Dirichlet forms on infinite-dimensional "manifold-like" state spaces: a survey of recent results and some prospects for the future

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Dedicated to our admired colleague, teacher, and friend Prof. Masatoshi Fukushima on the occasion of his 60th birthday.

Abstract

We give a (to some extent pedagogical) survey on recent results about Dirichlet forms on infinite-dimensional "manifold-like" state spaces including path and loop spaces as well as spaces of measures. The latter are associated with interacting Fleming-Viot processes resp. infinite particle systems. Also some new results, further developing the Dirichlet form approach to infinite particle systems, are enclosed. Finally, a brief summary of other research activities in the theory of Dirichlet forms is given and some prospects for the future are indicated.

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1 Introduction

The purpose of this paper is to give a survey of recent results on Dirichlet forms and their associated Markov processes on infinite-dimensional "manifold-like" state spaces. At the same time we try to indicate directions of future research in this area. The selection of topics presented here is very much biased and concentrates on results mostly achieved by the author in collaboration with a number of colleagues. The exposition is to some extent pedagogical, adressing the non-expert reader, supressing technicalities as much as possible, and giving more emphasis to ideas and prospects rather than proofs. We start with the classical case of gradient-type Dirchlet forms on finite-dimensional manifolds in Section 2, summarizing the necessary modifications to go to infinite-dimensional situations. The latter are presented in the subsequent Sections 3 to 6 in the shape of three *model-case studies*. The first concerns gradient-type Dirichlet forms on Banach (or topological vector) spaces. Special emphasis is given here to the classical Wiener space (cf. Section 3). In the second and third model-case studies the underlying infinite-dimensional "manifolds" are *no longer flat*. More precisely, we consider Dirichlet forms on loop spaces over compact finite-dimensional Riemannian manifolds (cf. Section 4) resp. Dirichlet forms on state spaces of measures over a polish space (cf. Sections 5 and 6). The latter are related to Fleming-Viot processes resp.infinite particle systems. We emphasize that in Section 6 all underlying hard results are borrowed from [Y95], [Os95]. Our contribution here is only to provide the suitable geometric framework, i.e., we identify a natural underlying tangent bundle structure which, in particular, enables us to introduce more general Dirichlet forms than the ones studied in [Y95], [Os95] (e.g. we can allow non-constant diffusion coefficients). Thus, Section 6 contains really new results and should hence be of interest also for experts. Subsequently, in Section 7 we describe a recent application of gradient-type Dirichlet forms on Banach spaces (cf. Section 3) to the study of Gibbs states in statistical mechanics. Finally, we include a list of other topics presently being studied intensively within Dirichlet space theory. However, we do not claim any completeness of this list, neither w.r.t. the topics nor the scientists mentioned. On the contrary we would like to apologize at this point towards all people who should have been also mentioned or whose work should have been quoted. To keep the cited literature within bearable limits we mostly refer to the recent monographs [BH91], [MR92], [FOT94] and the references therein as well as the proceedings [FFGKRS93], [MRYa95].

Concerning the prospects for the future I do not wish to single out any special sub-branches in Dirichlet form theory that I personally think are of particular importance. I only want to express my sincere conviction that this theory and its applications bear a substantial amount of deep mathematics that only partially has been discovered. A lot of progress has been made, but we are far from being even only remotely close to the frontiers. I hope that this paper will contribute to attract more attention of both the working probabilist and analyst to the theory of Dirichlet forms, an area of mathematics which, as I am convinced, will be extremely active also in the next decade and, most probably, much further into the next century.

2 Review of the finite-dimensional case and motivations

Let (M, g) be an oriented complete Riemannian manifold with volume element dx. Consider the corresponding Laplace-Beltrami operator $L := \frac{1}{2}\Delta$ on (real) $L^2(M; dx)$ defined as the closure of the restriction of L to $C_0^\infty(M)$ (:= the set of all smooth compactly supported functions on M). Hence L is a self-adjoint operator on $L^2(M; dx)$ and let us denote its corresponding operator semigroup by $(T_t)_{t\geq 0}$, i.e., $T_t := e^{\frac{1}{2}\Delta t}$, $t \geq 0$, on $L^2(M; dx)$. As is well-known, L generates the Brownian motion $\mathbb{M} := (P_x)_{x\in M}$ on M, i.e., each P_x is a probability measure on $\Omega :=$ up to the life time ζ continuous paths $\omega : [0, \zeta(\omega)[\longrightarrow M$ with $\omega(0) = x$, such that for all $u \in L^2(M; dx)$, $t \geq 0$,

$$\begin{aligned} &x \mapsto \int_\Omega u(\omega(t))\ P_x(d\omega) \text{ is a } dx\text{-version} \\ &\text{of (the } L^2(dx)\text{-class) } T_t u\ . \end{aligned} \tag{2.1}$$

Integrating by parts for $u, v \in C_0^\infty(M)$ we obtain

$$\begin{aligned} -\int_M Lu\ v\ dx &= \frac{1}{2}\int_M g(\nabla u, \nabla v)\ dx \\ &=: \mathcal{E}(u, v) \end{aligned} \tag{2.2}$$

and taking the completion $D(\mathcal{E}) := \overline{C_0^\infty(M)}$ of $C_0^\infty(M)$ w.r.t. the norm $(\mathcal{E} + (\ ,\ )_{L^2(M;dx)})^{1/2}$ we get the associated *Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$. Note that $D(\mathcal{E}) \subset L^2(M; dx)$ and that $D(\mathcal{E})$ is nothing but $H_0^{1,2}(M; dx)$, the classical Sobolev space on M of order 1 in $L^2(M; dx)$. $(\mathcal{E}, D(\mathcal{E}))$ is also sometimes called *Dirchlet space*. $(\mathcal{E}, D(\mathcal{E}))$ contains the entire information about L, hence about $(T_t)_{t\geq 0}$ and thus about $\mathbb{M}$. In fact, they are all "equivalent objects":

$$\mathcal{E} \longleftrightarrow L \longleftrightarrow (T_t)_{t\geq 0} \longleftrightarrow \mathbb{M} \tag{2.3}$$

The advantage of $\mathcal{E}$ over $(T_t)_{t\geq 0}$ is that it is more explicit, and w.r.t. L it is a "first order" object and hence (in particular, in regard to domain questions) easier to analyse resp. to construct in much more general situations, as we shall see below. The heart of Dirichlet space theory is to use this simpler object $(\mathcal{E}, D(\mathcal{E}))$ to study the corresponding Markov process, that is, the Brownian motion $\mathbb{M}$ directly, and vice versa, to use $\mathbb{M}$ to deduce information about the two analytic objects L and $(T_t)_{t\geq 0}$ via the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. In other words the theory of Dirichlet forms can be described as classical potential theory via a Hilbert space approach (i.e., within an L^2-framework).

As mentioned in the introduction the main purpose of this paper is to explain how to "lift" this scheme resp. machinery resp. theory to cases where the manifold M is infinite-dimensional. The first problem one meets is that there is no reasonable exact analogue of the volume element dx (resp. the Lebesgue measure in the flat case) and that, nevertheless, one *needs* an underlying *reference measure*. So, let us first modify the above finite-dimensional situation by inserting a density $\rho : M \to [0, \infty)$ in front of dx. The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ can still be written down explicitly:

$$\begin{aligned}
&\mathcal{E}(u, v) := \frac{1}{2} \int g(\nabla u \ , \nabla v) \ \rho \ dx \ , \\
&u, v \in D(\mathcal{E}) := H_0^{1,2}(M; \rho dx) := \\
&\qquad \text{completion of } C_0^\infty(M) \text{ w.r.t. } (\mathcal{E} + (\ ,\ )_{L^2(M;\rho dx)})^{1/2}
\end{aligned} \tag{2.4}$$

However, one has to make sure that this completion $H_0^{1,2}(M; \rho dx)$ still embeds (*one-to-one*) into $L^2(M; \rho dx)$. This problem is known as the *closability problem* and is a key problem of the theory. Above it can easily be overcome by imposing very weak conditions on ρ (as e.g. lower-semi-continuity, see the references in [MR92]). Again, as comes out of the general theory (cf. below) we have the correspondence (2.3). E.g., the operator L has the heuristic representation

$$L = \frac{1}{2}\Delta + \frac{1}{2}\nabla \log \rho \cdot \nabla \ . \tag{2.5}$$

Though we have quite limited information about the corresponding semigroup $(T_t)_{t\geq 0}$, the associated process $\mathbb{M}$, the so-called *distorted Brownian motion*, has been completely analysed through the Dirichlet form (2.4) by M. Fukushima (see the references in [FOT94]).

The advantage of this modification is that infinite-dimensional analogues of ρdx *do* exist. Before we make the transition to infinite dimensions let us first briefly summarize the underlying abstract scheme resp. theory.

We confine ourselves to *symmetric Dirichlet forms* since essentially only these will appear in the concrete model-cases presented in subsequent sections (see [MR92] for the general case).

We arrive at the abstract scheme of Dirichlet space theory as follows: replace the manifold M by a general Hausdorff topological space E and dx (resp. ρdx) by a σ-finite measure m on its Borel σ-algebra $\mathcal{B}(E)$. An abstract (symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E; m)$ is a positive definite symmetric bilinear form $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ on a dense linear domain

$D(\mathcal{E}) \subset L^2(E;m)$ which is *closed* (i.e., $D(\mathcal{E})$ is complete w.r.t. the norm $(\mathcal{E} + (\ , \ )_{L^2(E;m)})^{1/2}$) such that the following property (called *contraction* or *Dirichlet property*) holds:

$$u^{\#} := \min(\max(u,0),1) \in D(\mathcal{E}) \text{ and } \mathcal{E}(u^{\#},u^{\#}) \le \mathcal{E}(u,u) \text{ for all } u \in D(\mathcal{E}). \tag{2.6}$$

In order to have the complete correspondence (2.3) including the process $\mathbb{M}$ as well as all the theory of Dirichlet forms available in this general case, we need, however, an additional assumption on $(\mathcal{E}, D(\mathcal{E}))$. It is clear from (2.1) that the existence of $\mathbb{M}$ implies some sort of regularity of $(T_t)_{t\ge 0}$ hence of $(\mathcal{E}, D(\mathcal{E}))$. This additional property of $(\mathcal{E}, D(\mathcal{E}))$ is called *quasi-regularity*. It is an analytic property that characterizes all Dirichlet forms which are associated with right continuous strong Markov processes. Its discovery was the final step of a long development within the theory which originates from early purely analytic work by A. Beurling and J. Deny which was given its probabilistic counterpart through fundamental work by M. Fukushima and M.L. Silverstein under some additional regularity hypothesis. The above complete characterization was finally achieved in [AMR93] (see also [MR92]). But in the meantime numerous people (as e.g. A. Ancona, N. Bouleau, S. Carrillo–Menendez, E.B. Dynkin, P. Fitzsimmons, F. Hirsch, R. Høegh–Krohn, S. Kusuoka, Y. LeJan, T. Lyons, Y. Oshima, M. Takeda, B. Schmuland, ...) made important contributions to the development (cf. the "Notes" in [BH91], [MR92], and [FOT94]). So, if the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, the full correspondence (2.3) (including the theory that goes with it) is at our disposal, in particular, there exists an associated process $\mathbb{M}$. We refer to [MR92, Diagrams 1,2,3 on pp. 14, 27, 39 respectively] for the precise description of the connecting arrows "$\longleftrightarrow$" in (2.3) in this general case.

Though not complicated, we shall not give the precise definition of quasi-regularity, but instead look into the following more concrete situation, sufficient to understand the examples later. Let us assume that in (2.3) $\mathbb{M}$ has continuous sample paths and no path with a finite life-time terminates inside E. Then we have the following representation (cf. [FOT94, Chap. 5] and [MR92, Chap. VI]) for our abstract Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$:

$$\mathcal{E}(u,v) = \frac{1}{2}\int d\mu_{\langle u,v\rangle} \ , \quad u,v \in D(\mathcal{E}) \ , \tag{2.7}$$

where $\mu_{\langle u,v\rangle}$ are signed measures (so-called *energy-measures*) on $\mathcal{B}(E)$ satisfying the chain rule

$$\mu_{\langle uw,v\rangle} = u\mu_{\langle w,v\rangle} + w\mu_{\langle u,v\rangle} \ , \tag{2.8}$$

$u, v, w \in D(\mathcal{E})$, bounded. In the classical situation above and all cases in the subsequent sections we have $\mu_{\langle u,v\rangle} \ll m$, i.e.,

$$\mu_{\langle u,v\rangle} = \Gamma(u,v) \cdot m \tag{2.9}$$

(where $\Gamma(\cdot,\cdot)$ is called *square-field operator*), and more precisely even

$$\Gamma(u,v) = g(\nabla u, \nabla v)\ . \tag{2.10}$$

Dirichlet forms satisfying (2.7) – (2.10) are called *of gradient-type.*

Our aim below is to present a "recepy" how to identify resp. construct the Dirichlet form (of gradient-type for simplicity) which is appropriate for the respective application one is interested in, and make the above scheme (i.e., correspondence (2.3)) available and the theory behind it work. Recall we still fix E, m as above.

The following ingredients are sufficient:

(G) We need to find a suitable *gradient* (satisfying the chain rule) on a space D of sufficiently many continuous test functions in $L^2(E;m)$. In particular, we need to find a corresponding ("tangent"-) bundle structure. This will give us what is called a *pre-Dirichlet form*, namely $\mathcal{E}(u,v)$ for $u, v \in D$ on $L^2(E;m)$.

(C) We need to check whether $(\mathcal{E}, D)$ is *closable* on $L^2(E;m)$, i.e., whether the continuous extension of the inclusion map $D \subset L^2(E;m)$ to the completion $D(\mathcal{E}) := \overline{D}$ of D w.r.t. $(\mathcal{E} + (\ ,\ )_{L^2(E;m)})^{1/2}$ is one-to-one. $(\mathcal{E}, D(\mathcal{E}))$, called the *closure* of $(\mathcal{E}, D)$ on $L^2(E;m)$, is then automatically a Dirichlet form, since the contraction property (2.6) immediately follows from the chain rule. In analogy to (2.4), $D(\mathcal{E})$ can be considered as a generalized Sobolev space $H_0^{1,2}(E;m)$ on E.

(T) We have to show whether the following *tightness* condition holds: there exist compact $K_n \subset E$, $n \in \mathbb{N}$, such that $\{u \in D(\mathcal{E}) | u = 0$ on $E \setminus K_n$ for some $n \in \mathbb{N}\}$ is dense in $D(\mathcal{E})$ w.r.t. $(\mathcal{E} + (\ ,\ )_{L^2(E;m)})^{1/2}$. This implies that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular.

We shall not formally prove that (G), (C), (T) imply that $(\mathcal{E}, D(\mathcal{E}))$ is in fact a quasi-regular Dirichlet form, though we already gave some indication concerning the rôle of (G) and (C). We rather ask the reader to simply believe us at this point or to consult [MR92, Chap. I and Chap. IV, Sections 1–3]. Instead, we present a number of examples where (G), (C), and (T) have been verified in detail and where the underlying geometry and the resulting processes are of particular interest. The latter will be strong Markov processes which (because of (G)) automatically have continuous sample paths, i.e., they are *diffusions.* Furthermore, in all examples below (except for the last in Section 6 if b or d or c is not identically equal to zero) these diffusions have m as *symmetrizing* and *invariant measure.*

Remark 2.1 Summarizing the above we know that if we can check (G), (C), (T), then the correspondence (2.3) holds (in particular there exists an associated diffusion) and the entire theory of Dirichlet forms is available. We emphasize that this also means that (essentially) all results in [FOT94] are applicable though they are proved there only for *regular* Dirichlet forms on *locally compact* separable metric *state spaces*. These results, however, generalize immediately to quasi-regular Dirichlet forms on arbitrary state spaces by the so-called *transfer method* developed in [MR92, Chap. VI]. Thus, we can e.g. apply spectral synthesis, capacity-methods, quasi-sure analysis, Beurling-Deny-LeJan formulae, Revuz correspondence, stochastic analysis by additive functionals, Itô-Fukushima decomposition, forward/backward martingale decomposition, perturbation theory, traces, general criteria for conservativity, transience, recurrence, and for tightness of path space measures, etc.

We conclude this section with some comments on (G): one might ask the question whether we are interested in rather having a gradient than just a square field operator Γ (cf. (2.9)). It is well-known and, in fact, quite trivial (cf. [BH91, Chap. V, Exercise 5.9]) that for each such Γ one can construct various (of course, non-canonical) abstract bundle structures and gradients so that (2.10) holds. But this is almost useless for practical purposes (e.g., it does not help at all in solving the key problem (C) in concrete applications). A gradient-type Dirichlet form on a natural tangent bundle adapted to the respective situation, however, is very well suited for a deep analysis and, in particular, for constructing new Dirichlet forms from it. Thus, one can fully exploit the robustness of the theory w.r.t. all kinds of perturbations and apply the respective very well-developed techniques. We shall see this in some detail in Section 6 below.

3 Infinite-dimensional state space — flat case: Banach spaces

Let us restrict to a well-studied special case. We assume that

$$E := C_0([0,1] \to \mathbb{R}^d) := \{z : [0,1] \to \mathbb{R}^d | z \text{ is continuous and } z(0) = 0\}$$

equipped with the uniform norm, i.e., we replace our finite-dimensional manifold M in Section 2 by the *classical Wiener space*. Instead of the volume element dx we take

$$m := \textit{Wiener measure} \text{ on } E\ .$$

To realize condition (G) we take as test function space D the space $\mathcal{F}C_b^\infty$ of bounded smooth cylinder functions, i.e.,

$$\mathcal{F}C_b^\infty := \{f(l_1,\ldots,l_n) | n \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^n),\ l_1,\ldots,l_n \in E'\}$$

where E' denotes the dual of E and $C_b^\infty(\mathbb{R}^n)$ the space of all infinitely differentiable functions on $\mathbb{R}^n$ with all partial derivatives bounded. The bundle structure is in this case trivial, namely we take as "tangent space" to E at every $z \in E$ the same space, namely the classical *Cameron-Martin space*

$$\begin{aligned} H := \{h \in E \mid\ & h \text{ is absolutely continuous and} \\ & \|h\|_H^2 := \int_0^1 |h'|^2_{\mathbb{R}^d} dt < \infty\}. \end{aligned}$$

Then $E' \subset H' \equiv H \subset E$ densely and continuously. Let us denote the embedding $E' \subset H$ by j_H. Then we can define the gradient for $u \in \mathcal{F}C_b^\infty$ by

$$\nabla u(z) := j_H(u'(z))\ (\in H) \text{ for } z \in E\ ,$$

where $u'(z)$ $(\in E')$ is the Fréchet derivative of u in z, and we obtain the pre-Dirichlet form

$$\begin{aligned} \mathcal{E}(u,v) &:= \frac{1}{2}\int_E \langle \nabla u(z), \nabla v(z)\rangle_H\ m(dz)\ , \\ & u,v \in \mathcal{F}C_b^\infty\ . \end{aligned} \tag{3.1}$$

Thus, (G) is verified. It is also well-known that by the choice of H $(\mathcal{E},\mathcal{F}C_b^\infty)$ is closable on $L^2(E;m)$ (cf. e.g. [MR92, Chap. II, Corollary 3.13]), i.e., (C) holds. The corresponding closure $(\mathcal{E},D(\mathcal{E}))$ is of fundamental importance in the *Malliavin calculus* (cf. [MR92, Chap. II, Remark 3.14], [BH91, Chap. III] and the references therein). Also (T) holds for the Dirichlet form $(\mathcal{E},D(\mathcal{E}))$ (e.g. by the general result in [RS92] or [MR92, Chap. IV, Subsection 4b)]). Hence the above scheme (cf. (2.3)) applies. The corresponding diffusion is nothing but the Ornstein–Uhlenbeck process on Wiener space. It solves the stochastic differential equation

$$dX_t = dW_t^H - \frac{1}{2}X_t\,dt\ ,$$

where W^H is the Brownian motion on E with covariance $\langle\ ,\ \rangle_H$ (cf. e.g. [AR91, Section 7.I]).

A more complicated situation arises if above in case $d = 2,3$ instead of Wiener measure one takes m to be equal to the *polymer measure*. Since m is no longer Gaussian and not absolutely continuous w.r.t. Wiener measure if $d = 3$, it has only been discovered recently that (C) (and also (T)) holds so that everything above extends to this case except that the stochastic differential equation for the associated process cannot be written down. (cf. [AHRZho95] and [ARZho95] for details).

Also topological vector spaces E other than $C_0([0,1],\mathbb{R}^d)$ replacing our manifold M of Section 2 have been studied, as e.g. $E := \mathcal{S}'(\mathbb{R}^2)$ (i.e., the space of tempered distributions on $\mathbb{R}^2$). In this case again a fixed tangent space at each point of $\mathcal{S}'(\mathbb{R}^2)$ has been chosen, namely $H := L^2(\mathbb{R}^2; dx)$. Thus,

$$\mathcal{S}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2; dx) \subset \mathcal{S}'(\mathbb{R}^2) \ .$$

m was taken to be, for example, a *Euclidean (infinite volume) Φ_2^4-quantum field*. Defining the gradient similarly as above one thus realizes (G) and obtains the pre-Dirichlet form

$$\begin{aligned} \mathcal{E}(u,v) &= \int_{\mathcal{S}'(\mathbb{R}^2)} \langle \nabla u(z), \nabla v(z)\rangle_{L^2(\mathbb{R}^2;dx)}\, m(dz) \ , \\ & \qquad u, v \in \mathcal{F}C_b^\infty \ , \end{aligned} \tag{3.2}$$

on $L^2(\mathcal{S}'(\mathbb{R}^2); m)$. Both (C) and (T) have also been verified (cf. [AR90], [AR89] resp.). In this case the corresponding process is a weak solution of the non-linear stochastic differential equation

$$dX_t = dW_t^H + \frac{1}{2}((\Delta - 1)(X_t) - 4 : X_t^3 :)dt \ , \tag{3.3}$$

where $: z^3 :$ is the usual renormalized power of $z \in \mathcal{S}'(\mathbb{R}^2)$. We refer to [AR91] for details.

One main remaining open problem here is, whether the above process is the unique weak solution of (3.3). There has been a lot of progress also in this direction (see e.g. [RZ92, RZ94], [AKR95a]), but in cases where the drift is as singular as in (3.3) all results achieved so far do not apply. It seems that a completely new approach is required.

4 Infinite-dimensional state space — non-flat case: loop spaces over manifolds

Let (M, g) be an oriented compact (finite-dimensional) Riemannian manifold without boundary. Let $x_0 \in M$ be fixed and

$$E := \{\sigma : [0,1] \to M \mid \sigma \text{ continuous}, \sigma(0) = \sigma(1) = x_0\}$$

equipped with the topology of uniform convergence. Let m be *pinned Wiener measure* on E. So, E is our "manifold-like" state space, but to explain the tangent bundle in this case we first have to look at the "pinned" Cameron-Martin space "at x_0", i.e.,

$$H_0 := \{h : [0,1] \to T_{x_0}M \quad | \quad h \text{ absolutely continuous, } h(0) = h(1) = 0 \\ \text{and } \|h\|_{H_0}^2 := \int_0^1 g_{x_0}(h'(s), h'(s))ds < \infty\} .$$

Let $\tau_t(\sigma) : T_{x_0}M \to T_{\sigma(t)}M$, $t \in [0,1]$, be the stochastic parallel transport along σ (w.r.t. the Levy-Cevita connection on M). Now, we define the tangent space to E at $\sigma \in E$ by

$$H_\sigma := T_\sigma E := \{(\tau_t(\sigma)h(t))_{t\in[0,1]} \mid h \in H_0\} ,$$

that is, H_σ consists of all vector fields along σ obtained by stochastic parallel translation from any $h \in H_0$. Our test function space in this case is

$$\mathcal{F}C^\infty := \{\sigma \mapsto f(\sigma(s_1), \dots, \sigma(s_n)) \mid n \in \mathbb{N},\ f \in C^\infty(M^n),\ s_1, \dots, s_n \in [0,1]\} .$$

In order to obtain a gradient we first define the *directional derivative* of $\sigma \mapsto u(\sigma) := f(\sigma(s_1), \dots, \sigma(s_n))$ w.r.t. $(\tau_t(\sigma)h(t))_{t\in[0,1]}$ by

$$\partial_h u(\sigma) := \sum_{i=1}^n g_{\sigma(s_i)} \left(\nabla_i f(\sigma(s_1), \dots, \sigma(s_n)), \tau_{s_i}(\sigma)h(s_i)\right) ,\ \sigma \in E ,$$

where $\nabla_i f$ is the gradient of f on (M,g) w.r.t. to its i-th coordinate. Clearly, for $\sigma \in E$,

$$h \mapsto \partial_h u(\sigma)$$

is continuous on H_0 w.r.t. $\| \ \|_{H_0}$. Hence we can define the gradient $\nabla u(\sigma)$ as the unique element in H_0 such that

$$\langle \nabla u(\sigma), h\rangle_{H_0} = \partial_h u(\sigma) \quad \text{for all } h \in H_0 ,$$

and condition (G) is realized. We obtain a (w.r.t. (M,g) intrinsic) pre-Dirichlet form

$$\mathcal{E}(u,v) = \frac{1}{2} \int_E \langle \nabla u(\sigma), \nabla v(\sigma)\rangle_{H_0}\, m(d\sigma) , \\ u, v \in \mathcal{F}C_b^\infty ,$$

on $L^2(E;m)$ which (by construction of ∇ or more precisely the choice of the tangent space H_σ, $\sigma \in E$) has been shown to be closable (see e.g. [D92]), so (C) is fulfilled.

(T) has been proved in [DR92]. So, the scheme of Dirichlet space theory applies. The corresponding diffusion is the analogue of an Ornstein-Uhlenbeck process, but with values in a loop space (cf. [DR92]). So far, the Dirichlet form approach gives the only construction of this process.

Remark 4.1 A general method how to prove (T) in (not necessarily flat) infinite-dimensional situations including the above example, as well as the free loop space (cf. [ALR93]) was presented in [RS95].

To describe all open problems in this model-case would be beyond the scope of this paper. Let us only mention that the study of the geometry and, in particular, the topology of the underlying manifold through the above Dirichlet form or its associated process is more than fascinating.

For example, by a recent result of S. Aida and S. Kusuoka it follows that the Dirichlet form (or equivalently the process) is ergodic if and only if the manifold is simply connected. However, it is not known so far whether in this case the corresponding operator L (cf. (2.3)), which can be interpreted as a kind of Laplace-Beltrami operator on the loop space E, has a spectral gap. There is a a lot of activity going on in this direction (S. Aida, B. Driver, K.D. Elworthy, E. Hsu, Z.M. Ma,...) and also in developing a natural geometry on the infinite-dimensional manifold E itself (A.B. Cruzeiro, O. Enchev, S. Fang, P. Malliavin, D.W. Stroock,...).

5 Infinite-dimensional state space — non flat case: spaces of probability measures on a polish space

Let S be a polish space (i.e., a complete separable metric space) with Borel σ-algebra $\mathcal{B}(S)$, and replace our manifold M of Section 2 by

$$E := \mathcal{M}_1(S) := \text{all probability measures on } \mathcal{B}(S)$$

equipped with the weak topology. As a substitute for the volume element dx we take a probability measure $m := m_{\theta,\nu_0}$ on $\mathcal{B}(E)$ defined as follows:

$$m(A) := P\Big[\sum_{i=1}^{\infty} \rho_i \varepsilon_{\xi_i} \in A\Big]\ ,\ A \in \mathcal{B}(E)\,,$$

where ε_x, $x \in S$, denotes Dirac measure in x, $\xi_i : \Omega \to S$, $i \in \mathbb{N}$, are i.i.d. with distribution $\nu_0 \in \mathcal{M}_1(S)$ and $(\rho_1, \rho_2, \ldots) : \Omega \to [0,\infty)^{\mathbb{N}}$ is Poisson-Dirichlet distributed with θ (cf. [EK93, Theorem 8.1]). As the “tangent space” at $\mu \in E$ we take

$$T_\mu E := L^2(S;\mu)$$

and as test functions

$$\mathcal{F}C^\infty := \Big\{\mu \ \mapsto\ f\Big(\int \varphi_1 d\mu, \ldots, \int \varphi_n d\mu\Big)\Big|\ n \in \mathbb{N},\\ f \in C^\infty(\mathbb{R}^n),\ \varphi_1, \ldots, \varphi_n \in \mathcal{B}_b(S)\Big\}\ ,$$

where $\mathcal{B}_b(S)$ denotes the set of all bounded $\mathcal{B}(S)$-measurable functions on S. Now we define the gradient for $u \in \mathcal{F}C^\infty$ and $\mu \in E$ by

$$\nabla u(\mu) := (x \mapsto D_x u(\mu),\ x \in S)\ ,$$

where

$$D_x u(\mu) := \frac{d}{ds} u((1-s)\mu + s\,\varepsilon_x)\Big|_{s=0}\ .$$

If $u(\mu) = f(\int \varphi_1 d\mu, \dots, \int \varphi_n d\mu)$, $\mu \in E$, then by the chain rule for all $x \in S$, $\mu \in E$,

$$D_x u(\mu) = \sum_{i=1}^n \partial_i f\Big(\int \varphi_1 d\mu, \dots, \int \varphi_n d\mu\Big)(\varphi_i(x) - \int \varphi_i d\mu)\ ,$$

where ∂_i denotes the derivative w.r.t. the i-th coordinate. This shows that $x \mapsto D_x u(\mu)$ is in $\mathcal{B}_b(S)$, hence in $L^2(S;\mu)$. In other words ∇u is a vector field on E (w.r.t. the tangent bundle $(L^2(S;\mu))_{\mu\in E}$). Now we have the pre-Dirichlet form

$$\begin{aligned} \mathcal{E}(u,v) &= \frac{1}{2}\int_E \langle \nabla u(\mu), \nabla v(\mu)\rangle_{L^2(S;\mu)}\ m(d\mu) \\ & \quad u, v \in \mathcal{F}C^\infty \end{aligned}$$

on $L^2(E;m)$ and (G) is realized. Also (C) holds, which immediately follows from the identity

$$\mathcal{E}(u,v) = \int_E (-Lu)(\mu)\ v(\mu)\ m(d\mu) \tag{5.1}$$

for all $u, v \in \mathcal{F}C^\infty$, where

$$\begin{aligned} Lu(\mu) &= \frac{1}{2}\int_S\int_S \frac{\partial}{\partial \varepsilon_x}\Big(\frac{\partial u}{\partial \varepsilon_y}\Big)(\mu)(\varepsilon_x(dy) - \mu(dy))\ \mu(dx) \\ &\quad + \frac{\theta}{2}\int_S\int_S \Big(\frac{\partial u}{\partial \varepsilon_y}(\mu) - \frac{\partial u}{\partial \varepsilon_x}(\mu)\Big)\nu_0(dy)\mu(dx) \end{aligned}$$

with $\frac{\partial u}{\partial \varepsilon_x}(\mu) := \frac{d}{ds} u(\mu + s\varepsilon_x)\Big|_{s=0}$. Hence the closure $(\mathcal{E}, D(\mathcal{E}))$ of $(\mathcal{E}, \mathcal{F}C^\infty)$ on $L^2(E;m)$ is a Dirichlet form, and again $D(\mathcal{E})$ is a kind of first order Sobolev space $H_0^{1,2}(E;m)$ on $E = \mathcal{M}_1(S)$. The general method from [RS95] to show (T) also applies here. So, the entire theory of Dirichlet forms described in Section 2 applies here as well (cf. [ORS95] for details). The corresponding diffusion is just the well-studied *Fleming-Viot process* (*with parent independent mutation*) appearing in population genetics (cf. [EK93]).

Remark 5.1 We would like to emphasize that by definition $(\mathcal{E}, D(\mathcal{E}))$ above depends in fact only on the σ-algebra $\mathcal{B}(S)$ and not on the topology of S.

One challenging problem in this model-case is to study the geometry of $(\mathcal{E}, D(\mathcal{E}))$ respectively of the Fleming-Viot process on $\mathcal{M}_1(S)$ in more detail. In [OR94] a few steps in this direction have been done by analysing the corresponding intrinsic metric. It turns out that the latter is equivalent to the variation norm. Furthermore, it is shown in [OR94], that in some sense the Fleming-Viot process is a (time-changed) Brownian motion when considered w.r.t. the "correct" geometry on $\mathcal{M}_1(S)$. In finite dimensions, i.e., the case where $S = \{1, \dots, d\}$ this can be proved explicitly, and if ν_0 is the uniform distribution on $\{1, \dots, d\}$ the Fleming-Viot is just the Brownian motion on the $(d-1)$-dimensional sphere (cf. [OR94, Remark 3.3]).

6 Infinite-dimensional state space — non flat case: spaces of integer-valued Radon measures on $\mathbb{R}^d$

In this section we replace our manifold M of Section 2 by the space E of all $\mathbb{Z}^+$-valued Radon measures on $\mathbb{R}^d$ equipped with the vague topology (or a closed subset thereof). As the "tangent space" at $\mu \in E$ in this case we take

$$T_\mu E := L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu) \ ,$$

i.e., the space of all μ-square integrable vector fields on $\mathbb{R}^d$. As test functions on E we choose

$$\begin{aligned} \mathcal{F}C_b^\infty \ := \ & \Big\{ \mu \mapsto f\left(\int \varphi_1 d\mu, \dots, \int \varphi_n d\mu \right) \mid n \in \mathbb{N}, \\ & \qquad f \in C_b^\infty(\mathbb{R}^n),\ \varphi_1, \dots, \varphi_n \in C_0^1(\mathbb{R}^d) \Big\} \ , \end{aligned}$$

where $C_0^1(\mathbb{R}^d)$ denotes the set of all continuously differentiable functions on $\mathbb{R}^d$ with compact support.

Now we define our gradient for $u \in \mathcal{F}C_b^\infty$, $\mu \in E$, by

$$\nabla u(\mu) := (x \mapsto \vec{D}_x u(\mu),\ x \in \operatorname{supp}[\mu]) \tag{6.1}$$

with

$$\vec{D}_x u(\mu) := \left(\frac{d}{ds} u(\mu + \varepsilon_{x_i(s)} - \varepsilon_x)|_{s=0} \right)_{1 \le i \le d} \ , \tag{6.2}$$

where for $1 \le i \le d$

$$x_i(s) = x + s\, e_i$$

with $\{e_i \mid 1 \le i \le d\}$ the canonical basis of $\mathbb{R}^d$. From (6.4) below we see that for given $\mu \in E$ it is really enough to define $\vec{D}_x u(\mu)$ for $x \in \mathbb{R}^d$ such that μ has mass in x. (Note that (6.2) is really an intrinsic definition.)

By the chain rule we obtain that for all $x \in \operatorname{supp}[\mu]$, $\mu \in E$,

$$\vec{D}_x u(\mu) = \Big(\sum_{j=1}^{n} \partial_i f \Big(\int \varphi_1 d\mu, \ldots, \int \varphi_n d\mu \Big) \partial_i \varphi_j(x) \Big)_{1 \le i \le d} , \tag{6.3}$$

which shows that $x \mapsto \vec{D}_x u(\mu)$ is in $L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$. This implies that ∇u is a vector field on E (w.r.t. the tangent bundle $(L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu))_{\mu \in E}$). For any finite measure m on the Borel σ-algebra $\mathcal{B}(E)$ we thus have a pre-Dirichlet form

$$\begin{aligned} \mathcal{E}(u,v) \;&:=\; \frac{1}{2} \int_E \langle \nabla u(\mu), \nabla v(\mu) \rangle_{L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)} \; m(d\mu) \\ &\qquad u, v \in \mathcal{F}C_b^\infty , \end{aligned} \tag{6.4}$$

on $L^2(E;m)$ and (G) is realized. In [Y95] (see also [Os95]) a whole class of measures m have been analysed which come from *superstable lower regular pair potentials* Φ (see [Y95] for details) so that (C) holds. (More precisely, an operator L has been identified such that

$$\mathcal{E}(u,v) = -\int Lu \; v \; dm \quad \text{for all } u, v \in \mathcal{F}C_b^\infty.)$$

And also (T) has been verified for these type of measures (cf. [Y95], [Os95]).

Remark 6.1 The reader should note that (6.4) is a priori different from the definition of $\mathcal{E}$ in [Y95]. A more "coordinate based" definition has been given there without identifying an appropriate underlying tangent bundle. But it is easily verified that both definitions are in fact the same.

Now, as explained in Section 2 correspondence (2.3) is also valid in this case (more precisely for the closure $(\mathcal{E}, D(\mathcal{E}))$ of (6.4) on $L^2(E;m)$) and the complete theory applies. The corresponding diffusion (at least heuristically) weakly solves a stochastic differential equation of type

$$dX_t^i = -\frac{1}{2} \sum_{j \neq i} \nabla \Phi(X_t^i - X_t^j) dt + dW_t^i \; , \; i \in \mathbb{N} \; , \tag{6.5}$$

where $(W^i)_{i \in \mathbb{N}}$ is a family of independent standard Brownian motions (see [Y95] for the precise meaning of (6.5) in this case.). Hence it models an interacting infinite particle system.

We conclude this section with briefly explaining how to construct new (even non-symmetric) Dirichlet forms from gradient-type Dirichlet forms. This is completely standard and as easy as in the case of finite-dimensional state spaces. We shall do so starting from the closure $(\mathcal{E}, D(\mathcal{E}))$ of (6.4) for a fixed m as in [Y95]. (But in all examples from the previous sections this can be carried out in exactly the same way; cf. [MR92, Chap. II, Exercises 3.6, 3.9 and Subsection 3e)] and [ORS95, Sect. 2]). In particular, non-constant diffusion coefficients can occur.

For $\mu \in E$ let $A(\mu)$ be a self-adjoint bounded operator on $L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$ such that:

(i) There exists $c \in (0, \infty)$ such that $A(\mu) \geq c\, 1$ for all $\mu \in E$.

(ii) $\mu \mapsto \langle A(\mu) f(\mu, \cdot), g(\mu, \cdot)\rangle_{L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)}$ is $\mathcal{B}(E)$-measurable for all bounded continuous functions $f, g : E \times \mathbb{R}^d \to \mathbb{R}^d$.

Remark 6.2 Condition (i) is used to prove closability for the form $(\mathcal{E}^{(1)}, \mathcal{F}C_b^\infty)$ defined in (6.7) below. However, it can be weakened considerably. By the same arguments as e.g. those in [AR91, Sect. 3] or [MR92, Chap. II, Subsection 2b)] one can allow degenerate cases. The respective comparison function ρ in these references can e.g. be taken to be equal to φ^2, where according to [Eb94, Theorem 1.1] φ can be an arbitrary function in $(\mathcal{E}, D(\mathcal{E}))$ such that $\varphi \neq 0$ m-a.e.

Furthermore, let $c \in L^\infty(E; m)$ and $b : \mathbb{R}^d \times E \to \mathbb{R}^d$ be $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(E)$-measurable such that

$$\sup_{\mu \in E} \|b(\mu)\|_{L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)} < \infty , \tag{6.6}$$

where $b(\mu)(x) := b(\mu, x)$. Let d be another such function, but c and d should be such that

$$\int \Big(\langle \nabla u, d \rangle_{L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)} + u\ c \Big) d\mu \geq 0 \quad \text{for all } u \in \mathcal{F}C_b^\infty \text{ with } u \geq 0 .$$

Then for some $\alpha \in (0, \infty)$

$$\begin{aligned} \mathcal{E}^{(1)}(u, v) \ &:= \ \int \langle A(\mu) \nabla u(\mu), \nabla v(\mu) \rangle_{L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)}\ m(d\mu) \\ &\quad + \int \langle \nabla u(\mu), b(\mu) \rangle_{L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)} v(\mu)\ m(d\mu) \\ &\quad + \int \langle \nabla v(\mu), d(\mu) \rangle_{L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)} u(\mu)\ m(d\mu) , \\ &\quad + \int u(\mu)\ v(\mu) (c(\mu) + \alpha)\ m(d\mu) \\ &\quad u, v \in \mathcal{F}C_b^\infty , \end{aligned} \tag{6.7}$$

is closable and the closure is a (non-symmetric) *semi-Dirichlet form* in the sense of [MOR95, Definition 2.1]. The proof is exactly analogous to [MR92, Chap II, Subsection 3e)] (see also [MOR95, Section 3.4]).

Also for (non-symmetric) semi-Dirichlet forms a correspondence like (2.3) holds and the corresponding theory is equally well-developed (cf. [MOR95] and also [MR92]). Hence this entire theory can be applied in this case. In particular, there exists a corresponding diffusion, which solves a martingale problem given by the corresponding generator. If $b \equiv d \equiv c \equiv 0$, this diffusion has m as symmetrizing and invariant measure.

The main problem for future research is to analyse these processes more precisely and to understand the kind of particle systems they describe. Also questions in how far and what sense this approach gives a unique solution to (6.5) resp., in the last more general case, to the corresponding martingale problem, are of interest. Also the very difficult problem concerning the ergodicity of the resulting processes should be considered. In the case of lattice models this will be discussed in the next section.

7 One application to lattice Gibbs states

In this section we want to describe one (as I think) particularly striking application of Dirichlet space theory to lattice Gibbs states (which is contained in [AKR95b]). For simplicity, we shall, however, introduce the Gibbs states in question not, as usual, via a local specification, but as symmetrizing measures of the solution for the corresponding infinite-dimensional stochastic differential equation.

Consider the following weighted ℓ^2-space of sequences over the lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$:

$$E := H_{-p} := \left\{ z \in \mathbb{R}^{\mathbb{Z}^d} \;\middle|\; \|z\|_{-p}^2 := \sum z_k^2 (1+|k|)^{-p} < \infty \right\}$$

where $p > d$ is fixed. Here $|k|$ denotes the Euclidean norm of $k \in \mathbb{Z}^d \subset \mathbb{R}^d$. Define for $k \in \mathbb{Z}^d$ and $z = (z_j)_{j\in\mathbb{Z}^d} \in E$

$$b_k(z) := -4z_k^3 - 2 \sum_{j\in\mathbb{Z}^d:|j-k|=1} (z_k - z_j)$$

and $b := (b_k)_{k\in\mathbb{Z}^d}$. (However, all below applies for much more general b, cf. [AKR95b]; we made this choice of b for simplicity.) Let $X^z := (X_k^z)_{k\in\mathbb{Z}^d}$ be the strong solution of

$$\begin{aligned} dX^z(t) &= \frac{1}{2} b(X^z(t))dt + dW_t \\ X^z(0) &= z \in E \ , \end{aligned}$$

where $W = (W_k)_{k\in\mathbb{Z}^d}$ is a family of independent real-valued Wiener processes. (For existence and uniqueness see [DoRo79]). Define for $f : E \to \mathbb{R}_+$, $\mathcal{B}(E)$-measurable, and $t \geq 0$

$$p_t f(z) := E[f(X^z(t))] \ , \ z \in E \ ,$$

and let $\mathcal{M}$ denote the set of all *tempered* probability measures m on $\mathcal{B}(E)$ w.r.t. which $(p_t)_{t\geq 0}$ is *symmetric*, i.e., for all $t > 0$

$$\int_E p_t f \; g \; dm = \int_E f \; p_t g \; dm$$

for all $f, g : E \to \mathbb{R}_+$, $\mathcal{B}(E)$-measurable. "m is tempered" means that

$$\left(\int_E |z_k| \; m(dz) \right)_{k \in \mathbb{Z}^d} \in H_{-p'}$$

for some $p' \in \mathbb{N}$.

Remark 7.1 It can be shown (cf. [AKR95b], in particular Proposition 5.9) that $\mathcal{M}$ consists of exactly the tempered Gibbs states of a local specification which are nothing but the Φ_d^4-*Gibbs states on the lattice* $\mathbb{Z}^d$ (cf. also the end of Section 3 above for the continuous Φ_d^4-model in case $d = 2$).

Now the question arises whether, after having defined the convex set $\mathcal{M}$ in terms of $(p_t)_{t>0}$, we can also characterize its extreme points $\mathcal{M}_{\mathrm{ex}}$ using $(p_t)_{t>0}$. This question is answered by the following theorem. First we recall that for each $m \in \mathcal{M}$, $(p_t)_{t>0}$ extends to a strongly continuous semigroup $(T_t^m)_{t\geq 0}$ of symmetric contractions on $L^2(E; m)$ (cf. [MR92, Chap. II, Subsection 4a)]).

Theorem 7.2 *Let $m \in \mathcal{M}$. Then $m \in \mathcal{M}_{\mathrm{ex}}$ if and only if $(T_t^m)_{t\geq 0}$ is* ergodic *(i.e., $T_t^m u \underset{t\to\infty}{\longrightarrow} \int u \; dm$ in $L^2(E; m)$).*

This was first proved in [AKR95b] (in much more generality; see Theorem 5.15 therein which covers the above special case) using one of the main results in [BoR95] (namely, Theorem 6.15) in an essential way. It extends a classical result by R.A. Holley and D.W. Stroock ([HoSt76]) for the Ising model to this case with non-compact continuous single spin spaces.

So far, no Dirichlet form has appeared in this section, but it plays the central rôle in the method of proof for Theorem 7.2. We remind the reader of the correspondence (2.3), and this time we enter it through the semigroup $(T_t^m)_{t\geq 0}$ (where $m \in \mathcal{M}$ is given as in Theorem 7.2) which comes by definition from the transition semigroup of a nice process. Hence we know by (2.3) that there exists an associated quasi-regular (symmetric) Dirichlet form $(\mathcal{E}_m, D(\mathcal{E}_m))$ on $L^2(E; m)$. This knowledge alone is of not much use, but here $(\mathcal{E}_m, D(\mathcal{E}_m))$ can be identified explicitly as the closure of

$$\begin{aligned} \mathcal{E}_m(u, v) = \int_E \langle \nabla u(z), \nabla v(z) \rangle_{\ell^2(\mathbb{Z}^d)} \; m(dz) \; , \\ u, v \in \mathcal{F}C_b^\infty \; , \end{aligned} \tag{7.1}$$

i.e., we have a gradient-type Dirichlet form as in Section 3. (Note that here $\ell^2(\mathbb{Z}^d)$ is the tangent space at each point $z \in E$ and that $\ell^2(\mathbb{Z}^d) \subset E$

continuously and densely, hence $\nabla u(z)$ can be defined exactly as in Section 3). The proof that $\mathcal{E}_m$ is given by (7.1) on $\mathcal{F}C_b^\infty$ is trivial, but it is extremely hard to show that $D(\mathcal{E}_m)$ is really the completion of $\mathcal{F}C_b^\infty$ w.r.t. $(\mathcal{E}_m + (\ ,\ )_{L^2(E;m)})^{1/2}$. Recent results on the Markov uniqueness problem mentioned at the end of Section 3 had to be used (cf. [AKR95a,b]).

The ergodicity of $(T_t^m)_{t\geq 0}$ is equivalent to the more tractable *irreducibility* of $(\mathcal{E}^m, D(\mathcal{E}^m))$ (i.e., $u =$ const. whenever $u \in D(\mathcal{E}^m)$ with $\mathcal{E}^m(u,u) = 0$; by the explicit form of $(\mathcal{E}^m, D(\mathcal{E}^m))$ this is equivalent to $u =$ const. whenever $u \in D(\mathcal{E}^m)$ with $\nabla u = 0$, where we also denote the closure of the gradient by ∇). Thus, it remains to show that this property of $(\mathcal{E}^m, D(\mathcal{E}^m))$ holds if and only if $m \in \mathcal{M}_{\text{ex}}$. Also this second step of the proof is quite tough. The key part in it was already done in [BoR95, Theorem 6.15]. We refer to [AKR95b, Sect. 2] for details.

Remark 7.3 Since a log-Sobolev inequality for $(\mathcal{E}^m, D(\mathcal{E}^m))$ implies a mass gap, hence ergodicity of $(T_t^m)_{t\geq 0}$, because of Theorem 7.2 one can thus only hope to prove a log-Sobolev inequality (resp. a mass gap) if m is extreme.

Future research in this direction will focus on extending Theorem 7.2 within the framework of (mathematical) statistical mechanics to more general models than the ones studied in [AKR95b], in particular, cases where the single spin spaces are themselves (finite or) infinite-dimensional manifolds. Also an extension to particle systems as in the previous section seems feasible and would certainly be of substantial interest.

8 Other recent developments

So far, we have tried to show that in many concrete infinite-dimensional situations there are natural underlying “geometric structures” and (intrinsically) associated gradient-type Dirichlet forms to be discovered which determine and help to analyse stochastic processes of substantial interest resp. give means to construct them. With such a wide range of possible applications in prospect there are, of course, also enormous research activities within Dirichlet space theory itself. In addition, areas of applications other than those described above are being explored. It is impossible to describe them here in a way that meets the importance of all these developments. We just want to give a (as already emphasized in the introduction highly incomplete) list of corresponding topics (in the shape of key words) to give an impression of the variety of ongoing research in this area of mathematics. We also mention some names of colleagues working specifically with emphasis on Dirichlet forms in the respective directions. However, we do not include topics that have already been discussed in previous sections.

- boundary theory (Chen, Silverstein, Williams, Zheng,...)
- convergence/approximations (Dal Maso, Dell' Antonio, Kuwae, Lyons, Mosco, Posilicano, Uemura, Zhang,...)
- Dirichlet operators (Albeverio, Eberle, Kondratiev, Liskevich, Semenov, Takeda,...)
- fractals (Fukushima, Kusuoka, LeJan, Metz, Sabot, Shima,...)
- geometry induced by Dirichlet forms (Ma, Jost, Sturm,...)
- large deviations (Fang, Fukushima, Mück, Takeda,...)
- Markov uniqueness (Albeverio, Eberle, Fitzsimmons, Kondratiev, Shigekawa, Song,...)
- non-commutative case (Albeverio, Lindsay,...)
- parabolic case (Oshima, Stannat,...)
- partial differential equations (Biroli, Mosco,...)
- pseudo-differential operators (Hoh, Jacob,...)
- statistical mechanics (Albeverio, Kondratiev, Park, Yoo,...)
- stochastic partial differential equations (Albeverio, Mück, Stannat,...)
- theory of Markov processes (Fitzsimmons, Getoor, Ma,...)

Finally, we would like to draw attention to an entirely new framework of so-called *generalized Dirichlet forms* presently being developed (see [Sta95a, b, 96]). It includes both the elliptic (semi-) Dirichlet forms in [MR92] (resp. [MOR95]) and the parabolic ones. In particular, the fairly restrictive *sector condition* (cf. [MR92]) in non-symmetric situations is dropped. This theory covers all cases known so far, and also includes examples which may even have no dual theory at all, but are still related to some (possibly highly non-symmetric) *notion of energy.* This new framework widens the scope of applications enormously and (e.g., in particular, w.r.t. stochastic partial differential equations) a lot of progress is expected.

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