Exactly definable categories

Henning Krause

Given a skeletally small abelian category $\mathcal{C}$ we denote by $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the category of exact additive functors from $\mathcal{C}^{\mathrm{op}}$ to the category Ab of abelian groups. We call a category exactly definable provided that it is equivalent to $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ for some skeletally small abelian category $\mathcal{C}$.

This definition and our interest in this type of categories is motivated by the following example. Let Λ be a ring and denote by $C(\Lambda) = \mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ the category of finitely presented functors from the category of finitely presented Λ^{op}-modules to the category of abelian groups. Then $C(\Lambda)$ is skeletally small abelian. In fact, it is the free abelian category over Λ viewed as a category with a single object [5], and the functor

$$\mathrm{Mod}(\Lambda) \to \mathrm{Ex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab}), \quad M \mapsto \mathrm{Hom}(\ , M \otimes_\Lambda -)$$

is an equivalence. Thus we can replace a Λ-module by an exact functor $C(\Lambda)^{\mathrm{op}} \to \mathrm{Ab}$. Of course, since the work of Auslander it has been common practice to identify a Λ-module M either with $\mathrm{Hom}_\Lambda(\ , M)$ or $M \otimes_\Lambda -$ [1]. However, these functors are not exact in general. The aim of this paper is to show that certain properties of Λ-modules become more transparent if one views them as exact functors. In particular, one can use the machinery of localization theory for locally coherent Grothendieck categories because $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is a subcategory of the locally coherent category $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ of left exact functors. Another reason for introducing the concept of an exactly definable category is the fact that for any ring Λ there are plenty of categories related to $\mathrm{Mod}(\Lambda)$ which are exactly definable but not necessarily module categories.

We give now a brief survey of the basic notions and results of this paper. To this end fix an exactly definable category $\mathcal{A}$ and we may assume that $\mathcal{A} = \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ for some skeletally small abelian category $\mathcal{C}$. The first sections are devoted to properties of the inclusion $d\colon \mathcal{A} \to D(\mathcal{A}) = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. This leads to a notion of purity for $\mathcal{A}$ which coincides with the usual definition if $\mathcal{A}$ is a category of modules. A sequence $0 \to L \to M \to N \to 0$ of morphisms in $\mathcal{A}$ is pure exact if the sequence $0 \to L(X) \to M(X) \to N(X) \to 0$ is exact for every $X \in \mathcal{C}$, and it is shown that an object M is pure-injective if and only if for every set I the summation morphism $\coprod_I M \to M$ factors through the canonical morphism $\coprod_I M \to \prod_I M$. Moreover, the functor d identifies pure-exactness and pure-injectivity in $\mathcal{A}$ with exactness and injectivity in $D(\mathcal{A})$.

Call a full subcategory of $\mathcal{A}$ definable if it consists of those functors in $\mathcal{A}$ which vanish on some fixed subcategory of $\mathcal{C}$. This is a natural concept because a definable subcategory

is again exactly definable. Also it is closed under forming direct limits, products and pure subobjects, and we shall prove that this property characterizes a definable subcategory. The dual concept is that of a definable quotient category which is discussed in the following section. We describe the full subcategory of pure-injective objects of a definable quotient category and use a result about localizing subcategories which seems to be of independent interest.

Next we exhibit the Ziegler spectrum $\mathrm{Zsp}(\mathcal{A})$ of $\mathcal{A}$ which is, by definition, the set of isomorphism classes of indecomposable pure-injective objects in $\mathcal{A}$. The subsets which are of the form $\mathcal{B} \cap \mathrm{Zsp}(\mathcal{A})$ for some definable subcategory $\mathcal{B}$, are precisely the closed sets of a topology on $\mathrm{Zsp}(\mathcal{A})$.

Exactly definable categories have direct limits and products and it is therefore natural to study functors between exactly definable categories which commute with direct limits and products. We prove that any such functor $f\colon \mathcal{A} \to \mathcal{B} = \mathrm{Ex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ is of the form $X \mapsto X \circ g$ for some exact functor $g\colon \mathcal{D} \to \mathcal{C}$, and we deduce several properties from this fact. For instance, f induces a continuous and closed map $\mathrm{Zsp}(\mathcal{A}) \to \mathrm{Zsp}(\mathcal{B})$ provided that f preserves indecomposability.

Given any object $M \in \mathcal{A}$, the endomorphism ring $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$ acts on $M(X)$ for every $X \in \mathcal{C}$, and we obtain therefore an induced functor $\bar{M}\colon (\mathcal{C}/\mathrm{Ker}(M))^{\mathrm{op}} \to \mathrm{Mod}(\Gamma)$ which is faithful and exact. We call its image the endocategory $\mathcal{E}_M$ of M. This is a useful invariant since several local properties of M depend only on $\mathcal{E}_M$. For example, M is Σ-pure-injective if and only if every object in $\mathcal{E}_M$ is artinian, and M is endofinite if and only if every object in $\mathcal{E}_M$ is of finite length.

The next part of the paper is devoted to various finiteness conditions for exactly definable categories. Some of them are related to filtrations of the form $(\mathcal{A}_\alpha)_\alpha$ where each $\mathcal{A}_\alpha$ is a definable subcategory of $\mathcal{A}$. We give a complete classification of the indecomposable pure-injective objects in $\mathcal{A}$ under the assumption that a certain filtration of $\mathcal{A}$ exists.

Our discussion of exactly definable categories is mainly inspired and motivated by results from module theory. In fact the category of modules over a ring is the prototype of an exactly definable category, even though we have choosen a more abstract level for our exposition. To illustrate this point we devote the final section of this paper to epimorphisms in the category of rings. Using techniques and results from the previous sections it is shown that any epimorphism $\Lambda \to \Gamma$ is induced by a suitable localization of the Grothendieck category $D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$. We shall also discuss some consequences. For instance, we show that each epimorphism $\Lambda \to \Gamma$ is determined by a closed subset of the Ziegler spectrum of $\mathrm{Mod}(\Lambda)$. Further applications and examples of the material presented in this paper, in

particular from representation theory of artin algebras, can be found in [13, 14].

1. Locally finitely presented categories

We recall some terminology and some well-known facts about locally finitely presented categories and Grothendieck categories. Let $\mathcal{A}$ be an additive category with direct limits. An object X in $\mathcal{A}$ is *finitely presented* provided that $\mathrm{Hom}(X, \ )$ commutes with direct limits and we denote by $\mathrm{fp}(\mathcal{A})$ the full subcategory of finitely presented objects in $\mathcal{A}$. The category $\mathcal{A}$ is said to be *locally finitely presented* if the isomorphism classes of $\mathrm{fp}(\mathcal{A})$ form a set and every object in $\mathcal{A}$ is a direct limit of objects in $\mathrm{fp}(\mathcal{A})$. Given a full additive subcategory $\mathcal{C}$ of $\mathrm{fp}(\mathcal{A})$ we denote by $\vec{\mathcal{C}}$ the full subcategory of $\mathcal{A}$ which consists of direct limits $\varinjlim X_i$ with $X_i \in \mathcal{C}$ for all i. Note that this subcategory is closed under direct limits in $\mathcal{A}$ and that the finitely presented objects in $\vec{\mathcal{C}}$ are precisely the direct summands of objects in $\mathcal{C}$ [3, Theorem 4.1]. An abelian category $\mathcal{A}$ is locally finitely presented if and only if it is a Grothendieck category with a generating set of finitely presented objects [2, Satz 1.5], [3, 2.4].

Fix now a locally finitely presented abelian category $\mathcal{A}$. An object in $\mathcal{A}$ is *finitely generated* if it is a quotient of some finitely presented object and it is easily seen that any object in $\mathcal{A}$ is a direct limit of finitely generated subobjects. The category $\mathcal{A}$ is completely determined by $\mathrm{fp}(\mathcal{A})$. In fact, $\mathrm{fp}(\mathcal{A})$ is additive, has cokernels and the functor

$$\mathcal{A} \to \mathrm{Lex}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathrm{Hom}(\ , X)|_{\mathrm{fp}(\mathcal{A})}$$

from $\mathcal{A}$ into the category of additive left exact functors from $\mathrm{fp}(\mathcal{A})^{\mathrm{op}}$ to Ab is an equivalence [2, Satz 2.4]. Finally, we recall that $\mathcal{A}$ is *locally coherent* provided that finitely generated subobjects of finitely presented objects are finitely presented, equivalently if $\mathrm{fp}(\mathcal{A})$ is abelian [19, Proposition 2.2]. Note, that

$$\mathcal{A} \mapsto \mathrm{fp}(\mathcal{A}) \qquad \text{and} \qquad \mathcal{C} \mapsto \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$$

induces, up to equivalence, a bijective correspondence between locally coherent categories and skeletally small abelian categories. We shall frequently use the following facts about functors between locally coherent categories.

Lemma 1.1 *Let $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and $\mathcal{B} = \mathrm{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ be a pair of locally coherent categories and suppose that $f\colon \mathcal{C} \to \mathcal{D}$ is an exact functor. Then there is, up to isomorphism, a unique exact functor $f^*\colon \mathcal{A} \to \mathcal{B}$ which commutes with direct limits and sends $\mathrm{Hom}(\ , X)$ to $\mathrm{Hom}(\ , f(X))$ for every $X \in \mathcal{C}$. The restriction functor $f_*\colon \mathcal{B} \to \mathcal{A}$, $X \mapsto X \circ f$, is a right adjoint of f^*. Moreover f^* is (fully) faithful iff f is (fully) faithful.*

Proof: See [11, Lemma 2.7].

2. Exactly definable and locally coherent categories

In this section we discuss the relation between exactly definable categories and locally coherent categories which is fundamental for our approach. Recall that an object M of a Grothendieck category $\mathcal{A}$ is *fp-injective* provided that $\mathrm{Ext}^1(Z, M) = 0$ for all $Z \in \mathrm{fp}(\mathcal{A})$. We denote the full subcategory of fp-injective objects in $\mathcal{A}$ by $\mathrm{fpinj}(\mathcal{A})$. Note that $\mathrm{fpinj}(\mathcal{A})$ is automatically closed under products. The following lemma is taken from [3].

Lemma 2.1 *Let $\mathcal{A}$ be a locally finitely presented abelian category. For $Z \in \mathrm{fp}(\mathcal{A})$ and $M \in \mathcal{A}$ the following are equivalent.*

(1) $\mathrm{Ext}^1(Z, M) = 0$.

(2) $0 \to \mathrm{Hom}(Z, M) \to \mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M) \to 0$ *is exact for all exact sequences* $0 \to X \to Y \to Z \to 0$ *in $\mathcal{A}$ with Y finitely presented.*

Proof: Straightforward.

Proposition 2.2 *A category is exactly definable if and only if it is equivalent to the full subcategory* $\mathrm{fpinj}(\mathcal{B})$ *of fp-injective objects of some locally coherent category $\mathcal{B}$.*

Proof: The preceding lemma shows that for any locally coherent category $\mathcal{B}$ the equivalence $\mathcal{B} \to \mathrm{Lex}(\mathrm{fp}(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto \mathrm{Hom}(\ , X)$ induces an equivalence between $\mathrm{fpinj}(\mathcal{B})$ and $\mathrm{Ex}(\mathrm{fp}(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab})$.

For any abelian category $\mathcal{A}$ we denote by $\mathrm{inj}(\mathcal{A})$ the full subcategory of injective objects. The category $\mathcal{A}$ has *sufficiently many injectives* provided that for any object in $\mathcal{A}$ there exists a monomorphism into some injective object.

Proposition 2.3 *Let $\mathcal{A} = \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ be an exactly definable category.*

(1) *The Yoneda functor $\mathcal{C} \to (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto \mathrm{Hom}(\ , X)$ induces an equivalence between* $\mathrm{inj}(\mathcal{C})$ *and* $\mathrm{fp}(\mathcal{A})$.

(2) *$\mathcal{A}$ is locally finitely presented if and only if $\mathcal{C}$ has sufficiently many injectives.*

Proof: See [12, Lemma 9.5 and Theorem 9.6].

Having characterized those exactly definable categories which are locally finitely presented, we turn now to locally finitely presented categories which are exactly definable.

Proposition 2.4 *A locally finitely presented category is exactly definable if and only if it has products, equivalently if* $\mathrm{fp}(\mathcal{A})$ *has pseudo-cokernels.*

Proof: See [3, Theorem 3.3 and Theorem 2.1].

Our next aim is to show that any exactly definable category $\mathcal{A}$ determines, up to equivalence, uniquely an abelian category $\mathcal{C}$ such that $\mathcal{A}$ and $\mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ are equivalent. We need some criterion for injectivity which is due to Jensen and Lenzing [9, Proposition 7.32]. We start with the following analogue of Baer's criterion.

Lemma 2.5 *Let $\mathcal{A}$ be a locally finitely presented abelian category. For an object M in $\mathcal{A}$ the following are equivalent.*

(1) *M is injective, i.e.* $\mathrm{Ext}^1(Z, M) = 0$ *for all $Z \in \mathcal{A}$.*

(2) $\mathrm{Ext}^1(Z, M) = 0$ *for all finitely generated $Z \in \mathcal{A}$.*

Proof: Suppose that $\mathrm{Ext}^1(Z, M) = 0$ for all finitely generated $Z \in \mathcal{A}$ and let $\varphi\colon M \to E$ be an injective envelope. If $\psi\colon Y \to \mathrm{Coker}(\varphi)$ is a morphism with $Y \in \mathrm{fp}(\mathcal{A})$, then $Z = Y/\mathrm{Ker}(\psi)$ is a subobject of $\mathrm{Coker}(\varphi)$. Now φ can be written as composition $M \to M \coprod Z \to E$ of two monomorphisms since $\mathrm{Ext}^1(Z, M) = 0$ by assumption. We conclude from the minimality of E that $Z = 0$. Thus $\mathrm{Coker}(\varphi) = 0$ and M is injective.

Theorem 2.6 *Let $\mathcal{A}$ be a locally finitely presented abelian category. An fp-injective object M in $\mathcal{A}$ is injective if and only if for every set I the summation morphism $\coprod_I M \to M$ factors through the canonical monomorphism $\coprod_I M \to \prod_I M$.*

Proof: Using Baer's criterion as stated in Lemma 2.5 it suffices to show that for any $Y \in \mathrm{fp}(\mathcal{A})$ and any subobject $X \subseteq Y$ a morphism $\varphi\colon X \to M$ can be extended to a morphism $\psi\colon Y \to M$. To this end let $\{X_i \mid i \in I\}$ be the set of finitely generated subobjects of Y and suppose that I is the set of ordinals $\alpha < \kappa$ for some ordinal κ. Clearly, any subobject X of Y can be written as $X = \sum_{\alpha<\kappa} X_{i_\alpha}$ with $i_\alpha \in I$ for all $\alpha < \kappa$. We denote by Λ those ordinals $\lambda \leq \kappa$ such that any morphism $\varphi\colon X \to M$ with $X = \sum_{\alpha<\lambda} X_{i_\alpha}$ and $i_\alpha \in I$ for all $\alpha < \lambda$ can be extended to a morphism $\psi\colon Y \to M$. Using transfinite induction we show that $\kappa \in \Lambda$. Therefore fix an ordinal $\lambda \leq \kappa$, an object $X = \sum_{\alpha<\lambda} X_{i_\alpha}$ and a morphism $\varphi\colon X \to M$.

(1) $\lambda = 0 \in \Lambda$. Clear, since $X = 0$.

(2) $\lambda = 1 \in \Lambda$. This follows from the fp-injectivity of M since $Y/X \in \mathrm{fp}(\mathcal{A})$.

(3) $\lambda = \mu + 1 > 1$ and $\mu \in \Lambda$ implies $\lambda \in \Lambda$. A finite sum of finitely generated objects is finitely generated. Thus we may replace X_{i_0} by $X_{i_0} + X_{i_\mu}$ such that $X = \sum_{\alpha<\mu} X_{i_\alpha}$.

(4) λ a limit ordinal and $\mu \in \Lambda$ for all $\mu < \lambda$ implies $\lambda \in \Lambda$. We write $Y_\mu = \sum_{\alpha<\mu} X_{i_\alpha}$ for every $\mu < \lambda$ and denote by $\psi_\mu\colon Y \to M$ a morphism extending $\varphi|_{Y_\mu}$. For an ordinal $\mu \leq \lambda$ and a family $\chi = (\chi_\alpha)_{\alpha<\mu}$ of morphisms in $\mathrm{Hom}(Y, M)$ we denote by $\sum_{\alpha<\mu} \chi_\alpha$

the composition $Y \xrightarrow{\chi} \prod_{\alpha<\mu} M \to \prod_{\alpha<\lambda} M \to M$ where the second morphism is the natural monomorphism and the third $\prod_{\alpha<\lambda} M \to M$ is supposed to extend the summation morphism $\coprod_{\alpha<\lambda} M \to M$ which exists by our assumption. Now we define a family $(\chi_\mu)_{\mu<\lambda}$ of morphism in $\mathrm{Hom}(Y, M)$ inductively as follows. Put $\chi_0 = \psi_0$ and $\chi_\mu = \psi_\mu - \chi_\nu$ if $\mu = \nu + 1$. Finally, put $\chi_\mu = \psi_\mu - \sum_{\nu<\mu} \chi_\nu$, if μ is a limit ordinal. It is not hard to check that χ_μ satisfies the following condition for every $\mu < \lambda$.

(1) $\psi_\mu = \sum_{\alpha\leq\mu} \chi_\alpha$.

(2) $\chi_\mu(Y_\nu) = 0$, if $\mu = \nu + 1$, and $\chi_\mu(Y_\mu) = 0$, if μ is a limit ordinal.

To finish the proof let $\psi = \sum_{\mu<\lambda} \chi_\mu$. It is an easy consequence of (1) – (2) that $(\psi - \psi_\mu)(Y_\mu) = 0$ for all $\mu < \lambda$. Thus ψ extends each $\varphi|_{Y_\mu}$ and therefore also φ. We conclude that $\lambda \in \Lambda$ and this completes the proof.

Let $\mathcal{C}$ be any additive category. Recall that a functor $F: \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is said to be *finitely presented* provided that there exists an exact sequence $\mathrm{Hom}(\ , X) \to \mathrm{Hom}(\ , Y) \to F \to 0$ of functors in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. The finitely presented functors form an additive category with cokernels which we denote by $\mathrm{fp}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

Lemma 2.7 *Let $\mathcal{A}$ be an abelian category with sufficiently many injectives. Then the functor $\mathcal{A} \to \mathrm{fp}(\mathrm{inj}(\mathcal{A}), \mathrm{Ab})^{\mathrm{op}}$, $X \mapsto \mathrm{Hom}(X, \ )|_{\mathrm{inj}(\mathcal{A})}$ is an equivalence.*

Proof: Straightforward.

Let $\mathcal{A}$ be an exactly definable category. For reasons which hopefully will become clear we call an object M in $\mathcal{A}$ *pure-injective* provided that for every set I the summation morphism $\coprod_I M \to M$ factors through the canonical morphism $\coprod_I M \to \prod_I M$. We denote by $\mathrm{pinj}(\mathcal{A})$ the full subcategory of pure-injective objects in $\mathcal{A}$. We define $D(\mathcal{A}) = \mathrm{fp}(\mathrm{pinj}(\mathcal{A}), \mathrm{Ab})^{\mathrm{op}}$ and $C(\mathcal{A}) = \mathrm{fp}(D(\mathcal{A}))$. Then the following functors are of interest:

$$d: \mathcal{A} \to D(\mathcal{A}), \quad X \mapsto \mathrm{Hom}(X, \ )|_{\mathrm{pinj}(\mathcal{A})} \qquad \text{and}$$

$$h: D(\mathcal{A}) \to \mathrm{Lex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathrm{Hom}(\ , X)|_{C(\mathcal{A})}.$$

The next result summarizes our discussion. It is an analogue of [3, Theorem 3.3].

Theorem 2.8 (1) *The category $D(\mathcal{A})$ is locally coherent and the functor $d: \mathcal{A} \to D(\mathcal{A})$ induces an equivalence between $\mathcal{A}$ and the full subcategory* $\mathrm{fpinj}(D(\mathcal{A}))$ *of fp-injectives in $D(\mathcal{A})$. Moreover, d identifies the pure-injectives in $\mathcal{A}$ with the injectives in $D(\mathcal{A})$.*

(2) *The category $C(\mathcal{A})$ is skeletally small abelian and the composition $h \circ d$ induces an equivalence $\mathcal{A} \to \mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$.*

Corollary 2.9 *There is, up to equivalence, a bijective correspondence between exactly definable categories and skeletally small abelian categories. The correspondence is given by*

$$\mathcal{A} \mapsto C(\mathcal{A}) \qquad \text{and} \qquad \mathcal{C} \mapsto \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}).$$

The final result of this section expresses the relation between exactly definable and locally coherent categories in a form which we shall frequently use.

Corollary 2.10 *Let $\mathcal{A}$ be an exactly definable category and identify $\mathcal{A}$ with a category of functors via the canonical equivalence $\mathcal{A} \to \mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$. For every pair of objects $M \in \mathcal{A}$ and $X \in C(\mathcal{A})$ there is an isomorphism $\mathrm{Hom}(X, d(M)) \cong M(X)$ which is natural in both M and X.*

Proof: We have $\mathrm{Hom}(X, d(M)) \cong \mathrm{Hom}(\mathrm{Hom}(\ , X), M) \cong M(X)$, where the first isomorphism is induced by the equivalence $h\colon D(\mathcal{A}) \to \mathrm{Lex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ and the second follows from Yoneda's lemma.

3. Purity

Let $\mathcal{A}$ be an exactly definable category. We call a sequence $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ of morphisms in $\mathcal{A}$ *pure-exact* provided that the induced sequence $0 \to \mathrm{Hom}(N, Q) \to \mathrm{Hom}(M, Q) \to \mathrm{Hom}(L, Q) \to 0$ is exact for every pure-injective object Q in $\mathcal{A}$. In this case φ is called a *pure monomorphism* and ψ is called a *pure epimorphism*. Now suppose that $\mathcal{A} \to \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is an equivalence for some abelian category $\mathcal{C}$ which we view as an identification.

Lemma 3.1 *For a sequence $\varepsilon\colon 0 \to L \to M \to N \to 0$ of morphisms in $\mathcal{A}$ the following are equivalent.*

(1) $0 \to L \to M \to N \to 0$ *is pure-exact in* $\mathcal{A}$.

(2) $0 \to L \to M \to N \to 0$ *is exact in* $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

(3) $0 \to L(X) \to M(X) \to N(X) \to 0$ *is exact for every* $X \in \mathcal{C}$.

Moreover, if $\mathcal{A}$ is locally finitely presented, then (1) – (3) *are equivalent to the following.*

(4) $0 \to \mathrm{Hom}(X, L) \to \mathrm{Hom}(X, M) \to \mathrm{Hom}(X, N) \to 0$ *is exact for all* $X \in \mathrm{fp}(\mathcal{A})$.

(5) ε *is a direct limit of split exact sequences* $0 \to L_i \to M_i \to N_i \to 0$ *in* $\mathcal{A}$.

Proof: (1) $\Leftrightarrow$ (2) We use the fact that in any category with sufficiently many injectives a sequence $0 \to X \to Y \to Z \to 0$ is exact if and only if $0 \to \mathrm{Hom}(Z, Q) \to \mathrm{Hom}(Y, Q) \to$

$\mathrm{Hom}(X,Q) \to 0$ is exact for every injective object Q. The assertion follows from this fact since the pure-injectives in $\mathcal{A}$ are identified with the injectives in $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

(2) $\Leftrightarrow$ (3) Condition (3) says that $0 \to L \to M \to N \to 0$ is exact in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. The inclusion $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ has an exact left adjoint. Thus (3) implies (2). Now suppose (2). A right adjoint preserves kernels and therefore $0 \to L \to M \to N$ is exact in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. But the snake lemma shows that the cokernel of $L \to M$ in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is an exact functor and therefore coincides with the cokernel in $\mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

(3) $\Leftrightarrow$ (4) We use the fact mentioned in Proposition 2.3 that the Yoneda functor $X \mapsto \mathrm{Hom}(\ ,X)$ identifies $\mathrm{inj}(\mathcal{C})$ with $\mathrm{fp}(\mathcal{A})$. Now, (3) $\Rightarrow$ (4) is an immediate consequence of Yoneda's lemma. Therefore suppose (4) and let $X \in \mathcal{C}$. We choose an exact sequence $0 \to X \to Y_0 \to Y_1 \to Y_2$ with $Y_i \in \mathrm{inj}(\mathcal{C})$ for all i. We obtain the following commutative diagram

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
L(Y_2) & \to & L(Y_1) & \to & L(Y_0) & \to & L(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M(Y_2) & \to & M(Y_1) & \to & M(Y_0) & \to & M(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
N(Y_2) & \to & N(Y_1) & \to & N(Y_0) & \to & N(X) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & &
\end{array}$$

in which rows are exact and the first three columns are exact by assumption. Using the snake lemma the exactness of the last column follows. This shows (3).

(4) $\Rightarrow$ (5) Writing $N = \varinjlim N_i$ as a direct limit of finitely presented objects we obtain ε as a direct limit of exact sequences $\varepsilon_i\colon 0 \to L \to M \to N_i \to 0$ which are split exact by assumption.

(5) $\Rightarrow$ (4) If $0 \to L_i \to M_i \to N_i \to 0$ is split exact, then $0 \to \mathrm{Hom}(X, L_i) \to \mathrm{Hom}(X, M_i) \to \mathrm{Hom}(X, N_i) \to 0$ is exact. The fact that $\mathrm{Hom}(X,\ )$ commutes with direct limits implies (4).

Suppose again that $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is a sequence of morphisms in $\mathcal{A}$. The preceding lemma shows that the morphisms φ and ψ form a kernel-cokernel pair in $\mathcal{A}$ if the sequence is pure-exact. Moreover, φ is a pure monomorphism iff $\mathrm{Hom}(M,Q) \to \mathrm{Hom}(L,Q) \to 0$ is exact for every pure-injective object Q in $\mathcal{A}$.

Corollary 3.2 *For every object in $\mathcal{A}$ there is a pure monomorphism into some pure-injective object. In particular, an object in $\mathcal{A}$ is pure-injective if and only if every pure monomorphism starting in it splits.*

4. Definable subcategories

Let $\mathcal{A}$ be an exactly definable category and fix an equivalence $\mathcal{A} \to \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ for some abelian category $\mathcal{C}$ which we view as an identification. We call a full subcategory $\mathcal{B}$ of $\mathcal{A}$ a *definable subcategory* provided that there exists a subcategory $\mathcal{S}$ of $\mathcal{C}$ such that an object M is in $\mathcal{B}$ if and only if the functor M vanishes on $\mathcal{S}$. In this section we discuss some properties of definable subcategories. In particular we give a characterization which shows that the definition does not depend on the identification $\mathcal{A} = \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. We begin with some preparations.

Recall that a full subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ is a *Serre subcategory* provided that for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ the object X belongs to $\mathcal{S}$ iff X' and X'' belong to $\mathcal{S}$. Given a Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ one can form the *quotient category* $\mathcal{C}/\mathcal{S}$ which is abelian and admits an exact *quotient functor* $q{:}\mathcal{C} \to \mathcal{C}/\mathcal{S}$ with $\mathrm{Ker}(q) = \mathcal{S}$ [6, III, Proposition 1.1]. Here, the *kernel* $\mathrm{Ker}(f)$ of a functor f is, by definition, the full subcategory of all objects X such that $f(X) = 0$. Note that $\mathrm{Ker}(f)$ is a Serre subcategory if f is exact. The quotient functor q is characterized by the property that every exact functor $f{:}\mathcal{C} \to \mathcal{D}$ between abelian categories with $f(\mathcal{S}) = 0$ induces, up to isomorphism, a unique exact functor $\bar{f}{:}\mathcal{C}/\mathcal{S} \to \mathcal{D}$ such that $f = \bar{f} \circ q$. The subcategory $\mathcal{S}$ is called *localizing* if q has a right adjoint.

Let $\mathcal{B}$ be a definable subcategory of $\mathcal{A} = \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and suppose that $M \in \mathcal{B}$ if and only if M vanishes on $\mathcal{S}$ for some subcategory $\mathcal{S}$ of $\mathcal{C}$. Replacing $\mathcal{S}$ by the Serre subcategory of $\mathcal{C}$ which is generated by $\mathcal{S}$ we may assume that $\mathcal{S}$ is a Serre subcategory. This has the following consequence.

Proposition 4.1 *A definable subcategory of an exactly definable category is again an exactly definable category.*

Proof: We keep our notation from above and denote by $q{:}\mathcal{C} \to \mathcal{C}/\mathcal{S}$ the corresponding quotient functor. The assignment $M \mapsto M \circ q$ induces a fully faithful functor $\mathrm{Ex}((\mathcal{C}/\mathcal{S})^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ which provides an equivalence between $\mathrm{Ex}((\mathcal{C}/\mathcal{S})^{\mathrm{op}}, \mathrm{Ab})$ and $\mathcal{B}$.

Let I be a non-empty set. Recall that a set $\mathcal{F}$ of non-empty subsets of I is a *filter* on I provided that for each pair of subsets $J_1, J_2 \subseteq I$

(i) $J_1, J_2 \in \mathcal{F}$ implies $J_1 \cap J_2 \in \mathcal{F}$;

(ii) $J_1 \in \mathcal{F}$ and $J_1 \subseteq J_2$ implies $J_2 \in \mathcal{F}$.

Note, that the set $\mathcal{F}$ is directed if one defines $J_1 \leq J_2$, if $J_2 \subseteq J_1$.

Let $\mathcal{A}$ be a category with direct limits and products. Given a set of objects X_i, $i \in I$, in $\mathcal{A}$ and a filter $\mathcal{F}$ on I the canonical morphisms $\prod_{i\in J_1} X_i \to \prod_{i\in J_2} X_i$, $J_1 \leq J_2$ in $\mathcal{F}$, form a directed system and we denote by $\prod_{i\in I} X_i/\mathcal{F} = \varinjlim_{J\in\mathcal{F}} \prod_{i\in J} X_i$ the *reduced product* of the X_i's with respect to $\mathcal{F}$. A reduced product of the form $\prod_I X/\mathcal{F}$ is called a *reduced power* of X.

Lemma 4.2 *Let $\mathcal{A}$ be an additive category with left exact direct limits and products. Then any direct limit $\varinjlim_{i\in I} X_i$ in $\mathcal{A}$ is subobject of a reduced product $\prod_{i\in I} X_i/\mathcal{F}$ for some filter $\mathcal{F}$ on I.*

Proof: For each $i \in I$ let $H_i = \{j \in I \mid i \leq j\}$. Let $\mathcal{F}$ be the filter on I consisting of all subset $J \subseteq I$ such that $H_i \subseteq J$ for some $i \in I$. The map $I \to \mathcal{F}$, $i \mapsto H_i$ identifies I with a cofinal subset of $\mathcal{F}$ and therefore the canonically defined monomorphisms $X_i \to \prod_{j\in H_i} X_j$, $i \in I$, induce a monomorphism $\varinjlim_{i\in I} X_i \to \prod_{i\in I} X_i/\mathcal{F}$ since direct limits in $\mathcal{A}$ are left exact.

We reformulate the preceding lemma as follows.

Lemma 4.3 *Let $\mathcal{A}$ be an additive category with left exact direct limits and products. For a class $\mathcal{C}$ of objects of $\mathcal{A}$ which is closed under subobjects the following are equivalent.*

(1) *$\mathcal{C}$ is closed under products and direct limits.*

(2) *$\mathcal{C}$ is closed under reduced products.*

Proof: (1) implies (2) just by the definition of a reduced product. Therefore suppose that $\mathcal{C}$ is closed under reduced products. Any product $\prod_{i\in I} X_i$ has the form $\prod_{i\in I} X_i/\mathcal{F}$ where $\mathcal{F} = \{I\}$ and a direct limit $\varinjlim_{i\in I} X_i$ is subobject of a reduced product of the X_i's by the preceding lemma. Thus $\mathcal{C}$ is closed under products and direct limits.

Let $\varphi_i\colon X_i \to Y_i$, $i \in I$, be a set of morphisms in $\mathcal{A}$ and let $\mathcal{F}$ be a filter on I. Then there is an obvious morphism $\prod_{i\in I} \varphi_i/\mathcal{F}\colon \prod_{i\in I} X_i/\mathcal{F} \to \prod_{i\in I} Y_i/\mathcal{F}$. If direct limits in $\mathcal{A}$ are left exact, then $\prod_{i\in I} \varphi_i/\mathcal{F}$ is a monomorphism provided that all φ_i's are monomorphism. This observation has the following consequence.

Lemma 4.4 *Let $\mathcal{A}$ be an additive category with products and left exact direct limits. If $\mathcal{C}$ is a class of objects of $\mathcal{A}$ which is either closed under direct limits and products, or which is closed under reduced products, then the class $\mathcal{F} = \{X \in \mathcal{A} \mid X \subseteq Y \text{ for some } Y \in \mathcal{C}\}$ is also closed under direct limits, products and reduced products.*

Proof: $\mathcal{C}$ is closed under reduced products and therefore so is $\mathcal{F}$ by the remark above. The assertion now follows from the preceding lemma.

Let $\mathcal{A}$ be a locally coherent category. We shall view the functor $\mathcal{A} \to \mathrm{Lex}(\mathrm{fp}(\mathcal{A}), \mathrm{Ab})$, $X \mapsto \mathrm{Hom}(\ , X)$ as an identification. Thus a full subcategory $\mathcal{B}$ of $\mathrm{fpinj}(\mathcal{A})$ is definable provided that the class of objects is of the form $\{X \in \mathrm{fpinj}(\mathcal{A}) \mid \mathrm{Hom}(\mathcal{S}, X) = 0\}$ for some subcategory $\mathcal{S}$ of $\mathrm{fp}(\mathcal{A})$.

Proposition 4.5 *Let $\mathcal{A}$ be a locally coherent category. For a full subcategory $\mathcal{B}$ of* $\mathrm{fpinj}(\mathcal{A})$ *the following are equivalent.*

(1) *$\mathcal{B}$ is definable.*

(2) *$\mathcal{B}$ is closed under direct limits, products and fp-injective subobjects.*

(3) *$\mathcal{B}$ is closed under reduced products and fp-injective subobjects.*

(4) *$\mathcal{B} = \mathcal{F} \cap \mathrm{fpinj}(\mathcal{A})$ for some hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ of finite type for $\mathcal{A}$.*

Proof: Recall that a torsion theory $(\mathcal{T}, \mathcal{F})$ for $\mathcal{A}$ is *hereditary* provided that $\mathcal{T}$ is closed under subobjects and $(\mathcal{T}, \mathcal{F})$ is said to be of *finite type* if $\mathcal{F}$ is closed under direct limits. We shall use the fact that $(\mathcal{T}, \mathcal{F})$ is hereditary of finite type if and only if it is of the form $(\vec{\mathcal{S}}, \mathcal{F})$ for some Serre subcategory $\mathcal{S}$ of $\mathrm{fp}(\mathcal{A})$ [11, Corollary 2.10].

(1) $\Rightarrow$ (2) follows immediately from the definition of a definable subcategory.

(2) $\Leftrightarrow$ (3) follows from Lemma 4.3.

(2) $\Rightarrow$ (4) Consider the class $\mathcal{F} = \{X \in \mathcal{A} \mid X \subseteq Y \text{ for some } Y \in \mathcal{B}\}$. We claim that $\mathcal{F}$ is the torsionfree class for some hereditary torsion theory of finite type for $\mathcal{A}$. First observe that $\mathcal{F}$ is closed under direct limits, products and subobjects by Lemma 4.4. Now consider $\mathcal{F}$ as a full subcategory of $\mathcal{A}$. The inclusion functor $\mathcal{F} \to \mathcal{A}$ has a left adjoint $f\colon \mathcal{A} \to \mathcal{F}$ which is constructed as follows. For $X \in \mathcal{A}$ let Y_i, $i \in I$, be the set of quotient objects of X which are in $\mathcal{F}$. Define $f(X)$ to be the image and $t(X)$ the kernel of the canonical morphism $X \to \prod_i Y_i$. Also define $\mathcal{T} = \{X \in \mathcal{A} \mid \mathrm{Hom}(X, \mathcal{F}) = 0\} = \{X \in \mathcal{A} \mid \mathrm{Hom}(X, \mathcal{B}) = 0\}$ and note that $\mathcal{S} = \mathcal{T} \cap \mathrm{fp}(\mathcal{A})$ is a Serre subcategory of $\mathrm{fp}(\mathcal{A})$. We claim that $t(X) \in \vec{\mathcal{S}}$ for $X \in \mathrm{fp}(\mathcal{A})$. To this end write $t(X) = \varinjlim U_i$ as direct limit of finitely generated subobjects. We need to show that $U_i \in \mathcal{S}$ for all i. Suppose that $U = U_i \notin \mathcal{S}$. Then there is a non-zero morphism $\varphi\colon U \to Y$ for some $Y \in \mathcal{B}$ and φ extends to a morphism $\psi\colon X \to Y$ since Y is fp-injective. But the adjointness property of f implies that ψ factors through $X \to f(X)$. Therefore $\varphi(U) = 0$, a contradiction to our assumption. Thus $t(X) \in \vec{\mathcal{S}}$. Now let $X = \varinjlim X_i$ be an arbitrary object in $\mathcal{A}$ written as direct limit of finitely presented objects. We obtain an exact sequence $0 \to \varinjlim t(X_i) \to X \to \varinjlim f(X_i) \to 0$ with $\varinjlim t(X_i) \in \vec{\mathcal{S}}$ and $\varinjlim f(X_i) \in \mathcal{F}$

since both $\vec{\mathcal{S}}$ and $\mathcal{F}$ are closed under direct limits. We conclude that $(\vec{\mathcal{S}}, \mathcal{F})$ is a torsion theory for $\mathcal{A}$ which is hereditary of finite type.

(4) $\Rightarrow$ (1) Suppose we have $\mathcal{B} = \mathcal{F} \cap \mathrm{fpinj}(\mathcal{A})$ for some hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ of finite type for $\mathcal{A}$. The assertion follows immediately if we take $\mathcal{S} = \mathcal{T} \cap \mathrm{fp}(\mathcal{A})$ as defining subcategory for $\mathcal{B}$ since $\mathcal{F} = \{X \in \mathcal{A} \mid \mathrm{Hom}(\mathcal{S}, X) = 0\}$.

The following consequence has been proved by Crawley-Boevey for module categories using some different arguments.

Corollary 4.6 *Let $\mathcal{A}$ be an exactly definable category. For a full subcategory $\mathcal{B}$ of $\mathcal{A}$ the following are equivalent.*

(1) *$\mathcal{B}$ is definable.*

(2) *$\mathcal{B}$ is closed under direct limits, products and pure subobjects.*

(3) *$\mathcal{B}$ is closed under reduced products and pure subobjects.*

We present another consequence of Proposition 4.5.

Corollary 4.7 *Let $\mathcal{A}$ be an exactly definable category and view the canonical equivalence $\mathcal{A} \to \mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ as an identification. There is a bijective correspondence between Serre subcategories of $C(\mathcal{A})$ and definable subcategories of $\mathcal{A}$. The correspondence is given by*

$$\mathcal{S} \mapsto \{M \in \mathcal{A} \mid M(\mathcal{S}) = 0\} \qquad \textit{and} \qquad \mathcal{B} \mapsto \cap_{M \in \mathcal{B}} \mathrm{Ker}(M).$$

Proof: The assignment $\mathcal{S} \mapsto (\vec{\mathcal{S}}, \{X \mid \mathrm{Hom}(\mathcal{S}, X) = 0\})$ yields a bijection between the Serre subcategories of $C(\mathcal{A})$ and the hereditary torsion theories of finite type for $D(\mathcal{A})$ by [11, Corallary 2.10]. The assertion follows from the fact that any hereditary torsion theory is completely determined by the class of injective objects which are torsion-free.

We proceed with two results which allow us to construct definable subcategories. The following observation is of independent interest.

Proposition 4.8 *Let $\prod_{i \in I_j} X_i/\mathcal{F}_j$, $j \in J$, be a set of reduced products and let $\mathcal{G}$ be a filter on J. Let $I = \cup_{j \in J} I_j$ be the disjoint union of the I_j's and define $\mathcal{F} = \{K \subseteq I \mid \{j \mid K \cap I \in \mathcal{F}_j\} \in \mathcal{G}\}$. Then $\mathcal{F}$ is a filter on I and $\prod_{j \in J}(\prod_{i \in I_j} X_i/\mathcal{F}_j)/\mathcal{G} \cong \prod_{i \in I} X_i/\mathcal{F}$.*

Proof: It is well-known that the assertion holds for abelian groups [10, Theorem 6.8]. Thus the assertion holds for any functor category $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and an equivalence $\mathcal{A} \to \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ allows us to deduce this result for an exactly definable category.

Remark 4.9 *If $\mathcal{G}$ and all $\mathcal{F}_j$'s are ultrafilters, then $\mathcal{F}$ is also an ultrafilter.*

Corollary 4.10 *Let $\mathcal{C}$ be a class of objects of an exactly definable category. Then the pure subobjects of reduced products of objects in $\mathcal{C}$ form a definable subcategory of $\mathcal{A}$.*

Proof: The reduced products of objects in $\mathcal{C}$ form a class of objects which is closed under reduced products by the preceding proposition. Taking the full subcategory of pure subobjects the assertion follows from Lemma 4.4 and the characterization of definable subcategories in Corollary 4.6.

Corollary 4.11 *Let M be an object of an exactly definable category. Then there is a pure monomorphisms from M into a pure-injective object which is a direct summand of a reduced power $\prod_I M/\mathcal{F}$ of M.*

Proof: Apply the preceding result with $\mathcal{C} = \{M\}$.

To give another application of Corollary 4.6 we need the following lemma.

Lemma 4.12 *Let $\mathcal{A}$ be a locally coherent category. If $\mathcal{C}$ is a full additive subcategory of* $\mathrm{fp}(\mathcal{A})$*, then the subcategory $\vec{\mathcal{C}}$ of $\mathcal{A}$ is closed under subobjects from* $\mathrm{fpinj}(\mathcal{A})$.

Proof: Let $\varphi: X \to \varinjlim C_i$ be a monomorphism in $\mathcal{A}$ with $X \in \mathrm{fpinj}(\mathcal{A})$ and $C_i \in \mathcal{C}$ for all i. To show that $X \in \vec{\mathcal{C}}$ we need to check that any morphism $\psi: Y \to X$ with $Y \in \mathrm{fp}(\mathcal{A})$ factors through some object in $\mathcal{C}$ [3, Lemma 4.1]. Let $\chi: Y \to C_i$ be a morphism such that $\varphi \circ \psi$ factorizes through χ. The morphism ψ factorizes through the canonical morphism $Y \to \mathrm{Im}(\chi)$ since φ is a monomorphism, and the corresponding morphism $\mathrm{Im}(\chi) \to X$ can be extended to a morphism $C_i \to X$ since X is fp-injective. Thus $X \in \vec{\mathcal{C}}$

Recall that a subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be *covariantly finite* provided that every object $X \in \mathcal{C}$ has a *left $\mathcal{D}$-approximation*, i.e. a morphism $X \to Y$ with $Y \in \mathcal{D}$ such that the induced map $\mathrm{Hom}(Y, Y') \to \mathrm{Hom}(X, Y')$ is an epimorphism for every $Y' \in \mathcal{D}$.

Proposition 4.13 *Let $\mathcal{A}$ be a locally finitely presented category with products and suppose that $\mathcal{C}$ is a full additive subcategory of* $\mathrm{fp}(\mathcal{A})$*. Then $\vec{\mathcal{C}}$ is a definable subcategory of $\mathcal{A}$ if and only if $\mathcal{C}$ is covariantly finite in* $\mathrm{fp}(\mathcal{A})$.

Proof: The category $\mathcal{A}$ is exactly definable by Proposition 2.4. The subcategory $\vec{\mathcal{C}}$ is closed under direct limits and also closed under pure subobjects by the preceding lemma. Furthermore, $\vec{\mathcal{C}}$ is closed under products if and only if $\mathcal{C}$ is covariantly finite in $\mathrm{fp}(\mathcal{A})$ by a result of Crawley-Boevey [3, Theorem 4.2]. Thus the assertion follows from Corollary 4.6.

5. Definable quotient categories

Let $\mathcal{A}$ be an exactly definable category. A category $\mathcal{B}$ together with a functor $q\colon \mathcal{A} \to \mathcal{B}$ is called a *definable quotient category* of $\mathcal{A}$ if there exists a Serre subcategory $\mathcal{S}$ of $C(\mathcal{A})$ and an equivalence $f\colon \mathrm{Ex}(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{B}$ such that $q(M) = f(M|_{\mathcal{S}})$ for all $M \in \mathcal{A} = \mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$. We shall give a categorical explanation of this definition in Section 7. The aim of this section is a description of the full subcategory $\mathrm{pinj}(\mathcal{B})$ of pure-injective objects in $\mathcal{B}$. We shall also describe the full subcategory $\mathrm{fp}(\mathcal{B})$ of finitely presented objects in case that $\mathcal{A}$ is locally finitely presented. The following definition is needed.

Let $\mathcal{C}$ be an additive category and suppose that $\mathcal{S}$ is an additive subcategory of $\mathcal{C}$. Then one can form the *stable quotient category* $\mathcal{C}\lceil\mathcal{S}$ which is additive and admits an additive *quotient functor* $q\colon \mathcal{C} \to \mathcal{C}\lceil\mathcal{S}$ such that $q(\mathcal{S}) = 0$. The objects of $\mathcal{C}\lceil\mathcal{S}$ are those of $\mathcal{C}$ and for every pair X, Y of objects $\mathrm{Hom}(X, Y)$ is the group of morphisms in $\mathcal{C}$ modulo those which factor through an object in $\mathcal{S}$. The quotient functor q is characterized by the property that every additive functor $f\colon \mathcal{C} \to \mathcal{D}$ between additive categories with $f(\mathcal{S}) = 0$ induces, up to isomorphism, a unique additive functor $\bar{f}\colon \mathcal{C}\lceil\mathcal{S} \to \mathcal{D}$ such that $f = \bar{f} \circ q$.

Theorem 5.1 *Let $\mathcal{A}$ be an exactly definable category and fix a Serre subcategory of $C(\mathcal{A})$. Denote by $\mathcal{A}'$ the corresponding definable subcategory and by $\mathcal{A}''$ the corresponding definable quotient category. Then the canonical functors $\mathcal{A}' \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}''$ send pure-injective objects to pure-injective objects and induce an equivalence* $\mathrm{pinj}(\mathcal{A})\lceil\mathrm{pinj}(\mathcal{A}') \to \mathrm{pinj}(\mathcal{A}'')$.

We postpone a proof of this theorem and begin with some preparations. Let $\mathcal{C}$ be an additive category and suppose that the category $\mathrm{fp}(\mathcal{C}, \mathrm{Ab})$ of finitely presented functors $\mathcal{C} \to \mathrm{Ab}$ is abelian. Suppose that $i\colon \mathcal{D} \to \mathcal{C}$ is the inclusion of a full additive subcategory and let $q\colon \mathcal{C} \to \mathcal{C}\lceil\mathcal{D}$.

Lemma 5.2 *The following are equivalent.*

(1) *$\mathcal{D}$ is covariantly finite.*

(2) *The left exact functor $i^*\colon \mathrm{fp}(\mathcal{D}, \mathrm{Ab})^{\mathrm{op}} \to \mathrm{fp}(\mathcal{C}, \mathrm{Ab})^{\mathrm{op}}$ which extends* $\mathrm{Hom}(X, \ ) \mapsto \mathrm{Hom}(i(X), \ )$ *has a left adjoint.*

(3) *The restriction $X \mapsto X \circ i$ induces a functor $i_*\colon \mathrm{fp}(\mathcal{C}, \mathrm{Ab})^{\mathrm{op}} \to \mathrm{fp}(\mathcal{D}, \mathrm{Ab})^{\mathrm{op}}$.*

In this case i_* is exact and $q_*\colon \mathrm{fp}(\mathcal{C}\lceil\mathcal{D}, \mathrm{Ab})^{\mathrm{op}} \to \mathrm{fp}(\mathcal{C}, \mathrm{Ab})^{\mathrm{op}}$ induces an equivalence between $\mathrm{fp}(\mathcal{C}\lceil\mathcal{D}, \mathrm{Ab})^{\mathrm{op}}$ *and* $\mathrm{Ker}(i_*)$. *Moreover* $\mathrm{Ker}(i_*)$ *is a localizing subcategory and i_* induces an equivalence* $\mathrm{fp}(\mathcal{C}, \mathrm{Ab})^{\mathrm{op}} / \mathrm{Ker}(i_*) \to \mathrm{fp}(\mathcal{D}, \mathrm{Ab})^{\mathrm{op}}$.

Proof: See [12, Theorem 3.4] for most parts. The rest follows easily using the fact that i_* is a left adjoint of the fully faithful functor i^*.

Now let $\mathcal{A}$ be an abelian category and denote by $\mathrm{inj}(\mathcal{A})$ the full subcategory of injective objects in $\mathcal{A}$. Recall from Lemma 2.7 that $\mathcal{A} \to \mathrm{fp}(\mathrm{inj}(\mathcal{A}), \mathrm{Ab})^{\mathrm{op}}$, $X \mapsto \mathrm{Hom}(X, \ )|_{\mathrm{inj}(\mathcal{A})}$ is an equivalence provided that $\mathcal{A}$ has sufficently many injectives. We use this fact to obtain our next result.

Theorem 5.3 *Let $\mathcal{A}$ be an abelian category with injective envelopes. There is a bijective correspondence between localizing subcategories of $\mathcal{A}$ and covariantly finite full additive subcategories of* $\mathrm{inj}(\mathcal{A})$ *which are closed under direct summands. The correspondence is given by*

$$\mathcal{C} \mapsto \{X \in \mathrm{inj}(\mathcal{A}) \mid \mathrm{Hom}(\mathcal{C}, X) = 0\} \qquad \textit{and} \qquad \mathcal{I} \mapsto \{X \in \mathcal{A} \mid \mathrm{Hom}(X, \mathcal{I}) = 0\}.$$

If $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$, then the right adjoint $\mathcal{A} \to \mathcal{C}$ of the inclusion and the right adjoint $\mathcal{A}/\mathcal{C} \to \mathcal{A}$ of the quotient functor send injective objects to injective objects and induce an equivalence $\mathrm{inj}(\mathcal{A}) \amalg \mathrm{inj}(\mathcal{A}/\mathcal{C}) \to \mathrm{inj}(\mathcal{C})$.

Proof: If $\mathcal{C}$ is a localizing subcategory, then the quotient category $\mathcal{A}/\mathcal{C}$ has sufficently many injectives and the section functor $\mathcal{A}/\mathcal{C} \to \mathcal{A}$ identifies $\mathrm{inj}(\mathcal{A}/\mathcal{C})$ with the full subcategory $\mathcal{I}$ of objects $X \in \mathrm{inj}(\mathcal{A})$ satisfying $\mathrm{Hom}(\mathcal{C}, X) = 0$ [6, III, Corollaire 3.2]. Thus the section functor is of the form $i^*\colon \mathrm{fp}(\mathcal{I}, \mathrm{Ab})^{\mathrm{op}} \to \mathrm{fp}(\mathrm{inj}(\mathcal{A}), \mathrm{Ab})^{\mathrm{op}}$ and $\mathcal{I}$ is covariantly finite by the preceding lemma since the quotient functor is a left adjoint of i^*. The rest of the assertion follows similarly from the preceding lemma.

Our result about exactly definable quotient categories is now a consequence.

Proof of Theorem 5.1: We shall use the locally coherent category $D(\mathcal{A})$. We fix a Serre subcategory $\mathcal{S}$ of $C(\mathcal{A})$ and the corresponding sequence of functors

$$\mathcal{S} \xrightarrow{i} C(\mathcal{A}) \xrightarrow{q} C(\mathcal{A})/\mathcal{S}$$

induces a sequence

$$D(\mathcal{A}'') \xrightarrow{i^*} D(\mathcal{A}) \xrightarrow{q^*} D(\mathcal{A}')$$

by Lemma 1.1. It is shown in [11, Theorem 2.8] that i^* induces an equivalence between $D(\mathcal{A}'')$ and $\mathrm{Ker}(q^*)$. Moreover $\mathrm{Ker}(q^*)$ is a localizing subcategory of $D(\mathcal{A})$ and q^* induces an equivalence between $D(\mathcal{A})/\mathrm{Ker}(q^*)$ and $D(\mathcal{A}')$. Restricting the right adjoints of i^* and q^* to the full subcategories of fp-injective objects we obtain, up to equivalence, the functors $\mathcal{A} \to \mathcal{A}''$ and $\mathcal{A}' \to \mathcal{A}$. The assertion now follows from the preceding theorem.

We have seen in the previous section a method to construct definable subcategories for locally finitely presented categories. Our next result discusses the corresponding definable quotient categories.

Theorem 5.4 *Let $\mathcal{A}$ be a locally finitely presented category with products and suppose that $\mathcal{C}$ is a covariantly finite full additive subcategory of* $\mathrm{fp}(\mathcal{A})$. *Then $\mathcal{A}' = \vec{\mathcal{C}}$ is a definable subcategory which is locally finitely presented. The corresponding definable quotient category $\mathcal{A}''$ is also locally finitely presented and the canonical functor $\mathcal{A} \to \mathcal{A}''$ induces an equivalence* $\mathrm{fp}(\mathcal{A})/\mathrm{fp}(\mathcal{A}') \to \mathrm{fp}(\mathcal{A}'')$.

Proof: We have already shown in Proposition 4.13 that $\mathcal{A}'$ is definable. Also we know from Proposition 2.3 and Lemma 2.7 that an exactly definable category $\mathcal{B}$ is locally finitely presented iff $C(\mathcal{B})$ is equivalent to $\mathrm{fp}(\mathcal{D}, \mathrm{Ab})^{\mathrm{op}}$ for some additive category $\mathcal{D}$. In this case $\mathrm{fp}(\mathcal{B})$ and $\mathcal{D}$ are canonically equivalent and the assertion now follows from Lemma 5.2.

6. The Ziegler spectrum

Let $\mathcal{A}$ be an exactly definable category and recall that there is a fully faithful functor $\mathcal{A} \to D(\mathcal{A})$ into a locally coherent category which identifies the pure-injectives in $\mathcal{A}$ with the injectives in $D(\mathcal{A})$. We shall use this fact in studying the *Ziegler spectrum* $\mathrm{Zsp}(\mathcal{A})$ of $\mathcal{A}$ which is by definition the set of isomorphism classes of indecomposable pure-injective objects in $\mathcal{A}$. It will be convenient to identify each isomorphism class in $\mathrm{Zsp}(\mathcal{A})$ with a representative belonging to it. We follow Ziegler [21] and introduce a topology on $\mathrm{Zsp}(\mathcal{A})$. To this end identify $\mathcal{A} = \mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$. Let $\mathcal{X}$ be any class of objects in $\mathcal{A}$. We shall frequently identify $\mathcal{X} = M$ if $\mathcal{X}$ consists of a single object M. Denote by $\Sigma(\mathcal{X})$ the Serre subcategory $\cap_{M \in \mathcal{X}} \mathrm{Ker}(M)$ of $C(\mathcal{A})$. For a subcategory $\mathcal{S}$ of $C(\mathcal{A})$ let $\Upsilon(\mathcal{S}) = \{M \in \mathrm{Zsp}(\mathcal{A}) \mid M(\mathcal{S}) = 0\}$. We call $\overline{\mathcal{X}} = \Upsilon \circ \Sigma(\mathcal{X})$ the *Ziegler closure* of $\mathcal{X}$.

Lemma 6.1 *The assignment*

$$\mathcal{U} \mapsto \overline{\mathcal{U}} = \Upsilon \circ \Sigma(\mathcal{U})$$

is a closure operator on the Ziegler spectrum $\mathrm{Zsp}(\mathcal{A})$ *of $\mathcal{A}$, i.e. the subsets $\mathcal{U} \subseteq \mathrm{Zsp}(\mathcal{A})$ satisfying $\overline{\mathcal{U}} = \mathcal{U}$ form the closed sets of a topology on* $\mathrm{Zsp}(\mathcal{A})$.

Proof: See [11, Lemma 4.1].

We refer the reader to [11] for a further discussion of this topology. We point out, however, the following result.

Theorem 6.2 *Let $\mathcal{A}$ be an exactly definable category. There is a bijective inclusion reversing correspondence between Serre subcategories of $C(\mathcal{A})$ and closed subsets of* $\mathrm{Zsp}(\mathcal{A})$. *The correspondence is given by*

$$\mathcal{S} \mapsto \Upsilon(\mathcal{S}) \qquad \textit{and} \qquad \mathcal{U} \mapsto \Sigma(\mathcal{U}).$$

Proof: See [11, Theorem 4.2].

Corollary 6.3 *Let $\mathcal{A}'$ be a definable subcategory of $\mathcal{A}$ and denote by $\mathcal{A}''$ the corresponding definable quotient category. Then the canonical functor $\mathcal{A}' \to \mathcal{A}$ induces a homeomorphism $\operatorname{Zsp}(\mathcal{A}') \to \operatorname{Zsp}(\mathcal{A})$ onto a closed subset of $\operatorname{Zsp}(\mathcal{A})$ and the canonical functor $\mathcal{A} \to \mathcal{A}''$ induces a homeomorphism $\operatorname{Zsp}(\mathcal{A}) \setminus \operatorname{Zsp}(\mathcal{A}') \to \operatorname{Zsp}(\mathcal{A}'')$.*

Proof: See [11, Corollary 4.4].

7. Functors between exactly definable categories

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of exactly definable categories. In this section we exhibit the properties of certain functors between $\mathcal{A}$ and $\mathcal{B}$. Given an exact functor $f\colon C(\mathcal{B}) \to C(\mathcal{A})$ the corresponding restriction functor $(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}) \to (C(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto X \circ f$ induces a functor $f_*\colon \mathcal{A} \to \mathcal{B}$ if one identifies $\mathcal{A} = \operatorname{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ and $\mathcal{B} = \operatorname{Ex}(C(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab})$. We call a functor $\mathcal{A} \to \mathcal{B}$ *definable* if it is naturally equivalent to a functor of the form f_* for some exact $f\colon C(\mathcal{B}) \to C(\mathcal{A})$. Our aim in this section is to discuss the basic properties of such functors. We start with some categorical properties.

Proposition 7.1 *The exactly definable categories together with the definable functors form a category with kernels and cokernels. If $f\colon \mathcal{A} \to \mathcal{B}$ is a definable functor, then $\operatorname{Ker}(f)$ is the definable subcategory of all objects $X \in \mathcal{A}$ such that $f(X) = 0$, and $\operatorname{Coker}(f)$ is a definable quotient category of $\mathcal{B}$. The kernel of the canonical functor $\mathcal{B} \to \operatorname{Coker}(f)$ is the smallest definable subcategory containing $f(X)$ for all $X \in \mathcal{A}$.*

Proof: The assertion follows from the corresponding statement for the category of skeletally small abelian categories. To this end let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor. Let $\operatorname{Ker}(f)$ be the full subcategory of all objects $X \in \mathcal{C}$ such that $f(X) = 0$. Denote by $\mathcal{S}$ the smallest Serre subcategory of $\mathcal{D}$ containing $f(X)$ for all $X \in \mathcal{C}$ and define $\operatorname{Coker}(f) = \mathcal{D}/\mathcal{S}$. Then it is easily checked that the inclusion $\operatorname{Ker}(f) \to \mathcal{C}$ is a kernel and that the quotient functor $\mathcal{D} \to \operatorname{Coker}(f)$ is a cokernel in the categorical sense.

We proceed now with a characterization of definable functors. It is obvious from the definition that a definable functor commutes with direct limits and products. The next result discusses the converse of this observation.

Theorem 7.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of exactly definable categories and suppose that $\mathcal{A}$ is locally finitely presented. There is, up to isomorphism, a natural and bijective correspondence*

between exact functors $C(\mathcal{B}) \to C(\mathcal{A})$ *and functors* $\mathcal{A} \to \mathcal{B}$ *commuting with direct limits and products. The correspondence is given by*

$$f\colon C(\mathcal{B}) \to C(\mathcal{A}) \;\mapsto\; f_*\colon \mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ex}(C(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab}).$$

Proof: We construct an inverse for $f \mapsto f_*$ as follows. Suppose that $f\colon \mathcal{A} \to \mathcal{B}$ commutes with direct limits and products and fix the fully faithful functors $d_{\mathcal{A}}\colon \mathcal{A} \to D(\mathcal{A})$ and $d_{\mathcal{B}}\colon \mathcal{B} \to D(\mathcal{B})$. It is shown in [12, Property 10.4] that $d_{\mathcal{B}} \circ f$ extends, up to isomorphism, uniquely to a left exact functor $g\colon D(\mathcal{A}) \to D(\mathcal{B})$ which commutes with direct limits and products. It follows from the adjoint functor theorem [15, V, Corollary 3.2] that g has a left adjoint $D(\mathcal{B}) \to D(\mathcal{A})$ which is exact since g preserves injectives by our characterization in Theorem 2.6. Moreover, a left adjoint sends finitely presented to finitely presented objects by [12, Lemma 6.1] since the right adjoint commutes with direct limits. Thus we obtain an exact functor $f^*\colon C(\mathcal{B}) \to C(\mathcal{A})$ since $C(\mathcal{A}) = \mathrm{fp}(D(\mathcal{A}))$ and $C(\mathcal{B}) = \mathrm{fp}(D(\mathcal{B}))$ by definition. We leave it to the reader to check that the assignments $f \mapsto f_*$ and $f \mapsto f^*$ are mutually inverse to each other. Moreover, using [15, V, Proposition 2.1] it is not hard to check that these assignments are natural, i.e. there are natural bijections $\mathrm{Hom}(f, g) \to \mathrm{Hom}(g_*, f_*)$ for each pair of exact functors f and g from $C(\mathcal{B})$ to $C(\mathcal{A})$.

The preceding proof shows that one needs to study certain functors between locally coherent categories in order to understand the definable functors between exactly definable categories. We begin with some preparations. To this end fix a locally coherent category $\mathcal{A}$. For $X \in \mathrm{fp}(\mathcal{A})$ we denote by $\mathrm{Latt}_{\mathrm{fp}}(X)$ the lattice of subobjects in $\mathrm{fp}(\mathcal{A})$.

Lemma 7.3 *Let* $X \in \mathrm{fp}(\mathcal{A})$*. There is a bijective correspondence between subobjects of* X *in* $\mathcal{A}$ *and ideals in* $\mathrm{Latt}_{\mathrm{fp}}(X)$*. The correspondence is given by*

$$U \mapsto \mathcal{I} = \{Y \in \mathrm{Latt}_{\mathrm{fp}}(X) \mid Y \subseteq U\} \qquad \text{and} \qquad \mathcal{I} \mapsto U = \textstyle\sum_{Y \in \mathcal{I}} Y.$$

Proof: Straightforward.

For each subset $\mathcal{I} \subseteq \mathrm{Latt}_{\mathrm{fp}}(X)$ we denote by $\mathcal{I}^-$ its complement. Recall that an object X is *irreducible* if it is not the sum of two proper subobjects. Dually, X is *coirreducible* if any two non-zero subobjects have non-zero intersection. Note that X/U is coirreducible iff U cannot be be written as the intersection of two strictly bigger subobjects of X.

Lemma 7.4 *Let* U *be a subobject of* $X \in \mathrm{fp}(\mathcal{A})$ *and denote by* $\mathcal{I}$ *the corresponding ideal in* $\mathrm{Latt}_{\mathrm{fp}}(X)$*. Then* X/U *is coirreducible iff for each pair* $Y_1, Y_2 \in \mathcal{I}^-$ *there is* $Y \in \mathcal{I}$ *such that* $(Y_1 + Y) \cap (Y_2 + Y) \in \mathcal{I}^-$*.*

Proof: We show one direction, the other one being similar. So suppose that X/U is coirreducible and let $Y_1, Y_2 \in \mathcal{I}^-$. It follows that $(Y_1 + U) \cap (Y_2 + U)$ is strictly bigger than U. Thus there is a finitely generated subobject $W \subseteq (Y_1 + U) \cap (Y_2 + U)$ which lies in $\mathcal{I}^-$. Writing $W = Y_i + U_i$ for some $U_i \in \mathcal{I}$, $i = 1, 2$, we obtain $(Y_1 + Y) \cap (Y_2 + Y) = W \in \mathcal{I}^-$ for $Y = U_1 + U_2 \in \mathcal{I}$.

Lemma 7.5 *Let $(\mathcal{I}, \mathcal{F})$ be a pair of non-empty subsets of $\mathrm{Latt}_{\mathrm{fp}}(X)$ such that for each pair $Y_1, Y_2 \in \mathcal{F}$ there is $Y \in \mathcal{I}$ with $(Y_1 + Y) \cap (Y_2 + Y) \in \mathcal{F}$. If $\mathcal{J}$ is a maximal ideal such that $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{J} \cap \mathcal{F} = \emptyset$, then X/U with $U = \sum_{Y \in \mathcal{J}} Y$ is coirreducible.*

Proof: Let $Y_1, Y_2 \in \mathcal{J}^-$. It follows from the maximality of $\mathcal{J}$ that there are $U_i \in \mathcal{J}$ and $V_i \in \mathcal{F}$ such that $V_i \subseteq Y_i + U_i$ for $i = 1, 2$. Choose $V \in \mathcal{I}$ such that $(V_1 + V) \cap (V_2 + V) \in \mathcal{F}$. We obtain for $Y = V + (U_1 + U_2) \in \mathcal{J}$ that

$$(V_1 + V) \cap (V_2 + V) \subseteq (Y_1 + Y) \cap (Y_2 + Y) \in \mathcal{J}^-$$

since $\mathcal{F} \in \mathcal{J}^-$.

Given any Grothendieck category $\mathcal{A}$, we denote by $\mathrm{Sp}(\mathcal{A})$ the set of isomorphism classes of indecomposable injective objects in $\mathcal{A}$. The next lemma is based on [17, Proposition 2].

Lemma 7.6 *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a faithful exact functor between locally coherent categories which preserves direct limits and finitely presented objects. Denote by $g\colon \mathcal{B} \to \mathcal{A}$ a right adjoint. If $M \in \mathrm{Sp}(\mathcal{A})$, then there is a monomorphism $\varphi\colon M \to g(N)$ for some $N \in \mathrm{Sp}(\mathcal{B})$.*

Proof: We may assume that $\mathcal{A}$ is a subcategory of $\mathcal{B}$, and we refer to Lemma 1.1 for the existence of a right adjoint g for the inclusion. Now let X/U be a finitely generated non-zero subobject of $M \in \mathcal{A}$ with $X \in \mathrm{fp}(\mathcal{A})$ and denote by $(\mathcal{I}, \mathcal{I}^-)$ the corresponding pair of subsets of $\mathrm{Latt}_{\mathrm{fp}}(X)$ with $\mathcal{I} = \{Y \in \mathrm{fp}(\mathcal{A}) \mid Y \subseteq U\}$ and $\mathcal{I}^- = \mathrm{Latt}_{\mathrm{fp}}(X) \setminus \mathcal{I}$. Passing to the lattice of finitely presented subobjects of X in $\mathcal{B}$ we may apply Zorn's lemma to find a maximal ideal $\mathcal{I} \subseteq \mathcal{J}$ missing $\mathcal{I}^-$. Thus $U \subseteq V = \sum_{Y \in \mathcal{J}} Y$ and X/V is coirreducible in $\mathcal{B}$ by the preceding lemma. Moreover, for any non-zero morphism $\varphi\colon Y \to X/U$ in $\mathcal{A}$ the composition $\pi \circ \varphi$ with the canonical morphism $\pi\colon X/U \to X/V$ is again non-zero. Now let $\chi\colon X/V \to N$ be an injective envelope and let $\psi\colon M \to N$ be a morphism which extends $\chi \circ \pi$. Also denote by $\xi_X\colon g(X) \to X$ the functorial morphism defined for each $X \in \mathcal{B}$ which reflects the adjointness of g. In particular there is a morphism $\eta_X\colon X \to g(X)$ in $\mathcal{A}$ for each $X \in \mathcal{A}$ such that $\xi_X \circ \eta_X = \mathrm{id}_X$. We claim that $\varphi = g(\psi) \circ \eta_M$ is a monomorphism. To this end assume that $\mathrm{Ker}(\varphi) \neq 0$. Thus $X/U \cap \mathrm{Ker}(\varphi) \neq 0$ and therefore $\pi(X/U \cap \mathrm{Ker}(\varphi)) \neq 0$.

This implies that $\psi(X/U \cap \mathrm{Ker}(\varphi)) \neq 0$, contradicting the fact that $\psi = \xi_N \circ \varphi$. Thus the proof is complete.

Given a definable functor $f\colon \mathcal{A} \to \mathcal{B}$ we can now formulate some relation between $\mathrm{Zsp}(\mathcal{A})$ and $\mathrm{Zsp}(\mathcal{B})$ induced by f.

Proposition 7.7 *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a definable functor. Then for every $M \in \mathrm{Zsp}(\mathcal{B})$ there is a pure monomorphism $M \to f(N)$ for some $N \in \mathrm{Zsp}(\mathcal{A})$ if and only if $\mathrm{Coker}(f) = 0$.*

Proof: Suppose first that $\mathrm{Coker}(f) = 0$. Thus the exact functor $C(\mathcal{B}) \to C(\mathcal{A})$ corresponding to f is faithful and extends therefore to a faithful exact functor $g\colon D(\mathcal{B}) \to D(\mathcal{A})$ which commutes with direct limits by Lemma 1.1. The assertion now follows from the preceding lemma since a right adjoint of g is naturally equivalent to f when restricted to the fp-injective objects. If $\mathrm{Coker}(f) \neq 0$, then the pure subobjects of objects in $f(\mathcal{A})$ belong to a proper definable subcategory $\mathcal{B}'$ of $\mathcal{B}$ by Proposition 7.1. Thus $\mathrm{Zsp}(\mathcal{B}) \setminus \mathcal{B}' \neq \emptyset$ by Theorem 6.2 and this finishes the proof.

We are now in position to prove the main result of this section. Note that part (4) of the following theorem generalizes a result of Prest [17, Theorem 3].

Theorem 7.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of exactly definable categories. A definable functor $f\colon \mathcal{A} \to \mathcal{B}$ has the following properties.*

(1) *f sends pure-injective objects to pure-injective objects and pure-exact sequences to pure-exact sequences.*

(2) *If $\mathcal{X}$ is a definable subcategory of $\mathcal{B}$, then the full subcategory $f^{-1}(\mathcal{X})$ of objects $X \in \mathcal{A}$ with $f(X) \in \mathcal{X}$ is a definable subcategory of $\mathcal{A}$.*

(3) *If $\mathcal{X}$ is a class of objects in $\mathcal{A}$, then the Ziegler closure $\overline{f(\mathcal{X})}$ equals the set of indecomposable direct summands of objects in $f(\overline{\mathcal{X}})$.*

(4) *If f induces a map $\mathrm{Zsp}(f)\colon \mathrm{Zsp}(\mathcal{A}) \to \mathrm{Zsp}(\mathcal{B})$, then this map is continuous and closed.*

Proof: We may assume that f is of the form

$$\mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ex}(C(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto X \circ g$$

for some exact functor $g\colon C(\mathcal{B}) \to C(\mathcal{A})$.

(1) Follows from the definition of pure-injectivity and Lemma 3.1.

(2) Recall that $\mathcal{X}$ is definable iff it is closed under direct limits, products and pure subobjects. The assertion follows since f preserves direct limits, products and pure monomorphisms by (1).

(3) Let $\mathcal{A}'$ be the smallest definable subcategory of $\mathcal{A}$ containing $\mathcal{X}$ and let $\mathcal{B}'$ be the smallest definable subcategory of $\mathcal{B}$ containing $f(\mathcal{X})$. Note that $\mathcal{B}'$ is also the smallest definable subcategory containing $f(\mathcal{A}')$. The functors f and g induce commutative diagrams

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{f'} & \mathcal{B}' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \mathcal{A} & \xrightarrow{f} & \mathcal{B} \end{array} \qquad\qquad \begin{array}{ccc} C(\mathcal{A}') & \xleftarrow{g'} & C(\mathcal{B}') \\ \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle q} \\ C(\mathcal{A}) & \xleftarrow{g} & C(\mathcal{B}) \end{array}$$

where i and j are inclusions and p and q are isomorphic to the quotient functors $C(\mathcal{A}) \to C(\mathcal{A})/\mathcal{S}$ and $C(\mathcal{B}) \to C(\mathcal{B})/\mathcal{T}$ such that $\mathcal{S}$ corresponds to $\mathcal{A}'$ and $\mathcal{T}$ corresponds to $\mathcal{B}'$. Furthermore, the composition $p \circ g$ induces the functor g' making the above diagram commutative since $\mathrm{Ker}(p \circ g) = \mathrm{Ker}(q)$ by construction. The functor g' is faithful and extends therefore to a faithful exact functor $D(\mathcal{B}') \to D(\mathcal{A}')$ which commutes with direct limits by Lemma 1.1. The restriction functor $\mathrm{Lex}(C(\mathcal{A}')^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Lex}(C(\mathcal{B}')^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto X \circ g'$ provides a right adjoint $D(\mathcal{A}') \to D(\mathcal{B}')$ which coincides with f' when restricted to the exact functors. Now the assertion follows from Lemma 7.6.

(4) We say that f induces a map $\mathrm{Zsp}(f)\colon \mathrm{Zsp}(\mathcal{A}) \to \mathrm{Zsp}(\mathcal{B})$ provided that f preserves indecomposable pure-injectives. Such a map is continuous by (2) and closed by (3).

8. The endocategory of an object

In this section we concentrate on properties of a single object M of an exactly definable category $\mathcal{A}$. We shall view the object M as an exact functor $M\colon C(\mathcal{A})^{\mathrm{op}} \to \mathrm{Ab}$ via the identification $\mathcal{A} = \mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab})$ and we denote by $\mathcal{S}_M = \mathrm{Ker}(M)$ the corresponding Serre subcategory of $C(\mathcal{A})$. Let $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$ so that M becomes naturally a functor $C(\mathcal{A})^{\mathrm{op}} \to \mathrm{Mod}(\Gamma)$. It induces a faithful exact functor $\bar{M}\colon (C(\mathcal{A})/\mathcal{S}_M)^{\mathrm{op}} \to \mathrm{Mod}(\Gamma)$ and we call the image $\mathcal{E}_M$ of $\bar{M}$ the *endocategory* of M. We shall tacitly assume that the subcategory $\mathcal{E}_M$ of $\mathrm{Mod}(\Gamma)$ is closed under isomorphisms.

Lemma 8.1 *The endocategory $\mathcal{E}_M$ of M is abelian and the inclusion $\mathcal{E}_M \to \mathrm{Mod}(\Gamma)$ is an exact functor.*

Let $\mathcal{A}_M$ be the definable subcategory of $\mathcal{A}$ which consists of all functors in $\mathcal{A}$ vanishing on $\mathcal{S}_M$. We call $\mathcal{A}_M$ *generated by* M since it is the smallest definable subcategory of $\mathcal{A}$ containing M.

Proposition 8.2 *The category $\mathcal{A}_M$ consists of all pure subobjects of reduced powers of M. The functor $C(\mathcal{A}_M)^{\mathrm{op}} \to \mathrm{Mod}(\Gamma)$ corresponding to M induces an anti-equivalence between $C(\mathcal{A}_M)$ and the endocategory $\mathcal{E}_M$ of M.*

Proof: It is shown in Corollary 4.10 that the pure subobjects of reduced powers of M form a definable subcategory of $\mathcal{A}$ which is clearly the smallest containing M, since $\mathcal{A}_M$ is closed under reduced products and pure subobjects. The second assertion follows from the fact that $C(\mathcal{A}_M)$ and $C(\mathcal{A})/\mathcal{S}_M$ are equivalent.

The next result shows that pure-injectivity depends only on the inclusion $\mathcal{E}_M \to \mathrm{Mod}(\Gamma)$.

Proposition 8.3 *An object M is pure-injective if and only if for every object X of the endocategory $\mathcal{E}_M$ and for every codirected family X_i, $i \in I$, of subobjects of X in $\mathcal{E}_M$ the canonical morphism $X \to \varprojlim_{i\in I} X/X_i$ in $\mathrm{Mod}(\Gamma)$ is an epimorphism.*

Proof: We view M as an object of the locally coherent category $D(\mathcal{A})$. Using Baer's criterion as stated in Lemma 2.5, M is pure-injective iff for every $Y \in C(\mathcal{A})$ any subobject U of Y induces an epimorphism $\mathrm{Hom}(Y, M) \to \mathrm{Hom}(U, M)$, equivalently if any directed family of subobjects U_i, $i \in I$, in $C(\mathcal{A})$ induces an epimorphism $\mathrm{Hom}(Y, M) \to \mathrm{Hom}(\varinjlim U_i, M) \cong \varprojlim \mathrm{Hom}(U_i, M)$. Replacing $X = \mathrm{Hom}(Y, M)$ and $X_i = \mathrm{Hom}(Y/U_i, M)$ the assertion follows since $X \in \mathcal{E}_M$ iff $X \cong \mathrm{Hom}(Y, M)$ for some $Y \in C(\mathcal{A})$ by Corollary 2.10.

Our next aim is to show that the construction of the endocategory is somehow functorial.

Theorem 8.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of exactly definable categories and suppose that $f\colon \mathcal{A} \to \mathcal{B}$ is a definable functor. If M is an object in $\mathcal{A}$ and $\varphi\colon \Gamma_M \to \Gamma_{f(M)}$ is the ring homomorphism induced by f where $\Gamma_X = \mathrm{End}(X)^{\mathrm{op}}$ for any object X, then $\varphi_*\colon \mathrm{Mod}(\Gamma_{f(M)}) \to \mathrm{Mod}(\Gamma_M)$ induces an exact and faithful functor $\mathcal{E}_{f(M)} \to \mathcal{E}_M$.*

Proof: We proceed as in the proof of Theorem 7.8. Thus we may assume that f is of the form $\mathrm{Ex}(C(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ex}(C(\mathcal{B})^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto X \circ g$ for some exact functor $g\colon C(\mathcal{B}) \to C(\mathcal{A})$. The functors f and g induce commutative diagrams

$$\begin{array}{ccc} \mathcal{A}_M & \longrightarrow & \mathcal{B}_{f(M)} \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{f} & \mathcal{B} \end{array} \qquad\qquad \begin{array}{ccc} C(\mathcal{A}_M) & \longleftarrow & C(\mathcal{B}_{f(M)}) \\ \uparrow & & \uparrow \\ C(\mathcal{A}) & \xleftarrow{g} & C(\mathcal{B}) \end{array}$$

where the functor $C(\mathcal{B}_{f(M)}) \to C(\mathcal{A}_M)$ is faithful and exact. Using the anti-equivalences $C(\mathcal{A}_M) \to \mathcal{E}_M$ and $C(\mathcal{B}_{f(M)}) \to \mathcal{E}_{f(M)}$ induced by M and $f(M)$, respectively, the assertion follows.

9. Σ-pure-injectives

In this section we give various characterizations of objects M having the property that the endocategory $\mathcal{E}_M$ is artinian, i.e. every object in $\mathcal{E}_M$ is artinian. This generalizes well-known results about Σ-pure-injective modules over a ring. We shall prove our results in the context of locally coherent categories. To this end recall that a Grothendieck category $\mathcal{A}$ is said to be *locally noetherian* (*locally finite*) provided that $\mathcal{A}$ has a generating set of noetherian (finite length) objects. Note that in such a category $\mathrm{fp}(\mathcal{A})$ consists precisely of the noetherian (finite length) objects of $\mathcal{A}$. In particular $\mathcal{A}$ is locally coherent. We start with a series of preliminary results.

Proposition 9.1 *Let $\mathcal{A}$ be a locally noetherian category. Then every fp-injective object in $\mathcal{A}$ is injective.*

Proof: The assertion follows from Baer's criterion as stated in Lemma 2.5 since any finitely generated object is finitely presented.

Proposition 9.2 *Let $\mathcal{A}$ be a locally noetherian category. Then every fp-injective object in $\mathcal{A}$ is a coproduct of indecomposable objects with local endomorphism ring.*

Proof: Let M be a non-zero fp-injective object and choose a finitely generated non-zero subobject X of M. Using Zorn's lemma there exists a maximal fp-injective subobject U of M which does not contain X. The subobject U is injective by the preceding result and it follows that $M = U \coprod V$ for some object V. Moreover, V is indecomposable since a decomposition of V would contradict the maximality of U. Thus M has an indecomposable direct summand. Using Zorn's lemma we can find a maximal set $(U_i)_{i\in I}$ of indecomposable fp-injective subobjects of M such that the sum $N = \sum_i U_i$ is direct. Again, N is a direct summand of M and its complement is zero since otherwise an indecomposable direct summand could be adjoined to I. Now the assertion follows since indecomposable injective objects have local endomorphism ring.

To simplify the next proofs we introduce the following notation. We fix a locally coherent category $\mathcal{A}$ and some fp-injective object $M \in \mathcal{A}$. Suppose also that M cogenerates $\mathrm{fp}(\mathcal{A})$, i.e. $\mathrm{Hom}(X, M) \neq 0$ for every non-zero $X \in \mathrm{fp}(\mathcal{A})$. Let $\varphi = (X_i)_{i\in\mathbb{N}}$ be an ascending chain $X_1 \subseteq X_2 \subseteq \ldots$ of finitely presented subobjects of some $X \in \mathrm{fp}(\mathcal{A})$. Given any object $N \in \mathcal{A}$, we view $N_{\varphi_i} = \mathrm{Hom}(X/X_i, N)$ as a subgroup of $\mathrm{Hom}(X, N)$ and obtain therefore a descending chain $\ldots \subseteq N_{\varphi_2} \subseteq N_{\varphi_1}$ in $\mathrm{Hom}(X, N)$.

Proposition 9.3 *If the canonical morphism $\coprod_{\mathbb{N}} M \to \prod_{\mathbb{N}} M$ is a split mono, then $\mathcal{A}$ is locally noetherian.*

Proof: Let $\varphi = (X_i)_{i\in\mathbb{N}}$ be a proper ascending chain $X_1 \subseteq X_2 \subseteq \ldots$ of finitely presented subobjects of some $X \in \mathrm{fp}(\mathcal{A})$, which induces a proper descending chain $\ldots \subseteq M_{\varphi_2} \subseteq M_{\varphi_1}$. Choose $\theta_i \in M_{\varphi_i} \setminus M_{\varphi_{i+1}}$ for all $i \in \mathbb{N}$ and denote by $\theta = (\theta_i)$ the corresponding morphism in $\mathrm{Hom}(X, \prod_{\mathbb{N}} M)$. By assumption the canonical monomorphism $\coprod_{\mathbb{N}} M \to \prod_{\mathbb{N}} M$ has an inverse $\pi\colon \prod_{\mathbb{N}} M \to \coprod_{\mathbb{N}} M$. The composition $\pi\circ\theta\colon X \to \coprod_{\mathbb{N}} M = \sum_{i\in\mathbb{N}} \coprod_{j<i} M$ factors through some finite sum $\coprod_{j<n} M$ for some $n \in \mathbb{N}$ since X is finitely presented. Now write $\theta = \theta_{<n} + \theta_n + \theta_{>n}$ with $(\theta_{<n})_i \neq 0$ iff $i < n$, $(\theta_n)_i \neq 0$ iff $i = n$ and $(\theta_{>n})_i \neq 0$ iff $i > n$. Note that $\theta_{>n} \in (\prod_{\mathbb{N}} M)_{\varphi_{n+1}}$. We obtain $\pi\circ\theta = \pi\circ\theta_{<n}$ and therefore $\pi\circ\theta_n = -\pi\circ\theta_{>n} \in (\coprod_{\mathbb{N}} M)_{\varphi_{n+1}} = \coprod_{\mathbb{N}}(M)_{\varphi_{n+1}}$. Thus $\theta_n \in M_{\varphi_{n+1}}$, contradicting the choice of θ_n, and this finishes the proof.

Lemma 9.4 *Let $(M_n)_{n\in\mathbb{N}}$ be a family of objects in $\mathcal{A}$ and suppose that $M = \prod_{n\in\mathbb{N}} M_n = \coprod_{i\in I} N_i$ for some family $(N_i)_{i\in I}$ of objects in $\mathcal{A}$. If $\varphi = (X_i)_{i\in\mathbb{N}}$ is an ascending chain in $\mathrm{fp}(\mathcal{A})$, then there exists $m \in \mathbb{N}$ such that $\pi_i(\prod_{n\geq m} M_n)_{\varphi_m} \subseteq \cap_{n\in\mathbb{N}}(N_i)_{\varphi_n}$ for almost all $i \in I$, where $\pi_i\colon M \to N_i$ denotes the canonical projection.*

Proof: Assume the conclusion to be false. We obtain a sequence $(n_j)_{j\in\mathbb{N}}$ of natural numbers with $n_{j+1} > n_j$ and sequences of pairwise different elements $i_j \in I$ and $\theta_j \in (\prod_{n\geq n_j} M_n)_{\varphi_{n_j}}$, respectively, satisfying $\pi_{i_j}(\theta_j) \notin (N_{i_j})_{\varphi_{n_{j+1}}}$ and $\pi_{i_j}(\theta_k) = 0$ for $k < j$. Now let $\theta = \sum_{j\in\mathbb{N}} \theta_j \in \mathrm{Hom}(X, \prod_{n\in\mathbb{N}} M_n)$. We have $\pi_{i_j}(\theta) = \pi_{i_j}(\theta_j) + \pi_{i_j}(\sum_{k>j} \theta_k) \neq 0$ since the second summand lies in $(N_{i_j})_{\varphi_{n_{j+1}}}$ whereas the first does not. On the other hand the morphism θ factors through a finite sum $\coprod_{i\in J} N_i$ for some $J \subseteq I$ since X is finitely presented. This contradiction finishes the proof.

Proposition 9.5 *If any countable product of copies of M is a direct sum of objects with local endomorphism ring, then $\mathcal{A}$ is locally noetherian.*

Proof: Let $\prod_{\mathbb{N}} M = \coprod_{i\in I} N_i$, where all N_i have a local endomorphism ring. We may apply the Krull-Remak-Schmidt-Azumaya Theorem. Thus any indecomposable direct summand N of M occurs, up to isomorphism, an infinite number of times in the family $(N_i)_{i\in I}$. Now let $\varphi = (X_i)_{i\in\mathbb{N}}$ be an ascending chain of finitely presented subobjects of some $X \in \mathrm{fp}(\mathcal{A})$. With the previous observation it follows immediately from the preceding lemma that there exists $m \in \mathbb{N}$ such that $N_{\varphi_m} \subseteq \cap_{n\in\mathbb{N}} N_{\varphi_n}$ for all indecomposable direct summands N of M. Therefore $M_{\varphi_m} \subseteq \cap_{n\in\mathbb{N}} M_{\varphi_n}$, and $X_m = X_{m+1} = \ldots$ follows since M cogenerates $\mathrm{fp}(\mathcal{A})$. This finishes the proof.

Recall that an object M is Σ-*injective* if any coproduct of copies of M is injective. Similarly, M is Σ-*pure-injective* if any coproduct of copies of M is pure-injective. Combining the results of this section we obtain the following characterization of a Σ-pure-injective object.

Theorem 9.6 *Let $\mathcal{A}$ be an exactly definable category. Then the following are equivalent for an object M in $\mathcal{A}$.*

(1) *M is Σ-pure-injective.*

(2) *Any countable coproduct of copies of M is pure-injective.*

(3) *The canonical morphism $\coprod_{\mathbb{N}} M \to \prod_{\mathbb{N}} M$ is a split mono.*

(4) *Any product of copies of M is a direct sum of indecomposable objects with local endomorphism ring.*

(5) *Any countable product of copies of M is a direct sum of indecomposable objects with local endomorphism ring.*

(6) *The endocategory $\mathcal{E}_M$ of M is artinian.*

Proof: We pass from $\mathcal{A}$ to the definable subcategory generated by M. The category $D(\mathcal{A}_M)$ is locally coherent and we view M as an fp-injective object which cogenerates $C(\mathcal{A}_M)$. The assertion now follows from the preceding propositions since each condition characterizes the fact that $D(\mathcal{A}_M)$ is locally noetherian. In particular, one uses for the last condition the fact that there is an anti-equivalence between $C(\mathcal{A}_M)$ and $\mathcal{E}_M$ by Proposition 8.2.

Remark 9.7 *A decomposition into indecomposables with local endomorphism ring is essentially unique by the Krull-Remak-Schmidt-Azumaya Theorem.*

We continue with some properties of Σ-pure-injective objects. Our first result is a characterization in terms of the Ziegler spectrum. Recall that $\overline{M}$ denotes the Ziegler closure of an object $M \in \mathcal{A}$.

Corollary 9.8 *The following are equivalent for an object $M \in \mathcal{A}$.*

(1) *M is Σ-pure-injective.*

(2) *Every product of copies of M is a coproduct of copies of objects in $\overline{M}$.*

(3) *The coproducts of copies of objects in $\overline{M}$ form a definable subcategory of $\mathcal{A}$.*

Moreover, in this case $\overline{M}$ consists precisely of all isoclasses of indecomposable direct summands of products of copies of M.

Proof: The equivalence of the conditions (1) – (3) is a direct consequence of the preceding theorem. The last assertion is shown as follows. If $N \in \mathrm{Zsp}(\mathcal{A})$ is a direct summand of some

product of copies of M, then N belongs to $\mathcal{A}_M$ and therefore to $\overline{M}$. Conversely, any $N \in \overline{M}$ embeds purely in some product of copies of M since the canonical functor $d\colon \mathcal{A}_M \to D(\mathcal{A}_M)$ identifies M with an injective cogenerator by the lemma below.

Lemma 9.9 *If $M \in \mathcal{A}$ is Σ-pure-injective, then the canonical functor $d\colon \mathcal{A}_M \to D(\mathcal{A}_M)$ identifies M with an injective cogenerator for $D(\mathcal{A}_M)$.*

Proof: The object $d(M)$ cogenerates $C(\mathcal{A}_M)$. Also, $D(\mathcal{A}_M)$ is locally noetherian and therefore finitely generated objects are finitely presented. It follows that $d(M)$ cogenerates $D(\mathcal{A}_M)$.

The rest of this section is devoted to several properties of the endocategory of a Σ-pure-injective object. We need some lemmas.

Lemma 9.10 *Let $\mathcal{A}$ be a locally coherent category. Let $M \in \mathcal{A}$ be an injective object and $X \in \mathrm{fp}(\mathcal{A})$. Then every finitely generated $\mathrm{End}(M)^{\mathrm{op}}$-submodule of $\mathrm{Hom}(X, M)$ is an intersection of subgroups of the form $\mathrm{Hom}(X/U, M)$ with U a finitely presented subobject of X.*

Proof: A finitely generated submodule has the form $\mathrm{Hom}(X/U, M)$ for some $U \subseteq X$ since M is injective. In fact one takes $U = \cap_{i=1}^{n} \mathrm{Ker}(\varphi_i)$, if $\varphi_1, \ldots, \varphi_n$ are generators of the submodule. Writing $U = \sum_i U_i$ as sum of finitely generated subobjects we obtain $\mathrm{Hom}(X/U, M) = \cap_i \mathrm{Hom}(X/U_i, M)$ and the assertion follows.

Lemma 9.11 *Let $\mathcal{A}$ be an abelian category and suppose that $M \in \mathcal{A}$ is an injective object which cogenerates $\mathcal{A}$. Let $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$.*

(1) *If $X \in \mathcal{A}$, then $\mathrm{Hom}(X, M)$ is finitely generated over Γ iff there is a monomorphism $X \to \coprod_{i=1}^{n} M$ for some natural n.*

(2) *The functor $\mathrm{Hom}(\ , M)\colon \mathcal{A} \to \mathrm{Mod}(\Gamma)$ is exact and faithful. Moreover, the induced map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}_\Gamma(\mathrm{Hom}(Y, M), \mathrm{Hom}(X, M))$ is an isomorphism, if $\mathrm{Hom}(Y, M)$ is finitely generated over Γ.*

Proof: (1) and the first assertion of (2) are clear. For the surjectivity of the maps induced by $\mathrm{Hom}(\ , M)$ we refer to the argument of [19, Lemma 4.1].

We have the following consequence.

Proposition 9.12 *Let M be a Σ-pure-injective object with $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$. The endocategory $\mathcal{E}_M$ of M contains the full subcategory of all finitely generated Γ-modules which are submodules of objects in $\mathcal{E}_M$.*

Proof: Pass from $\mathcal{A}$ to $D(\mathcal{A}_M)$ and use the preceding lemmas.

Objects having the property that the endocategory contains a simple object are of some interest. Note that Σ-pure-injectives have this property.

Proposition 9.13 *Let M be a pure-injective object with $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$ and let $X \in \mathcal{E}_M$.*

(1) *X is a simple object in $\mathcal{E}_M$ if and only if X is a simple Γ-module.*

(2) *If M is indecomposable and X is simple, then $\Gamma/\operatorname{rad}(\Gamma) \cong \mathrm{End}_{\mathcal{E}_M}(X) = \mathrm{End}_\Gamma(X)$.*

Proof: We shall view M as an object in $D(\mathcal{A}_M)$ so that $X = \mathrm{Hom}(S, M)$ for some object $S \in C(\mathcal{A}_M) \cong (\mathcal{E}_M)^{\mathrm{op}}$ by Corollary 2.10.

(1) One direction is clear. So suppose that X is simple in $\mathcal{E}_M$. By Lemma 9.10, any finitely generated Γ-submodule of X is an intersection of subgroups of the form $\mathrm{Hom}(S/U, M)$ with U a subobject of S. But S is simple and therefore X is a simple Γ-module.

(2) The functor $\mathrm{Hom}(\ , M)$ induces the desired isomorphism. This follows from the lemma below.

Lemma 9.14 *Let M be an injective envelope of a simple object S in any abelian category and let $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$.*

(1) *$X = \mathrm{Hom}(S, M)$ is a simple Γ-module.*

(2) *The functor $\mathrm{Hom}(\ , M)$ induces an isomorphism $\Gamma/\operatorname{rad}(\Gamma) \cong \mathrm{End}(S)^{\mathrm{op}} \cong \mathrm{End}_\Gamma(X)$.*

Proof: Straightforward.

Our final result on objects having a simple object in their endocategory is the following.

Proposition 9.15 *Let $M \in \mathrm{Zsp}(\mathcal{A})$. If $\mathcal{E}_M$ contains a simple object, then $\{M\}$ is an open subset of $\overline{\{M\}}$ with respect to the induced topology.*

Proof: We may assume that $\mathcal{A} = \mathcal{A}_M$ since $\overline{\{M\}} = \mathrm{Zsp}(\mathcal{A}_M)$. Note also, that the topology of $\mathrm{Zsp}(\mathcal{A}_M)$ coincides with the topology of $\overline{\{M\}}$ induced by that of $\mathrm{Zsp}(\mathcal{A})$. Let $S \in C(\mathcal{A}) \cong (\mathcal{E}_M)^{\mathrm{op}}$ be a simple object. The canonical functor $d\colon \mathcal{A} \to D(\mathcal{A})$ identifies M with an injective envelope of S so that $\mathrm{Hom}(S, d(N)) = 0$ for all $N \in \mathrm{Zsp}(\mathcal{A})$ different from M. It follows from the definition of the topology on $\mathrm{Zsp}(\mathcal{A})$ that the points different from M form a closed subset. Thus $\{M\}$ is open.

10. Endofinite objects

Let $\mathcal{A}$ be an exactly definable category and fix $M \in \mathcal{A}$ with $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$. This section is devoted to studying the property that the endocategory $\mathcal{E}_M$ of M is a length category, i.e. every object in $\mathcal{E}_M$ is of finite length. This leads to the concept of an endofinite object which has been introduced by Crawley-Boevey [3, 3.6]. Call M *endofinite*, if every object of $\mathcal{E}_M$ is a Γ-module of finite length.

Proposition 10.1 *Suppose that the endocategory $\mathcal{E}_M$ of M is a length category. Then $\mathcal{E}_M$ is a full subcategory of* $\mathrm{Mod}(\Gamma)$ *which consists of modules of finite length. Moreover, $\mathcal{E}_M$ is closed under submodules and factor modules.*

Proof: We may apply Proposition 9.12 since M is Σ-pure-injective by Theorem 9.6. Thus $\mathcal{E}_M$ contains the full subcategory of all finitely generated Γ-modules which are submodules of objects in $\mathcal{E}_M$. Using the ascending chain condition in $\mathcal{E}_M$ it follows that every Γ-submodule of an object in $\mathcal{E}_M$ belongs to $\mathcal{E}_M$. The assertion of the proposition is now a consequence.

Theorem 10.2 *An object $M \in \mathcal{A}$ is endofinite if and only if the endocategory of M is a length category.*

Proof: One direction is trivial and the other follows from the preceding proposition.

Remark 10.3 *An endofinite object is Σ-pure-injective and therefore it decomposes into a coproduct of indecomposable objects which are endofinite.*

We continue with some further characterizations of endofinite objects.

Lemma 10.4 *Let $M \in \mathrm{Zsp}(\mathcal{A})$ and suppose that $\mathcal{E}_M$ contains a simple object. Then M is endofinite iff $\overline{\{M\}} = \{M\}$.*

Proof: Let $\mathcal{S}$ be the Serre subcategory of all finite length objects in $C(\mathcal{A}_M) \cong (\mathcal{E}_M)^{\mathrm{op}}$. Using the argument of Proposition 9.15 the canonical functor $d\colon \mathcal{A}_M \to D(\mathcal{A}_M)$ identifies $\overline{\{M\}} \setminus \{M\}$ with the set of indecomposable injective objects N satisfying $\mathrm{Hom}(\mathcal{S}, N) = 0$. We apply now the bijective correspondence between Serre subcategories and closed subsets from Theorem 6.2 and conclude that $\mathcal{S} = C(\mathcal{A}_M)$ iff $\overline{\{M\}} \setminus \{M\} = \emptyset$. Thus $\mathcal{E}_M$ is a length category iff $\overline{\{M\}} = \{M\}$.

Corollary 10.5 *For $M \in \mathrm{Zsp}(\mathcal{A})$ the following conditions are equivalent:*

(1) *M is endofinite.*

(2) *Every product of copies of M is a coproduct of copies of M.*

(3) *The coproducts of copies of M form a definable subcategory of $\mathcal{A}$.*

(4) *$\{M\}$ is closed in* $\mathrm{Zsp}(\mathcal{A})$ *and $\mathcal{E}_M$ contains a simple object.*

Proof: Combine the corresponding characterization of indecomposable Σ-pure-injectives and the preceding lemma.

Further characterizations of endofinite objects, using the Ziegler spectrum, can be found in Corollary 11.3 and Corollary 12.13.

11. Locally coirreducible categories

We exhibit two finiteness conditions for exactly definable categories and discuss their relation. To this end recall that a Grothendieck category $\mathcal{A}$ is *locally coirreducible* if it satisfies the following equivalent conditions.

(i) Any non-zero injective object in $\mathcal{A}$ is the injective envelope of some coproduct of indecomposable injective objects.

(ii) Any non-zero injective object has an indecomposable direct summand.

Note that for any localizing subcategory $\mathcal{C}$ of $\mathcal{A}$ the category $\mathcal{A}$ is locally coirreducible if and only if $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are locally coirreducible.

Lemma 11.1 *Let $\mathcal{A}$ be a locally coherent category with $\mathrm{Sp}(\mathcal{A}) = \{M\}$. Then $\mathcal{A}$ is locally coirreducible iff $\mathcal{A}$ is locally finite.*

Proof: Suppose first that $\mathcal{A}$ is locally coirreducible. Clearly, $\mathcal{A}$ contains a simple object and we claim that this is finitely presented. To this end choose any non-zero $X \in \mathrm{fp}(\mathcal{A})$. By our assumption the injective envelope of X has an indecomposable direct summand. This implies that X has a simple subobject which is therefore finitely presented. Now consider the Serre subcategory $\mathcal{S}$ of objects of finite length in $\mathrm{fp}(\mathcal{A})$. We shall use the bijective correspondence between Serre subcategories of $\mathrm{fp}(\mathcal{A})$ and closed subsets of $\mathrm{Sp}(\mathcal{A})$ from Theorem 6.2. We have $\mathcal{S} \neq 0$ and we conclude that $\mathcal{S} = \mathrm{fp}(\mathcal{A})$ since $\mathrm{Sp}(\mathcal{A})$ has no proper closed subset. Thus $\mathcal{A}$ is locally finite. To prove the converse suppose that $\mathcal{A}$ is locally finite and let $0 \neq N \in \mathcal{A}$ be injective. By definition, N has a subobject of finite length and we can even choose a simple subobject S. Taking an injective envelope of S one obtains an indecomposable direct summand of N. Thus $\mathcal{A}$ is locally coirreducible.

Now let $\mathcal{A}$ be an exactly definable category and consider the following conditions for $M \in \mathrm{Zsp}(\mathcal{A})$.

(i) The endocategory of M contains a simple object.

(ii) $\{M\}$ is an open subset of $\overline{\{M\}}$ whith respect to the induced topology.

We have shown in Proposition 9.15 that (i) implies (ii), and we say that $\mathcal{A}$ has the *isolation*

property if (i) and (ii) are equivalent for every $M \in \mathrm{Zsp}(\mathcal{A})$. This property has been studied by Prest and the next result is an analogue of [16, Theorem 10.16].

Theorem 11.2 *If every non-zero pure-injective object of $\mathcal{A}$ has an indecomposable direct summand, then $\mathcal{A}$ has the isolation property.*

Proof: Let $M \in \mathrm{Zsp}(\mathcal{A})$ and suppose that $\{M\}$ is open in $\overline{\{M\}}$. Let $\mathcal{S}$ be the Serre subcategory of $C(\mathcal{A}_M)$ which corresponds to the closed set $\overline{\{M\}} \setminus \{M\}$. Then $\mathcal{B} = \vec{\mathcal{S}}$ is a localizing subcategory of $D(\mathcal{A}_M)$ [11, Theorem 2.8] and therefore locally coirreducible by our assumption on $\mathcal{A}$. Moreover, the right adjoint of the inclusion $\mathcal{B} \to D(\mathcal{A}_M)$ identifies $\{M\}$ with $\mathrm{Sp}(\mathcal{B})$; see [11] for details. The preceding lemma shows that $\mathcal{B}$ is locally finite and a simple object of $\mathcal{S} = \mathrm{fp}(\mathcal{B})$ is also simple in $C(\mathcal{A}_M) \cong (\mathcal{E}_M)^{\mathrm{op}}$. This finishes the proof.

Corollary 11.3 *An object $M \in \mathrm{Zsp}(\mathcal{A})$ is endofinite if and only if $\{M\}$ is closed and every non-zero pure-injective object belonging to the definable subcategory generated by M has an indecomposable direct summand.*

Proof: The assertion follows from the characterization of endofiniteness in Corollary 10.5 together with the preceding theorem.

12. Filtrations and dimensions

In this section we introduce a method to study the complexity of an exactly definable category in terms of filtrations and ordinal valued functions. We try to cover several concepts (e.g. see [7, 16, 21]). In fact Prest's book [16] contains a lot of material on this topic.

Let $\mathcal{C}$ be a skeletally small abelian category and denote for each object X by $\mathrm{Latt}(X)$ the lattice of its subobjects in $\mathcal{C}$. Note that $\mathrm{Latt}(X)$ is a modular lattice with 0 and 1. Now suppose that $\mathcal{L}$ is a class of modular lattices which is closed under intervals, quotients and isomorphisms. An object X is called an *$\mathcal{L}$-object* of $\mathcal{C}$ if there exists a finite filtration

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

of X with $\mathrm{Latt}(X_i/X_{i-1}) \in \mathcal{L}$ for all $i \in \{1, \ldots, n\}$. Note that the $\mathcal{L}$-objects form a Serre subcategory of $\mathcal{C}$. The *$\mathcal{L}$-filtration* $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ is defined recursively as follows.

– $\mathcal{C}_{-1} = 0$;

– if α is an ordinal of the form $\alpha = \beta + 1$, then $\mathcal{C}_\alpha$ is the the Serre subcategory of all objects in $\mathcal{C}$ which become $\mathcal{L}$-objects in $\mathcal{C}/\mathcal{C}_\beta$;

– if α is a limit ordinal, then $\mathcal{C}_\alpha$ is the smallest Serre subcategory containing all $\mathcal{C}_\gamma$ with $\gamma < \alpha$.

We denote by $\mathcal{C}_\infty$ the smallest Serre subcategory containing all $\mathcal{C}_\alpha$ and define the $\mathcal{L}$-*dimension* of $\mathcal{C}$ as follows. If $\mathcal{C}_\infty = \mathcal{C}$, then the least ordinal α with $\mathcal{C}_\alpha = \mathcal{C}$ is denoted by $\dim_{\mathcal{L}} \mathcal{C}$. Otherwise let $\dim_{\mathcal{L}} \mathcal{C} = \infty$. Let α and β be ordinals. Taking the appropriate definition of the ordinal $\alpha \oplus \beta$ (e.g. $\alpha \oplus \beta = \alpha + \beta + 1$, if α and β are finite) one obtains the following property of $\mathcal{L}$-dimensions (see [6, IV, Proposition 1.1]).

Lemma 12.1 *Suppose that $\mathcal{D}$ is a Serre subcategory of $\mathcal{C}$. Then*

$$\sup(\dim_{\mathcal{L}} \mathcal{D}, \dim_{\mathcal{L}} \mathcal{C}/\mathcal{D}) \leq \dim_{\mathcal{L}} \mathcal{C} \leq \dim_{\mathcal{L}} \mathcal{D} \oplus \dim_{\mathcal{L}} \mathcal{C}/\mathcal{D}.$$

Now let $\mathcal{A}$ be an exactly definable category and we may assume that $\mathcal{A} = \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. The $\mathcal{L}$-filtration $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ induces the $\mathcal{L}$-*filtration* $(\mathcal{A}_\alpha)_\alpha$ of $\mathcal{A}$ if we define $\mathcal{A}_\alpha = \{M \in \mathcal{A} \mid M(\mathcal{C}_\alpha) = 0\}$ for each ordinal α. Thus we obtain the following chain of definable subcategories of $\mathcal{A}$.

$$\ldots \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_0 \subseteq \mathcal{A}_{-1} = \mathcal{A}$$

The $\mathcal{L}$-*dimension* of $\mathcal{A}$, denoted by $\dim_{\mathcal{L}} \mathcal{A}$, is the $\mathcal{L}$-dimension of $\mathcal{C}$. The $\mathcal{L}$-dimension $\dim_{\mathcal{L}} M$ of an object $M \in \mathcal{A}$ is by definition the $\mathcal{L}$-dimension $\dim_{\mathcal{L}} \mathcal{A}_M$ of the definable subcategory generated by M.

We state some basic properties of $\mathcal{L}$-dimensions and consider for $\mathcal{L}$ the following examples to illustrate the notion of $\mathcal{L}$-dimension.

– fin = the class of all lattices of finite length.

– noeth = the class of all lattices having the ascending chain condition.

– ser = the class of all lattices which are chains.

Proposition 12.2 *Let $\mathcal{A}'$ be a definable subcategory of $\mathcal{A}$ and denote by A'' the corresponding definable quotient category. Then*

$$\sup(\dim_{\mathcal{L}} \mathcal{A}', \dim_{\mathcal{L}} \mathcal{A}'') \leq \dim_{\mathcal{L}} \mathcal{A} \leq \dim_{\mathcal{L}} \mathcal{A}' \oplus \dim_{\mathcal{L}} \mathcal{A}''.$$

Proposition 12.3 *Let $\mathcal{A}$ be an exactly definable category and let $0 \neq M \in \mathcal{A}$.*

(1) $\dim_{\mathrm{fin}} M = 0$ *if and only if M is endofinite.*

(2) $\dim_{\mathrm{noeth}} M = 0$ *if and only if M is Σ-pure-injective.*

(3) $\dim_{\mathrm{noeth}} \mathcal{A} \leq \dim_{\mathrm{fin}} \mathcal{A}$, *and* $\dim_{\mathrm{noeth}} \mathcal{A} = \infty$ *if and only if* $\dim_{\mathrm{fin}} \mathcal{A} = \infty$.

(4) $\dim_{\mathrm{ser}} \mathcal{A} \leq \dim_{\mathrm{fin}} \mathcal{A}$.

Theorem 12.4 *If* $\dim_{\mathrm{fin}} \mathcal{A} < \infty$ *or* $\dim_{\mathrm{ser}} \mathcal{A} < \infty$, *then every non-zero pure-injective object in $\mathcal{A}$ has an indecomposable direct summand.*

Proof: Let $0 \neq M \in \mathcal{A}$ be pure-injective. Clearly, we may assume by Proposition 12.2 that $\mathcal{A} = \mathcal{A}_M$. We identify M with the corresponding object in $D(\mathcal{A})$ and we can choose a non-zero morphism $\varphi: X \to M$ with $X \in C(\mathcal{A})_0$ by Corollary 2.10. By definition there exists a finite filtration $0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$ of X such that X_i/X_{i-1} is of finite length (uniserial) in $C(\mathcal{A})$, hence of finite length (uniserial) in $D(\mathcal{A})$ for all i. Choose i minimal with $\varphi(X_i) \neq 0$ and denote by $\psi: X_i/X_{i-1} \to M$ the induced morphism. Taking $Y = (X_i/X_{i-1})/\operatorname{Ker}(\psi)$ this morphism induces a monomorphism $Y \to M$ and Y is of finite length (uniserial). In fact we may assume that Y is simple or uniserial. Thus Y has an indecomposable injective envelope which is isomorphic to a direct summand of M. This finishes the proof.

Corollary 12.5 *If* $\dim_{\text{fin}} \mathcal{A} < \infty$ *or* $\dim_{\text{ser}} \mathcal{A} < \infty$*, then* $\mathcal{A}$ *has the isolation property.*

Proof: Combine the preceding theorem with Theorem 11.2.

Remark 12.6 *A stronger result holds in case that* $\dim_{\text{fin}} \mathcal{A} < \infty$*. In fact, the definition of* $\dim_{\text{fin}}$ *immediately implies that the endocategory of every non-zero object contains a simple object.*

Our next aim is a classification of the indecomposable pure-injective objects provided that $\dim_{\text{fin}}$ exists. We shall derive this result from a corresponding result for locally coherent categories. To this end let $\mathcal{A}$ be a locally coherent category with $\mathcal{C} = \operatorname{fp}(\mathcal{A})$. The $\mathcal{L}$-filtration $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ induces the *$\mathcal{L}$-filtration* $(\mathcal{A}_\alpha)_\alpha$ of $\mathcal{A}$ if we define $\mathcal{A}_\alpha = \vec{\mathcal{C}}_\alpha$ for each ordinal α. Note that each $\mathcal{A}_\alpha$ is a localizing subcategory of finite type by [11, Theorem 2.8]. We denote for each α by $s_\alpha: \mathcal{A}/\mathcal{A}_\alpha \to \mathcal{A}$ the corresponding section functor, and we call an object of the form $s_\alpha(X)$ *α-simple* if X is simple in $\operatorname{fp}(\mathcal{A}/\mathcal{A}_\alpha)$. The next result is due to Jensen and Lenzing [9, Theorem 8.53]. It is based on Gabriels's analysis of abelian categories with Krull dimension [6, IV, Proposition 1.2].

Theorem 12.7 *Let* $\mathcal{A}$ *be a locally coherent category and suppose that* $\dim_{\text{fin}} \mathcal{A} < \infty$*. Then the map which assigns to each* α*-simple object its injective envelope in* $\mathcal{A}$ *induces a bijection between the isomorphism classes of simple objects in* $\operatorname{fp}(\mathcal{A}/\mathcal{A}_\alpha)$*,* $-1 \leq \alpha < \dim_{\text{fin}} \mathcal{A}$*, and* $\operatorname{Sp}(\mathcal{A})$*.*

Proof: We shall use freely some results from localization theory for locally coherent categories for which we refer to [11]. However, most basic is the fact that for any localizing subcategory $\mathcal{C}$ of $\mathcal{A}$, the section functor $s: \mathcal{A}/\mathcal{C} \to \mathcal{A}$ identifies $\operatorname{Sp}(\mathcal{A}/\mathcal{C})$ with $\{M \in \operatorname{Sp}(\mathcal{A}) \mid \operatorname{Hom}(\mathcal{C}, M) = 0\}$ [6, III, Corollary 3.2]. We construct now a simple object for each indecomposable injective

object in $\mathcal{A}$ as follows. Let $M \in \mathrm{Sp}(\mathcal{A})$ and choose β minimal such that M has a non-zero subobject contained in $\mathcal{A}_\beta$. Clearly, $\beta = \alpha + 1$ for some ordinal α and we can identify M with an object in $\mathrm{Sp}(\mathcal{A}/\mathcal{A}_\alpha)$. The category $\mathcal{A}_\beta/\mathcal{A}_\alpha$ is a localizing subcategory of finite type of $\mathcal{A}/\mathcal{A}_\alpha$ which is locally finite since $\mathrm{fp}(\mathcal{A}_\beta/\mathcal{A}_\alpha) \cong \mathrm{fp}(\mathcal{A}_\beta)/\mathrm{fp}(\mathcal{A}_\alpha)$ [11, Theorem 2.6]. It follows that M is the injective envelope of some simple object $X \in \mathrm{fp}(\mathcal{A}_\beta/\mathcal{A}_\alpha)$. The composition of the inclusion $\mathcal{A}_\beta/\mathcal{A}_\alpha \to \mathcal{A}/\mathcal{A}_\alpha$ with the section functor $s_\alpha\colon \mathcal{A}/\mathcal{A}_\alpha \to \mathcal{A}$ sends X to an α-simple object and M is isomorphic to its injective envelope.

Corollary 12.8 *Let $\mathcal{A}$ be an exactly definable category and suppose that $\dim_{\mathrm{fin}} \mathcal{A} < \infty$. Then there exists a bijection between $\mathrm{Zsp}(\mathcal{A})$ and the isomorphism classes of simple objects in $C(\mathcal{A})/C(\mathcal{A})_\alpha$, $-1 \leq \alpha < \dim_{\mathrm{fin}} \mathcal{A}$.*

Let $\mathcal{A}$ be an exactly definable category. Then any $\mathcal{L}$-filtration $(\mathcal{A}_\alpha)_\alpha$ of $\mathcal{A}$ induces the *$\mathcal{L}$-filtration* of $\mathcal{U} = \mathrm{Zsp}(\mathcal{A})$ if we define $\mathcal{U}_\alpha = \mathrm{Zsp}(\mathcal{A}) \cap \mathcal{A}_\alpha$ for every ordinal α. Note that $\mathcal{U}_\alpha$ is a closed subset of $\mathcal{U}$ for each ordinal α.

Now let $\mathcal{V}$ be an arbitrary topological space. Recall that an element $X \in \mathcal{V}$ is *isolated* provided that $\{X\}$ is open. The *Cantor-Bendixson filtration* $(\mathcal{V}_\alpha)_\alpha$ of $\mathcal{V}$ is defined recursively as follows.

- $\mathcal{V}_{-1} = \mathcal{V}$;
- $\mathcal{V}_\alpha = \{X \in \mathcal{V} \mid X \text{ is not isolated in } \mathcal{V}_\beta\}$, if α is an ordinal of the form $\alpha = \beta + 1$;
- $\mathcal{V}_\alpha = \cap_{\gamma<\alpha} \mathcal{V}_\gamma$, if α is a limit ordinal.

We denote by $\mathcal{V}_\infty$ the intersection of all $\mathcal{V}_\alpha$ and define the *Cantor-Bendixson rank* of $\mathcal{V}$ as follows. If $\mathcal{V}_\infty = \emptyset$, then the least ordinal with this property is denoted by $\mathrm{rank}_{\mathrm{CB}} \mathcal{V}$. Otherwise let $\mathrm{rank}_{\mathrm{CB}} \mathcal{V} = \infty$. Note that $\mathcal{V}_\alpha$ is a closed subset of $\mathcal{V}$ for each ordinal α.

We want to exhibit the relation between the fin-filtration and the Cantor-Bendixson filtration of $\mathrm{Zsp}(\mathcal{A})$. We may assume that $\mathcal{A} = \mathrm{Ex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

Lemma 12.9 *Let $M \in \mathrm{Zsp}(\mathcal{A})$ and suppose that $M(X) \neq 0$ for some object $X \in \mathcal{C}$ of finite length. Then M is isolated.*

Proof: We may assume that X is simple. Thus the canonical functor $\mathcal{A} \to D(\mathcal{A})$ identifies M with the injective envelope of a simple object by Corollary 2.10 and therefore $N(X) = 0$ for every $N \in \mathrm{Zsp}(\mathcal{A})$ different from M. It follows that $\{M\}$ is open.

Proposition 12.10 *Let $(\mathcal{V}_\alpha)_\alpha$ be the Cantor-Bendixson filtration and $(\mathcal{U}_\alpha)_\alpha$ be the* fin*-filtration of $\mathrm{Zsp}(\mathcal{A})$. Then $\mathcal{V}_\alpha \subseteq \mathcal{U}_\alpha$ for each ordinal α.*

Proof: The assertion follows by transfinite induction from the preceding lemma.

Lemma 12.11 *Suppose that $\mathcal{A}$ has the isolation property and let $M \in \operatorname{Zsp}(\mathcal{A})$. If M is isolated, then $M(X) \neq 0$ for some object $X \in \mathcal{C}$ of finite length.*

Proof: Denote by $q:\mathcal{C} \to \mathcal{C}/\operatorname{Ker}(M) \cong C(\mathcal{A}_M)$ the quotient functor corresponding to the Serre subcategory $\operatorname{Ker}(M)$. It follows from the assumption on M that there exists $X \in \mathcal{C}$ such that $M(X) \neq 0$ and $N(X) = 0$ for every $N \in \operatorname{Zsp}(\mathcal{A})$ different from M. We can choose a maximal subobject U of $q(X)$ in $D(\mathcal{A}_M)$ since $q(X)$ is finitely generated. Then M corresponds in $D(\mathcal{A}_M)$ to an injective envelope of $S = q(X)/U$ and S is finitely presented since $\mathcal{A}$ has the isolation property. Thus $U \in C(\mathcal{A}_M)$ and we find a subobject V of X in $\mathcal{C}$ such that $q(V) \cong U$ and therefore $q(T) \cong S$ for $T = X/V$. Clearly, $M(T) \neq 0$, and we claim that T is simple in $\mathcal{C}$. To see this let $0 \to T' \to T \to T'' \to 0$ be an exact sequence in $\mathcal{C}$. Then either $q(T') = 0$ or $q(T'') = 0$ since $q(T)$ is simple. Thus $M(T') = 0$ or $M(T'') = 0$. On the other hand, $N(T') = 0 = N(T'')$ for every $N \in \operatorname{Zsp}(\mathcal{A})$ different from M since M is isolated, and we conclude that $T' = 0$ or $T'' = 0$. Thus T is simple and this finishes the proof.

The following consequence is our variant of [21, Theorem 8.6] and [16, Proposition 10.19].

Corollary 12.12 *Suppose that $\mathcal{A}$ has the isolation property. Then the Cantor-Bendixson filtration and the* fin*-filtration of* $\operatorname{Zsp}(\mathcal{A})$ *coincide. In particular,* $\operatorname{rank}_{\mathrm{CB}} \operatorname{Zsp}(\mathcal{A}) = \dim_{\mathrm{fin}} \mathcal{A}$.

Proof: The assertion follows by transfinite induction from Lemma 12.9 and Lemma 12.11.

We conclude this section with a useful application of the preceding result.

Corollary 12.13 *An object M in $\mathcal{A}$ is endofinite if and only if there is a closed subset $\mathcal{U}$ of* $\operatorname{Zsp}(\mathcal{A})$ *satisfying*

(i) *$\mathcal{U}$ carries the discrete topology.*

(ii) *Every product of copies of M is a coproduct of copies of objects in $\mathcal{U}$.*

Proof: We may assume that $M \neq 0$. Suppose first that M is endofinite so that $\dim_{\mathrm{fin}} M = 0$ by Proposition 12.3. We take $\mathcal{U} = \overline{M}$. Clearly $\mathcal{A}_M$ has the isolation property and therefore $\operatorname{rank}_{\mathrm{CB}} \mathcal{U} = \dim_{\mathrm{fin}} M = 0$ follows from the preceding result. Thus (i) holds and (ii) follows from Corollary 9.8. The converse is shown similarly and therefore the proof is complete.

13. Epimorphisms of rings

The final section of this paper is devoted to epimorphisms in the category of rings. Our motivation is mainly to illustrate the techniques developed in the previous sections. However,

there are some new results. We shall prove that any epimorphism $\Lambda \to \Gamma$ is induced by a suitable localization of the Grothendieck category $D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$. This allows us to deduce a new proof of the fact that the epimorphisms starting in a fixed ring form a set [4, p. 93]. Moreover, in analogy with a classical result for localizations of rings, we show that any epimorphism $\Lambda \to \Gamma$ is equivalent to the canonical homomorphism $\Lambda \to \mathrm{Biend}_{\Lambda}(M)$ for some Λ-module M.

Let Λ be a ring (associative with identity) which we view as a category with one object, and denote by $\mathrm{Mod}(\Lambda) = (\Lambda^{\mathrm{op}}, \mathrm{Ab})$ the category of (right) Λ-modules. Let $C(\Lambda) = \mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ be the category of finitely presented functors from the category $\mathrm{mod}(\Lambda^{\mathrm{op}})$ of finitely presented Λ^{op}-modules to the category of abelian groups. The functor

$$h\colon \Lambda \to C(\Lambda), \quad \Lambda \mapsto \Lambda \otimes_{\Lambda} -$$

is fully faithful and $C(\Lambda)$ is an abelian category. In fact, $C(\Lambda)$ is the free abelian category over Λ in the following sense [8, Lemma 1].

Lemma 13.1 *Let $f\colon \Lambda \to \mathcal{C}$ be an additive functor into an abelian category $\mathcal{C}$. Then there exists, up to isomorphsm, a unique exact functor $f^*\colon C(\Lambda) \to \mathcal{C}$ such that $f = f^* \circ h$.*

This lemma has the following consequence if we take for $\mathcal{C}$ the category of abelian groups.

Lemma 13.2 *The functor*

$$(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab}) \to (\Lambda^{\mathrm{op}}, \mathrm{Ab}) = \mathrm{Mod}(\Lambda), \quad X \mapsto X \circ h$$

induces an equivalence between $\mathrm{Ex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab})$ *and* $\mathrm{Mod}(\Lambda)$. *An inverse is*

$$\mathrm{Mod}(\Lambda) \to \mathrm{Ex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathrm{Hom}(\ , X \otimes_{\Lambda} -).$$

The preceding lemma shows that $\mathrm{Mod}(\Lambda)$ is an exactly definable category and the equivalence

$$D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}) \to \mathrm{Lex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathrm{Hom}(\ , X)|_{C(\Lambda)}$$

gives an alternative description of $D(\mathrm{Mod}(\Lambda))$.

Now suppose that $\varphi\colon \Lambda \to \Gamma$ is a ring homomorphism. The functor φ induces a unique exact functor $\varphi'\colon C(\Lambda) \to C(\Gamma)$ according to Lemma 13.1, and φ' extends uniquely to an exact functor $\varphi^*\colon D(\Lambda) \to D(\Gamma)$ which commutes with direct limits by Lemma 1.1. Identifying $D(\Lambda) = \mathrm{Lex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab})$ and $D(\Gamma) = \mathrm{Lex}(C(\Gamma)^{\mathrm{op}}, \mathrm{Ab})$, respectively, we obtain a right adjoint $D(\Gamma) \to D(\Lambda)$, $X \mapsto X \circ \varphi'$. Note that this right adjoint restricts to the functor $\varphi_*\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Lambda)$, $X \mapsto X \circ \varphi$.

We shall now characterize the property of $\varphi\colon \Lambda \to \Gamma$ to be an epimorphism in the category of rings. To this end define $\mathcal{C} = \mathrm{Ker}(\varphi^*)$ and $\mathcal{S} = \mathrm{Ker}(\varphi')$. Note that $\mathcal{C}$ is a localizing subcategory of $D(\Lambda)$ and that $\mathcal{S}$ is a Serre subcategory of $C(\Lambda)$.

Theorem 13.3 *Let $\varphi\colon \Lambda \to \Gamma$ be a ring homomorphism. Then the following conditions are equivalent.*

(1) *φ is an epimorphism.*

(2) *The functor $\varphi^*\colon D(\Lambda) \to D(\Gamma)$ induces an equivalence $D(\Lambda)/\mathcal{C} \to D(\Gamma)$.*

(3) *The functor $\varphi'\colon C(\Lambda) \to C(\Gamma)$ induces an equivalence $C(\Lambda)/\mathcal{S} \to C(\Gamma)$.*

(4) *The restriction functor $\varphi_*\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Lambda)$ induces an equivalence between $\mathrm{Mod}(\Gamma)$ and a definable subcategory of $\mathrm{Mod}(\Lambda)$.*

We need some preparations in order to prove this result. We start with the following technical lemma about localization in Grothendieck categories.

Lemma 13.4 *Let $f\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between two Grothendieck categories. Let $\mathcal{C} = \mathrm{Ker}(f)$ and suppose there exists a right adjoint $g\colon \mathcal{B} \to \mathcal{A}$. Then the following are equivalent.*

(1) *f induces an equivalence $\mathcal{A}/\mathcal{C} \to \mathcal{B}$.*

(2) *g is fully faithful.*

(3) *g induces a fully faithful functor $\mathrm{inj}(\mathcal{B}) \to \mathrm{inj}(\mathcal{A})$ between the full subcategories of injective objects in $\mathcal{B}$ and $\mathcal{A}$, respectively.*

Proof: $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ and the assertion follows easily from the discussion of localizing subcategories in [6].

Lemma 13.5 *$\varphi\colon \Lambda \to \Gamma$ is an epimorphism iff the restriction functor $\varphi_*\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Lambda)$ is fully faithful.*

Proof: See [20, XI, Proposition 1.2].

Proof of Theorem 13.3: (1) $\Rightarrow$ (2) Let φ be an epimorphism. It follows from the preceding lemma that φ_* induces a fully faithful functor $\mathrm{pinj}(\mathrm{Mod}(\Gamma)) \to \mathrm{pinj}(\mathrm{Mod}(\Lambda))$ between the full subcategories of pure-injective objects in $\mathrm{Mod}(\Gamma)$ and $\mathrm{Mod}(\Lambda)$, respectively. Therefore the right adjoint $D(\Gamma) \to D(\Lambda)$ of φ^* induces a fully faithful functor $\mathrm{inj}(D(\Gamma)) \to \mathrm{inj}(D(\Lambda))$ and (2) is a consequence of Lemma 13.4.

(2) $\Rightarrow$ (3) The composition $C(\Lambda) \to D(\Lambda) \to D(\Lambda)/\mathcal{C}$ induces an equivalence between $C(\Lambda)/S$ and the the full subcategory of finitely presented objects in $D(\Lambda)/\mathcal{C}$ by [11, Theorem 2.6]. Composition with the equivalence $D(\Lambda)/\mathcal{C} \to D(\Gamma)$ from (2) induces the equivalence $C(\Lambda)/S \to C(\Gamma)$.

(3) $\Rightarrow$ (4) Identifying $\mathrm{Mod}(\Lambda) = \mathrm{Ex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab})$ and $\mathrm{Mod}(\Gamma) = \mathrm{Ex}(C(\Gamma)^{\mathrm{op}}, \mathrm{Ab})$, respectively, the restriction functor $\varphi_*\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Lambda)$ sends $M \in \mathrm{Ex}(C(\Gamma)^{\mathrm{op}}, \mathrm{Ab})$ to $M \circ \varphi'$. It follows from (3) that φ_* induces an equivalence between $\mathrm{Mod}(\Gamma)$ and the full subcategory of all functors $M \in \mathrm{Ex}(C(\Lambda)^{\mathrm{op}}, \mathrm{Ab})$ vanishing on $\mathcal{S}$. But this is a definable subcategory of $\mathrm{Mod}(\Lambda)$.

(4) $\Rightarrow$ (1) Condition (4) implies that φ_* is fully faithful. Therefore φ is an epimorphism according to the lemma above, and this completes the proof of the theorem.

Denote by $\mathrm{Zsp}(\Lambda)$ the Ziegler spectrum of $\mathrm{Mod}(\Lambda)$. Furthermore, we denote for any epimorphism $\varphi\colon \Lambda \to \Gamma$ by $\mathcal{U}_\varphi$ the subset of all $M \in \mathrm{Zsp}(\Lambda)$ such that $M \cong \varphi_*(N)$ for some $N \in \mathrm{Mod}(\Gamma)$. The following consequence of Theorem 13.3 is due to Prest [18, Corollary 8].

Corollary 13.6 *The subset $\mathcal{U}_\varphi$ of $\mathrm{Zsp}(\Lambda)$ is closed and φ_* induces a homeomorphism $\mathrm{Zsp}(\Gamma) \to \mathcal{U}_\varphi$.*

We shall see that $\mathcal{U}_\varphi$ determines φ uniquely in the following sense. Call two ring homomorphisms $\varphi_1\colon \Lambda \to \Gamma_1$ and $\varphi_2\colon \Lambda \to \Gamma_2$ *equivalent*, if there is an isomorphism $\psi\colon \Gamma_1 \to \Gamma_2$ such that $\varphi_2 = \psi \circ \varphi_1$.

Corollary 13.7 (1) *Two epimorphisms $\varphi_1\colon \Lambda \to \Gamma_1$ and $\varphi_2\colon \Lambda \to \Gamma_2$ are equivalent if and only if $\mathcal{U}_{\varphi_1} = \mathcal{U}_{\varphi_2}$.*

(2) *The equivalence classes of epimorphisms starting in Λ form a set of cardinality at most $2^{\aleph_0 + \mathrm{card}(\Lambda)}$.*

Proof: (1) We shall apply the bijective correspondence between Serre subcategories of $C(\Lambda)$ and closed subsets of $\mathrm{Zsp}(\Lambda)$. If $\varphi\colon \Lambda \to \Gamma$ is an epimorphism, then the corresponding Serre subcategory $\mathcal{S} = \mathrm{Ker}(\varphi')$ is precisely the full subcategory of objects $X \in C(\Lambda)$ satisfying $\mathrm{Hom}(X, M \otimes_\Lambda -) = 0$ for all $M \in \mathcal{U}_\varphi$. Moreover, φ is equivalent to the ring homomorphism $\Lambda \cong \mathrm{End}(\Lambda \otimes_\Lambda -) \to \mathrm{End}(q(\Lambda \otimes_\Lambda -))$ induced by the quotient functor $q\colon C(\Lambda) \to C(\Lambda)/\mathcal{S}$ according to Theorem 13.3. Thus $\mathcal{U}_{\varphi_1} = \mathcal{U}_{\varphi_2}$ implies $\mathcal{S}_1 = \mathcal{S}_2$ for the corresponding Serre subcategories of $C(\Lambda)$, and therefore φ_1 and φ_2 are equivalent.

(2) The cardinality of the set of isomorphism classes of objects in $C(\Lambda)$ is bounded by $\aleph_0 + \mathrm{card}(\Lambda)$ since each finitely presented functor $\mathrm{mod}(\Lambda^{\mathrm{op}}) \to \mathrm{Ab}$ is determined by a finite matrix with entries in Λ. Thus the Serre subcategories of $C(\Lambda)$ form a set of cardinality at most $2^{\aleph_0 + \mathrm{card}(\Lambda)}$. The assertion now follows from our argument in (1).

Let M be a Λ-module with $\Sigma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ and view M in the natural way as a Σ-module. We call $\mathrm{Biend}_\Lambda(M) = \mathrm{End}_\Sigma(M)^{\mathrm{op}}$ the *biendomorphism ring* of M and denote by

$\varepsilon_M\colon \Lambda \to \mathrm{Biend}_\Lambda(M)$ the canonical ring homomorphism which is induced by the functor $\mathrm{Hom}_\Lambda(\ , M)$. The next result shows that an epimorphism $\varphi\colon \Lambda \to \Gamma$ can be reconstructed from the corresponding closed subset $\mathcal{U}_\varphi$ of $\mathrm{Zsp}(\Lambda)$.

Corollary 13.8 *Let $\varphi\colon \Lambda \to \Gamma$ be an epimorphism and denote by M a product of κ copies of $M_\varphi = \prod_{N \in \mathcal{U}_\varphi} N$ where $\kappa = \mathrm{card}(M_\varphi)$. Then φ is equivalent to the canonical ring homomorphism $\varepsilon_M\colon \Lambda \to \mathrm{Biend}_\Lambda(M)$.*

Proof: Let $N_\varphi = \prod_{N \in \mathrm{Zsp}(\Gamma)} N$ and $N = \prod_\kappa N_\varphi$. The homomorphism φ is equivalent to the homomorphism $\Lambda \cong \mathrm{End}(\Lambda \otimes_\Lambda -) \to \mathrm{End}(\varphi^*(\Lambda \otimes_\Lambda -))$ induced by the functor $\varphi^*\colon D(\Lambda) \to D(\Gamma)$, and we shall construct an isomorphism $\mathrm{End}(\varphi^*(\Lambda \otimes_\Lambda -)) \cong \mathrm{Biend}_\Lambda(M)$ such that the composition is equivalent to ε_M. The right adjoint $D(\Gamma) \to D(\Lambda)$ of φ^* identifies the injective cogenerator $N_\varphi \otimes_\Gamma -$ of $D(\Gamma)$ with $M_\varphi \otimes_\Lambda -$ and therefore

$$\mathrm{Hom}(\varphi^*(\Lambda \otimes_\Lambda -), N_\varphi \otimes_\Gamma -) \cong \mathrm{Hom}(\Lambda \otimes_\Lambda -, M_\varphi \otimes_\Lambda -) \cong M_\varphi.$$

Thus there exists a monomorphism $\varphi^*(\Lambda \otimes_\Lambda -) \to N \otimes_\Gamma -$ and $\mathrm{Hom}(\ , N \otimes_\Gamma -)$ induces an isomorphism

$$\mathrm{End}(\varphi^*(\Lambda \otimes_\Lambda -)) \cong \mathrm{End}(\mathrm{Hom}(\varphi^*(\Lambda \otimes_\Lambda -), N \otimes_\Gamma -))^{\mathrm{op}}$$

by Lemma 9.11. The adjointness gives

$$\mathrm{End}(\mathrm{Hom}(\varphi^*(\Lambda \otimes_\Lambda -), N \otimes_\Gamma -))^{\mathrm{op}} \cong \mathrm{End}(\mathrm{Hom}(\Lambda \otimes_\Lambda -, M \otimes_\Lambda -))^{\mathrm{op}} \cong \mathrm{Biend}_\Lambda(M)$$

and we obtain therefore a homomorphism $\Lambda \to \mathrm{Biend}_\Lambda(M)$ which is induced by $\mathrm{Hom}_\Lambda(\ , M)$ since the right adjoint of φ^* identifies $N \otimes_\Gamma -$ with $M \otimes_\Lambda -$ and M, respectively.

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Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
E-mail address: `henning@mathematik.uni-bielefeld.de`

January 1996