All Ramsey numbers $r(K_3, G)$ for connected graphs of order 7 and 8

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Abstract

In this paper we will give all Ramsey numbers $r(K_3, G)$ for connected graphs G of order 7 and 8. For smaller orders and not connected graphs of order up to 8, the number of graphs with given Ramsey number is listed for all possible values.

Introduction

All graphs are assumed to be finite undirected simple graphs. The Ramsey number $r(G_1, G_2)$ of two graphs G_1 and G_2 is defined as the minimum n so that every graph on n vertices contains an isomorphic copy of G_1 as a (weak) subgraph or its complement contains an isomorphic copy of G_2. The existence of this number follows directly from the fundamental theorem of Ramsey (cf. [1]). With $|G|$ we denote the number of vertices in G.

In the literature several collections of Ramsey numbers for specific sets of graphs can be found. All numbers $r(G_1, G_2)$ with $|G_1| \leq 4$ and $|G_2| \leq 4$ were already listed by Chvatal and Harary [4],[5]. Clancy [6] extended these results to $|G_1| \leq 4$ and $|G_2| \leq 5$ except for 5 entries. The last missing case $G_1 = K_4$ and $G_2 = K_5$ was solved by McKay and Radziszowski in 1995 [13].

Special attention was paid to the case $G_1 = K_3$: In 1980 Faudree, Rousseau and Schelp [7] determined all numbers $R(K_3, G)$ with $|G| = 6$. Schelten and Schiermeyer [17] determined the Ramsey numbers $R(K_3, G)$ – in the sequel called triangle Ramsey numbers – for 812 connected graphs (out of 853) of order 7. Xia Jin [11] computed the triangle Ramsey numbers for all connected graphs of order 7. Unfortunately there were some errors in his thesis, so that this problem remained open to some extend. The triangle Ramsey numbers for the most important cases $G = K_n$ or $G = K_n - e$ have been determined by various authors for all $n \leq 9$, see [9],[8], [10], [12], [15]. Further information on small Ramsey numbers can be found in [16].

In the following we will give all Ramsey numbers $R(K_3, G)$ for connected graphs G of order up to 8. The results will be given in the form of maximal and minimal elements of the subgraph chains belonging to a certain number.

The algorithm

The triangle Ramsey numbers were obtained with the aid of a computer. All numbers given except $R(K_3, K_8)$ have been computed, giving an independent check for those numbers already known before.
The basic tool was a computer program developed together with Stephan Brandt and Thomas Harmuth [2] to generate *maximal triangle free* graphs. A maximal triangle free graph (in short MTF) is defined as a triangle free graph so that the insertion of any additional edge gives a triangle.
The lists generated by that program were sufficient to determine the triangle Ramsey numbers because of the following lemma, which follows directly from the definition:

Lemma 1

For every graph G and number n we have $R(K_3, G) \leq r$ if and only if the complement of every MTF with r vertices contains a copy of G as a weak subgraph.

Table 1 gives the numbers of MTFs with up to 20 vertices. Up to 17 vertices the lists were checked by a completely independent generation program for MTFs and up to 14 vertices they were additionally checked by generating all triangle free graphs using the graph generator *makeg* [14] and filtering the output for graphs with diameter 2.

vertex number	number of graphs	vertex number	number of graphs
3	1	12	147
4	2	13	392
5	3	14	1 274
6	4	15	5 036
7	6	16	25 617
8	10	17	164 796
9	16	18	1 337 848
10	31	19	13 734 745
11	61	20	178 587 364

Table 1: Numbers of MTFs

Since the number of MTFs is not too large for small vertex numbers, for small triangle Ramsey numbers one could simply apply Lemma 1. Obviously no connected graph of order n can be

contained in the complement of $K_{n-1,n-1}$, so the smallest possible triangle Ramsey number for a connected 7 vertex graph is 13 and for an 8 vertex graph it is 15.
Because of the large number of graphs that had to be processed in the case $|G| = 8$, one further (easy) observation had to be used:

Lemma 2

If a graph G' is contained in a graph G'' as a weak subgraph, then $R(G, G') \leq R(G, G'')$ for all graphs G.

Since during the first runs it turned out that only relatively few MTFs were used as Ramsey graphs showing $R(K_3, G) > r$, this observation was included in the algorithm.
The algorithm works as follows:
When testing whether a set of graphs with n vertices has triangle Ramsey number at most r, the program uses 3 internal lists:

- A list *maxlist* of graphs with n vertices that have triangle Ramsey number r. This list contains the maximal elements of the subgraph chains of the graphs with triangle Ramsey number r. When the program starts, this list is empty.
- A list *problemlist* of MTFs that were used as Ramsey graphs during the run. This list is also empty when the program starts.
- A list *minlist* of graphs with fewer vertices than the graphs tested and triangle Ramsey number larger than r. In general, this list is not empty at the beginning. To be exact, the list does not contain all such graphs, but they are ordered to subgraph chains and only the minimal elements are taken. The list is not altered during the program run.

Furthermore the program uses 2 external lists to output the graphs: (*List M* for the graphs with triangle Ramsey number more than r and *list A* for graphs with triangle Ramsey number at most r), one external list *MTFlist* that contains all the MTFs with r vertices and of course the list *testlist* of graphs the triangle Ramsey number shall be determined of. These graphs are generated using the program *makeg* and are sorted by decreasing number of edges. So if $G' \subset G$, then G' appears in *testlist* after G.
The program takes one element G of *testlist* after the other and tests it as described in steps 1 to 4. If in one of the steps it is determined whether the triangle Ramsey number of G is at most r, the following steps are omitted.

step 1 Check whether or not G is a weak subgraph of one of the graphs in *maxlist*. If this is the case, G has triangle Ramsey number at most r and the graph G is written to *list A*.

step 2 Test whether the graph is contained in all elements of *problemlist*. If there is an element of *problemlist* that does not contain G, the graph has triangle Ramsey number larger than r and is written to *list M*.

step 3 Check whether one of the graphs in *minlist* is contained in G. This would imply that the triangle Ramsey number is larger than r and the graph can be written to *list M*.

step 4 Check whether G is contained in all graphs of *MTFlist*. If this is the case, G is written to *list A* as well as to *maxlist* (note that G is a new maximal element of the subgraph chains). If in *MTFlist* a graph G_0 is found that does not contain G, the test is interrupted, G is written to *list M* and G_0 is written to *problemlist*.

This is the most expensive test and it is only carried out if steps 1 to 3 fail. In step 4 it is always determined whether the triangle Ramsey number is at most r.

The subgraph test used is straightforward. In case of dense graphs G it is not checked whether the graph G is contained in the complement of the MTF, but whether a subset S of size n of the vertices of the MTF exists, so that the induced subgraph $G(S)$ is contained in G^c.

This method is fast enough for the cases with comparatively small triangle Ramsey numbers, since the number of MTFs is not too large and relatively few graphs have to be checked against the list of MTFs. When e.g. checking all 11117 connected graphs of order 8 whether their triangle Ramsey number is 15, only 27 graphs had to be checked in step 4.
The method was used to reduce the list of connected graphs, until only graphs with triangle Ramsey number larger than 18 remained. For some of these graphs (those with triangle Ramsey number larger than 20) it wouldn't even be possible to generate and store the lists of MTFs. The triangle Ramsey number of the remaining 2 connected graphs on 7 vertices and 36 connected graphs on 8 vertices (except $R(K_3, K_8)$) were calculated one by one using a modified generation program that generates only (maximal) Ramsey graphs for the specific graph.
To understand the changes, it is not necessary to know exactly how the generation program works (e.g. the isomorphism rejection), but only how larger graphs are constructed from smaller ones:
All graphs are generated starting from the unique MTF with 3 vertices. A graph G on, say, m vertices is enlarged to a graph G' on $m+1$ vertices in the following way:

1. Choose a *good dominating set* S of vertices of G – that is a subset of the vertices so that all vertices of G are contained in S or adjacent to a vertex in S (*dominating*) and even after removing all interior edges of S, every additional edge between vertices of S and S^C would constitute a triangle (*good*).

2. Insert a new vertex (labelled $m+1$) and connect it to all vertices in S. Delete all interior edges in S.

It is easy to see that – choosing S in all possible ways – every MTF will be generated.
So once a graph on m vertices is established, we never **add** edges between the vertices $1,\ldots,m$, we only sometimes delete edges. For the complement this means that sometimes additional edges between the vertices $1,\ldots,m$ are added, but no edges with both endpoints in $\{1\ldots m\}$ are deleted in graphs that are generated out of G. Hence, if a graph G_0 is contained in the complement of G, it will be contained in the complement of every descendant of G and – trying to construct just Ramsey graphs for G_0 – we don't need to continue the generation in this direction.
This procedure was used for the remaining cases. Nevertheless the calculation of $R(K_3,K_8)$ with this method would need more time than the computation of all the other triangle Ramsey numbers altogether. The calculation of all triangle Ramsey numbers for connected graphs of order 7 took much less than one day on an SGI workstation – including the time to generate the lists containing all MTFs with up to 18 vertices. Still by far most of the time was needed to determine $R(K_3,K_7)$. Testing e.g. all connected graphs on 7 vertices for triangle Ramsey number 13 takes less than 3 seconds.
In table 2 the numbers of connected graphs with up to 8 vertices and a certain triangle Ramsey number are listed. In table 3 the same statistic is given for graphs that are not connected. These numbers were computed by the same algorithm.
In table 4 the values $f(3,n)$ and $g(3,n)$ are given as defined in [3]. The function $f(3,n)$ gives the maximal number of edges, so that every connected graph on n vertices with $f(3,n)$ edges has the smallest possible triangle Ramsey number $2n-1$. The function $g(3,n)$ gives the maximal number of edges, so that at least one connected graph on n vertices with $g(3,n)$ edges has triangle Ramsey number $2n-1$. These numbers can also be read directly from the subgraph chains given. For $n=9$ some data that was obtained with the same methods was used. Determining all triangle Ramsey numbers for connected graphs of order 9 would not be possible with the methods described above. Nevertheless combining theoretical arguments for some graphs with improved algorithms (and a lot of computation time) for the rest, we hope to be able to do so in the future.

	$\lvert G\rvert = 3$	$\lvert G\rvert = 4$	$\lvert G\rvert = 5$	$\lvert G\rvert = 6$	$\lvert G\rvert = 7$	$\lvert G\rvert = 8$
$r = 5$	1					
$r = 6$	1					
$r = 7$		5				
$r = 8$						
$r = 9$		1	18			
$r = 10$						
$r = 11$			2	98		
$r = 12$				6		
$r = 13$				2	772	
$r = 14$			1	4	40	
$r = 15$						9024
$r = 16$					13	1440
$r = 17$				1	19	498
$r = 18$				1	7	119
$r = 19$						
$r = 20$						
$r = 21$					1	28
$r = 23$					1	6
$r = 25$						1
$r = 28$						1

Table 2: Numbers of connected graphs with Ramsey number $R(K_3, G) = r$.

	$\lvert G\rvert = 3$	$\lvert G\rvert = 4$	$\lvert G\rvert = 5$	$\lvert G\rvert = 6$	$\lvert G\rvert = 7$	$\lvert G\rvert = 8$
$r = 3$	2					
$r = 4$		2				
$r = 5$		2	4			
$r = 6$		1	3	7		
$r = 7$			5	11	18	
$r = 8$				3	5	23
$r = 9$			1	20	50	60
$r = 10$						36
$r = 11$				2	102	225
$r = 12$					6	12
$r = 13$					2	776
$r = 14$				1	6	52
$r = 15$						
$r = 16$						13
$r = 17$					1	21
$r = 18$					1	9
$r = 19$						
$r = 20$						
$r = 21$						1
$r = 23$						1

Table 3: Numbers of not connected graphs with Ramsey number $R(K_3, G) = r$.

	$f(3,n)$	$g(3,n)$
n=3	2	2
n=4	5	5
n=5	7	8
n=6	8	12
n=7	11	16
n=8	11	20
n=9	16	27

Table 4: Values for $f(3,n)$ and $g(3,n)$.

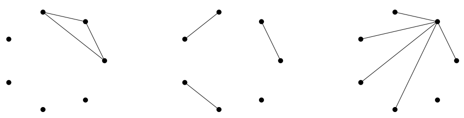
The triangle Ramsey number for connected graphs of order 7

Except in those cases where only one graph has a certain triangle Ramsey number, the results are presented by giving minimal and maximal elements of the subgraph chains of the **complements** of the graphs with this triangle Ramsey number.

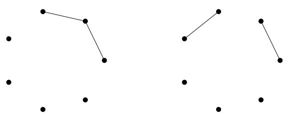
$R(K_3, G) = 23$ if and only if $G = K_7$.

$R(K_3, G) = 21$ if and only if $G = K_7 - e$.

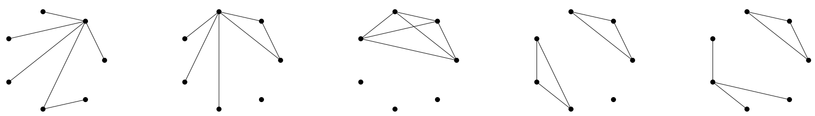
$R(K_3, G) = 18$ if and only if G^c is contained in one of the graphs



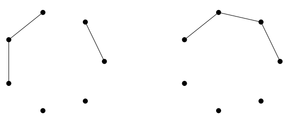
and contains one of the graphs



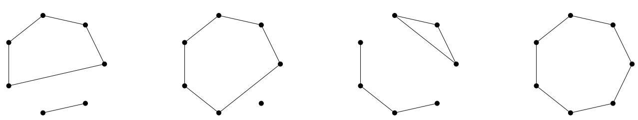
$R(K_3, G) = 17$ if and only if G^c is contained in one of the graphs



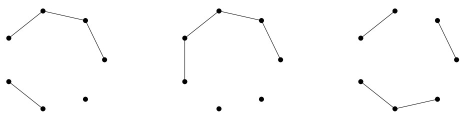
and contains one of the graphs



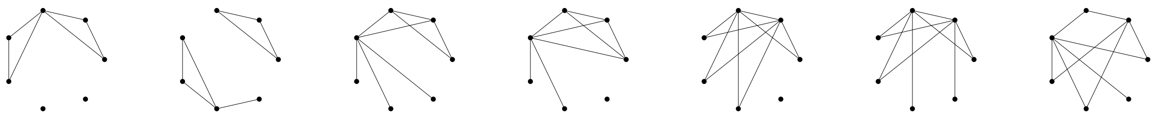
$R(K_3, G) = 16$ if and only if G^c is contained in one of the graphs



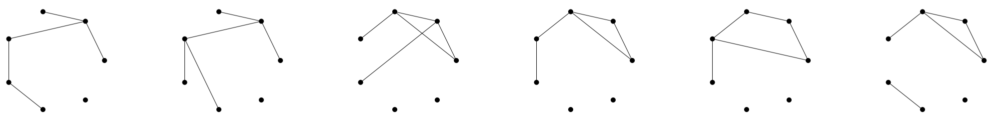
and contains one of the graphs



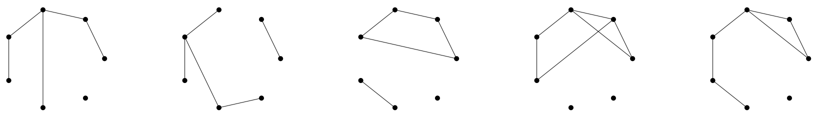
$R(K_3, G) = 14$ if and only if G^c is contained in one of the graphs



and contains one of the graphs



$R(K_3, G) = 13$ if and only if G^c contains one of the graphs

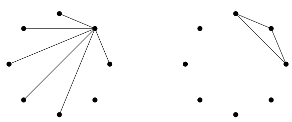


The triangle Ramsey number for connected graphs of order 8

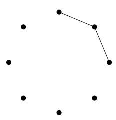
$R(K_3, G) = 28$ if and only if $G = K_8$.

$R(K_3, G) = 25$ if and only if $G = K_8 - e$.

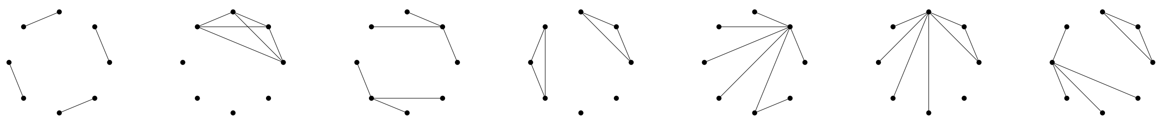
$R(K_3, G) = 23$ if and only if G^c is contained in one of the graphs



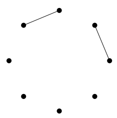
and contains the graph



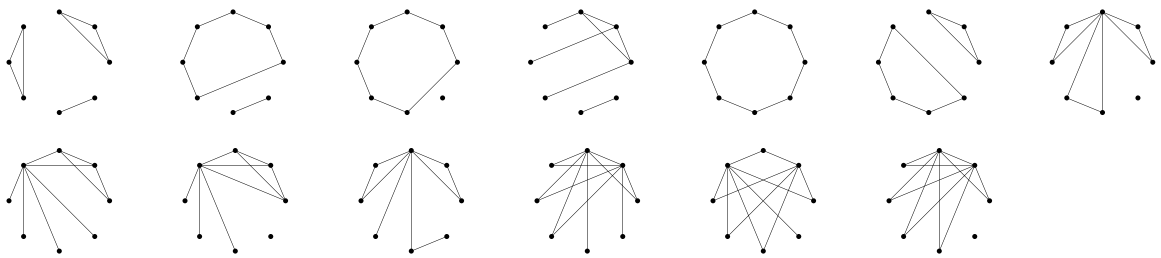
$R(K_3, G) = 21$ if and only if G^c is contained in one of the graphs



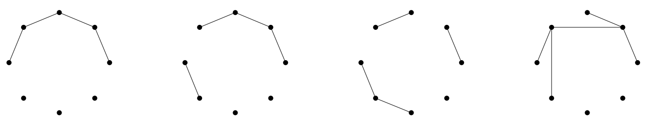
and contains the graph



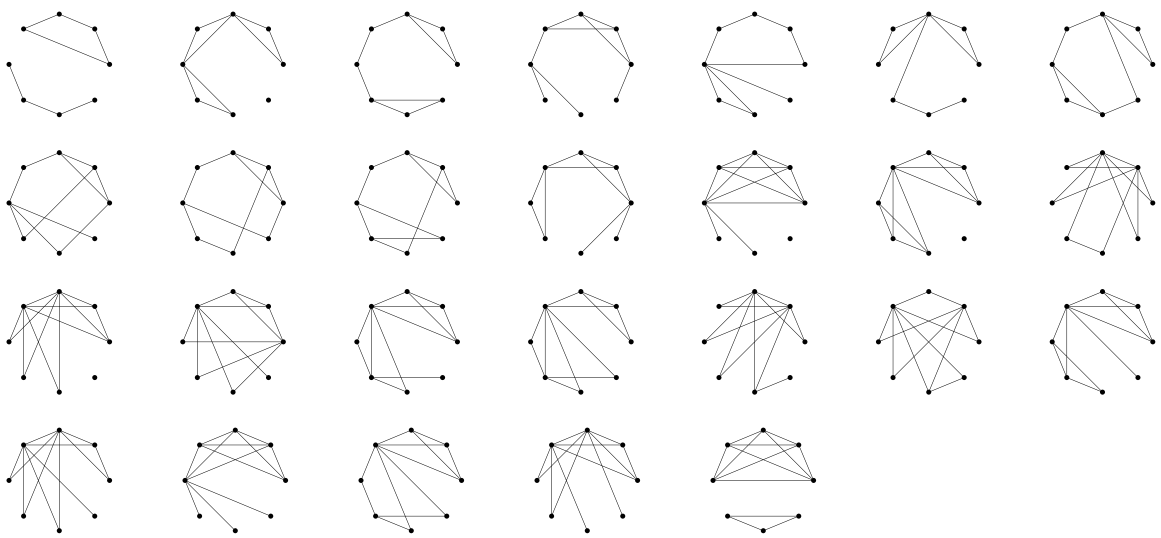
$R(K_3, G) = 18$ if and only if G^c is contained in one of the graphs



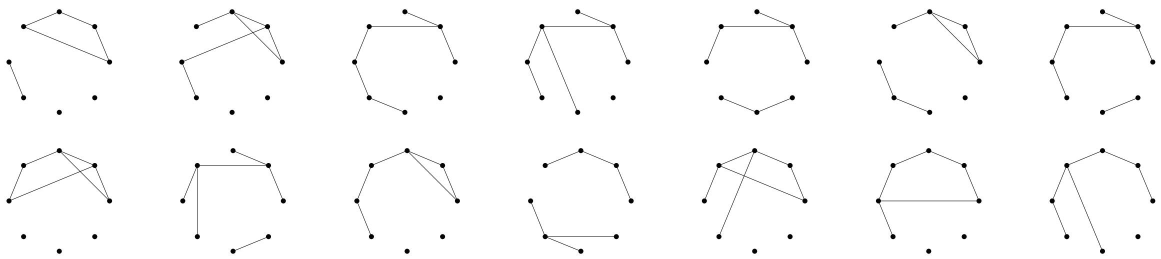
and contains one of the graphs



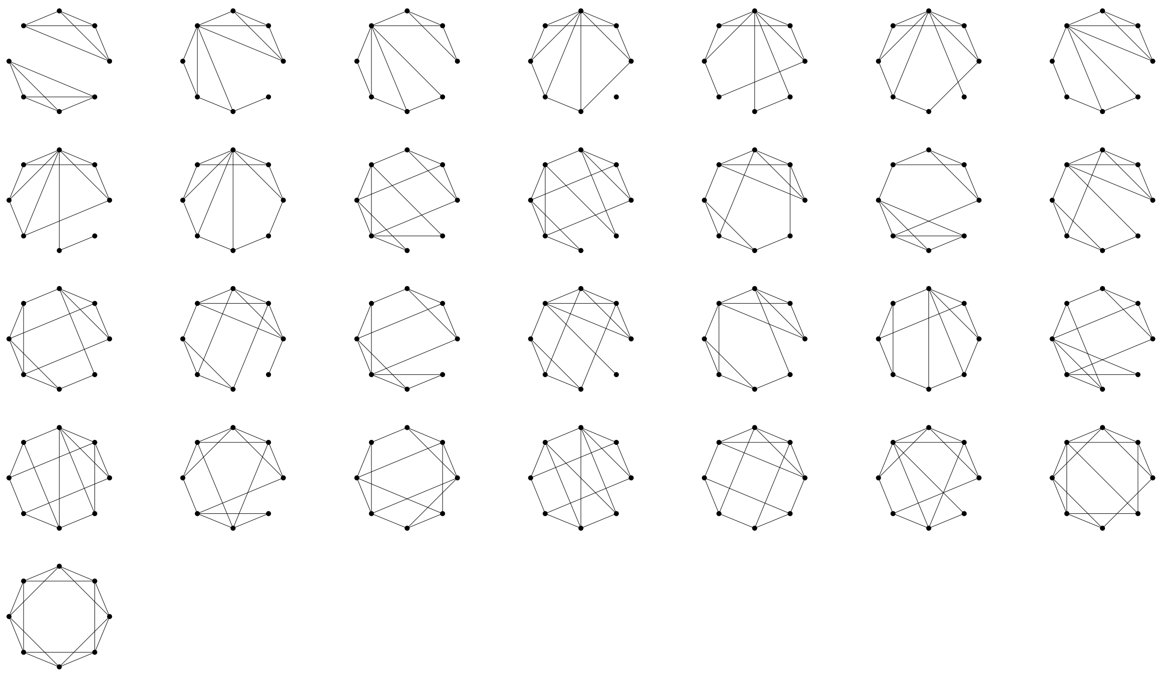
$R(K_3, G) = 17$ if and only if G^c is contained in one of the graphs



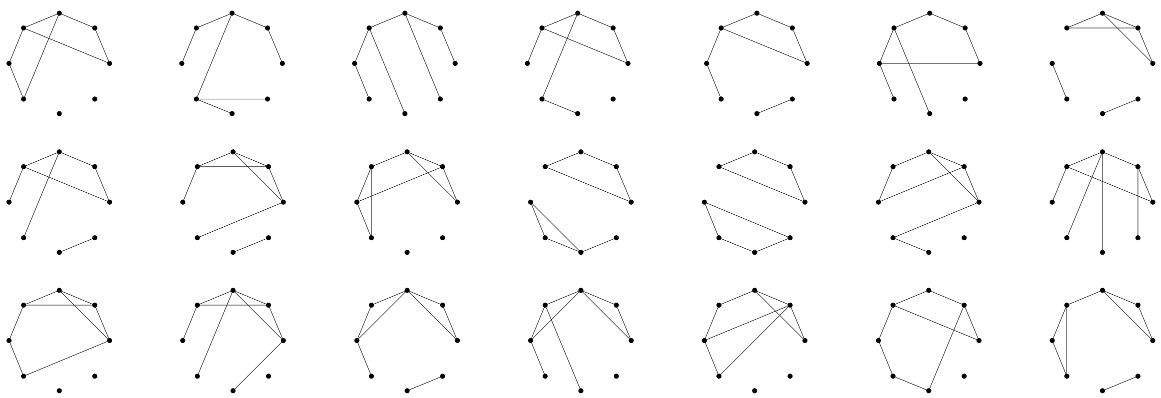
and contains one of the graphs



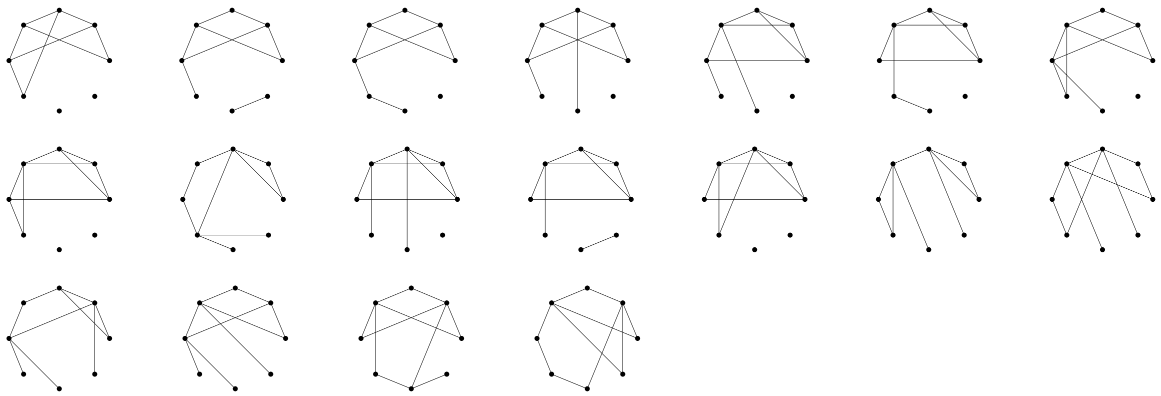
$R(K_3, G) = 16$ if and only if G^c is contained in one of the graphs



and contains one of the graphs



$R(K_3, G) = 15$ if and only if G^c contains one of the graphs



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