Semilinear stochastic equations for symmetric diffusions

Stefan Mück*

Institut für Angewandte Mathematik, Wegelerstraße 6, D-53115 Bonn, Germany.
(Fax: +49-0228/737864 ; e-mail: mueck@wiener.iam.uni-bonn.de)

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Summary. In this paper we provide new sufficient conditions for the existence of weak solutions to a class of semilinear stochastic equations, which at the same time guarantee that the solution is a symmetric ergodic right process. The invariant measure (which is also the symmetrizing measure) is given explicitly. The emphasis is on the fact that due to the symmetry in our situation, the assumptions on the nonlinear term can be relaxed considerably.

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1 Introduction

Let $\mathbb{M} = \left(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\triangle}, \zeta\right)$ be a stochastic process with lifetime ζ, semigroup $(p_t)_{t\geq 0}$ and state space E. Then $\mathbb{M}$ is called symmetric, if there is a measure μ on E, such that $\int p_t f \cdot g \, d\mu = \int f \cdot p_t g \, d\mu$ for all bounded measurable f, g. If, moreover, $p_t 1 = 1$ μ-a.s., then $\mathbb{M}$ is stationary and reversible w.r.t. the measure $P_\mu := \int P_x \, \mu(dx)$. The aim of this paper is to give sufficient conditions for the existence of a symmetric conservative diffusion $\mathbb{M}$ which is a weak solution (weak in the probabilistic sense) to the stochastic differential equation

$$
\begin{aligned}
dX_t &= -(\tfrac{1}{2} A X_t + F(X_t))\, dt + dW_t, \\
X_0 &= x \in E,
\end{aligned}
\tag{1}
$$

where E is a separable Hilbert space, F a non-linear map, $(A, D(A))$ a linear operator on E and $(W_t)_{t\geq 0}$ a Wiener process with covariance $Q \geq 0$. For the construction of the solution we use Dirichlet form techniques (the fundamental results were obtained in [AR 91]), which

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are combined with semigroup arguments in order to show that the solution $(X_t)_{t\geq 0}$ is not only a weak solution in the functional analytic sense, but that also $\int_0^t X_s\, ds \in D(A)$ under rather mild assumptions. So far, the symmetric case in infinte dimensions has been dealt with explicitly by J.Zabczyk [Z 89], who constructs mild solutions in the case $Q = Id$ and by I.Shigekawa [Shi 87, Section 4], who considers Girsanov transforms of the infinite dimensional Ornstein-Uhlenbeck process (i.e. $A = Id$). Symmetric mild solutions for $Q = Id$ have also been constructed by R.Marcus in [Mar 74], [Mar 78] and by Gatarek, Goldys [GaGo 95b]. Related results in the Dirichlet form approach have been obtained by S.Albeverio, M.Röckner [AR 91] and by T.S. Zhang [Zh 92]. Here, the task is split up into two parts. The first step is to characterize those operators A, Q for which the linear equation

$$dX_t = -\tfrac{1}{2}AX_t\, dt + dW_t, \quad X_0 = x \in E, \tag{2}$$

has a symmetric solution on E, cf. Theorem 4 below. The special feature of our approach to the second step is that the admissible drift terms F are characterized in terms of the symmetrizing measure μ, which in turn depends on A and Q. It seems that one thus obtains the weakest possible assumptions on F. Some restrictions on F, however, are independent of A, Q. If $Q = Id$, then F has to be a gradient ∇U. This is the classical characterization of Kolmogorov (see [Ko 37]) for the drift term of a stationary process, which can be extended to infinite dimensions, see [BR 95]. If $Q \neq Id$, then at least any drift term F, which can be generated by a Girsanov transformation, necessarily has to be of type $Q\nabla U$, as can be easily seen in the one-dimensional situation. Since these conditions are necessary, they occur in any approach. Apart from these restrictions, however, further conditions are needed that guarantee the existence of a solution. Here, we basically require that U is lower bounded and bounded by an even polynomial q such that $\int \|p(z)\|_E^2\, \mu(dz) < \infty$ for all polynomials p with degree(p) $\leq$degree(q^2). It is very interesting to note that in the case $Q = Id$ T.Funaki obtained a very similar condition [Fu 83, Assumptions 2 and 3 in Section 4], although his techniques are different. He uses a polygonal approximation to obtain a finite dimensional solution and a tightness argument to construct the infinite dimensional one.

In the "semigroup approach", i.e., the construction of mild solutions, different additional regularity assumptions on F are common, which need not be satisfied here; e.g. (local) Lipschitz continuity [Mar 74], [Z 89] or monotonicity [Mar 78]. In the non-symmetric case, mild solutions have been proved to exist also for less regular non-linearities. Among other conditions, it is often assumed that F satisfies a dissipativity condition [dPZ 92], or that F is polynomially bounded and satisfies a one-sided linear growth condition [I 87], [BaGyPa 94], [GyPa 94], or the semigroup generated by $(-A)$ has to be compact (e.g. [GaGo 94]). Here, we give examples, where a *weak* solution exists on E, although none of the latter two conditions is satisfied, cf. Examples 3(iii), 17.

Our second main result is that all the solutions constructed below which are started with initial condition μ are time-shift ergodic, without any further assumption. This is equivalent to the $L^2(E;\mu)$-ergodicity of the semigroup $(p_t)_{t\geq 0}$. We emphasize that this is a weaker result than the pointwise ergodicity of $(p_t)_{t\geq 0}$, investigated now by some other authors (cf. e.g. [MasSei 94], [Sei 94]).
It should be noted also that we obtain examples for diffusions with an unbounded non-linearity, such that none of the known criteria for the existence of an invariant measures is applicable, but which do have an invariant (and symmetrizing) measure on E, cf. Remark 19, below.
Finally, we would like to mention that the large deviation results obtained in [Mü 94] apply to all the processes constructed in this paper, cf. Remark 21 below.

The paper is organized as follows. Our main results are presented in Sections 3.1 and 4.1. In Section 2 we recall some basic facts on Dirichlet forms and right processes (which we consider on a canonical path space, also defined there). In Section 3 on the linear equation, we have to be somewhat detailed. First of all, because the existence and uniqueness results of this section are needed for the semi-linear case, which thus becomes shorter. But then also, because the usual Dirichlet form arguments have to be modified such that a comparison and combination of the results with the "semigroup approach" is possible. In the fourth section, we use an analytic analogue of the Girsanov transformation to obtain the desired additional drift term. The regularity result that $\int_0^t X_s\,ds \in D(A)$ then follows by absolute continuity. In the last section we apply our results to operators A, Q that are powers of $(m-\triangle)$, $m \geq 1$, on the Lebesgue L^2-space over a rectangle in $I\!R^d$.

2 Preliminaries

Let E be a separable Hilbert space, $C(E)$ the space of continuous real valued functions on E, $\mathcal{B}(E)$ the Borel-σ-Algebra on E and let $\mathcal{B}_b(E)$ denote the bounded $\mathcal{B}(E)/\mathcal{B}(I\!R)$-measurable functions. Furthermore, let μ be a probability measure on $(E,\mathcal{B}(E))$ and let $(\cdot,\cdot)$ denote the inner product in $L^2(E;\mu)$.

Definition 1 A symmetric bilinear form $\mathcal{E}(\cdot,\cdot)$ on $L^2(E;\mu)$ with $D(\mathcal{E})$dense in $L^2(E;\mu)$ is called a *Dirichlet form* if it is closed (i.e., $D(\mathcal{E})$ is complete with respect to $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)$) and if every normal contraction operates on $(\mathcal{E}, D(\mathcal{E}))$, i.e., $\mathcal{E}(T(u),T(u)) \leq \mathcal{E}(u,u)$ for every $u \in D(\mathcal{E})$ and every $T : I\!R \to I\!R$ satisfying $T(0) = 0$, $|T(x) - T(y)| \leq |x-y|, x, y \in I\!R$.

¿From now on we fix a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;\mu)$. Let $(L, D(L))$ denote the unique selfadjoint negative definite operator such that $D(L) \subset D(E)$ and $\mathcal{E}(u,v) = (-Lu, v), u \in$

$D(L), v \in D(\mathcal{E})$. The generator $(L, D(L))$ is in one-to-one correspondence with a strongly continuous semigroup of symmetric contractions $T_t = e^{tL}, t \geq 0$, and a strongly continuous symmetric contraction resolvent $G_\alpha = (\alpha - L)^{-1}, \alpha > 0$, on $L^2(E;\mu)$ (cf., e.g., [MR 92, Diagram 3, remarks after I.2.22]).
Here, it will be always assumed that $T_t 1 = 1$ for all $t > 0$, i.e., $(\mathcal{E}, D(\mathcal{E}))$ is *conservative.* So we have in particular $1 \in D(L)$. Let us recall the definition of the *(1-) Capacity* associated with $(\mathcal{E}, D(\mathcal{E}))$: For $U \subseteq E, U$ open, set

$$Cap(U) := \inf\{\mathcal{E}_1(u,u) : u \geq 1 \ \mu - a.e. \text{ on } U\},$$

and for $A \subseteq E$ arbitrary $Cap(A) := \inf\{Cap(U) : A \subseteq U,\ U \text{ open}\}$. A set $N \subseteq E$ is called *$\mathcal{E}$-exceptional,* iff $Cap(N) = 0$. An increasing sequence $(F_n)_{n\in I\!N}$ of closed sets is called an *$\mathcal{E}$-nest,* iff $\lim_{n\to\infty} Cap(E \setminus F_n) = 0$. We say that a property holds *$\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.)* iff there is an $\mathcal{E}$-exceptional set N, such that it holds everywhere on $E \setminus N$. Since we assume $T_t 1 = 1$, this capacity is the capacity $Cap_{h,g}$ defined in [MR 92, III.2.4] for $h = g = G_1 1 = 1$. By virtue of [MR 92, III.2.11] the definitions of $\mathcal{E}$-nests and $\mathcal{E}$-exceptional sets given here are equivalent to those in [MR 92]. Most important in the study of Dirichlet forms is a characterisation of those, that are associated to a stochastic process. There is a one-to-one correspondance between so-called quasi-regular Dirichlet forms and right processes, cf. [MR 92, Thm.IV.3.5, Thm.IV.5.1] (cf. [Sh 88] for the definition of a right process). Let us give the relevant details.
The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is called *quasi-regular,* iff (i) there is an $\mathcal{E}$-nest $(E_n)_{n\in I\!N}$ consisting of compact sets, (ii) there is an $\mathcal{E}_1^{1/2}$-dense subset of $D(\mathcal{E})$, whose elements have $\mathcal{E}$-quasi-continuous μ-versions and (iii) there exist $u_n \in D(\mathcal{E}), n \in I\!N$, having $\mathcal{E}$-quasi-continuous μ-versions $\tilde{u}_n, n \in I\!N$, and an $\mathcal{E}$-exceptionel set $N \subseteq E$, such that $\{\tilde{u}_n | n \in I\!N\}$ separates the points of $E \setminus N$.
We adjoin a point $\triangle$ (the cemetery) as an isolated point to E, if E is not locally compact, else we take $E_\triangle := E \cup \{\triangle\}$ to be the one point compactification of E. ¿From now on we assume that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Then, by [MR 92, IV.3.5], there exists a a right process $I\!M = \big(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\triangle}, \zeta\big)$ with state space E and life time ζ, which is *properly associated to* $(\mathcal{E}, D(\mathcal{E}))$, i.e., $p_t f(x) := E_x[f(X_t)]$ is an $\mathcal{E}$-quasi-continuous μ-version of $T_t f$ for all $f \in \mathcal{B}_b(E),\ t > 0$. Here, $E_x[\,\cdot\,]$ denotes expectation w.r.t. P_x. In particular, $I\!M$ is μ-tight, μ-special standard.
The canonical model. W.l.o.g. we *always* assume that we are in the following canonical situation. Let Ω' be the set of all maps $\omega : I\!R_0^+ \to E_\triangle$ and define $\zeta(\omega) := \inf\{t > 0 : \omega(t) = \triangle\},\ \omega \in \Omega'$. Then let Ω be the set of all $\omega \in \Omega'$ with $\omega(t) = \triangle$ for all $t \geq \zeta$, that are right continuous on $[0,\infty)$ and have left limits in E on $(0,\zeta)$. Further let $X_t(\omega) := \omega(t)$, $\mathcal{F}_t^0 := \sigma(X_s; s \leq t)$, and let $\mathcal{F}$ be a universally complete σ-algebra (i.e., $\mathcal{F}$ equals the intersection of all P_ν-completions of $\mathcal{F}$, as ν runs over all probability measures on $(E_\triangle, \mathcal{B}(E_\triangle))$) such

that $\mathcal{F} \supset \mathcal{F}_0^t$. Then let $\mathcal{F}_t$ be the universal completion of $\mathcal{F}_t^0$, in $\mathcal{F}$. Furthermore, let $\{\theta_t : \Omega \to \Omega, t \geq 0\}$, denote the time-shift semigroup defined by $X_s(\omega \circ \theta_t) = X_{s+t}(\omega)$.
Finally, let $\overline{X_0^t}(\omega)$ denote the closure of $\{X_s(\omega) : 0 \leq s \leq t\}$ in E. A Borel set $S \in \mathcal{B}(E)$ is called *$I\!\!M$-invariant,* iff there exists $\Omega_{E\setminus S} \in \mathcal{F}$ such that

$$\Omega_{E\setminus S} \supseteq \{\overline{X_0^t} \cap (E \setminus S) \neq \emptyset \text{ for some } 0 \leq t < \zeta\}$$

and $P_x[\Omega_{E\setminus S}] = 0$ for all $x \in S$. We will call $\Omega \setminus \Omega_{E\setminus S}$ a *defining set for S.*

Definition 2 *The right process $I\!\!M$ is called a* diffusion, *if there is an $\mathcal{E}$- exceptional set $N \subseteq E$, such that $P_x[t \mapsto X_t$ is continuous for all $t \geq 0] = 1$ for all $x \in E \setminus N$. It is called a* conservative diffusion, *if in addition $P_x[\zeta = \infty] = 1$ for all $x \in E \setminus N$.*

This notion coincides with the ordinary definition of a diffusion apart from the fact that continuity holds for $\mathcal{E}$-q.e. $x \in E$, but not for all $x \in E$. However, the process does not "see" the $\mathcal{E}$-exceptional set N in the following sense. For $B \subseteq E$ the *first touching time τ_B of B* is defined by

$$\tau_B := \inf\{0 \leq t \leq \zeta : \overline{X_t^0} \cap B \neq \emptyset\} \wedge \zeta,$$

where $\overline{X_t^0}$ is the closure of $\{X_s(\omega) : 0 \leq s \leq t\}$ in E. Then a subset N of E is called *$I\!\!M$-exceptional,* if there exists $\tilde{N} \subseteq \mathcal{B}(E)$ such that $N \subseteq \tilde{N}$ and $P_\mu[\tau_{\tilde{N}} < \zeta] = 0$. It is an important result that a set $N \subseteq E$ is $\mathcal{E}$-exceptional, if and only if it is $I\!\!M$-exceptional, cf. [AR 91, Thm.5.29]. We will often use this result without explicitly referring to it.

3 Symmetric Solutions to Linear Equations

3.1 Introduction and main results

In the Dirichlet form approach, one has to guess or construct the invariant measure μ for the linear equation first and one has to guess an appropriate Hilbert space H, $H \subseteq E$ densely and continuously, on which a gradient is to be defined. Then one checks that the process which is properly associated to the symmetric gradient Dirichlet form on $L^2(E;\mu)$ indeed solves the equation. Let now E be a separable real Hilbert space with inner product $\langle\cdot,\cdot\rangle_E$, let $(A, D(A))$ be a linear densely defined non-negative definite self-adjoint operator on E and let $Q \geq 0$ be a symmetric bounded operator on E (with domain $D(Q) = E$). Consider the following assumption.

Assumption (A): *$(A, D(A))$ and Q have discrete spectrum with a common set of normalized eigenvectors $B_0 := \{e_n : n \in I\!\!N\}$, which form an orthonormal basis of E, with* *corresponding finite eigenvalues* $\alpha_n > 0$, $\lambda_n > 0$, $n \in I\!N$, $(Ae_n = \alpha_n e_n,\ Qe_n = \lambda_n e_n)$, *such that*

$$\sum_{n=1}^{\infty} \alpha_n^{-1} \lambda_n < \infty.$$

Example 3 *(i) For operators* $(A, D(A))$ *and* Q *as above it is a classical result that assumption (A) is satisfied, if a)* Q *is 1-1 and* $\operatorname{tr} Q < \infty$*, b)* $A \geq c Id_E$, $c > 0$*, and if c)* $(Q^{1/2}A^{-1/2}, D(Q^{1/2}A^{-1/2})) = (A^{-1/2}Q^{1/2}, D(A^{-1/2}Q^{1/2}))$*. For a proof cf., e.g., [Mü 95].*
(ii) Let $\tilde{A} > 0$ *be a linear densely defined selfadjoint operator on* E *such that* $\tilde{A}^{-\theta}$ *is trace class for some* $\theta > 0$*. Then assumption (A) is satisfied, if we set, e.g.,*

a) $A := c\, Id;\quad Q := \tilde{A}^{-\gamma},\ \gamma \geq \theta,$

b) $A := \tilde{A};\quad Q := \tilde{A}^{-\gamma},\ \gamma \geq \theta - 1,$

c) $A := \tilde{A}^{\gamma},\quad Q := Id,\ \gamma \geq \theta,$ *compare [Z 89, Thm.1].*

(iii) Let $\gamma_n > 0$, $n \in I\!N$*, be a sequence such that* $\sum_{n=1}^{\infty} \gamma_n < \infty$*. Then define* Q *by* $\lambda_n := \gamma_n^{3/2}$ *and* A *by* $\alpha_n := \gamma_n^{1/2}$*, if* n *is odd and* $\alpha_n := \gamma_n^{-1/2}$ *else. In this case* $\inf_{n \in I\!N} \alpha_n = 0$ *and* $\sup_{n \in I\!N} \alpha_n = \infty$*. The important feature of this example is that the semigroup generated by* $(-A, D(A))$ *is not compact.*

Let now μ be the mean zero Gaussian measure on $(E, \mathcal{B}(E))$ having characteristic function

$$\hat{\mu}(y) := \int e^{i\langle y,z\rangle_E}\, \mu(dz) = \exp[-\frac{1}{2} \sum_{n=1}^{\infty} \alpha_n^{-1} \lambda_n \langle e_n, y\rangle_E^2],\ y \in E. \tag{3}$$

Note that the measure μ exists on $(E, \mathcal{B}(E))$ by assumption (A) and the Theorem of Sazonov, cf., e.g., [dPZ 92, Prop.2.18].

Theorem 4 *Assume that (A) holds. Then there is a conservative* μ*-symmetric diffusion* $I\!M = \left(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t \geq 0}, (P_x)_{x \in E_\Delta}, \zeta\right)$ *with state space* E*, and for each* $k \in \operatorname{span} B_0$ *there is an* $I\!M$*-exceptional set* $N_k \subseteq E$ *such that*

$$\langle k, X_t\rangle_E = \langle k, x\rangle_E - \frac{1}{2}\int_0^t \langle Ak, X_s\rangle_E\, ds + W_t^k,\ k \in \operatorname{span} B_0, \tag{4}$$

P_x*-a.s. for all* $x \in E \setminus N_k$*, where* $(\Omega, (\mathcal{F}_t)_{t \geq 0}, (W_t^k)_{t\geq 0}, P_x)$ *is a one-dimensional mean-zero Gaussian process starting in zero with quadratic variation* $\langle W^k\rangle_t = t\langle Qk, k\rangle_E$*. Moreover,* $I\!M$ *is the unique weak solution to the stochastic differential equation (2) in the class of all* μ*-symmetric right processes. If, in addition,*

$$\sum_{n=1}^{\infty} (1 + \alpha_n^{-2}) \lambda_n < \infty, \tag{5}$$

then there is an $I\!M$-exceptional set $N \subseteq E$ such that firstly, the Bochner-integral $\int_0^t X_s\,ds$, $t \geq 0$, exists and is in $D(A)$ P_x-a.s. for all $x \in E \setminus N$ and, secondly, there is a map $W : \Omega \to C([0,\infty); E)$ such that $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0}, P_x)$ is a Q-Wiener process for all $x \in E \setminus N$ and such that, finally,

$$X_t = x - \frac{1}{2} A \int_0^t X_s\,ds + W_t, \quad t \geq 0, \tag{6}$$

P_x-a.s. for all $x \in E \setminus N$.

Remark 5 *(i) Note that the additional condition (5) is satisfied in Examples 3 (i), (ii) a), (iii) and also in (ii) b), if $\gamma \geq \theta$.*
(ii) W.l.o.g. we may always assume that we are in the canonical situation, cf. Section 2. What we construct, is in fact the strong markovian family $(P_x)_{x\in E}$ on Ω. Thus we call the solution a weak solution. *It might be called a* weak solution of strong type, *if the Bochner integral $\int_0^t X_s\,ds$ exists and is in $D(A)$. If we set $P_\mu := \int_E P_z\,\mu(dz)$, then $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, P_\mu)$ is a* martingale solution *in the sense of [dPZ 92, Chapter 8].*

Theorem 6 *The measure $P_\mu = \int_E P_z\,\mu(dz)$ is (time-shift-) ergodic.*

3.2 Construction of the gradient Dirichlet form

In the following it will always be assumed that assumption **(A)** holds. First, we have to discuss the basic triple $(E' \subseteq H \subseteq E)$. For $s > 0$ let $H_{Q^{-s}}$ denote the set $D(Q^{-s})$ endowed with the inner product $\langle Q^{-s}\cdot, Q^{-s}\cdot\rangle_E$. Then $H_{Q^{-s}}$ is a separable real Hilbert space which is densely and continuously contained in E. Choose now $H := H_{Q^{-1/2}}$ and identify H with its dual H' via the Riesz-isomorphism $j_H : H' \to H$. Then $E' \subseteq H'$ densely and continuously, where the injection is given by the restriction mapping $i_{H'}(l) = l|_{H'}$. Let finally i_E denote the trivial inclusion from H to E, i.e., $i_E(h) = h$. One thus obtains the chain of inclusions $E' \xrightarrow{i_{H'}} H' \xrightarrow{j_H} H \xrightarrow{i_E} E$. We need to identify E' and the embedding $E' \subseteq E$. Let therefore $i_H : H_{Q^{-1}} \to H$ denote the trivial inclusion $i_H(k) = k$.

Proposition 7 *The linear map $j_{E'} : H_{Q^{-1}} \to E'$ given by $H_{Q^{-1}} \xrightarrow{i_H} H \xrightarrow{j_H^{-1}} H' \xrightarrow{i_{H'}^{-1}} E'$ is defined on all of $H_{Q^{-1}}$ and is a surjective isomorphism. In particular,*

$$_{E'}\langle j_{E'}(k)\,,\,u\,\rangle_E \;=\; \langle Q^{-1}k, u\rangle_E, \quad k \in H_{Q^{-1}},\ u \in E. \tag{7}$$

Proof: Let $k \in H_{Q^{-1}}$. Then clearly $k \in H$. It is sufficient to consider $u = e_n \in B_0$. Since Q is self-adjoint and 1-1, also $Q^{-1/2}$ is self-adjoint. Hence

$$\langle Q^{-1}k, e_n\rangle_E = \langle Q^{-1/2}k, Q^{-1/2}e_n\rangle_E = \langle i_H(k), e_n\rangle_H = j_H^{-1} \circ i_H(k)(e_n).$$

Thus, in particular, $j_H^{-1} \circ i_H(k) \in i_{H'}(E')$ and therefore

$$\langle Q^{-1}k, e_n \rangle_E = {}_{E'}\Big\langle i_{H'}^{-1} \circ j_H^{-1} \circ i_H(k)\, ,\, e_n \Big\rangle_E = {}_{E'}\langle j_{E'}(k)\, ,\, e_n \rangle_E \, .$$

This shows in particular that $j_{E'}$ is 1-1. Moreover, $\|k\|_{H_{Q^{-1}}} = \|Q^{-1}k\|_E = \|j_{E'}(k)\|_{E'}$, hence $j_{E'}$ is an isometry. In particular, $j_{E'}(H_{Q^{-1}})$ is closed. It remains to prove that $j_{E'}(H_{Q^{-1}})$ is dense in E'. Let therefore $u \in E$ be such that ${}_{E'}\langle j_{E'}(k)\, ,\, u \rangle_E = 0$ for all $k \in H_{Q^{-1}}$. Then $0 = {}_{E'}\langle j_{E'}(k)\, ,\, u \rangle_E = \langle Q^{-1}k, u \rangle_E$, which implies $u \in D((Q^{-1})^*)$, henceforth $u \in D(Q^{-1})$ by self-adjointness. Thus $\langle k, Q^{-1}u \rangle_E = 0$ for all $k \in H_{Q^{-1}}$. Since $H_{Q^{-1}}$ is dense in E and Q is 1-1, we conclude $u = 0$. ///

From now on we neglect the trivial embedding $i_H : H_{Q^{-1}} \to H$. The domain of $j_{E'}$ (resp. the range of $(j_{E'}^{-1})$) will thus be identified with the subset $H_{Q^{-1}} \subseteq H$.
We turn to the definition of the gradient Dirichlet form on $L^2(E;\mu)$, where μ is the measure defined in (3). Since this is a special case of the general theory, the following results are well-known and the proofs are standard. We proceed as in [MR 92, Section II.3a, 3b]. Define for $K \subseteq E'$ the linear space

$$\begin{aligned} \mathcal{F}C_b^\infty(K) := \{u : E \to I\!\!R \ :\ & u = f({}_{E'}\langle l_1\, ,\, \cdot \rangle_E, \ldots, {}_{E'}\langle l_n\, ,\, \cdot \rangle_E), \\ & n \in I\!\!N,\, l_i \in K, 1 \le i \le n, f \in C_b^\infty(I\!\!R^n)\}, \end{aligned}$$

where $C_b^\infty(I\!\!R^n)$ is the set of all infinitely differentiable functions on $I\!\!R^n$ such that all partial derivatives are bounded. Set $\mathcal{F}C_b^\infty := \mathcal{F}C_b^\infty(E')$. Let now $k \in E \setminus 0$ and define for $u = f(l_1, \ldots, l_m) \in \mathcal{F}C_b^\infty$ the partial derivative in direction k by

$$\frac{\partial}{\partial k} u(z) := \frac{d}{ds} u(z + sk)\Big|_{s=0} = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(l_1(z), \ldots, l_m(z))\ {}_{E'}\langle l_i\, ,\, k \rangle_E\, ,\ z \in E.$$

Since $h \mapsto \frac{\partial u}{\partial h}(z)$, $h \in H$, is a continuous linear functional on H, we can define $\nabla u(z) \in H$ by $\langle \nabla u(z), h \rangle_H = \frac{\partial u}{\partial h}(z)$, $h \in H$. Then $(\mathcal{E}, \mathcal{F}C_b^\infty)$ defined by

$$\mathcal{E}(u,v) := \frac{1}{2}\int_E \langle \nabla u, \nabla v \rangle_H d\mu, \quad u, v \in \mathcal{F}C_b^\infty\, ,$$

is a densely defined positive definite symmetric bilinear form on $L^2(E;\mu)$. We show next that it is closable and that its closure is a quasi-regular symmetric Dirichlet form. Let therefore $T_k : E \to E$ be the translation map $T_k(z) := z + k$, $z \in E$, and let $R^\mu : D(Q^{-1}) \to L^2(E;\mu)$ be defined by $k \mapsto R_k^\mu := {}_{E'}\langle j_{E'}(k)\, ,\, \cdot \rangle_E = \langle Q^{-1}k, \cdot \rangle_E$. Finally, we fix an orthonormal basis $K_0 := \{\sqrt{\lambda_n} e_n : n \in I\!\!N\}$ of H.

Lemma 8 *Let $k \in \operatorname{span} K_0$. Then the measure μ is k-invariant, i.e., $\mu \circ T_k^{-1} \ll \mu$. The density is given by*

$$\exp\left[R_{Ak}^\mu - \frac{1}{2}\|A^{1/2}k\|_H^2\right].$$

Proof: Let $k \in \operatorname{span} K_0$. Then clearly $Ak \in D(Q^{-1})$ and $k \in D(Q^{-1/2}A^{1/2})$. Hence the density is well-defined. Now the proof is analogous to that of [MR 92, Lemma II.3.12] apart from the following modifications. Firstly, the test function $l \in E'$ can be replaced by $j_{E'}(\tilde{k})$, $\tilde{k} \in \operatorname{span} K_0$, since $\operatorname{span} K_0$ is dense in $D(Q^{-1})$ and, secondly, the parameter α in the proof of [MR 92, Lemma II.3.12] has to be replaced by $\alpha := \frac{\langle A^{-1/2}\tilde{k}, A^{1/2}k\rangle_H}{\|A^{1/2}k\|_H^2}$. Note that $A^{-1/2}\tilde{k}$ is well-defined and in H, since $\tilde{k}$ has a representation as a finite sum w.r.t. the eigenvalues of A, Q. ///

The proof of closability relies on the existence of a partial integration formula. An element $k \in E$ is called *well-μ-admissible,* if there exists $\beta_k \in L^2(E;\mu)$ such that

$$\int \frac{\partial u}{\partial k}\, d\mu = -\int u\beta_k \, d\mu \tag{8}$$

for all $u \in \mathcal{F}C_b^\infty$. Now Lemma 8 implies that each $k \in \operatorname{span} K_0$ is well-μ-admissible with μ having partial logarithmic derivative

$$\beta_k := -R^\mu_{Ak} = -\, {}_{E'}\langle j_{E'}(Ak)\,,\,\cdot\,\rangle_E\,. \tag{9}$$

This follows in the same way as [MR 92, Thm.3.11] follows by [MR 92, Lemma 3.12]. Thus we arrive at

Theorem 9 *$(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable on $L^2(E;\mu)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular symmetric Dirichlet form.*

Proof: Since K_0 is a complete orthonormal set in H consisting of well-μ-admissible elements, [MR 92, Prop.II.3.8] implies that $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable and that its closure is a symmetric Dirichlet form. Let for the moment j denote the injection $j : E' \to H$. Then, since

$$\sum_{k\in K_0} {}_{E'}\langle\, l\,,\, k\,\rangle_E^2 = \sum_{k\in K_0} \langle j(l), k\rangle_H^2 = \|j(l)\|_H^2 < \infty, \tag{10}$$

[MR 92, IV.4.b)] implies that $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. ///

By the closure $(\mathcal{E}, D(\mathcal{E}))$ of $(\mathcal{E}, \mathcal{F}C_b^\infty)$ we always mean the minimal closed extension. In the special case considered here it coincides with the maximal closed extension, which shall be denoted by $(\mathcal{E}^+, D(\mathcal{E}^+))$. To make this precise, one has to consider the concept of Markov uniqueness ([RZ 92], [RZ 94], [ARZ 93b]). Let $(L, D(L))$ denote the generator of $(\mathcal{E}, D(\mathcal{E}))$. It is immediate, that if $u = f(l_1, \dots, l_m) \in \mathcal{F}C_b^\infty(K)$ and $K_1 \subseteq K$ is an orthonormal basis of H having $l_1, \dots, l_m$ in its linear span, then $u \in D(L)$ and $Lu = \sum_{k\in K_0}(\frac{\partial}{\partial k}(\frac{\partial u}{\partial k}) + \beta_k \frac{\partial u}{\partial k})$. Note that the right hand side is a finite sum and defines a linear operator $S_{\mu,H,K}$ on $L^2(E;\mu)$ with dense domain $\mathcal{F}C_b^\infty(K)$. Then it is said that *Markov uniqueness* holds for $(S_{\mu,H,K},\ \mathcal{F}C_b^\infty\,(K))$ if $(L, D(L))$ is the only Dirichlet operator on $L^2(E;\mu)$ extending $(S_{\mu,H,K},\ \mathcal{F}C_b^\infty\,(K))$. If there is no ambiguity concerning μ, H, K, we also say that Markov uniqueness holds for $(\mathcal{E}, D(\mathcal{E}))$. Here, we always consider $K = \operatorname{span} K_0$.

Proposition 10 *Markov uniqueness holds for the gradient Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ over $(E' \subseteq H \subseteq E)$ on $L^2(E;\mu)$.*

Proof: Let k_n denote the element $\sqrt{\lambda_n}\, e_n$ of K_0 and let $\beta_n := \beta_{k_n}$ denote the partial logarithmic derivative of μ w.r.t. k_n. Then $\beta_n = -\alpha_n \ {}_{E'}\langle j_{E'}(k_n)\,,\,\cdot\,\rangle_E$. Hence

$$\sum_{n=1}^{N} E_\mu[(\beta_n - E_\mu[\beta_n|\mathcal{B}_n])^2] = 0 \tag{11}$$

and Markov uniqueness holds by [ARZ 93b, Thm.1.4]. ///

3.3 Stochastic analysis related to the weak solution

Now, by [MR 92, IV.3.5], there is a right process $I\!M = \big(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta}, \zeta\big)$ which is properly associated to the quasi-regular symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;\mu)$. We are going to show that it is the desired solution.

Proposition 11 *The process $I\!M$ is a μ-symmetric conservative diffusion. Moreover, it is stationary and reversible under the measure $P_\mu = \int_E P_x\,\mu(dx)$.*

Proof: By [MR 92, Ex.V.1.12(ii)] $I\!M$ is a diffusion. Moreover, $1 \in D(\mathcal{E})$ and $\mathcal{E}(1,1) = 0$. Then [MR 92, Prop. 2.16] implies that $1 \in D(L)$ and $L1 = 0$. Henceforth $T_t 1 = e^{tL} = 1$. Now it follows as in [Mü 94, Lemma 5] and ff. that $I\!M$ is conservative. Symmetry of T_t and $T_t 1 = 1$ imply that $I\!M$ is stationary and reversible. ///

Proposition 12 *Let $k \in \operatorname{span} B_0$. Then there is an $I\!M$-exceptional set $N^k \subseteq E$ such that*

$$\langle k, X_t\rangle_E = \langle k, x\rangle_E - \frac{1}{2}\int_0^t \langle Ak, X_s\rangle_E\, ds + W_t^k, \tag{12}$$

where $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t^k)_{t\geq 0}, P_x)$ is a one-dimensional mean-zero Gaussian process starting in zero with quadratic variation $\langle W^k\rangle_t = t\langle Qk, k\rangle_E$ for all $x \in E \setminus N^k$.

Proof: Let $k \in \operatorname{span} K_0$. By [AR 91, Thm.5.3] and (9) there is an $\mathcal{E}$-exceptional, hence, by [MR 92, Thm.IV.5.29], $I\!M$-exceptional set N^k such that

$${}_{E'}\langle j_{E'}(k)\,,\, X_t\,\rangle_E = {}_{E'}\langle j_{E'}(k)\,,\, x\,\rangle_E - \frac{1}{2}\int_0^t {}_{E'}\langle j_{E'}(Ak)\,,\, X_s\,\rangle_E\; ds + B_t^k,$$

where $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (B_t^k)_{t\geq 0}, P_x)$ is a one-dimensional mean-zero Gaussian process starting in zero with quadratic variation $\langle B^k\rangle_t = t\|k\|_H^2$ for all $x \in E \setminus N^k$. Since clearly $k \in D(Q^{-1})$, relation (7) yields

$$\langle Q^{-1}k, X_t\rangle_E = \langle Q^{-1}k, x\rangle_E - \frac{1}{2}\int_0^t \langle Q^{-1}(Ak), X_s\rangle_E\, ds + B_t^k,$$

P_x-a.s. Set $k' := Q^{-1}k$. Then $Q^{-1}AQk' = \sum_{n=0}^N \langle k', e_n\rangle_E Q^{-1}AQe_n = Ak'$ for some $N \in I\!N$ and hence

$$\langle k', X_t\rangle_E = \langle k', x\rangle_E - \frac{1}{2}\int_0^t \langle Ak', X_s\rangle_E\, ds + B_t^{Qk'} \tag{13}$$

P_x-a.s. Since Q^{-1} maps $\operatorname{span} K_0$ onto $\operatorname{span} K_0$, it follows that (13) holds for all $k' \in \operatorname{span} K_0$. Note now that $\operatorname{span} K_0 = \operatorname{span} B_0$. Thus the assertion follows, if we set $W_t^k := B_t^{Qk}, k \in \operatorname{span} B_0$. ///

For the following intermediate steps we need to modify the above constructed Wiener process. Therefore, let

$$N^W := \bigcup_{k\in B_0} N^k.$$

Clearly, N^W is $\mathcal{E}$-exceptional, hence $I\!M$-exceptional by [MR 92, Thm.IV.5.29]. Since $I\!M$ is a right process, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous by definition. Fix now $x \in E\setminus N^W$. Let $\mathcal{F}_t^{P_x}$ be the P_x-completion of $\mathcal{F}_t$. Then $\mathcal{F}_t^{P_x}$, $t \geq 0$, is right-continuous, too, by [MR 92, IV.1.1], $(X_t)_{t\geq 0}$ is adapted to $\mathcal{F}_t^{P_x}$ and

$$(\Omega, (\mathcal{F}_t^{P_x})_{t\geq 0}, \{(W_t^k)_{t\geq 0} : k \in B_0\}, P_x)$$

is the representation of a cylindrical Wiener process with covariance operator Q (some authors prefer the notation as a formal sum $\sum_{k\in B_0} W_t^k$). For short, the family $\{(W_t^k)_{t\geq 0} : k \in B_0\}$ shall be denoted by $(W_t)_{t\geq 0}$. To complete the proof of Theorem 4 we compare the mild solution to equation (2), constructed upon $(W_t)_{t\geq 0}$, with the process $I\!M$, which yields the desired regularity result $\int_0^t X_s\, ds \in D(A)$. Therefore, set $S_t := e^{-\frac{t}{2}A}$, $t \geq 0$. Then $(S_t)_{t\geq 0}$ is a strongly continuous semigroup (cf., e.g., [F 80, Lemma 1.3.2]). Moreover, S_t, $t \geq 0$, is a family of bounded operators with $\|S_t\| \leq 1$ and $(S_t)_{t\geq 0}$ is analytic (see [Ka 66, IX, Example 1.25]).

Lemma 13 *Assume (A). Then* $\operatorname{tr}(S_tQS_t) \leq C(\frac{e}{t})$ *and* $\sup_{t\geq 0} \operatorname{tr} \int_0^t S_rQS_r\, dr < \infty$.

Proof: Since $\operatorname{tr}(B_1B_2) = \operatorname{tr}(B_2B_1) \leq \|B_1\| \operatorname{tr}(B_2)$ for a bounded linear operator B_1 and a trace class operator B_2, ([Pe 88, Lemma 3.4.10]) and since $(S_t)_{t\geq 0}$ is analytic, we obtain

$$\operatorname{tr}(S_tQS_t) \leq \operatorname{tr}(QS_t) \leq \operatorname{tr}(QA^{-1})\|AS_t\| \leq \operatorname{tr}(QA^{-1})\frac{Ce}{t}$$

for some constant $C > 0$. Hence it is finite, since by (A) $\mathrm{tr}(QA^{-1}) < \infty$. Now the last assertion follows since $\sup_{t\geq 0} \mathrm{tr} \int_0^t S_r Q S_r \, dr = \sum_{n=0}^{\infty} \lambda_n \int_0^t e^{-\alpha_n r} \, dr = \mathrm{tr}(QA^{-1})$. ///

The fact that $(S_t)_{t\geq 0}$ is strongly continuous and Lemma 13 imply that the Hilbert space valued stochastic convolution

$$W_A(t) := \int_0^t S_{t-s} \, dW_s, \; t \geq 0, \tag{14}$$

is well defined (cf. [dPZ 92, p.91, 94]; therefore we had to enlarge the filtration).

Proposition 14 *Assume (A) and let $x \in E \setminus N^W$. Then the process $(\Omega, (\mathcal{F}_t^{P_x})_{t\geq 0}, (X_t - S_t x)_{t\geq 0}, P_x)$ is a continuous modification of $(W_A(t))_{t\geq 0}$, i.e.,*

$$P_x[\, \|W_A(t) - (X_t - S_t x)\|_E = 0\,] = 1, \; t \geq 0.$$

Proof: Let $x \in E \setminus N^W$ and $k \in D(A)$. Then it follows by Lemma 13 and [dPZ 92, Thm.5.4] that, given $(\Omega, (\mathcal{F}_t^{P_x})_{t\geq 0}, (W_t)_{t\geq 0}, P_x)$, there is exactly one predictable process $(\Omega, (\mathcal{F}_t^{P_x})_{t\geq 0}, (Y_t)_{t\geq 0}, P_x)$, satisfying

$$\langle k, Y_t \rangle_E = \int_0^t \langle (-\tfrac{1}{2}A)k, Y_s \rangle_E \, ds + W_t^k, \; k \in D(A), \tag{15}$$

which is given by the predictable version of $(\Omega, (\mathcal{F}_t^{P_x})_{t\geq 0}, (W_A(t))_{t\geq 0}, P_x)$. Let now $k \in \operatorname{span} B_0$. Then $k \in D(A)$ and selfadjointness implies

$$\langle k, x \rangle_E - \langle k, S_t x \rangle_E = \langle k - S_t k, x \rangle_E = \langle \int_0^t (\tfrac{1}{2}A) S_r k \, dr, x \rangle_E = \int_0^t \langle (\tfrac{1}{2}A)k, S_r x \rangle_E \, dr.$$

Hence (12) implies that $(X_t - S_t x)_{t\geq 0}$ satisfies relation (15) P_x-a.s., too. Clearly, $(\Omega, (\mathcal{F}_t^{P_x})_{t\geq 0}, (X_t - S_t x)_{t\geq 0}, P_x)$ is adapted and has a continuous version, since $(X_t)_{t\geq 0}$ is continuous P_x-a.s. and since $(S_t)_{t\geq 0}$ is strongly continuous. Therefore $(\Omega, (\mathcal{F}_t^{P_x})_{t\geq 0}, (X_t - S_t x)_{t\geq 0}, P_x)$ has a predictable modification (cf., e.g., [dPZ 92, Prop.3.6(ii)]). Now uniqueness and the fact that $\operatorname{span} B_0$ is dense in E imply the assertion. ///

Proposition 15 *Let $x \in E \setminus N^W$ and $T > 0$. If (A) and (5) do hold, then P_x-a.s.*

$$\int_0^T W_A(t) \, dt \in D(A) \quad \text{and} \quad \int_0^T X_s \, ds \in D(A).$$

Proof: Let $0 \leq t \leq T$ and fix $x \in E \setminus N^W$. Set $\Phi(t) := \int_0^{T-t} S_r \, dr$. Then the conditions of [dPZ 92, Prop.4.15] are satisfied by (A) and (5), which implies $\int_0^T \Phi(s) \, dW_s \in D(A)$ P_x-a.s. Furthermore, (A) and the stochastic Fubini [dPZ 92, Thm.4.18, Rem.4.20] yield

$\int_0^T \Phi(s)\,dW_s = \int_0^T W_A(t)\,dt \in D(A)$ P_x-a.s. Clearly, the Bochner-integral $\int_0^t S_r x\,dr$ exists and $\int_0^t S_r x\,dr \in D(A)$ for all $x \in E$. By linearity and Propositon 14 we obtain therefore

$$\int_0^T X_s\,ds = \int_0^T W_A(t)\,dt + \int_0^T S_t x\,dt \in D(A)$$

P_x-a.s. ///

Proof: (of *Theorem 4*). The first part is already covered by Propositon 11 and 12. Assume now in addition that (5) holds and fix $x \in E \setminus N^W$. By Propositon 12 and Propositon 14 it follows that the Bochner integral $\int_0^t X_s\,ds$ exists and is in $D(A)$ P_x-a.s. and that for all $k \in \operatorname{span} B_0$

$$\langle k, X_t\rangle_E = \langle k, x\rangle_E - \langle k, \frac{1}{2}A\int_0^t X_s\,ds\rangle_E + W_t^k$$

P_x-a.s. Furthermore, [AR 91, Thm.6.2] implies that there is a map $W : \Omega \to C([0,\infty); E)$ such that $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0}, P_x)$ is a Q-Wiener process for all $x \in E \setminus \tilde{N}$ for some $\mathcal{E}$-exceptional, hence $I\!M$-exceptional set $\tilde{N}$ and such that $\langle k, W_t\rangle_E = W_t^k$ P_x-a.s. for all $x \in E \setminus \tilde{N^k}$ for some $I\!M$-exceptional set $\tilde{N^k}$. Hence (6) follows. We turn to the question of uniqueness. By Proposition 10 we know that Markov uniqueness holds for the Dirichlet form to which $I\!M$ is properly associated. By [ARZ 93b, Thm.1.9] this implies that $I\!M$ is the unique μ-symmetric right process with state space E which solves the martingale problem [ARZ 93b, Def.1.8] for $(S_{\mu,H,K},\ \mathcal{F}C_b^\infty\,(\operatorname{span} K_0))$, where $S_{\mu,H,K}u = \sum_{k\in K_0}(\frac{\partial}{\partial k}(\frac{\partial u}{\partial k}) + \beta_k\,\frac{\partial u}{\partial k})$, $u \in \mathcal{F}C_b^\infty(\operatorname{span} K_0)$. Since uniqueness in the martingale problem is uniqueness of the finite dimensional distributions, it follows in particular that $I\!M$ is the unique μ-symmetric Hunt process which satisfies (4) in Theorem 4. ///

3.4 Ergodicity of solutions

Here we prove Theorem 6, which relies on a criterion of Bogachev and Röckner, [BR 95, Thm.6.15].

Lemma 16 *Assume (A). Then* $\lim_{t\to\infty} S_t x = 0$ *for any* $x \in E$.

Proof: Let $x \in E$ and let $x_n := \langle x, e_n\rangle_E$. Then $\sum_{n=1}^\infty |x_n|^2 < \infty$ and $\|S_t x\|_E^2 = \sum_{n=1}^\infty |x_n|^2 e^{-t\alpha_n}$. Let now $\varepsilon > 0$ be given and set $\alpha_N^* := \min\{\alpha_n : 1 \leq n \leq N\} > 0$. Clearly, $\alpha_N^* > 0$ for all $N \in I\!N$. Then there exists $N = N(x,\varepsilon) \in I\!N$ such that $\sum_{n=N}^\infty |x_n|^2 < \varepsilon/2$ and there exists some $t = t(x,\varepsilon,N) \in I\!R$ such that

$$\sum_{n=1}^N |x_n|^2 e^{-t\alpha_n} \leq e^{-t\alpha_N^*}\sum_{n=1}^N |x_n|^2 < \varepsilon/2.$$

Hence $\|\lim_{t\to\infty} S_t x\|_E = 0$ and therefore $\lim_{t\to\infty} S_t x = 0$. ///

The proof of Theorem 6 requires a change of notation since we are going to consider a change of measure. Let $K = \operatorname{span} K_0$. As we know by (9), each $k \in K$ is well-μ-admissible with partial logarithmic derivative $\beta_k = -\,{}_{E'}\langle j_{E'}(Ak)\,,\,\cdot\,\rangle_E$. Set now $\beta_k^\mu := \beta_k,\ k \in K$. If ν is a second measure on $(E, \mathcal{B}(E))$, such that $k \in K$ is well-ν-admissible, then the corresponding partial logarithmic derivative shall be denoted by β_k^ν. Define $\mathcal{G}_{K,ac}^{\beta^\mu}$ to be the set of all probability measures ν on $(E, \mathcal{B}(E))$ such that ν is absolutely continuous w.r.t. μ and such that for every $k \in K$ the partial logarithmic derivative β_k^ν of ν along k exists and $\beta_k^\nu = \beta_k^\mu$ ν-a.e.

Proof: (of Theorem 6). Each $\nu \in \mathcal{G}_{K,ac}^{\beta^\mu}$ gives rise to a quasi-regular symmetric gradient Dirichlet form $(\mathcal{E}^\nu, D(\mathcal{E}^\nu))$ over $(E' \subseteq H \subseteq E)$ on $L^2(E;\nu)$. The properly associated right process $\mathbb{M}^\nu$ is again a weak solution to (2), since $\beta_k^\nu = \beta_k^\mu = -\,{}_{E'}\langle j_{E'}(Ak)\,,\,\cdot\,\rangle_E$ ν-a.s., compare the proof of Proposition 12. Moreover, it is a modification of the (unique) mild solution $(W_A(t) + S_t x)$, for the initial condition $X_0 = x$. Now [dPZ 92, Thm.11.11] and Corollary 16 imply that the invariant measure μ for the mild solution to (2) is unique. Thus we conclude that $\nu = \mu$ a.s. Hence $\mathcal{G}_{K,ac}^{\beta^\mu} = \{\mu\}$ and [BR 95, Thm.6.15] combined with Markov uniqueness implies that $(\mathcal{E}, D(\mathcal{E}))$ is irreducible. Hence P_μ is ergodic by [AKR 95, Thm.5.15] ///

4 Symmetric Solutions to Semilinear Equations

4.1 Introduction and main results

Assume from now on that E is a separable L^2-space of real valued functions, more precisely, let $(\Lambda, \mathcal{G})$ be a measurable space and let ρ be a finite positive measure on $(\Lambda, \mathcal{G})$. Assume that $L^2(\Lambda;\rho)$ is separable (which is, e.g., the case if $(\Lambda, \mathcal{G})$ is standard Borel, i.e., as a measurable space isomorphic to a Polish space) and set

$$E := L^2(\Lambda;\rho),\ \langle\cdot,\cdot\rangle_E := \langle\cdot,\cdot\rangle_{L^2(\Lambda;\rho)}.$$

Furthermore, let $(A, D(A))$ and Q as in Chapter 3. Assume, they satisfy **(A)** and

(B) The eigenvectors $e_n, n \in \mathbb{N}$, of A, Q are in $L^\infty(\Lambda;\rho)$.

Observe that any continuous map $F : \mathbb{R} \to \mathbb{R}$ induces a measurable map $F^* : E \to \mathbb{R}^\Lambda = \{y \mid y : \Lambda \to \mathbb{R}\}$ as follows. For $z \in E$ pick a ρ-version, again denoted by z, and define $F^*(z)$ by $F^*(z)(t) := F(z(t)) \in \mathbb{R},\ t \in \Lambda$. Fix now $F \in C^1(\mathbb{R})$ with derivative F' and assume that

F satisfies

$$\textbf{(F1)} \qquad \exists c \in I\!R : \inf\{F(\sigma) : \sigma \in I\!R\} \geq c > -\infty$$

$$\textbf{(F2)} \qquad \exists \text{ a polynomial q such that } |F(\sigma)| + |F'(\sigma)| \leq q(\sigma),\ \sigma \in I\!R.$$

Example 17 *Let q be a polynomial of even degree. Then $F_0(\xi) := q(\xi)$, $F_1(\xi) := q^4(\xi) + q(\xi) \cdot \sin^2(\xi)$ and $F_2(\xi) := q(\xi) \cdot \sin^2(q(\xi))$, are lower bounded. Moreover, since*

$$\begin{aligned} F_1' &= 4q^3q' + q' \cdot \sin^2 + q \cdot \sin\cos, \\ F_2' &= q' \cdot \sin^2(q) + 2qq' \cdot \sin(q)\cos(q), \end{aligned}$$

there are constants c, c' such that $|F_j(\xi)| + |F_j'(\xi)| \leq c\,\xi^{8\,deg(q)} + c'$, $j = 0, 1, 2$, i.e., (F2) is satisfied for all these functions. Note that $(F_0^*)'$ is not locally Lipschitz continuous on $L^2(E;\mu)$ and that F_2' does not satisfy a one-sided linear growth condition; it is strongly oszillating in the sense that $\limsup_{|\xi|\to\infty} F_2'(\xi) = +\infty$ and $\liminf_{|\xi|\to\infty} F_2'(\xi) = -\infty$.

Finally, we introduce the following assumption (P) on a polynomial p:

$$\textbf{(P)} \qquad \int_E \left(\int_\Lambda p^2(z(t))\, \rho(dt) \right)^2 \mu(dz) < \infty.$$

In the section on applications we give an example, where (P) holds for all polynomials. Let us now present our main results. Define $\varphi : E \to I\!R$ by

$$\varphi(z) := \exp\left[- \int_\Lambda F(z(t))\, \rho(dt) \right]. \tag{16}$$

Note that φ is bounded by (F1) and since $\rho(\Lambda) < \infty$ and that $\varphi > 0$, if (P) holds for the polynomial q in (F2). Define $\deg(F)$ to be the degree of the polynomial q in (F2).

Theorem 18 *(i) Assume that (A), (B) hold and assume (P) holds for any polynomial p with $deg(p) \leq 2deg(F)$. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0})$ as in Theorem 4. Then there is a conservative $\varphi^2\mu$-symmetric diffusion $I\!M^\varphi = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x^\varphi)_{x\in E}, \zeta)$ with state space E, there is an $I\!M^\varphi$-exceptional set $N \subseteq E$ such that the Bochner integrals $\int_0^t X_s\, ds$ and $\int_0^t (F')^*(X_s)\, ds$ exist P_x^φ-a.s. for all $x \in E \setminus N$, and for each $k \in \operatorname{span} B_0$ there is an $I\!M^\varphi$-exceptional set $N_k^\varphi \subseteq E$, $N_k^\varphi \supseteq N$, such that*

$$\langle k, X_t\rangle_E = \langle k, x\rangle_E - \langle \tfrac{1}{2}Ak, \int_0^t X_s\, ds\rangle_E - \langle k, Q\int_0^t (F')^*(X_s)\, ds\rangle_E + W_t^k \tag{17}$$

P_x^φ-a.s. for all $x \in E \setminus N_k^\varphi$, where $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t^k)_{t\geq 0}, P_x^\varphi)$ is a one-dimensional mean-zero Gaussian process starting in zero with quadratic variation $\langle W^k\rangle_t = t\langle Qk, k\rangle_E$. Moreover, $I\!M^\varphi$ is the unique weak solution to the stochastic differential equation (1) in the class of all $\varphi^2\mu$-symmetric right processes.

(ii) Each $\mathbb{M}$-exceptional set is $\mathbb{M}^\varphi$-exceptional and vice versa and $\mathbb{M}^\varphi$ is the Girsanov transform of $\mathbb{M}$, i.e., $P_x|_{\mathcal{F}_t} \sim P_x^\varphi|_{\mathcal{F}_t}$, $t \geq 0$, for all $x \in E \setminus \tilde{N}$.
(iii) If, in addition, (5) holds, then there is a map $W : \Omega \to C([0,\infty); E)$ such that $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0}, P_x^\varphi)$ is a Q-Wiener process and there is an $\mathbb{M}^\varphi$-exceptional set $N^\varphi \subseteq E$, $N^\varphi \supseteq N$, such that $\int_0^t X_s \, ds \in D(A)$ and

$$X_t = x - \tfrac{1}{2} A \int_0^t X_s \, ds - Q \int_0^t (F')^*(X_s) \, ds + W_t, \quad t \geq 0, \tag{18}$$

P_x^φ-a.s. for all $x \in E \setminus N^\varphi$.

Remark 19 *It is interesting to note that we obtain examples for diffusions with an unbounded non-linearity, such that none of the known criteria for the existence of an invariant measures is applicable, but which do have an invariant measure. For example, in [Shi 87] the drift has to be bounded, in [vV 89] it satisfies a kind of Novikov condition. In [Mas 89a], [dPZ 92], [dPGaZ 92], [Ch-MGo 94] and [GaGo 95a], it is, among other conditions assumed that the semigroup generated by $(-A/2)$ is compact. Our results apply even if this is not the case, cf. Example 3(iii). In [LRi 94], among other conditions, the (strong) Feller property is needed. But, there are linear examples, to which our results apply and which cannot have the strong Feller property; cf. [Mas 89a, Rem.2.12].*

Theorem 20 *Consider the situation of Theorem 18. Then the stationary law $P^\varphi_{\varphi^2\mu} := \int P_x^\varphi \varphi^2(x) \, \mu(dx)$ of the solution $\mathbb{M}^\varphi$ in Theorem 18 is (time-shift) ergodic.*

Remark 21 *By Theorem 20, the occupation density of $\mathbb{M}^\varphi$ converges towards the invariant measure $\varphi^2\mu$. The rate function, describing the large deviations from ergodic behaviour, is given by the Dirichlet form associated to $\mathbb{M}^\varphi$. We note that although $\mathbb{M}^\varphi$ is ergodic only if started with initial condition $\varphi^2\mu$, the author obtained "quasi-uniform" bounds (i.e. uniform bounds that hold for all starting points x outside of some capacity zero set) in [Mü 94], which apply to $\mathbb{M}^\varphi$. (See also [Mü 95] for a result in the case when the semigroup of $\mathbb{M}^\varphi$ is hypercontractive).*

4.2 Construction of the solution

We proceed as follows. First, we show that the gradient Dirichlet form $(\mathcal{E}^\varphi, D(\mathcal{E}^\varphi))$ over $(E' \subseteq H \subseteq E)$ on $L^2(E; \varphi^2\mu)$ exists and is quasi-regular. Then we show that the diffusion which is properly associated to $(\mathcal{E}^\varphi, D(\mathcal{E}^\varphi))$ is the solution we are looking for. So far, we have not assumed $(F')^* \in E$. Let us therefore begin with the following observation.

Lemma 22 *Let q as in (F2) and assume that (P) holds for q^2. Then*

$$(F')^*(z) \in E \;\; \text{for } \mu\text{-a.e. } z \text{ and } \; \|(F')^*\|_E^2 \in L^2(E; \mu). \tag{19}$$

Proof: Since (P) holds for q^2, there is a μ-zero set N such that $\int_\Lambda q(z(t))^2\, \rho(dt) < \infty$ for all $z \in E \setminus N$. Fix now $z \in E \setminus N$. Then, by (F2), $\int_\Lambda (q(z(t))^2\, \rho(dt) \geq \int_\Lambda (F')^2(z(t))\, \rho(dt)$ implies the assertion. ///

Lemma 23 *Consider the assumptions of Theorem 18 and let $k \in \operatorname{span} K_0$. Then*

$$\frac{\partial}{\partial k}(\ln \varphi)(z) = -\langle k, (F')^* \circ z\rangle_E \quad \mu - a.e. z \in E.$$

Proof: Let $k \in \operatorname{span} K_0$. Recall that φ^2 is bounded and note that it is μ-a.s. strictly positive under the above assumptions. Note furthermore that Cauchy-Schwarz and Lemma 22 imply $z \mapsto \langle k, (F')^*(z)\rangle_E \in L^2(E;\mu)$. Elementary calculations yield that there is a polynomial $\tilde{q}$, $\deg(\tilde{q}) = \deg(q)$, such that for ρ-a.e. $t \in \Lambda$ and all $\sigma \in [-1,1]$ (the ρ-zero set depends on k) $q(z(t) + \sigma k(t)) \leq C(k) \cdot \tilde{q}(z(t))$, where $C(k)$ is a constant depending on $\|k\|_{L^\infty(\Lambda,\rho)}$ and $\deg(q)$. W.l.o.g. we assume that also

$$|k(t)| q(z(t)) \leq C(k) \cdot \tilde{q}(z(t)) \;\; \rho - \text{a.e. } t \in \Lambda. \tag{20}$$

Since (P) holds for $\tilde{q}$, Cauchy-Schwarz implies $\|\tilde{q}(\cdot)\|_{L^1(\Lambda;\rho)} \in L^2(E;\mu)$. Hence there is an μ-zero set N such that for all $z \in E \setminus N$ and for each $|\sigma| \leq 1$

$$\|q \circ (z + \sigma k)\|_{L^1(\Lambda;\rho)} \leq C(k) \cdot \|\tilde{q} \circ z\|_{L^1(\Lambda;\rho)} < \infty. \tag{21}$$

Furthermore, differentiability of F implies that

$$\lim_{\sigma \to 0}\left(\frac{F^*(z + \sigma k)(t) - F^*(z)(t)}{\sigma}\right) = \lim_{\sigma \to 0}\left(\frac{F((z + \sigma k)(t)) - F(z(t))}{\sigma}\right) = k(t)\, F'(z(t)$$

for any $z \in E$, $t \in \Lambda$. Now the mean-value theorem implies that for each $0 < |\sigma| \leq 1$ there is a $\theta \in [-1,1]$ such that

$$\left(\frac{F^*(z + \sigma k)(t) - F^*(z)(t)}{\sigma}\right) \leq (F')^*(z + \theta\sigma k)(t) \leq C(k) \cdot \tilde{q} \circ z(t) \;\; \rho\text{-a.e. } t, \tag{22}$$

the latter by (F2) and (21). Moreover, (F2) and (21) imply that for all $z \in E \setminus N$ and $|\sigma| \leq 1$

$$\int_\Gamma |F_\Lambda^*(z + \sigma k)(t)|\, \rho(dt) \leq C(k) \cdot \|\tilde{q} \circ z\|_{L^1(\Lambda;\rho)}. \tag{23}$$

Finally, (20) and (F2) imply also $\langle k, (F')^*(z)\rangle_E \leq C(k) \cdot \|\tilde{q} \circ z\|_{L^1(\Lambda;\rho)}$, $z \in E \setminus N$. By the latter, (22) and (23), Lebesgue's theorem on dominated convergence implies that for all $z \in E \setminus N$

$$\frac{d}{d\sigma}\left(-\int_\Gamma F^*(z + \sigma k)(t)\, \rho(dt)\right)\Big|_{\sigma=0} = -\int_\Lambda \frac{d}{d\sigma}\, F^*(z + \sigma k)(t)|_{\sigma=0}\; \rho(dt) = -\langle k, (F')^*(z)\rangle_E.$$

By [AR 90, Prop.3.2] the term on the left is $\frac{\partial}{\partial k}(\ln \varphi)$. Thus $\frac{\partial}{\partial k}(\ln \varphi)$ is well-defined and the assertion is shown. ///

Proposition 24 *Consider the assumptions of Theorem 18. Let φ be defined as in (16). Then $\frac{\|\nabla\varphi\|_H}{\varphi} \in L^2(E;\mu)$ and $\varphi \in D(\mathcal{E})$.*

Proof: By Lemma 23 there is a μ-zero set $N_1(k) \subseteq E$ such that

$$(\frac{\partial}{\partial k}\varphi)\cdot\varphi^{-1}(z) = \frac{\partial}{\partial k}(\ln\varphi)(z) = -\langle k, (F')^*(z)\rangle_E$$

for all $z \in E \setminus N_1(k)$. Let now N_2 be the μ-zero set in Proposition 22 and set $N := N_2 \cup \bigcup_{k\in K_0} N_1(k)$. Fix $z \in E \setminus N$. Then Parseval's identity yields

$$\begin{aligned}
\|\nabla\varphi\|_H^2\,\varphi^{-2}(z) &= \sum_{k\in K_0} \langle k, \nabla\varphi\rangle_H^2\,\varphi^{-2}(z)\\
\text{([AR 91, Rem.3.2(ii)])} &= \sum_{k\in K_0} (\frac{\partial}{\partial k}\varphi)^2\,\varphi^{-2}(z)\\
\text{(Lemma 23)} &= \sum_{k\in K_0} \langle k, (F')^*(z)\rangle_E^2.
\end{aligned}$$

Hence $\|\nabla\varphi\|_H^2\,\varphi^{-2}(z) = \sum_{k\in K_0}\langle k, Q^{-1}Q(F')^*(z)\rangle_E^2 = \sum_{k\in K_0}\langle k, Q(F')^*(z)\rangle_H^2 = \|Q\,(F')^*(z)\|_H^2 = \|Q^{1/2}\,(F')^*(z)\|_E^2$. Boundedness of Q, Jensen's inequality (on $L^2(E;\mu)$) and Lemma 22 imply

$$\int \|\nabla\varphi\|_H^2\,\varphi^{-2}\,d\mu = \int \|Q^{1/2}\,(F')^*(z)\|_E^2\,\mu(dz) \le const\,(\int \|(F')^*(z)\|_E^2\,\mu(dz))^{1/2} < \infty,$$

which shows $\frac{\|\varphi\|_H}{\varphi} \in L^2(E;\mu)$. Since φ is bounded, this implies furthermore $\int\langle\nabla\varphi,\nabla\varphi\rangle_H\,d\mu < \infty$, hence $\varphi \in D(\mathcal{E}^+) = D(\mathcal{E})$ (the latter by Markov uniqueness, cf. Prop.10). ///

Theorem 25 *Consider the assumptions of Theorem 18. Then the form $(\mathcal{E}^\varphi,\ \mathcal{F}C_b^\infty)$ defined by*

$$\mathcal{E}^\varphi(u,v) := \frac{1}{2}\int_E \langle\nabla u, \nabla v\rangle_H\,\varphi^2\,d\mu,\ u,v \in\ \mathcal{F}C_b^\infty\ ,$$

is closable on $L^2(E;\varphi^2\mu)$ and its closure $(\mathcal{E}^\varphi, D(\mathcal{E}^\varphi))$ is a quasi-regular symmetric Dirichlet form. In particular, the partial logarithmic derivative of $\varphi^2\mu$ in direction $k \in \operatorname{span} K_0$ is given by

$$\beta_k^\varphi := \beta_k - 2\langle k, F'\circ z\rangle_E. \tag{24}$$

Proof: Lemma 23 implies that all $k \in \operatorname{span} K_0$ are well-$(\varphi^2\mu)$-admissible. Now the assertion follows by Proposition 24 and [MR 92, Section II.3d)]. ///

Corollary 26 *Markov uniqueness holds for the gradient Dirichlet form $(\mathcal{E}^\varphi, D(\mathcal{E}^\varphi))$ over $(E' \subseteq H \subseteq E)$ on $L^2(E;\varphi^2\mu)$.*

Proof: By Proposition 10 $D(\mathcal{E}) = D(\mathcal{E}^+)$ and $\varphi \in D(\mathcal{E})$ by Proposition 24. Since φ is bounded, (partint3) implies $\varphi\beta_k^\varphi \in L^2(E;\mu)$. Now (11) and [ARZ 93b, Cor.1.6] imply the assertion. ///

Proof: (proof of Theorem 18). By [MR 92, Thm.IV.3.5] there is a right process $\mathbb{M}^\varphi$, which is properly associated to $(\mathcal{E}^\varphi, D(\mathcal{E}^\varphi))$. W.l.o.g. we assume that we are in the canonical situation (cf. the Preliminaries). We conclude as in Section 3.3 that $\mathbb{M}^\varphi$ is in fact a symmetric conservative diffusion with invariant measure $\varphi^2\mu$. Let $(L^\varphi, D(L^\varphi))$ be the generator associated to $(\mathcal{E}^\varphi, D(\mathcal{E}^\varphi))$ on $L^2(E;\varphi^2\mu)$ and let $k \in \operatorname{span} K_0$. Then [AR 91, Lemma 5.1] implies that $u_k := {}_{E'}\langle j_{E'}(k)\,,\,\cdot\,\rangle_E \in D(L^\varphi)$ and $L^\varphi u_k(z) = \frac{1}{2}\beta_k^\varphi(z)$, hence

$$L^\varphi u_k(z) = -\frac{1}{2}\,{}_{E'}\langle j_{E'}(Ak)\,,\,z\,\rangle_E - \langle k, (F')^*(X_s)\rangle_E$$

by Theorem 25. Then [AR 91, Theorem 5.7] implies that there is an $\mathcal{E}$-exceptional set N_k such that P_x^φ-a.s. for all $x \in E \setminus N_k$

$$(25)\ B_t^k := {}_{E'}\langle k\,,\,X_t\,\rangle_E - {}_{E'}\langle k\,,\,x\,\rangle_E - \frac{1}{2}\int_0^t {}_{E'}\langle j_{E'}(Ak)\,,\,X_s\,\rangle_E\,ds - \int_0^t \langle k, (F')^*_\Lambda(X_s)\rangle_E\,ds$$

is a one-dimensional mean-zero Gaussian process starting in zero with quadratic variation $\langle B^k\rangle_t = t\|k\|_H^2 = t\langle Qk,k\rangle_E$. Let us show next that the Bochner integrals exists. Observe that by stationarity

$$\begin{aligned}\int_\Omega\int_0^t \|(F')^*(X_s(\omega))\|_E\,ds\,P^\varphi_{\varphi^2\mu}(d\omega) &\le C\int_0^t\int_\Omega \|q\circ X_s(\omega)\|_E\,P^\varphi_{\varphi^2\mu}(d\omega)\,ds\\ &= C\,t\int_E \|q\circ z\|_E\,\varphi^2(z)\,\mu(dz),\end{aligned}$$

which by (P) is finite for all $t>0$. Hence $\int_0^t \|(F')^*(X_s(\omega))\|_E\,ds < \infty$ $P^\varphi_{\varphi^2\mu}$-a.s. for all $t>0$. Fix now $T>0$ and let $t>0$. Set

$$\Omega_s := \{\omega\in\Omega : \int_0^s \|(F')^*(X_s(\omega))\|_E\,ds < \infty\}.$$

Clearly, $P^\varphi_{\varphi^2\mu}[I_{\Omega_T} = I_{\Omega_T}\circ\theta_t] \le P^\varphi_{\varphi^2\mu}[\Omega_{T+t}] = 1$, i.e., I_{Ω_T} is shift-invariant. Hence [Mü 94, Lemma 11] implies that there is an $\mathcal{E}$-exceptional set N_T such that $P_x^\varphi[\Omega_T] = 1$ for all $x\in E\setminus N_T$. Thus $P_x^\varphi[\int_0^s \|(F')^*(X_s)\|_E\,ds < \infty] = 1$ for all $s\in[0,T]$ and all $x\in E\setminus N_T$. Thus the Bochner integral $\int_0^s (F')^*(X_s(\omega))\,ds$ exists P_x^φ-a.s. Since Q is bounded, this implies $Q\int_0^s (F')^*(X_s(\omega))\,ds = \int_0^s Q(F')^*(X_s(\omega))\,ds$. The same argument shows that $\int_0^t X_s\,ds$ exists, since we may assume that $q(X_s)$ also dominates $|X_s|$. Setting $W_t^k = B_t^{Qk}$ in (25) yields (17). Ad (ii). By Proposition 10 Markov uniqueness holds for the gradient Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ over $(E'\subseteq H\subseteq E)$ on $L^2(E;\mu)$. Therefore the hypotheses of [ARZ 93a, Thm.1.3] are satisfied by assumption and Proposition 24 and we conclude that any $\mathcal{E}$-exceptional set is

$\mathcal{E}^{\varphi}$-exceptional and vice versa and that $P_x|_{\mathcal{F}_t} \sim P_x^{\varphi}|_{\mathcal{F}_t}$, $t \geq 0$, for any x outside of some $\mathcal{E}^{\varphi}$-exceptional set. Hence (ii) is clear, since any $\mathcal{E}^{(\varphi)}$-exceptional set is $\mathbb{M}^{(\varphi)}$-exceptional and vice versa.

Ad (iii). Assume now, in addition, that (5) holds. By assumption and Theorem 4 we already know that the Bochner integral $\int_0^t X_s\,ds$ exists and is in $D(A)$ P_x-a.s. for all $x \in E \setminus N^W$, $t \geq 0$. Hence absolute continuity and the equivalence of exceptional sets imply $\int_0^t X_s\,ds \in D(A)$ P_x^{φ}-a.s. for all $x \in E \setminus N^W$. Let now $N := N^W \cup N_T \cup \bigcup_{k \in K_0} N_k$. Then (25) implies (compare the proof of Proposition 12)

$$\begin{aligned}\langle k, X_t\rangle_E &= \langle k, x\rangle_E - \frac{1}{2}\langle k, A\int_0^t X_s\,ds\rangle_E - \int_0^t \langle k, Q(F')^*_{\Lambda}(X_s)\rangle_E\,ds + W_t^k \\ &= \langle k, x\rangle_E - \frac{1}{2}\langle k, A\int_0^t X_s\,ds\rangle_E - \langle k, Q\int_0^t (F')^*_{\Lambda}(X_s)\,ds\rangle_E + W_t^k\end{aligned}$$

P_x^{φ}-a.s. for any $x \in E \setminus N$, $k \in \operatorname{span} K_0$, $t \in [0,T]$. Then, by the choice of W_t^k, [AR 91, Thm.6.2] implies that there is a map $W : \Omega \to C([0,\infty); E)$ such that $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0}, P_x^{\varphi})$ is a Q-Wiener process for all $x \in E \setminus \tilde{N}$ for some $\mathcal{E}^{\varphi}$-exceptional, hence $\mathbb{M}^{\varphi}$-exceptional set $\tilde{N}$ and such that $\langle k, W_t\rangle_E = W_t^k$ P_x-a.s. for all $x \in E \setminus \tilde{N}^k$ for some $\mathbb{M}^{\varphi}$-exceptional set $\tilde{N}^k$. Thus the previous implies

$$X_t = x - \frac{1}{2}A\int_0^t X_s\,ds - Q\int_0^t (F')^*_{\Lambda}(X_s)\,ds + W_t,\ t \in [0,T],$$

P_x^{φ}-a.s. for all $x \in E \setminus N \cup \tilde{N} \cup \bigcup_{k \in K_0} \tilde{N}_k$. Since $t \mapsto X_t$ is continuous, the dependence of the $\mathbb{M}$-exceptional set N from T can be removed. Thus (18) follows. The uniqueness statement is implied by Markov uniqueness in the same way as in the proof of Theorem 4. ///

4.3 Ergodicity of solutions

Proof: (of Theorem 20). By Theorem 6 the stationary law P_μ of the process $\mathbb{M}$ which is properly associated to the gradient Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;\mu)$ is ergodic. Hence $(\mathcal{E}, D(\mathcal{E}))$ is irreducible by [AKR 95, Thm.5.15]. By Theorem 18 any $\mathcal{E}$-nest is an $\mathcal{E}^{\varphi}$-nest and vice versa. Hence [F 82, Corollary p.203] (or, more precisely, its generalization by virtue of the regularization method presented in [MR 92, Chapter VI]) implies that the new Dirichlet form $(\mathcal{E}^{\varphi}, D(\mathcal{E}^{\varphi}))$ is irreducible, too. Henceforth $P^{\varphi}_{\varphi^2\mu}$ is ergodic by [AKR 95, Thm.5.15]. ///

5 Applications

Consider now an open cube $\Lambda := \prod_{i=1}^d (a_i, b_i)$ in $I\!R^d$ and set $E := L^2(\Lambda; d\xi)$, where $d\xi$ denotes d-dimensional Lebesgue measure. Let $\Delta = \sum_{i=1}^d \frac{d^2}{d\xi_i^2}$ denote the d-dimensional Laplacian and let $(\Delta_\Lambda, D(\Delta_\Lambda))$ denote the Friedrich's extension of Δ on $C_0^\infty(\Lambda)$. Let $m \geq 1$ and set

$$A \;:=\; (m - \Delta_\Lambda), \quad D(A) := D(\Delta_\Lambda).$$

It is well known that $(A, D(A))$ is self-adjoint and has discrete spectrum. As before, let $F \in C^1(I\!R)$ and assume that F satisfies (F1), (F2). Finally, set

$$\varphi(z) := \exp\left[-\int_\Lambda F(z(\xi))\, d\xi\right].$$

Theorem 27 *Let $d \geq 1$, let $\gamma_A \geq 0$ and let $\gamma > d/2 - \gamma_A$. Let μ be the mean-zero Gaussian measure defined by the characteristic function*

$$\hat{\mu}(y) := \int_E e^{i\langle x,y\rangle_E}\, \mu(dx) = \exp[-\frac{1}{2}\|(m - \Delta_\Lambda)^{-\frac{\gamma+\gamma_A}{2}} y\|_E^2]. \tag{26}$$

Then there is a conservative $\varphi^2\mu$-symmetric diffusion $I\!M$ with state space E solving the stochastic (partial) differential equation

$$\begin{aligned} dX_t &= -\frac{1}{2}(m - \Delta_\Lambda)^{\gamma_A} X_t\, dt + (m - \Delta_\Lambda)^{-\gamma}(F')^*(X_t)dt + dW_t, \\ X_0 &= x \in E, \end{aligned}$$

in the sense of Theorem 18, where $(W_t)_{t\geq 0}$ is a cylindrical Brownian motion with covariance $Q = (m - \Delta_\Lambda)^{-\gamma}$. In particular, Theorem 18(ii) applies, if $\gamma > d/2$.

Note, in particular, that in the special case $d = 1$, there is a solution for the choice $\gamma_A = 1$ and $\gamma = 0$, i.e., for $A = (m - \Delta_\Lambda)$ and $Q = Id$. For the proof of Theorem 27 we use the following well-known facts. Let $(T_t)_{t\geq 0}$ denote the $L^2(\Lambda; d\xi)$-semigroup generated by $(m - \Delta_\Lambda)$. Then there are $\mathcal{B}(\Lambda) \otimes \mathcal{B}(\Lambda)$-measurable maps $p_t : \Lambda \times \Lambda \to I\!R_0^+$, $t \geq 0$, such that (for simplicity we do not distinguish between functions and classes)

$$T_t f(\eta) = \int_\Lambda p_t(\eta, \xi) f(\xi)\, d\xi \tag{27}$$

satisfying $p_t(\xi, \eta) = p_t(\eta, \xi)$, $\int p_t(\xi, \eta) p_s(\eta, \xi')\, d\eta = p_{t+s}(\xi, \xi')$ and

$$p_t(\xi, \eta) \;\leq\; const\, t^{-d/2} \exp[-\frac{|\xi - \eta|^2}{2t}]. \tag{28}$$

Moreover, $(m - \Delta_\Lambda)^{-\beta}$, $\beta > 0$, has an explicit expression in terms of $(T_t)_{t\geq 0}$, which is given by $(m - \Delta_\Lambda)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-mt} T_t\, dt$, where Γ is the gamma function.

Lemma 28 *The operator* $(m-\Delta_\Lambda)^{-\beta}$ *is Hilbert-Schmidt for* $\beta > d/4$.

Proof: Let $\beta > d/4$. By [ReS 90, Thm.VI.26] it is sufficient to show that the kernel $C^\beta(\xi,\xi') = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-mt}p_t(\xi,\xi')\,dt$. is in $L^2(\Lambda\times\Lambda; d\xi\otimes d\xi)$. Since $p_t(\xi,\xi') = p_t(\xi',\xi)$, and by the semigroup property it follows that

$$\begin{aligned}\int_\Lambda\int_\Lambda (C^\beta(\xi,\xi'))^2\,d\xi d\xi' &= \int_\Lambda\int_0^\infty\int_0^\infty (st)^{\beta-1}e^{-m(s+t)}p_{t+s}(\xi,\xi)\,ds\,dt\,d\xi \\ ((28)) \quad &\leq |\Lambda|\int_0^\infty\int_0^\infty (st)^{\beta-1}e^{-m(s+t)}(s+t)^{-d/2}\,ds\,dt.\end{aligned}$$

Now use $\int_0^\infty\int_0^\infty ds\,dt = 2\int_0^\infty\int_0^\infty I_{\{t<s\}}ds\,dt = 2\int_0^\infty dt\int_0^t ds$ and replace (st) by $(\max(s,t))^2$ as well as $(s+t)^{-d/2}$ by $(\max(s,t))^{-d/2}$, to obtain the upper bound

$$2|\Lambda|\int_0^\infty dt\int_0^t ds\; t^{2\beta-2-d/2}e^{-mt} = 2|\Lambda|\int_0^\infty t^{2\beta-1-d/2}e^{-ms}dt < \infty.$$

///

Proof: (of Theorem 27). It is sufficient to check the assumptions of Theorem 18. Note first, that the measure μ exists on E, since by Lemma 28 $(m-\Delta_\Lambda)^{-\frac{\gamma+\gamma_A}{2}}$ is Hilbert-Schmidt. It is well-known that $(-\Delta_\Lambda)$ is a positive-definite self-adjoint operator with discrete spectrum. Hence $A = (m-\Delta_\Lambda) \geq mId_E$. Moreover, the eigenvectors e_n of $(m-\Delta_\Lambda)$ are given by products of sine and cosine functions. Hence each e_n is bounded and assumption (B) is satisfied. Moreover, (A) is satisfied, since $A^{-1/2}Q^{1/2} = (m-\Delta_\Lambda)^{-(\gamma+\gamma_A)/2}$ and $A^{-1}Q^{1/2} = (m-\Delta_\Lambda)^{-\gamma/2-\gamma_A}$ are both Hilbert-Schmidt by Lemma 28. Thus it remains to check condition (P), which holds true by the following Proposition. ///

Proposition 29 *Let* μ *as in Theorem 27. Then* $z^n \in L^1(\Lambda; d\xi)$ *for all* $n \in I\!N$ *and* μ*-a.e.* $z \in E$. *Moreover,* $\int \|z^n\|_E^2\,\mu(dz) < \infty$.

For the proof we need a simple formula, which follows by Wick's theorem, cf. [S 74, Proposition I.2]. Namely, given $n \in I\!N$ and two real valued mean zero Gaussian random variables X, Y there are constants $c_1,\ldots,c_n \in I\!N\cup\{0\}$ such that

$$(29) \qquad E[X^nY^n] = \sum_{k=0}^n c_k\,E[XY]^k E[X^2]^{\frac{n-k}{2}}E[Y^2]^{\frac{n-k}{2}}.$$

Proof: (of Theorem 29). Set $z_\varepsilon := T_\varepsilon z$. The semigroup $(T_t)_{t\geq 0}$ is symmetric, hence analytic, which implies $T_\varepsilon z \in D((m-\Delta_\Lambda)^k)$ for arbitrary $k \in I\!N$. Thus the Sobolev embedding theorem implies that $\xi \mapsto z_\varepsilon(\xi)$ is continuous. Moreover, (27) implies $z_\varepsilon(\xi) = \langle z, p_\varepsilon(\xi,\cdot)\rangle_E$. Therefore, $z \mapsto z_\varepsilon(\xi)$ is $\mathcal{B}(E)$-measurable. These facts imply, as is well-known, that the map

$$(30) \qquad (z,\xi)\mapsto z_\varepsilon(\xi), \qquad E\times\Lambda \to I\!R,$$

is $\mathcal{B}(E) \otimes \mathcal{B}(\Lambda)$-measurable. Set now $\beta := (\gamma + \gamma_A)/2$ and let $E[\cdot]$ denote the integral w.r.t. μ. Then

$$
\begin{aligned}
E[z_\varepsilon(\xi) z_{\varepsilon'}(\xi')] &= \langle \{(m - \Delta_\Lambda)^{-\beta} p_\varepsilon(\xi, \cdot)\}(\cdot), p_{\varepsilon'}(\xi', \cdot)\rangle_E \\
&= T_{\varepsilon'}\{(m - \Delta_\Lambda)^{-\beta} p_\varepsilon(\xi, \cdot)\}(\xi') \\
&= T_{\varepsilon'}\{\frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-mt} (T_t p_\varepsilon(\xi, \cdot))(\cdot)\, dt\}(\xi').
\end{aligned}
$$

Then symmetry of $(p_t)_{t\geq 0}$ and the semigroup property imply $(T_t p_\varepsilon(\xi, \cdot))(\cdot) = \int p_t(\cdot, \eta) p_\varepsilon(\xi, \eta) d\eta = p_{t+\varepsilon}(\cdot, \xi)$. Clearly, the bounded operator $T_{\varepsilon'}$ interchanges with the Bochner-integral, hence

$$
\begin{aligned}
E[z_\varepsilon(\xi) z_{\varepsilon'}(\xi')] &= T_{\varepsilon'}\{\frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-mt} p_{t+\varepsilon}(\cdot, \xi)\, dt\}(\xi') \\
&= \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-mt} \{T_{\varepsilon'} p_{t+\varepsilon}(\cdot, \xi)\}(\xi')\, dt \\
&= \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-mt} p_{t+\varepsilon+\varepsilon'}(\xi, \xi')\, dt \\
&= \frac{1}{\Gamma(\beta)} e^{-m(\varepsilon+\varepsilon')} \int_{\varepsilon+\varepsilon'}^\infty (t - \varepsilon - \varepsilon')^{\beta-1} e^{-mt} p_{t+\varepsilon+\varepsilon'}(\xi, \xi')\, dt.
\end{aligned}
$$

Denote the last integrand by $I_{\varepsilon+\varepsilon'}$. By the bound (28) we have in case $d = 1$

$$
I_{\varepsilon+\varepsilon'} \leq \frac{const}{\sqrt{t}} e^{-mt}. \tag{31}
$$

In case $d \geq 2$

$$
I_{\varepsilon+\varepsilon'} \leq (t - \varepsilon - \varepsilon')^{\beta-1} \frac{const}{t^{d/2}} e^{-mt} \leq const\; t^{\beta-1-d/2} e^{-mt}. \tag{32}
$$

Since these bounds are integrable and uniform in ε, ε', Lebesgue's theorem on dominated convergence implies that

$$
\lim_{\varepsilon,\varepsilon'\to 0} E[z_\varepsilon(\xi) z_{\varepsilon'}(\xi')] = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-mt} p_t(\xi, \xi')\, dt \leq const \tag{33}
$$

for some constant which by (31), (32) is independent of (ξ, ξ'). Let now $n \in \mathbb{N}$. Then we obtain by Fubini (which is justified by (30))

$$
\begin{aligned}
&\int_E \int_\Lambda (z_\varepsilon^n - z_{\varepsilon'}^n)^2\, d\xi \mu(dz) = \int_\Lambda E[z_\varepsilon^{2n}(\xi)]\, d\xi - 2 \int_\Lambda E[z_\varepsilon^n(\xi) z_{\varepsilon'}^n(\xi)]\, d\xi + \int_\Lambda E[z_{\varepsilon'}^{2n}(\xi)]\, d\xi = \\
&= \sum_{j=0}^n c_j \Big\{ \int_\Lambda E[z_\varepsilon^2(\xi)]^j\, E[z_\varepsilon^2(\xi)]^{n-j}\, d\xi - 2 \int_\Lambda E[z_\varepsilon(\xi) z_{\varepsilon'}(\xi)]^j\, E[z_\varepsilon^2(\xi)]^{\frac{n-j}{2}}\, E[z_{\varepsilon'}^2(\xi)]^{\frac{n-j}{2}}\, d\xi \\
&\qquad + \int_\Lambda E[z_{\varepsilon'}^2(\xi)]^j\, E[z_\varepsilon^2(\xi)]^{n-j}\, d\xi \Big\}.
\end{aligned}
$$

Hence the uniform bound (33) and Lebesgue imply

$$
\lim_{\varepsilon,\varepsilon'\to 0} \int_E \int_\Lambda (z_\varepsilon^n(\xi) - z_{\varepsilon'}^n(\xi))^2\, d\xi \mu(dz) = 0. \tag{34}
$$

Thus there exists a map $F^n \in L^2(E \times \Lambda; \mu \otimes d\xi)$ such that $(z_\varepsilon(\xi))^n \to F^n(z,\xi)$ in $L^2(E \times \Lambda; \mu \otimes d\xi)$. Choosing a subsequence we obtain that there exists a set $N \subseteq E \times \Lambda$ such that $\mu \otimes d\xi(N) = 0$ and such that

$$(z_{\varepsilon_m}(\xi))^n \to F^n(z,\xi) \text{ for } \mu \otimes d\xi\text{-a.e. } (z,\xi).$$

Thus for μ-a.e. $z \in E$ $(z_{\varepsilon_m}(\xi))^n \to F^n(z,\xi)$ for $d\xi$-a.e. ξ. On the other hand $z_{\varepsilon_m} = p_{\varepsilon_m} z \to z$ in $L^2(\Lambda; d\xi)$. Hence for each $z \in E$ there is a subsequence of $(\varepsilon_m)_{m \in \mathbb{N}}$ such that $z_{\varepsilon_{m_k}} \to z \; d\xi - a.s.$, which implies $(z_{\varepsilon_{m_k}})^n \to z^n \; d\xi - a.s.$ Thus for μ-a.e. $z \in E$

$$F^n(z,\xi) = z^n(\xi) \; d\xi - a.s.,$$

which implies $z^n \in L^2(\Lambda; d\xi)$ for μ-a.e. $z \in E$. Now the Jensen inequality in $L^2(\Lambda; d\xi)$ implies $z^n \in L^1(\Lambda; d\xi)$ for μ-a.e. $z \in E$. Moreover, (34) also implies $\int \|z^n\|_E^2 \, \mu(dz) < \infty$. ///

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