Dualizing rings and a characterization of finite representation type [1]

Henning Krause

Abstract: Using the relation between indecomposable endofinite right and left Λ-modules, we introduce (right) dualizing rings and show that a ring Λ is of finite representation type if and only if Λ is right dualizing and every right Λ-module is a direct sum of (finitely presented) indecomposable Λ-modules. We present also criteria for the existence of non-finitely presented indecomposable endofinite modules over a dualizing ring.

Let Λ be a ring of finite representation type, i.e. Λ is right artinian and there is only a finite number of finitely presented indecomposable right Λ-modules. Then it has been shown by Auslander [1] and by Ringel and Tachikawa [7] that every right Λ-module is a direct sum of finitely presented indecomposable Λ-modules. In this note we shall prove the converse for rings Λ which satisfy the following condition:

(D) If M is a finitely presented indecomposable endofinite right Λ-module, then its dual DM is a finitely presented indecomposable endofinite left Λ-module.

Recall that a right Λ-module M is *endofinite* if it is of finite length when regarded in the natural way as a $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$-module. If M is finitely presented indecomposable and endofinite, we denote by DM the left Λ-module $\operatorname{Hom}_\Gamma(M, I)$, where I is an injective envelope of the unique simple Γ-module. Note that DM is automatically indecomposable and endofinite. We call Λ *right dualizing* if the condition (D) is satisfied. If Λ and Λ^{op} both satisfy (D), we call Λ *dualizing*. We include some examples.

(1) Any ring Λ of finite representation type is dualizing since Λ^{op} is also of finite representation type.
(2) If Λ is a noetherian algebra, i.e. the center $Z(\Lambda)$ is noetherian and Λ is a finitely generated $Z(\Lambda)$-module, then Λ is dualizing [3, Proposition 7.1].
(3) Let Λ be a two-sided artinian ring. If Λ is a PI-ring or, equivalently, if $\Lambda/\operatorname{rad}\Lambda$ is an artin algebra, then Λ is dualizing [8].

These examples show that the following theorem generalizes results of Auslander [2] and Herzog [5].

Theorem *For a ring Λ the following conditions are equivalent:*

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(1) Λ *is of finite representation type.*

(2) Λ *is right dualizing and every right* Λ*-module is a direct sum of (finitely presented) indecomposable* Λ*-modules.*

The proof of this result is based on an analysis of the category $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ of all additive functors from the category of finitely presented right Λ^{op}-modules to the category Ab of abelian groups, and we shall assume that the reader is familiar with the localization theory for such a Grothendieck category which may be found in [4]. The author would like to thank M. Prest for pointing out that an alternative proof can be obtained by using ideas from [5].

Throughout we fix a ring Λ and we denote by $\mathrm{Mod}(\Lambda)$ the category of (right) Λ-modules. The fully faithful functor

$$\mathrm{Mod}(\Lambda) \to D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}), \quad M \mapsto M \otimes_\Lambda -$$

is fundamental for our approach. It identifies the pure-injective Λ-modules with the injective objects in $D(\Lambda)$. The category $D(\Lambda)$ is locally coherent, i.e. the subcategory $C(\Lambda) = \mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ of finitely presented functors is abelian, and we can reformulate the conditions of the theorem as follows.

Lemma 1 (1) Λ *is of finite representation type iff* $D(\Lambda)$ *is locally finite, i.e. every object in* $C(\Lambda)$ *is of finite length.*

(2) *Every* Λ*-module is a direct sum of (finitely presented) indecomposable* Λ*-modules iff* $D(\Lambda)$ *is locally noetherian, i.e. every object in* $C(\Lambda)$ *is noetherian.*

Proof: (1) is shown in [1] and (2) follows from the well-known fact that a functor category is locally noetherian iff every injective object decomposes into a direct sum of indecomposable objects.

Remark *A ring* Λ *is said to be right pure-semisimple provided that every right* Λ*-module is pure-injective. An equivalent property is that* $D(\Lambda)$ *is locally noetherian.*

To use the condition (D) we recall the following result which is due to Crawley-Boevey.

Lemma 2 *Let* M *be a finitely presented indecomposable endofinite* Λ*-module. If* DM *is finitely presented, then* $M \otimes_\Lambda -$ *is the injective envelope of a simple functor in* $D(\Lambda)$*.*

Proof: The assumption implies that $M \cong DDM$ by [3, Proposition 6.3]. Thus $M \otimes_\Lambda -$ is the injective envelope of a simple functor by [3, Theorem 2.3].

Proof of the theorem: We need to show that (2) implies (1). Therefore assume (2). Thus $D(\Lambda)$ is locally noetherian by Lemma 1 and therefore its Krull dimension $\operatorname{Kdim} D(\Lambda)$ in the sense of Gabriel exists [4]. Recall, that the Krull filtration

$$0 = D(\Lambda)_{-1} \subseteq D(\Lambda)_0 \subseteq \ldots \subseteq D(\Lambda)_\alpha \subseteq \ldots \subseteq D(\Lambda)$$

consists of localizing subcategories of $D(\Lambda)$ indexed by the ordinals, and $\operatorname{Kdim} D(\Lambda)$ denotes the least ordinal α such that $D(\Lambda)_\alpha = D(\Lambda)$. Moreover, $\operatorname{Kdim} D(\Lambda) = \beta+1$ for some ordinal β since for any localizing subcategory $\mathcal{C}$ of $D(\Lambda)$ we have $\mathcal{C} = D(\Lambda)$ iff $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ ) \in \mathcal{C}$. We can choose an indecomposable injective object N from the quotient category $D(\Lambda)/D(\Lambda)_\beta$ and denote by $s_\beta(N) = M \otimes_\Lambda -$ the corresponding indecomposable injective object under the section functor $s_\beta \colon D(\Lambda)/D(\Lambda)_\beta \to D(\Lambda)$. We claim that M is endofinite. In fact, the section functor is a right adjoint of the quotient functor $q_\beta \colon D(\Lambda) \to D(\Lambda)/D(\Lambda)_\beta$ and this gives an isomorphism

$$M = \operatorname{Hom}(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ ), s_\beta(N)) \cong \operatorname{Hom}(q_\beta(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ )), N)$$

of Γ-modules where $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}} \cong \operatorname{End}(N)^{\mathrm{op}}$. The claim follows from the next lemma, using the fact that $X = q_\beta(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ ))$ is finitely presented and therefore of finite length since $D(\Lambda)/D(\Lambda)_\beta$ is locally finite according to our construction.

Lemma 3 *Let X and N be objects such that X is of finite length and N is injective. Then the $\operatorname{End}(N)^{\mathrm{op}}$-module $\operatorname{Hom}(X, N)$ is of finite length.*

Proof: Any finitely generated $\operatorname{End}(N)^{\mathrm{op}}$-submodule of $\operatorname{Hom}(X, N)$ has the form $\operatorname{Hom}(X/U, N)$ for some subobject U of X since N is injective. In fact, one takes $U = \bigcap_{i=1}^n \operatorname{Ker}(\varphi_i)$ if $\varphi_1, \ldots, \varphi_n$ are generators of the submodule. Now the finite length of X implies that $\operatorname{Hom}(X, N)$ has both chain conditions.

Having shown that M is endofinite, we can use Lemma 2 as follows. The module M is indecomposable and therefore finitely presented by our assumptions. Thus $M \otimes_\Lambda -$ has a simple subobject and therefore $\operatorname{Hom}(D(\Lambda)_\alpha, M \otimes_\Lambda -) \neq 0$ for all $\alpha \geq 0$ by definition of the Krull filtration of $D(\Lambda)$. On the other hand, $\operatorname{Hom}(D(\Lambda)_\beta, M \otimes_\Lambda -) = 0$ and therefore $\beta = -1$. We conclude that $\operatorname{Kdim} D(\Lambda) = 0$. Thus $D(\Lambda)$ is locally finite and Λ is of finite representation type by Lemma 1. This finishes the proof of the theorem.

The preceding proof is based on ideas which allow us to deduce criteria for the existence of generic modules over dualizing rings. Here we call a module *generic*, if it is indecomposable and endofinite but not finitely presented. Note that the existence of a generic module is crucial for the validity of the second Brauer-Thrall conjecture for artin algebras [3].

Two definitions are needed for the statement of our result. An additve subgroup of a Λ-module M is called of *finite definition* if it arises as the kernel of a morphism $M \to M \otimes_\Lambda X$, $m \mapsto m \otimes x$ for some finitely presented Λ^{op}-module X, and some element $x \in X$. We mention the well-known fact that M is Σ-pure-injective iff M has the descending chain condition on subgroups of finite definition, and M is endofinite iff M has the descending and the ascending chain condition on subgroups of finite definition. Given a partially ordered set, a *dense chain* is a subset isomorphic to $\{m \cdot 2^{-n} \in [0,1] \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ with its usual ordering.

Theorem *Let Λ be a right dualizing ring. Then there exists a generic Λ-module provided that one of the following conditions holds.*

(1) *There is a non-zero Λ-module having no finitely presented indecomposable endofinite direct summand and no dense chain of subgroups of finite definition.*

(2) *For some natural number n there is an infinite number of finitely presented indecomposable Λ-modules of endolength at most n.*

A proof of this result will appear in [6]. For artin algebras, the second criterion is due to Crawley-Boevey [3].

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Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
E-mail address: `henning@mathematik.uni-bielefeld.de`