mu) \ , \ f_i \in C(E)$$

Our approach is algebraically less complicated, because we only need very special f_i, $i = 1, \dots, l$. Their approach is based on the definition of the Fleming-Yiot process as a solution of a martingale problem, where the test functions are polynomials instead of the class $\mathcal{F}C_b^\infty(\mathcal{S}_b)$. That these two martingale problems yield the same process is proved in [EK93].

We shall now give a new proof for (part of) the fact just mentioned. Let us first reformulate the result precisely.

Proposition 4.2 *(i) If m is a symmetrizing measure for the Fleming-Viot process with parent-independent mutation (θ, ν_0), then $m = m_{\theta,\nu_0}$.*

(ii) The Fleming-Viot process with parent independent mutation (θ, ν_0) is symmetric with respect to m_{θ,ν_0}.

Proof. (i): As a distribution of a random measure m is uniquely determined by its finite-dimensional marginals

$$m(\{\mu \in E \mid (\mu(A_1), \ldots, \mu(A_{k-1})) \in \mathcal{B}\}) \ ,$$

where $A_i \cap A_j = \emptyset,\ i \neq j,\ \bigcup_{i=1}^k A_i = E,\ A_i \in \mathcal{B},\ B \in \mathcal{B}(\mathbb{R}^{k-1}),\ k \in \mathbb{N}$, let us take

$$u(\mu) := \Phi(\mu(A_1), \ldots, \mu(A_k)) \ , \ \ v(\mu) := \Psi(\mu(A_1), \ldots, \mu(A_k)) \ , \tag{4.6}$$

in (4.3). Then symmetry implies

$$\int \mathcal{A}u(\mu)\, v(\mu)\, m(d\mu) = \int u(\mu)\, \mathcal{A}v(\mu)\, m(d\mu) \tag{4.7}$$

for all $\Phi,\ \Psi \in C_0^\infty(\Delta)$. Let us compute

$$\begin{aligned}
\mathcal{A}u(\mu) &= \theta \int_E \mu(dy) \int_E \nu_0(dx) \sum_{i=1}^k \frac{\partial \Phi}{\partial x_i}(\mu(A_1), \ldots, \mu(A_k))(\mathbf{1}_{A_i}(x) - \mathbf{1}_{A_i}(y)) \\
&\quad + \int_E \mu(dy) \int_E (\delta_y(dx) - \mu(dx)) \sum_{i,j=1}^k \frac{\partial^2 \Phi}{\partial x_i\, \partial x_j}(\mu(A_1), \ldots, \mu(A_k)) \mathbf{1}_{A_i}(x) \mathbf{1}_{A_j}(y) \\
&= \theta \sum_{i=1}^k \frac{\partial \Phi}{\partial x_i}(\mu(A_1), \ldots, \mu(A_k))(\nu_0(A_i) - \mu(A_i)) \\
&\quad + \sum_{i,j=1}^k \frac{\partial^2 \Phi}{\partial x_i\, \partial x_j}(\mu(A_1), \ldots, \mu(A_k))(\delta_{ij}\mu(A_i) - \mu(A_i)\mu(A_j)) \\
&= \sum_{i=1}^k \frac{\partial \Phi}{\partial x_i}(\mu(A_1), \ldots, \mu(A_k))(\theta \nu_0(A_i) - \mu(A_i) \sum_{j=1}^k \theta \nu_0(A_j))
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{i,j=1}^{k} \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j}(\mu(A_1), \ldots, \mu(A_k))(\delta_{ij}\mu(A_i) - \mu(A_i)\mu(A_j)) \\
= \; & \sum_{i=1}^{k} \frac{\partial \Phi}{\partial x_i}(\mu(A_1), \ldots, \mu(A_k))(\theta\nu_0(A_i) - \theta\nu_0(A_i)\mu(A_i) - \sum_{\substack{l=1 \\ l \neq i}}^{k} \theta\nu_0(A_l)\,\mu(A_i)) \\
& + \sum_{i,j=1}^{k} \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j}(\mu(A_1), \ldots, \mu(A_k))(\delta_{ij}\mu(A_i) - \mu(A_i)\mu(A_j)) \\
= \; & \sum_{i=1}^{k} \frac{\partial \Phi}{\partial x_i}(\mu(A_1), \ldots, \mu(A_k)) \left(\sum_{\substack{l=1 \\ l \neq i}}^{k} \theta\nu_0(A_i)\,\mu(A_l) - \mu(A_i) \sum_{\substack{l=1 \\ l \neq i}}^{k} \theta\nu_0(A_l) \right) \\
& + \sum_{i,j=1}^{k} \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j}(\mu(A_1), \ldots, \mu(A_k))(\delta_{ij}\mu(A_i) - \mu(A_i)\mu(A_j)) \\
= \; & \sum_{i=1}^{k} \frac{\partial \Phi}{\partial x_i}(\mu(A_1), \ldots, \mu(A_k)) \sum_{l=1}^{k} q_{li}\,\mu(A_l) \\
& + \sum_{i,j=1}^{k} \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j}(\mu(A_1), \ldots, \mu(A_k))(\delta_{ij}\mu(A_i) - \mu(A_i)\mu(A_j)) \,,
\end{aligned}
$$

where

$$
\begin{cases}
q_{ji} \;=\; \theta\,\nu_0(A_i) \text{ for } i \neq j \text{ and} \\
q_{ii} \;=\; -\theta \sum_{\substack{l=1 \\ l \neq i}}^{k} \nu_0(A_l) \,.
\end{cases}
\tag{4.8}
$$

Hence

$$
\mathcal{A}u(\mu) = L_1 \Phi\,(\mu(A_1), \ldots, \mu(A_k)) \tag{4.9}
$$

where L_1 is defined in (1.2) with the Q-matrix defined in (4.8) above and (4.7) can be written as

$$
\begin{aligned}
& \int_\Delta \Psi L_1 \Phi(x_1, \ldots, x_k)\, m\Big(\mu(A_1) \in dx_1, \ldots, \mu(A_k) \in dx_k\Big) = \\
& \int_\Delta \Phi L_1 \Psi(x_1, \ldots, x_k)\, m\Big(\mu(A_1) \in dx_1, \ldots, \mu(A_k) \in dx_k\Big) \,.
\end{aligned}
$$

Hence Theorem 2.2 yields

$$
\begin{aligned}
& m\Big((\mu(A_1), \ldots, \mu(A_{k-1})) \;\in\; dy_1 \cdots dy_{k-1}\Big) = \\
& \quad y_1^{\theta\nu_0(A_1)-1} \cdots y_{k-1}^{\theta\nu_0(A_{k-1})-1} \left(1 - \sum_{i=1}^{k-1} y_i \right)^{\theta\nu_0(A_k)-1} dy_1 \cdots dy_{k-1}
\end{aligned}
$$

But the latter one is the finite-dimensional distribution of m_{θ,ν_0}, cf. Lemma 7.2 in [ORS95] or Lemma 2.2 in [EG92].

(ii): From the proof of (i) it is clear that (4.7) is true for u and v as in (4.6) and $m = m_{\theta,\nu_0}$. This obviously includes (4.7) for u and v based on simple functions $f_i = \sum_{j=1}^{n_i} a_j^i \, \mathbf{1}_{A_j^i}$, $i = 1, \ldots, k$. By bounded convergence (4.7) is therefore true for all $f_i \in \mathcal{S}_b$, if we can show that $\mathcal{A}u(\mu)$ is bounded. The first line in the computation of $\mathcal{A}u(\mu)$ above gives

$$\sup_{\mu \in E} \mathcal{A}u(\mu) \leq \theta \cdot 2k \|\nabla \Phi\|_\infty \sup_{1 \leq i \leq k} \|f_i\|_\infty + k^2 \sup_{1 \leq i,j \leq k} \left\| \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j} \right\|_\infty \sup_{1 \leq i,j \leq k} \|f_i f_j\|_\infty \, .$$

Hence, if simple functions f_i^n, $n \in \mathbb{N}$, $i = 1, \ldots, k$, converge bounded pointwise to f_i, $i = 1, \ldots, k$, as $n \to \infty$, we have that

$$\mathcal{A}u^n(\mu) \longrightarrow \mathcal{A}u(\mu)$$

bounded, where

$$u^n(\mu) = \Phi(\int f_1^n \, d\mu, \ldots, \int f_k^n \, d\mu).$$

$\square$

Remark 4.3 More complicated considerations based on the calculation of moments show that any member of an even larger class of Fleming-Viot processes has a unique stationary measure [EK93].

4.2 Approximation

Already in the very first paper about infinite-dimensional Fleming-Viot processes an approximation by a sequence of finite-dimensional diffusion processes is given, cf. [FV79] which applies also in the situation of the preceeding subsection. We shall now, however, give a different and shorter approach which, in addition, works for a more general class of Fleming-Viot processes, namely those *with symmetric interative selection* (for the definition of the latter cf. Theorem 4.5 (ii) below) by using the Lyons-Zheng decomposition of Dirichlet processes [LZhe88], [LZ94]. In the case without selection we will even show convergence in probability. Let us first generalize the processes considered in Subsection 4.1. We fix a finite (positive) measure m on $(E, \mathcal{B})$ (with $\mathcal{B}$ as specified in Definition 4.1 (ii)).

Remark 4.4 Note that in [ORS95] E was equipped with the Borel σ-algebra $\mathcal{B}_\tau(E)$ w.r.t. the weak topology on E coming from some Polish topology τ on S whose Borel σ-algebra is $\mathcal{S}$. But a simple monotone class argument implies that $\mathcal{B}_\tau(E) = \mathcal{B}$ i.e., $\mathcal{B}_\tau(E)$ is in particular independent of the Polish topology τ chosen on S.

Consider the following pre-Dirichlet form on $L^2(E;m)$

$$\mathcal{E}(u,v) = \int \mathrm{cov}_\mu(\nabla.u(\mu),\ \nabla.v(\mu))\, m(d\mu)\ ;\ \ u,v \in \mathcal{F}C^\infty(\mathcal{S}_b)\ . \tag{4.10}$$

where for $\mu \in E$,

$$\begin{aligned}\mathrm{cov}_\mu(\nabla.u(\mu), \nabla.v(\mu)) \ &= \ \int \nabla_x u(\mu)\nabla_x v(\mu)\, \mu(dx) \\ &\quad - \int \nabla_x u(\mu)\, \mu(dx) \int \nabla_x v(\mu)\, \mu(dx)\end{aligned} \tag{4.11}$$

(cf. (4.4)). In the special case $m := m_{\theta,\nu_0}$ (cf. (4.5)), we write $\mathcal{E}_{\theta,\nu_0}$ instead of $\mathcal{E}$.

By Remark 4.4 the following result is an immediate consequence of [ORS95, Theorem 2.4, Lemma 6.3, and Theorem 3.5]. Let $\mathcal{F}C^\infty(\mathcal{S}_b)\tilde{}$ denote the space of m-classes determined by $\mathcal{F}C^\infty(\mathcal{S}_b)$.

Theorem 4.5 *(i) Suppose that*

$$(\mathcal{E}, \mathcal{F}C^\infty(\mathcal{S}_b)\tilde{})\ \textit{is closable on}\ L^2(E;m)$$

and let $(\mathcal{E}, D(\mathcal{E}))$ be its closure. Then $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular, (strongly) local Dirichlet form *with state space E if E is equipped with the weak topology $\mathcal{W}_\tau$ coming from any Polish topology τ on S whose Borel σ-algebra is equal to $\mathcal{S}$. In particular, $(\mathcal{E}, D(\mathcal{E}))$ is* associated with *a diffusion process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_\mu)_{\mu\in E})$ with W_τ-continuous sample paths, which is conservative and has m as symmetrizing and invariant measure.*

(ii) If $\varphi \in (\mathcal{E}_{\theta,\nu_0}, D(\mathcal{E}_{\theta,\nu_0}))$, $\varphi > 0$, m_{θ,ν_0}-a.s. or $\varphi \in L^2(E; m_{\theta,\nu_0})$ with $\varphi^2 \geq \varepsilon > 0$, then the form

$$\mathcal{E}^\varphi(u,v) := \int \mathrm{cov}(\nabla u,\ \nabla v)\, \varphi^2\, dm_{\theta,\nu_0}\ ;\ \ u,v \in \mathcal{F}C^\infty(\mathcal{S}_b)$$

is closable. The associated process is a solution of the martingale problem with generator

$$L^\varphi u(\mu) = \mathcal{A}u(\mu) + \varphi^{-1}(\mu)\mathrm{cov}_\mu(\nabla\varphi(\mu), \nabla u(\mu)), \quad u \in \mathcal{F}C^\infty(S_b).$$

(A process of this type is called *Fleming-Viot process with interactive selection function* $\varphi^{-1}(\mu)\nabla_x\varphi(\mu)$.) If $\varphi \equiv 1$ this process coincides with the one in Definition 4.1 (i).

In Theorem 4.5 (i) and (ii) we may take $\Omega = C([0,\infty), E)$ and $X = (X_t)_{t\geq 0}$ to be the coordinate process. In order to be not to repetitious w.r.t. [ORS95] we do not recall the notions *quasi-regular, (strongly) local*, and *associated with* used in Theorem 4.5 here, but refer to [ORS95] and [MR92]. For the definition of a Dirichlet form we refer to Remark 2.4(i) above.)

Fix now τ and φ as in Theorem 4.5 (i), (ii) respectively, and let $(\mathcal{E}^\varphi, D(\mathcal{E}^\varphi))$ be the closure of $(\mathcal{E}^\varphi, \mathcal{F}C^\infty(\mathcal{S}_b)^\sim)$ on $L^2(E; \varphi^2 \cdot m_{\theta,\nu_0})$. Let

$$P^\varphi := \int P_\mu \ \varphi(\mu) \ m_{\theta,\nu_0}(d\mu) \ .$$

Since the measure $I(f) := \int \mu(f)\varphi^2(\mu) dm_{\theta,\nu_0}(\mu)$ is tight we can find a sequence $K_1 \subset K_2 \subset \ldots$ of compact subsets of S such that $I(K_n^c) \leq 1/n \ \forall n \in \mathbb{N}$. Then we may choose for each $n \in \mathbb{N}$ a partition $\{\tilde{A}_i^n\}_{i=1}^\infty$ of S such that $\mathrm{diam}(\tilde{A}_i^n) \leq \frac{1}{n} \ \forall i \in \mathbb{N}$ and a finite index k_n such that $K_n \subset \bigcup_{i=1}^{k_n} \tilde{A}_i^n$. Assume that the partitions are nested, i.e., for each $n, j \geq 1$ there exists $i \geq 1$ such that $\tilde{A}_j^{n+1} \subseteq \tilde{A}_i^n$. We set from now on $A_j^n := \tilde{A}_j^n, \ j = 1, \ldots, k_n$ and $A_{k_n+1}^n := \bigcup_{j=k_n+1}^\infty \tilde{A}_j^n$.

Remark 4.6 It is obvious that for $\varphi \equiv 1$ the $\mathbb{R}_+^{k_n+1}$-valued process Y^n defined by

$$Y_t^n := (X_t(A_1^n), \ldots, X_t(A_{k_n+1}^n))$$

is a Fleming-Viot process as considered in Subsection 2.3 with

$$\begin{aligned} q_{ji}^{(n)} &= \theta \nu_0(A_i^n) \ , \quad i \neq j \\ q_{ii}^{(n)} &= -\theta \sum_{\substack{l=1 \\ l\neq i}}^{k_n+1} \nu_0(A_l^n) \ . \end{aligned}$$

Theorem 4.7 *Define on* $C([0,\infty), E)$

$$X_t^n := \sum_{i=1}^{k_n+1} X_t(A_i^n)\delta_{x_i^n} \ , \tag{4.12}$$

where $\{x_i^n \mid i, n \in \mathbb{N}\}$ *is a fixed choice of points such that* $x_i^n \in A_i^n$ *for all* $i = 1, \ldots, k_n + 1, \ n \in \mathbb{N}$.

(i) If S *is locally compact then*

$$P^\varphi \circ (X^n)^{-1} \Longrightarrow P^\varphi.$$

(ii) If $\varphi = 1$ *then*

$$X^n \longrightarrow X \quad \textit{in } P^\varphi\textit{-probability.}$$

Proof. (i): By [MeRo94] and the fact that P^φ is supported by the continuous paths as well as the fact that all our processes take values in the space of probability measures it is enough to prove

$$P^\varphi \circ (X^n)^{-1} \Rightarrow P^\varphi \quad \text{in } M(D([0,\infty),(E,\text{vague})),$$

where (E, vague) denotes the space E equipped with the vague topology.

In order to show tightness of $(P^\varphi \circ (X^n)^{-1})_{n\in\mathbb{N}}$ we have to prove that $(X^n(f))_{n\in\mathbb{N}}$ is tight as a real-valued cadlag process for $f \in C_c(S)$. The last step is the identification of the limit point. It is known that for $u \in D(\mathcal{E}^\varphi)$ for each $T > 0$ one can give a decomposition of the real-valued process $(Z_s)_{0\le s\le T} = (u(X_s))_{0\le s\le T}$ into a forward and a backward martingale, i.e., for $t \in [0,T]$

$$Z_t = M_t^{[u]} + \hat{M}_T^{[u]} - \hat{M}_{T-t}^{[u]}$$

where $(M_t^{[u]})_{t\ge 0}$ is a martingale w.r.t. $\mathcal{F}_t := \sigma(X_s, s \le t)$ and $(\hat{M}_t^{[u]})_{t\ge 0}$ is a martingale w.r.t. $\hat{\mathcal{F}}_t := \sigma(X_{T-s}, s \le t)$ (cf. [LZhe88], more precisely [LZ94, Theorem 1.1] for the version needed here. We emphasize that the latter result immediately extends to arbitrary quasi-regular Dirichlet forms on general state space, i.e., particularly the ones considered here. The local compactness as well as the regularity of the Dirichlet form assumed in both these references are not necessary. This follows immediately from [MR92, Chapter VI].). To prove tightness of $(X_t^n(f))_{0\le t\le T}$, $n \in \mathbb{N}$, we consider its Lyons-Zheng decomposition, i.e., that of

$$\left(\sum_{i=1}^{k_n+1} f(x_i^n) X_t^{n,i}\right)_{t\ge 0}$$

with $X_t^{n,i} := X_t(A_i^n)$.

$$\sum_{i=1}^{k_n+1} f(x_i^n) X_t^{n,i} = \frac{1}{2}\sum_{i=1}^{k_n+1} f(x_i)(M_t^{i,n} - (\bar{M}_T^{i,n} - \bar{M}_{T-t}^{i,n})) \tag{4.13}$$

where $M^{i,n}$ (resp. $\bar{M}^{i,n}$) is the forward (resp. backward) martingale in the decomposition of $X^{i,n}$. Then the quadratic variation is computed as follows:

$$\left\langle \sum_{i=1}^{k_n+1} f(x_i^n) M^{i,n} \right\rangle_t = \sum_{i,j=1}^{k_n+1} f(x_i^n) f(x_j^n) \int_0^t X_s^{i,n}(\delta_{ij} - X_s^{j,n}) ds$$

and

$$\left\langle \sum_{i=1}^{k_n+1} f(x_i^n) \bar{M}^{i,n} \right\rangle_t = \sum_{i,j=1}^{k_n+1} f(x_i^n) f(x_j^n) \int_0^t X_{T-s}^{i,n}(\delta_{ij} - X_{T-s}^{j,n}) ds.$$

By the Hölder and Burkholder-Gundy inequalities for $p \in (2, \infty)$ and $0 \le s < t \le T$ we have

$$
\begin{aligned}
& E\Big[\Big(\sum_{i=1}^{k_n+1} f(x_i^n)(X_t^{i,n} - X_s^{i,n})\Big)^p\Big]^{1/p} \\
& \le c_p \Big\{ E\left[\left(\int_s^t \sum_{i,j=1}^{k_n+1} f(x_i^n) f(x_j^n) X_r^{i,n}(\delta_{ij} - X_r^{j,n}) dr\right)^{p/2}\right]^{1/p} \\
& \qquad + E\left[\left(\int_{T-t}^{T-s} \sum_{i,j=1}^{k_n+1} f(x_i^n) f(x_j^n) X_r^{i,n}(\delta_{ij} - X_r^{j,n}) dr\right)^{p/2}\right]^{1/p} \Big\} \\
& \le 2c_p\,(t-s)^{1/2} E\left[\left\{\sum_{i,j=1}^{k_n+1} f(x_i^n) f(x_j^n) X_0(A_i^n)(\delta_{ij} - X_0(A_j^n))\right\}^{p/2}\right]^{1/p} \\
& = 2c_p(t-s)^{1/2}\, E\left[\left\{ \mathrm{var}_{X_0}\left(\sum_{i=1}^{k_n+1} f(x_i^n)\, \mathbf{1}_{A_i^n}\right)\right\}^{p/2}\right]^{1/p} \\
& \le 2c_p(t-s)^{1/2}\, \|f\|_\infty\ ,
\end{aligned}
\tag{4.14}
$$

where $E[\,\cdot\,]$ denotes expectation w.r.t. P^φ. Tightness now follows by Kolmogorov's criterion, cf. [EK86].

We will now show that the finite-dimensional distributions of X^n converges in P^φ-probability to the finite-dimensional distributions of X. Take $0 \le t_1, \dots, t_l < \infty$ and define

$$
\|f\|_{BL} := \|f\|_\infty + \sup\left\{ \frac{|f(x) - f(y)|}{\varrho(x,y)} \,\Big|\, x, y \in S,\ x \ne y \right\}\ . \tag{4.15}
$$

Then

$$
\begin{aligned}
& P^\varphi\left[\sum_{i=1}^{l} \sup_{\|f\|_{BL}\le 1} \left|\int f(x)(X_{t_i}^n(dx) - X_{t_i}(dx))\right| > \varepsilon\right] \\
& = P^\varphi\left[\sum_{i=1}^{l} \sup_{\|f\|_{BL}\le 1} \left|\sum_{j=1}^{k_n} \int_{A_j^n} (f(x_j^n) - f(x)) X_{t_i}(dx) + \int_{A_{k_n+1}^n} (f(x_{k_n+1}^n) - f(x)) X_{t_i}(dx)\right| > \varepsilon\right] \\
& \le P^\varphi\left[\sum_{i=1}^{l} \sup_{\|f\|_{BL}\le 1} \left|\sum_{j=1}^{k_n} \int_{A_j^n} |x_j^n - x| X_{t_i}(dx) + 2X_{t_i}(A_{k_n+1}^n)\right| > \varepsilon\right]
\end{aligned}
$$

$$\leq P^{\varphi}\left[\sum_{i=1}^{l}|\frac{1}{n}+2X_{t_i}(A^n_{k_n+1})|>\varepsilon\right]\leq\frac{1}{\varepsilon}\sum_{i=1}^{l}\left(\frac{1}{n}+2I(A^n_{k_n+1})\right)$$
$$\leq\frac{1}{\varepsilon}\,\frac{1}{n}\,3l\longrightarrow 0\ ,$$

by the choice of $A^n_i,\ i=1,\ldots,k_n+1,\ n\in\mathbb{N}$.

(ii): In the case where $\varphi\equiv 1$, we can refine the last arguments for the identification of the limit point in part (i) to prove (ii). If $P:=P^{\varphi}$ for $\varphi\equiv 1$, we have that

$$P\left[\sup_{t\leq T}\sup_{\|f\|_{BL}\leq 1}\frac{1}{n}+2X_t(A^n_{k_n+1})>\varepsilon\right]\leq\frac{8}{\varepsilon^2}\int_0^T E\left[X_s(A^n_{k_n+1})(1-X_s(A^n_{k_n+1}))\right]ds\ . \quad (4.16)$$

This can be seen as follows: By (4.2), (4.3)

$$\tilde{M}_t:=X_t(A^n_{k_n+1})-X_0(A^n_{k_n+1})+\theta t\nu_0(A^n_{k_n+1})-\theta\int_0^t X_s(A^n_{k_n+1})ds,\ t\geq 0$$

is a martingale with $\langle\tilde{M}\rangle_t=\int_0^t X_s(A^n_{k_n+1})(1-X_s(A^n_{k_n+1}))ds$. Clearly,

$$X_t(A^n_{k_n+1})\leq X_0(A^n_{k_n+1})+\theta t\nu_0(A^n_{k_n+1})+\tilde{M}_t\ ,$$

hence with $\tilde{M}^0_t:=X_0(A^n_{k_n+1})+\tilde{M}_t$ we have for $n>\varepsilon$ that

$$\begin{aligned}
&P\left[\sup_{t\leq T}X_t(A^n_{k_n+1})>(\varepsilon-\frac{1}{n})\cdot\frac{1}{2}\right]\\
&\leq P\left[\sup_{t\leq T}X_t(A^n_{k_n+1})-\theta t\nu_0(A^n_{k_n+1})>(\varepsilon-\frac{1}{n})\cdot\frac{1}{2}-\theta T\nu_0(A^n_{k_n+1})\right]\\
&\leq P\left[\sup_{t\leq T}\tilde{M}^0_t>(\varepsilon-\frac{1}{n})\frac{1}{2}-\theta T\nu_0(A^n_{k_n+1})\right]\\
&\leq\frac{2}{((\varepsilon-\frac{1}{n})\frac{1}{2}-\theta T\nu_0(A^n_{k_n+1}))^2}E[\langle\tilde{M}^0\rangle_T]
\end{aligned}$$

which implies (4.16). Finally, the fact that $X_s(A^n_{k_n+1})(1-X_s(A^n_{k_n+1}))\leq X_s(A^n_{k_n+1})$ gives that the right hand side of (4.16) is bounded by

$$\frac{8}{\varepsilon^2}\int_0^T\nu_0(A^n_{k_n+1})ds\leq\frac{8}{\varepsilon^2}T\nu_0(A^n_{k_n+1})\leq\frac{2}{\varepsilon^2}\,T\,\frac{1}{n}$$

(since $A^n_{k_n+1}\subset K^c_n$). □

Remark 4.8 If S is locally compact, we shall now show that for all $\alpha \in]0, \frac{1}{2}[$ X is α-Hölder-continuous w.r.t. the metric

$$\varrho_v(\mu, \nu) := \sum_{i=1}^{\infty} \frac{1}{2^i} |\mu(f_i) - \nu(f_i)| \ .$$

Here $\{f_i | i \in \mathbb{N} = 1\}$ is a bounded set of continuous functions with compact support which is dense in $C_c(S)$. The metric ϱ_v is complete and the associated topology is the vague topology on E. Let $p \in (2, \infty)$ be the exponent in the proof of the tightness of $(X_t^n(f))_{t \geq 0}$ above. For $0 \leq s < t$ we then have

$$\begin{aligned}
&\left(\int \varrho_v(X_t^n, X_s^n)^p dP^\varphi \right)^{1/p} \leq E\left[\left(\sum_{i=1}^{\infty} \frac{1}{2^i} |X_t^n(f_i) - X_s^n(f_i)| \right)^p \right]^{1/p} \\
&\leq \sum_{i=1}^{\infty} \frac{1}{2^i} E[|X_t^n(f_i) - X_s^n(f_i)|^p]^{1/p} \\
&\leq 2c_p \sup_{i \in \mathbb{N}} \|f_i\|_\infty (t-s)^{1/2} \ ,
\end{aligned}$$

where we used (4.14) in the last step. This implies by a simple modification of standard arguments (see [StV79, Corollary 2.1.4]) the Hölder continuity for all $\beta \in]0, \frac{1}{2} - \frac{1}{p}[$. Because $p \in (2, \infty)$ was arbitrary, we have Hölder continuity for all $\alpha \in]0, \frac{1}{2}[$. □

In the case where $\varphi \equiv 1$, Theorem 4.7 and Subsection 3.2 justify to consider X as a (time-changed) Brownian motion with state-space E. However, we don't have a control about the limit behavior of the time-changes of the finite-dimensional approximations. In the following section we will analyze the Fleming-Viot process from the point of view of intrinsic metrics. This can be viewed as an interpretation of the Fleming-Viot process as a Brownian motion on E equipped with a suitable metric.

5 The intrinsic metric of the parent-independent Fleming-Viot process

In this section we study the intrinsic metrics associated with certain Dirichlet forms having a state of measures as state space (cf. [ORS95]). Our main application will be the Dirichlet form associated with Fleming-Viot process with PIM.

5.1 The intrinsic metric

Let $(S, \mathcal{S})$ be a standard measurable space. We keep the notation of Definition 4.1. We define the *intrinsic gradient* $Du(\mu)$ of $u \in \mathcal{F}C_b^\infty(\mathcal{S}_b)$ at $\mu \in E$ by

$$Du(\mu) := (D_x u(\mu))_{x \in S} \text{ with } D_x u(\mu) := \frac{d}{ds} u((1-s)\mu + s\varepsilon_x)\Big|_{s=0} \tag{5.1}$$

where ε_x denotes Dirac measure at $x \in S$. Clearly, by the (finite-dimensional) chain rule

$$D_x u(\mu) = \sum_{i=1}^k \frac{\partial \varphi}{\partial y_i} \Big(\int f_1\, d\mu, \ldots, f_k\, d\mu \Big) (f_i(x) - \int f_i\, d\mu) \tag{5.2}$$

if $x \in S,\ \mu \in E,\ u(\mu) = \varphi(\int f_1\, d\mu, \ldots, \int f_k\, d\mu)$. Note that

$$Du(\mu) \in \mathcal{S}_b \text{ for all } u \in \mathcal{F}C^\infty(\mathcal{S}_b),\ \mu \in E\ . \tag{5.3}$$

Hence we may set

$$\Gamma(u, v)(\mu) := (Du(\mu), Dv(\mu))_{L^2(S;\mu)} \text{ for } u, v \in \mathcal{F}C^\infty(\mathcal{S}_b)\ . \tag{5.4}$$

Remark 5.1 By (4.4), (4.11), and (5.2) we obviously have that

$$(Du(\mu), Dv(\mu))_{L^2(S;\mu)} = \mathrm{cov}_\mu(\nabla.u(\mu), \nabla.v(\mu))\ .$$

Define the following metric d on E by

$$d(\mu, \nu) := \sup\{u(\mu) - u(\nu) \mid u \in \mathcal{F}C^\infty(\mathcal{S}_b) \text{ with } \Gamma(u,u) \le 1\},\ \mu, \nu \in E\ . \tag{5.5}$$

Clearly, d is zero on the diagonal, symmetric, and satisfies the triangle inequality. We shall see below that it is finite and separating (cf. Theorem 5.1 below). We call the metric d *intrinsic metric*. This terminology is justified by the following "geometrical set-up". One should consider E as "the manifold" and $L^2(S;\mu)$ as the "tangent space" to E at $\mu \in E$. The "riemannian structure" is given by $(\ ,\ )_{L^2(S;\mu)}$. Du is then the "gradient vector field" of smooth function u on E and $\Gamma(u,u)^{1/2}$ its "norm".

The following theorem is the main result of this section. To state and prove it, we recall that the *variation norm* of a *signed* measure λ on $(S, \mathcal{S})$ is defined by

$$\begin{aligned} \|\lambda\|_{\mathrm{var}} \ := \ \sup\Big\{ & \textstyle\sum_{i=1}^n |\lambda(S_i)| \ \Big|\ n \in \mathrm{N},\ S_i \in \mathcal{S},\ S_i \cap S_j = \emptyset \\ & \text{if } i \ne j,\ 1 \le i, j \le n, \text{ and } S = \textstyle\bigcup_{i=1}^n S_i \Big\}\ . \end{aligned} \tag{5.6}$$

Clearly, if λ^+, λ^- denote the positive resp. negative part of λ in its Hahn-Jordan decomposition and if $|\lambda| := \lambda^+ + \lambda^-$, then

$$\|\lambda\|_{\mathrm{var}} = |\lambda|(S) = \sup\left\{ \int f \, d\lambda \;\middle|\; f \in \mathcal{S}_b, \; \|f\|_\infty \leq 1 \right\} . \tag{5.7}$$

Theorem 5.2 *For all ν, $\mu \in E$*

$$\|\mu - \nu\|_{\mathrm{var}} \leq d(\mu,\nu) \leq \sqrt{32} \|\mu - \nu\|_{\mathrm{var}}^{1/2} .$$

In particular, d is finite, separating, and complete.

Proof. Let $f \in \mathcal{S}_b$, $\|f\|_\infty \leq 1$, and

$$u(\mu) := \int f \, d\mu, \;\; \mu \in E .$$

Then $u \in \mathcal{F}C^\infty(\mathcal{S}_b)$ and by (5.2), $\Gamma(u,u) \leq 1$. Hence the first inequality follows by (5.7). To prove the second let $\mu, \nu \in E$ and $u \in \mathcal{F}C^\infty(\mathcal{S}_b)$ with $\Gamma(u,u) \leq 1$. Then, since $\mu(S) = \nu(S) = 1$,

$$\begin{aligned} u(\nu) - u(\mu) &= \int_0^1 \frac{d}{dt} u(\mu + t(\nu - \mu)) \, dt \\ &= \int_0^1 \int D_x u(\mu + t(\nu-\mu)) \, (\nu - \mu)(dx) \, dt \\ &\leq |\nu - \mu|(S)^{1/2} \int_0^1 \left[\int D_x u(\mu + t(\nu-\mu))^2 \, |\nu - \mu|(dx) \right]^{1/2} dt \\ &\leq \sqrt{2} \|\nu - \mu\|_{\mathrm{var}}^{1/2} \Big\{ \int_0^1 \left[\int D_x u(\mu + t(\nu-\mu))^2 \, \nu(dx) \right]^{1/2} dt \\ &\quad + \int_0^1 \left[\int D_x u(\nu + t(\mu-\nu))^2 \, \mu(dx) \right]^{1/2} dt \Big\} \end{aligned}$$

(where we substituted t by $(1-t)$ in the second integrand). Using that $\nu \leq \frac{1}{t}(\mu + t(\nu - \mu))$ and $\mu \leq \frac{1}{t}(\nu + t(\mu - \nu))$ as measures for $t \in [0,1]$ we see that the above is dominated by

$$\sqrt{2} \|\nu - \mu\|_{\mathrm{var}}^{1/2} \int_0^1 t^{-1/2} (\|Du(\mu + t(\nu-\mu))\|_{L^2(\mu + t(\nu - \mu))} + \|Du(\nu + t(\mu - \nu))\|_{L^2(\nu + t(\mu - \nu))}) \, dt .$$

Hence by assumption

$$u(\nu) - u(\mu) \leq \sqrt{2} \|\nu - \mu\|_{\mathrm{var}}^{1/2} \cdot 4 .$$

and the second inequality follows. The last assertion follows since the corresponding properties hold for $(\mu, \nu) \mapsto \|\mu - \nu\|_{\mathrm{var}}$. For the completeness see [DSch57, IV.5.1 and III.7.4]. $\square$

Remark 5.3 Let τ be a Polish topology on S such that its Borel σ-algebra conicides with $\mathcal{S}$. If in (5.5) we replace $\mathcal{S}_b$ by $C_{\tau,b}(S)$ (i.e., the bounded w.r.t. τ continuous functions on S) then by Theorem 5.1 we obtain a uniformly equivalent metric.

Corollary 5.4 *If S is countable, then (E,d) is complete and separable.*

Proof. Taking the discrete metric ϱ on S we see that for $\nu,\mu \in E$

$$\|\nu-\mu\|_{\text{var}} \le 3\sup\left\{\int f\,d\nu - \int f\,d\mu \,\Big|\, f\in\mathcal{S}_b,\ \|f\|_{BL}\le 1\right\} =: \beta(\nu,\mu)$$

where $\|\ \|_{BL}$ is as defined in (4.15). β is known to metrize the weak toplogy $\mathcal{W}_\varrho$ on E coming from ϱ. But $\mathcal{W}_\varrho$ is separable and (E,β) is complete (cf. [Du89, Theorem 11.3.3] and [B68, Appendix III]). $\square$

Remark 5.5 The intrinsic metric associated with general (strongly) local Dirichlet forms on locally compact separable metric spaces E has been studied intensively. We refer to [St93] and references therein. For instance it is proved in [St93] that if the intrinsic metric d is complete and compatible with the topology on E, then (E,d) is a geodesic space. Though E in our case is in general not locally compact, we expect that this and maybe some of the other results in modified form will hold in our situation (cf. [G94]).

5.2 Application to Fleming-Viot processes

We now consider the situation of Theorem 4.5 (ii) and the corresponding interactive Fleming–Viot processes. By Theorem 5.2 and [ORS95, Theorem 6.9] it follows that these Fleming–Viot processes have continuous paths w.r.t. the intrinsic metric d.

Remark 5.6 If $\varphi \equiv 1$ and $S=\{1,\dots,d\}$, then $E=\mathcal{M}_1(S)=\bar{\Delta}$ and the Dirichlet form in Theorem 4.5(ii) is just the Dirichlet form $(\mathcal{E},D(\mathcal{E}))$ in Subsection 2.3 with $q_i := \theta\nu_0(\{i\})$, $1\le i\le d$. Note that, since ν_0 is assumed to have full support (cf. Definition 4.1), $q_i>0$ for all $1\le i\le d$. By Theorem 5.2 the intrinsic metric d is uniformly equivalent to the metric coming from the norm $\|x\|=\sum_{i=1}^d |x_i|$, $x=(x_1,\dots,x_d)$, on $\mathbb{R}^d$, hence is equivalent with the Euclidean metric.

Appendix

Proof of Theorem 2.2. We first prove that the special form of the matrix Q and (2.6) are necessary. Taking $v = 1$ on $U \subset\subset A$ and $v \in C_0^2(\Delta)$ in the equality

$$\int_\Delta Lu \cdot v \, d\hat{\varrho} = \int_\Delta Lv \cdot u d\hat{\varrho} \ , \tag{A.0}$$

we see that the measure $\hat{\varrho}$ is a weak solution of the partial differential equation

$$L\hat{\varrho} = 0 \qquad \text{on } U.$$

Therefore, by elliptic regularity, $\hat{\varrho}$ is in the local Sobolev space $W_k(U)$ of each order k, hence has density ϱ which is in $C^\infty(U)$. Hence, (A.0) may be written as

$$\mathcal{E}^\varrho(u,v) := \int_\Delta Lu \cdot \ v\varrho \, dy = \int_\Delta Lv \ u\varrho \, dy \ .$$

Now we use Lemma 2.1 and integration by parts in order to obtain that for all $u, v \in C_0^\infty(\Phi(\Delta))$,

$$\begin{aligned}
&\mathcal{E}^\varrho(u,v) = \int \sum_{i,j=1}^{d-1} y_i(\delta_{ij} - y_j) \frac{\partial u}{\partial y_j}(y) \frac{\partial v}{\partial y_i}(y) \, \varrho(y) \, dy \\
&+ \int \sum_{i=1}^{d-1} v(y) \frac{\partial u}{\partial y_i}(y) \Big\{ y_i \frac{\partial \log \varrho}{\partial y_i}(y) - y_i \sum_{j=1}^{d-1} y_j \frac{\partial \log \varrho}{\partial y_i}(y) + \\
&(1 - dy_i) + \Big[q_{di} \sum_{l=1}^{d-1} y_l - \sum_{l=1}^{d-1} q_{li} y_l - q_{di} \Big] \Big\} \varrho(y) \, dy.
\end{aligned} \tag{A.1}$$

It is easy to see that suitable choices of u, v in the condition $\mathcal{E}^\varrho(u,v) = \mathcal{E}^\varrho(v,u)$ gives necessarily that for all $i = 1, \ldots d-1$ and all $y \in \Phi(\Delta)$

$$\begin{aligned}
0 \ = \ & \Big\{ y_i \frac{\partial \log \varrho}{\partial y_i}(y) - y_i \sum_{j=1}^{d-1} y_j \frac{\partial \log \varrho}{\partial y_i}(y) \Big\} \\
& + \Big\{ (1 - dy_i) + \Big[q_{di} \sum_{l=1}^{d-1} y_l - \sum_{l=1}^{d-1} q_{li} y_l - q_{di} \Big] \Big\}.
\end{aligned}$$

If we denote the second summand by $q_i(y)$ and $f_i(y) := \frac{\partial \log \varrho}{\partial y_i}(y), \ 1 \leq i \leq d-1$ then the vector $f(y)$ solves the matrix equation

$$A(y) \, f(y) = -q(y) \ , \tag{A.2}$$

where

$$A(y)_{i,j} := (y_i(\delta_{ij} - y_j))_{1 \le i,j \le d-1} \ , \quad q(y) := (q_i(y))_{1 \le i \le d-1} \ .$$

For every y with $y_i > 0,\ \sum_{i=1}^{d-1} y_i < 1$, the matrix A is invertible with

$$A^{-1}(y) = \left(\delta_{ij} \left(\frac{1}{y_i} \right) + \frac{1}{1 - \sum_{l=1}^{d-1} y_l} \right)_{1 \le i,j \le d-1} . \tag{A.3}$$

Hence $\log \varrho$ is a solution of a system of the following differential equation of first order:

$$\begin{aligned} \frac{\partial \log \varrho}{\partial y_i}(y) \ &= \ -\Bigg\{ \frac{1}{y_i} \left(1 - d y_i + q_{di} \sum_{l=1}^{d-1} y_l - \sum_{l=1}^{d-1} y_l q_{li} - q_{di} \right) \\ &+ \frac{1}{1 - \sum_{l=1}^{d-1} y_l} \left(d - 1 - d \sum_{l=1}^{d-1} y_l + \sum_{j=1}^{d-1} q_{dj} \sum_{l=1}^{d-1} y_l - \sum_{l=1}^{d-1} y_l \sum_{j=1}^{d-1} q_{lj} \sum_{j=1}^{d-1} q_{dj} \right) \Bigg\} , \end{aligned} \tag{A.4}$$

$1 \le i \le d-1$. This can be simplified to

$$\begin{aligned} \frac{\partial \log \varrho}{\partial y_i}(y) \ &= \ -\Bigg\{ \frac{1}{y_i} \left(1 - q_{di} + \sum_{l=1}^{d-1} y_l (q_{di} - q_{li} \right) \\ &\quad + \frac{1}{1 - \sum_{l=1}^{d-1} y_l} \left(\sum_{l=1}^{d-1} y_l \left(\sum_{j=1}^{d-1} (q_{dj} - q_{lj}) \right) - 1 - \sum_{j=1}^{d-1} q_d j \right) \Bigg\} . \end{aligned} \tag{A.5}$$

Integration yields for $i = 1, \ldots, d-1$

$$\begin{aligned} \log \varrho(y) \ &= \ c_i(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{d-1}) - \\ &\Bigg\{ \log y_i \cdot \left(1 - q_{di} + \sum_{l=1}^{d-1} (1 - \delta_{il}) y_l (q_{di} - q_{li}) \right) + y_i (q_{di} - q_{ii}) \\ &- \log \left(1 - \sum_{l=1}^{d-1} y_l \right) \left(\sum_{l=1}^{d-1} (1 - \delta_{il}) y_l \sum_{j=1}^{d-1} (q_{dj} - q_{lj}) - 1 - \sum_{j=1}^{d-1} q_{dj} \right) \\ &- y_i \sum_{j=1}^{d-1} (q_{dj} - q_{ij}) \\ &- \log \left(1 - \sum_{l=1}^{d-1} y_l \right) \left(1 - \sum_{j=1}^{d-1} y_l (1 - \delta_{il}) \right) \sum_{j=1}^{d-1} (q_{dj} - q_{ij}) \Bigg\} \\ &= \ c_i(y_1, \ldots, y_{i-1}, y_{i_1}, \ldots, y_{d-1}) - \\ &\Bigg\{ \log y_i \cdot \left(1 - q_{di} + \sum_{l=1}^{d-1} (1 - \delta_{il})(q_{di} - q_{li}) y_l \right) \end{aligned}$$

$$-y_i \sum_{j=1}^{d-1} (1-\delta_{ji})(q_{dj}-q_{ij})$$

$$+\log\left(1-\sum_{l=1}^{d-1} y_l\right)\left(1+\sum_{j=1}^{d-1} q_{ij}\right)\Bigg\}\ , \tag{A.6}$$

where c_i indicates a function which does not depend on y_i and which is obviously in C^1. Now we differentiate (A.6) for $i=1$ with respect ot y_2 and obtain using (A.5) that

$$\begin{aligned} -\Bigg\{ & \frac{1}{y_2}\left(1-q_{d2}+\sum_{l=1}^{d-1} y_l(q_{d2}-q_{l2})\right) \\ & +\frac{1}{1-\sum_{l=1}^{d-1} y_l}\left(\sum_{l=1}^{d-1} y_l\left(\sum_{j=1}^{d-1}(q_{dj}-q_{lj})\right)-1-\sum_{j=1}^{d-1} q_{dj}\right)\Bigg\} \\ = \; & \frac{\partial}{\partial y_2} c_1(y_2,\ldots,y_d-1)-\left\{\log y_1(q_{d1}-q_{21})-\frac{1}{1-\sum_{l=1}^{d-1} y_l}\left(1+\sum_{j=1}^{d-1} q_{1j}\right)\right\}\ . \end{aligned} \tag{A.7}$$

Differentiation of (A.7) with respect to y_1 gives

$$\begin{aligned} 0 = & \frac{1}{y_2}(q_{d2}-q_{l2})+\frac{\sum_{j=1}^{d-1}(q_{dj}-q_{1j})}{\left(1-\sum_{l=1}^{d-1} y_l\right)} \\ & +\frac{\sum_{l=1}^{d-1} y_l\left(\sum_{j=1}^{d-1}(q_d j-q_{lj})\right)+\sum_{j=1}^{d-1}(q_{1j}-q_{dj})}{\left(1-\sum_{l=1}^{d-1} y_l\right)^2} \\ & -\frac{1}{y_1}(q_{d1}-q_{21})\ . \end{aligned} \tag{A.8}$$

This is equivalent with

$$0 = y_1^3(q_{d2}-q_{12})\cdot\frac{1}{y_2}+\sum_{i=1}^{2}\tilde{c}_i(y_2,\ldots,y_{d-1})y_1^i\ . \tag{A.9}$$

In particular, this implies

$$q_{d2}=q_{12}\ . \tag{A.10}$$

The same procedure with $i=k$ in (A.6) and with differentiation with respect to $y_j,\ j\neq k$ proves that

$$q_{dj}=q_{kj}\quad \forall\ 1\leq j\neq k\leq d-1\ ,$$

i.e.,

$$Q = \begin{pmatrix} q_{11} & q_{d2} & \cdots & q_{dd-1} & q_{1d} \\ q_{d1} & q_{22} & \cdots & q_{dd-1} & q_{2d} \\ q_{d1} & q_{d2} & & \ddots & \\ \vdots & \vdots & & & \\ q_{d1} & q_{d2} & \cdots & q_{dd-1} & q_{dd} \end{pmatrix} . \tag{A.11}$$

Now we can use (A.11) and the fact that Q is a Q-matrix and obtain by (A.6) that

$$\begin{aligned} \log \varrho(y) &= c_i(y_1, \ldots, y_{i-1}, y_{i_1}, \ldots, y_{d-1}) \\ &\qquad - \left\{ \log y_i (1 - q_{di}) + \log \left(1 - \sum_{l=1}^{d-1} y_l\right)(1 - q_{id}) \right\} . \end{aligned} \tag{A.12}$$

We will show that

$$q_{1d} = \cdots = q_{d-1,d} =: q_d \ .$$

Again we consider (A.12) for $i = 1$ and differentiate with respect to y_2. Then (A.5) with $i = 2$ implies (again using (A.12) and that Q is a Q-matrix) that

$$\begin{aligned} &- \left\{ \frac{1}{y_2}(1 - q_{d2}) + q_{d2} - q_{22} + q_{dd} + \frac{1}{1 - \sum_{l=1}^{d-1} y_l} \left(\sum_{l=1}^{d-1} y_l q_{ld} - 1 \right) \right\} = \\ &= \frac{\partial}{\partial y_2}(y) c_1(y_2, \ldots, \ldots, y_{d-1}) + \frac{1}{1 - \sum_{l=1}^{d-1} y_l}(1 - q_{1d}) \ . \end{aligned} \tag{A.13}$$

Differentiation with respect to y_1 yields

$$\frac{1}{(1 - \sum_{l=1}^{d-1} y_l)^2} \sum_{l=1}^{d-1} y_l q_{ld} + \frac{1}{1 - \sum_{l=1}^{d-1} y_l} q_{1d} = \frac{1}{1 - \sum_{l=1}^{d-1} y_l} q_{1d}$$

which finally implies

$$q_{ld} = q_{1d}, \ l = 1, \ldots, d-1 \ .$$

If we set $q_i := q_{di}, \ i = 1, \ldots, d-1, \ q_d = q_{1d}$, then (A.5) simplifies to

$$\frac{\partial \log \varrho}{\partial y_i}(y) = \frac{1}{y_i}(q_i - 1) + (1 - q_d)\left(1 - \sum_{l=1}^{d-1} y_l\right)^{-1} . \tag{A.14}$$

Hence integrating (A.14) for $i = 1$ we obtain

$$\log \varrho(y) = c_1(y_2, \ldots, y_{d-1}) + (q_1 - 1)\log y_1 - (1 - q_d)\log\left(1 - \sum_{l=1}^{d-1} y_l\right) . \tag{A.15}$$

Differentiating with respect to y_2 and using (A.14) for $i=2$ we get

$$\frac{1}{y_2}(q_2-1)+(1-q_d)\left(1-\sum_{l=1}^{d-1}y_l\right)^{-1}=\frac{\partial}{\partial y_2}c_1(y_2,\dots,y_{d-1})+(1-q_d)\left(1-\sum_{l=1}^{d-1}y_l\right)^{-1}.$$

Hence

$$c_1(y_2,\dots,y_{d-1})=(q_2-1)\log y_2+c_{1,2}(y_3,\dots,y_{d-1})\ ,$$

where $c_{1,2}$ does not depend on y_1 and y_2.

Substituting this into (A.15) we are led to

$$\log\varrho(y)=-\log y_1(1-q_1)-\log y_2(1-q_2)\log\left(1-\sum_{l=1}^{d-1}y_l\right)(1-q_d)+c_{1,2}(y_3,\dots,y_{d-1})\ .$$

Proceeding succesively one arrives at

$$\log\varrho(y)=-\sum_{l=1}^{d-1}\log y_i(1-q_i)-(1-q_d)\log\left(\sum_{l=1}^{d-1}y_l\right)+c$$

for some $c\in\mathbb{R}$, which proves (2.6).

To see that (2.5), (2.6) imply (2.4) one simply uses (A.4), the product rule, and the fact that

$$\varrho^{-1}\frac{\partial\varrho}{\partial y_i}=\frac{q_i-1}{y_i}-\frac{q_{d-1}}{1-\sum_{l=1}^{d-1}y_l}\ .$$

The same argument implies (2.7) by Green's theorem provided $q_i>0,\ i=1,\dots,d$. (Note that, as is easily seen, $\tilde{\varrho}\,dy$ is indeed a finite measure.)

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