An axiomatic description of a duality for modules

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Given a ring Λ (associative with identity) there is no general procedure known which relates right and left Λ-modules to each other. However it is reasonable to ask for such a relation at least for certain classes of modules. Let us mention three properties which one might expect if one assigns to a right (left) Λ-module M a left (right) Λ-module DM.

(D1) $DDM \cong M$.

(D2) If M is indecomposable, then DM is indecomposable.

(D3) If I denotes an injective cogenerator for the centre C of Λ, then DM is isomorphic to a direct summand of $\mathrm{Hom}_C(M, I)$.

Take for instance a k-algebra Λ over a field k. Then the usual k-dual $DM = \mathrm{Hom}_k(M, k)$ of a finite dimensional Λ-module M satisfies all three properties. However for infinite dimensional Λ-modules the condition (D1) does not hold, and for most rings there is even on the level of the finite length modules no duality functor between right and left modules available.

The aim of this note is to describe without any restriction on the ring Λ a bijection between certain classes of right and left Λ-modules which satisfies (D1) – (D3) and which is uniquely determined by these properties. This bijection covers the k-duality mentioned above but usually there will be non-finitely generated modules on which D is defined. Our approach gives a new interpretation of a construction which is known as elementary duality amongst model theorists, and which is due to Herzog [6], see also [9].

We formulate now the main result. To this end denote by $\mathrm{Mod}(\Lambda)$ the category of (right) Λ-modules and by $\mathrm{mod}(\Lambda)$ the full subcategory of all finitely presented Λ-modules. We shall identify the category of left Λ-modules with $\mathrm{Mod}(\Lambda^{\mathrm{op}})$. Recall that a Λ-module M is pure-injective if every pure monomorphism starting in M splits. A map $\varphi\colon M \to N$ is a pure monomorphism if $\varphi \otimes_\Lambda X\colon M \otimes_\Lambda X \to N \otimes_\Lambda X$ is a monomorphism for every X in $\mathrm{mod}(\Lambda^{\mathrm{op}})$. We call an indecomposable pure-injective Λ-module M *simply reflexive* provided that there is a map $X \to Y$ in $\mathrm{mod}(\Lambda)$ such that the cokernel of the induced map $\mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M)$ is simple when it is regarded in the natural way as an $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module. We refer to the remark below for examples of simply reflexive modules.

Suppose now that M is an indecomposable pure-injective Λ-module which is simply reflexive.

Proposition *There exists, up to isomorphism, a unique indecomposable pure-injective* Λ^{op}*-module* DM *such that any map* φ *in* $\mathrm{mod}(\Lambda)$ *induces an epimorphism* $\mathrm{Hom}_\Lambda(\varphi, M)$ *if and*

only if it induces a monomorphism $\varphi \otimes_\Lambda DM$. *The module* DM *is again simply reflexive and satisfies* $DDM = M$.

This has been shown in [8]. However, we include a short proof at the end of this note. We point out that M is simply reflexive if and only if it is reflexive in the sense of Herzog [6] and that DM coincides with the dual of M in the sense of Herzog [6]. The following result describes an essential property of the module DM.

Theorem *Let* Γ *be a ring and suppose that* M *is a* Λ-Γ*-bimodule. Suppose also that* I *is an injective cogenerator for* $\mathrm{Mod}(\Gamma)$. *Then* DM *is isomorphic to a direct summand of the* Λ^{op}*-module* $\mathrm{Hom}_\Gamma(M, I)$.

To formulate an axiomatic description of the module DM we denote by $\mathrm{Ref}(\Lambda)$ the isomorphism classes of indecomposable pure-injective Λ-modules which are simply reflexive.

Corollary *The assignment* $M \mapsto DM$ *defines a bijection between* $\mathrm{Ref}(\Lambda)$ *and* $\mathrm{Ref}(\Lambda^{\mathrm{op}})$ *which satisfies the conditions* (D1) – (D3). *Moreover any assignment between right and left* Λ*-modules satisfying* (D1) – (D3) *sends* $M \in \mathrm{Ref}(\Lambda)$ *to* DM.

Remark 1 An assignment between right and left Λ-modules satisfying (D1) – (D3) is only possible for pure-injective modules. This follows from Lemma 4 below. Note that Λ is a ring of finite representation type iff every right or left Λ-module is pure-injective. In this case every Λ-module is a coproduct of indecomposable modules and every indecomposable module is simply reflexive.

Remark 2 Let M be an indecomposable Λ-module with $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$.

(1) If M is Σ*-pure-injective*, i.e. $\coprod_I M$ is pure-injective for any set I, then M is simply reflexive [6, 8]. In particular if Λ is a k-algebra over a commutative noetherian ring k, then M is simply reflexive if M is of finite length over k. However there are usually simply reflexive Λ-modules which are not finitely generated. Take for instance a Dedekind domain Λ and let $\Lambda_{\mathfrak{p}^\infty}$ be the Prüfer module corresponding to a prime ideal $\mathfrak{p}$. Then $D\Lambda_{\mathfrak{p}^\infty}$ is the corresponding $\mathfrak{p}$-adic module. Note that $D\Lambda_{\mathfrak{p}^\infty}$ is not Σ-pure-injective.

(2) Suppose that M is *endofinite*, i.e. of finite length when regarded in the natural way as a Γ-module. Then M is Σ-pure-injective [3] and therefore simply reflexive. In fact if M is pure-injective and simply reflexive, then M is endofinite iff M and DM are Σ-pure-injective.

(3) Let I be a minimal injective cogenerator for Γ. Then the Λ^{op}-module $\mathrm{Hom}_\Gamma(M, I)$ might be called the *local dual* of M. If M is finitely presented, then it is well-known that

$\mathrm{Hom}_\Gamma(M, I)$ is indecomposable iff Γ is local. If M is endofinite, then it has been shown by Crawley-Boevey [3] that $\mathrm{Hom}_\Gamma(M, I)$ is a coproduct of copies of DM.

(4) Recall from [8] that the *endocategory* $\mathcal{E}_M$ of M is the smallest abelian subcategory of $\mathrm{Mod}(\Gamma)$ containing M and all the endomorphisms of M induced by multiplication with an element from Λ. The modules M, DM and $\mathrm{Hom}_\Gamma(M, I)$ share several properties because they are related by a duality between $\mathcal{E}_M$ and $\mathcal{E}_{DM}$ and an equivalence between $\mathcal{E}_{DM}$ and $\mathcal{E}_{\mathrm{Hom}_\Gamma(M,I)}$ [8].

Remark 3 It would be interesting to know for which class of rings every indecomposable pure-injective module is simply reflexive. Note that there are rings with this property which are not of finite representation type. Take for instance a Dedekind domain. An example of similar nature is a finite dimensional hereditary algebra Λ which is tame. In fact an indecomposable pure-injective Λ-module M which is not finitely generated, is either a so-called Prüfer module, a p-adic module or M is endofinite and, up to isomorphism, uniquely determined by this property [10].

To give the proofs of the various assertions we need to recall some background material. Denote by $D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ the category of additive functors from the category of finitely presented Λ^{op}-modules to the category of abelian groups. The full subcategory of finitely presented functors is abelian and is denoted by $C(\Lambda)$. The fully faithful functor

$$\mathrm{Mod}(\Lambda) \longrightarrow D(\Lambda), \quad M \mapsto M \otimes_\Lambda -$$

will play an important role in our considerations. We refer to [3] for its basic properties. For instance we need the fact that, up to isomorphism, the injective objects in $D(\Lambda)$ are the functors $M \otimes_\Lambda -$ with M pure-injective. Given a module M we denote by $\mathcal{S} = \mathcal{S}_M$ the kernel of the exact functor

$$C(\Lambda) \longrightarrow \mathrm{Mod}(\Gamma), \quad X \mapsto \mathrm{Hom}(X, M \otimes_\Lambda -)$$

where $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ is identified with $\mathrm{End}(M \otimes_\Lambda -)^{\mathrm{op}}$. Furthermore, denote by $\vec{\mathcal{S}}$ the full subcategory of $D(\Lambda)$ which consists of all direct limits $\varinjlim X_i$ with $X_i \in \mathcal{S}$ for all i.

Recall that a full subcategory $\mathcal{T}$ of the functor category $D(\Lambda)$ is *localizing* if it is closed under subobjects, quotients, extensions and coproducts. For any localizing subcategory $\mathcal{T}$ one can form the *quotient category* $D(\Lambda)/\mathcal{T}$ which is abelian, has injective envelopes and admits an exact *quotient functor* $q\colon D(\Lambda) \to D(\Lambda)/\mathcal{T}$ with $\mathrm{Ker}(q) = \mathcal{T}$ [4].

Lemma 1 *The subcategory* $\vec{\mathcal{S}}$ *is localizing. The quotient functor* $q\colon D(\Lambda) \to D(\Lambda)/\vec{\mathcal{S}}$ *sends any injective object* N *satisfying* $\mathrm{Hom}(\mathcal{S}, N) = 0$ *to an injective object and* q *induces an isomorphism* $\mathrm{Hom}(X, N) \to \mathrm{Hom}(q(X), q(N))$ *for every* $X \in D(\Lambda)$.

Proof: The first statement is proved in [7]. The properties of q are well-known facts which may be found in [4].

Lemma 2 *Let $\varphi: X \to Y$ be a map in $\mathrm{mod}(\Lambda)$ and let $U = \mathrm{Ker}(\varphi \otimes_\Lambda -)$ in $D(\Lambda)$. Then $\mathrm{Hom}(U, M \otimes_\Lambda -) \cong \mathrm{Coker}(\mathrm{Hom}_\Lambda(\varphi, M))$ for any $M \in \mathrm{Mod}(\Lambda)$.*

Proof: The exact sequence $0 \to U \to X \otimes_\Lambda - \to Y \otimes_\Lambda -$ induces an exact sequence

$$\mathrm{Hom}(Y \otimes_\Lambda -, M \otimes_\Lambda -) \to \mathrm{Hom}(X \otimes_\Lambda -, M \otimes_\Lambda -) \to \mathrm{Hom}(U, M \otimes_\Lambda -) \to 0.$$

Lemma 3 *Let M be indecomposable pure-injective and suppose that the cokernel of $\mathrm{Hom}(\varphi, M)$ is a simple $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module for some φ in $\mathrm{mod}(\Lambda)$. Then the quotient functor $q: D(\Lambda) \to D(\Lambda)/\vec{\mathcal{S}}_M$ sends $U = \mathrm{Ker}(\varphi \otimes_\Lambda -)$ to a simple object and $M \otimes_\Lambda -$ to an injective envelope of $q(U)$.*

Proof: The object $q(U)$ is simple precisely if for any exact sequence $0 \to U' \to U \to U'' \to 0$ either $U' \in \vec{\mathcal{S}}_M$ or $U'' \in \vec{\mathcal{S}}_M$. Writing $U' = \varinjlim U_i$ as a direct limit of all its finitely generated submodules we obtain induced sequences $0 \to U_i \to U \to U/U_i \to 0$ in $C(\Lambda)$ since $C(\Lambda)$ is abelian. Now, for any i either $U_i \in \mathcal{S}_M$ or $U/U_i \in \mathcal{S}_M$ since there is an induced exact sequence

$$0 \to \mathrm{Hom}(U/U_i, M \otimes_\Lambda -) \to \mathrm{Hom}(U, M \otimes_\Lambda -) \to \mathrm{Hom}(U_i, M \otimes_\Lambda -) \to 0$$

of $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-modules and $\mathrm{Hom}(U, M \otimes_\Lambda -) \cong \mathrm{Coker}(\mathrm{Hom}_\Lambda(\varphi, M))$ is simple by assumption and Lemma 2. If $U/U_i \in \mathcal{S}_M$ for some i, then $U'' \in \vec{\mathcal{S}}_M$ since U'' is a quotient of U/U_i. Otherwise all $U_i \in \mathcal{S}_M$ and therefore $U' \in \vec{\mathcal{S}}_M$. Thus $q(U)$ is simple. Using again Lemma 2 there is a non-zero morphism $U \to M \otimes_\Lambda -$ and this is taken to a non-zero morphism $q(U) \to q(M \otimes_\Lambda -)$ by Lemma 1. Thus $q(M \otimes_\Lambda -)$ is an injective envelope of $q(U)$ since $q(M \otimes_\Lambda -)$ is indecomposable injective.

Lemma 4 *If M is a Λ-Γ-bimodule and $I \in \mathrm{Mod}(\Gamma)$ is injective, then $\mathrm{Hom}_\Gamma(M, I)$ is a pure injective Λ^{op}-module.*

Proof: See [1, I, Proposition 10.1].

Recall from [8] that a pair of modules $M \in \mathrm{Mod}(\Lambda)$ and $N \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ is *purely opposed* provided that any map φ in $\mathrm{mod}(\Lambda)$ induces an epi $\mathrm{Hom}_\Lambda(\varphi, M)$ iff it induces a mono $\varphi \otimes_\Lambda N$, equivalently if any map ψ in $\mathrm{mod}(\Lambda^{\mathrm{op}})$ induces an epi $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\psi, N)$ iff it induces a mono $\psi \otimes_{\Lambda^{\mathrm{op}}} M$.

Lemma 5 *If M is a Λ-Γ-bimodule and $I \in \mathrm{Mod}(\Gamma)$ is an injective cogenerator, then the Λ-module M and the Λ^{op}-module $\mathrm{Hom}_\Gamma(M, I)$ are purely opposed.*

Proof: If I is any injective Γ-module, then there is a well-known isomorphism

$$X \otimes_\Lambda \mathrm{Hom}_\Gamma(M, I) \longrightarrow \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(X, M), I)$$

for all $X \in \mathrm{mod}(\Lambda)$ which is functorial in X [2, VI, Proposition 5.2]. Taking a map φ in $\mathrm{mod}(\Lambda)$ it follows that $\varphi \otimes_\Lambda \mathrm{Hom}_\Gamma(M, I)$ is a mono iff $\mathrm{Hom}(\varphi, M)$ is an epi provided that I cogenerates $\mathrm{Mod}(\Gamma)$. Thus M and $\mathrm{Hom}_\Gamma(M, I)$ are purely opposed.

Recall from [8] that a pair of Λ-modules M and N is *purely equivalent* provided that $\mathcal{S}_M = \mathcal{S}_N$, equivalently if any map φ in $\mathrm{mod}(\Lambda)$ induces an epi $\mathrm{Hom}_\Lambda(\varphi, M)$ iff it induces an epi $\mathrm{Hom}_\Lambda(\varphi, N)$. Purely equivalent and purely opposed modules are related as follows.

Lemma 6 *There is a bijection $[M] \mapsto \Delta[M]$ between the equivalence classes of purely equivalent Λ-modules and the equivalence classes of purely equivalent Λ^{op}-modules. This bijection has the following properties.*

(Δ1) $\Delta\Delta[M] = [M]$.

(Δ2) *$M \in \mathrm{Mod}(\Lambda)$ and $N \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ are purely opposed iff $\Delta[M] = [N]$.*

Proof: Define Δ using (Δ2). It follows directly from the definitions that (Δ1) holds.

Lemma 7 *Let M and N be a pair of purely equivalent pure-injective Λ-modules. If M is indecomposable and simply reflexive, then M is isomorphic to a direct summand of N.*

Proof: We use again the quotient functor $q\colon D(\Lambda) \to D(\Lambda)/\vec{\mathcal{S}}_M$. By assumption and Lemma 1 the objects $q(M \otimes_\Lambda -)$ and $q(N \otimes_\Lambda -)$ are both injective and there are non-zero morphisms $q(U) \to q(M \otimes_\Lambda -)$ and $q(U) \to q(N \otimes_\Lambda -)$ where $q(U)$ is a simple object as defined in Lemma 3. We use now the fact that $q(M \otimes_\Lambda -)$ is an injective envelope of $q(U)$. Thus there is a split mono $q(M \otimes_\Lambda -) \to q(N \otimes_\Lambda -)$ which is the image of a split mono $M \otimes_\Lambda - \to N \otimes_\Lambda -$ by Lemma 1. We conclude that M is isomorphic to a direct summand of N.

Proof of the theorem: Let M be simply reflexive and suppose that M is a Λ-Γ-bimodule. Also let I be an injective cogenerator for $\mathrm{Mod}(\Gamma)$. The module DM is purely opposed to M and therefore purely equivalent to $\mathrm{Hom}_\Gamma(M, I)$ by Lemma 5. It follows from Lemma 7 that DM is isomorphic to a direct summand of $\mathrm{Hom}_\Gamma(M, I)$. Thus the proof is complete.

Proof of the corollary: It follows from the proposition and the theorem that $M \mapsto DM$ satisfies (D1) – (D3). Suppose now that $\mathcal{X}$ is a class of Λ-modules and that $\mathcal{Y}$ is a class of Λ^{op}-modules such that a pair $D'\colon \mathcal{X} \to \mathcal{Y}$ and $D'\colon \mathcal{Y} \to \mathcal{X}$ is defined which satisfies (D1) – (D3).

Lemma 8 *For M in $\mathcal{X}$ or $\mathcal{Y}$ the modules M and $D'M$ are purely opposed.*

Proof: We need (D1) and (D3). It follows from (D3) that $\mathcal{S}_{\mathrm{Hom}_C(M,I)} \subseteq \mathcal{S}_{D'M}$ and we obtain therefore the following chain of inclusions

$$\mathcal{S}_M = \mathcal{S}_{\mathrm{Hom}_C(\mathrm{Hom}_C(M,I),I)} \subseteq \mathcal{S}_{\mathrm{Hom}_C(D'M,I)} \subseteq \mathcal{S}_{D'D'M} = \mathcal{S}_M .$$

The first equality follows from Lemma 5 and Lemma 6; the last equality follows from (D1). Thus $\mathcal{S}_M = \mathcal{S}_{\mathrm{Hom}_C(D'M,I)}$ and a further application of Lemma 5 and Lemma 6 then shows that M and $D'M$ are purely opposed.

We continue with the proof of the corollary. It follows immediately from Lemma 4 that every module in $\mathcal{X}$ or $\mathcal{Y}$ is pure-injective. Now let M be a module in $\mathrm{Ref}(\Lambda)$ or $\mathrm{Ref}(\Lambda^{\mathrm{op}})$ and suppose that $D'M$ is defined. We claim that $D'M \cong DM$. It follows from (D2) that $D'M$ is indecomposable. Our assertion is therefore a consequence of Lemma 7 once we have shown that $D'M$ and DM are purely equivalent. But this follows from Lemma 8 together with Lemma 6 and therefore the proof is complete.

Proof of the proposition: Let M be a simply reflexive Λ-module. One uses the well-known duality $d\colon C(\Lambda) \to C(\Lambda^{\mathrm{op}})$ [5] to construct DM. Let $U = \mathrm{Ker}(\varphi \otimes_\Lambda -) \in C(\Lambda)$ be as in Lemma 3 and put $\mathcal{T} = d(\mathcal{S}_M)$. Adapting the argument of Lemma 3 one shows that the quotient functor $q\colon D(\Lambda^{\mathrm{op}}) \to D(\Lambda^{\mathrm{op}})/\vec{\mathcal{T}}$ sends $d(U)$ to a simple object. Now one uses the section functor $s\colon D(\Lambda^{\mathrm{op}})/\vec{\mathcal{T}} \to D(\Lambda^{\mathrm{op}})$ to find an indecomposable pure-injective Λ^{op}-module N such that $\mathrm{Hom}(\mathcal{T}, N \otimes_{\Lambda^{\mathrm{op}}} -) = 0$ and $q(N \otimes_{\Lambda^{\mathrm{op}}} -)$ is an injective envelope of $q(d(U))$. It is not hard to check that M and N are purely opposed. Finally, the uniqueness of N follows from the fact that an injective envelope of $q(d(U))$ is unique up to isomorphism. Thus $DM = N$.

References

[1] M. Auslander, Functors and morphisms determined by objects, in: Representation theory of algebras. Proc. conf. Philadelphia 1976, ed. R. Gordon, (Dekker, New York 1978), 1–244.

[2] H. Cartan and S. Eilenberg, Homological Algebra, (Princeton University Press, Princeton 1956).

[3] W. Crawley-Boevey, Modules of finite endolength over their endomorphism ring, in: Representations of algebras and related topics, eds. S. Brenner and H. Tachikawa, London Math. Soc. Lec. Note Series **168** (1992) 127–184.

[4] P. Gabriel, Des catégories abéliennes, Bull. Soc. math. France, **90** (1962), 323–448.

[5] L. Gruson, Simple coherent functors, in: Representations of algebras, Springer Lec. Notes **488** (1975) 156–159.

[6] I. Herzog, Elementary duality for modules, Trans. Am. Math. Soc. **340** (1993) 37–69.

[7] H. Krause, The spectrum of a locally coherent category, J. Pure Appl. Algebra, to appear.

[8] H. Krause, The endocategory of a module, Proc. ICRA VII 1994, eds. R. Bautista, R. Martinez-Villa and J.A. de la Pena, Can. Math. Soc., to appear.

[9] M. Prest, P. Rothmaler and M. Ziegler, Extensions of elementary duality, J. Pure Appl. Algebra **93** (1994) 33–56.

[10] C.M. Ringel, Infinite dimensional representations of finite dimensional hereditary algebras, Symp. Math. Ist. Naz. Alta Mat. **23** (1979) 321–412.

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