On the Storage Capacity of Hopfield Models with Weakly Correlated Patterns

Matthias Löwe
Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
email: loewe@mathematik.uni-bielefeld.de

Keywords: Hopfield model, neural networks, storage capacity, Markov chains, large deviations

AMS Subject Classification: 82C32, 82B44, 60K35

Abstract

We analyze the storage capacity of different forms of the Hopfield model with semantically correlated patterns ξ_i^ν (i.e. the patterns are correlated in ν but independent in i). We show that the standard Hopfield model of neural networks with N neurons can store $\frac{N}{\gamma \log N}$ or αN semantically correlated patterns (depending on which notion of storage is used), provided that the correlation comes from a homogeneous Markov chain. The quantity γ increases with an increasing correlation of the patterns, while α decreases with large correlations. This answers the open question whether the standard Hopfield can store any increasing number of correlated patterns at all in the affirmative. Moreover, we show that there is a version of the Hopfield model that can store a number of semantically correlated data that is even larger than the bound for the number of independent patterns that can be stored in the standard Hopfield model provided we require that the patterns are exactly reproduced. Finally we show that this model can store αN patterns (provided Newman's notion of storage capacity is used) with an α not depending on p.

1 Introduction

Neural networks are intensively studied in the recent probabilistic and physical literature. The so–called Hopfield model has an exposed position, since it connects the ideas of neural computing with those of statistical mechanics.

Originally the Hopfield model has been introduced by Pastur and Figotin [FP77] as a simplified model of a spin glass. Five years later it was rediscovered as a model of neural networks by Hopfield [Ho82]. The basic idea behind this model is to choose the information to be stored – which usually is referred to as patterns – as the local minima of an appropriate Hamiltonian (this is going to be made more formal in the next section) on the set $\{1, \dots, N\}$ which is thought of as the set of neurons. Since, according to the other rules of neural networks, this is possible only up to a certain accuracy depending on the number of patterns to be stored, one may introduce the notion of a storage capacity of the Hopfield model as the critical number of patterns (depending on N, of course) up to which the model can successfully reconstruct these patterns.

To be more precise, there are two definitions of storage capacity: The first one introduces the gradient dynamics on the energy landscape on $\{1, \dots, N\}$ associated with the Hamiltonian of the system and requires that the patterns are fixed points of this dynamics (and thus minima of the Hamiltonian). The second one going back to a seminal paper of Newman [N88] defines the notion of storage capacity less strictly requiring that a pattern is surrounded by an energy barrier of height proportional to N.

It is well known and first was announced by McEliece et al. [MPRV87] that the first definition leads to a storage capacity of at least $\frac{N}{6 \log N}$ for the Hopfield model with N neurons and randomly and independently chosen patterns (also see the recent survey paper by Petritis [P95]). On the other hand Newman [N88] gave a proof for the non-rigorous result of Amit, Gutfreund and Sompolinsky [AGS87] showing that the Hopfield model allows to store even αN patterns (for an $\alpha > 0$ small enough), if small errors are tolerated. The value for α found by Newman was $\alpha = 0.056$, which has been recently improved by Loukianova [L94] to $\alpha = 0.071$ by a refined large deviations analysis. The latest result concerning the critical storage capacity is due to Talagrand [T95] who has announced a bound of $\alpha = 0.085$, obtained via isoperimetric inequalities. Nevertheless, it is still an open question, whether the prediction of $\alpha = 0.14$ in [AGS87] on the basis of the non-rigorous replica method and computer simulations can be mathematically justified.

But its abilities as an auto-associative memory only partly motivated the recent interest in the Hopfield model. The other important feature is that it may serve as a sort of a simplified spin glass model, where the complexity of the interactions between two sites, hence the complexity of the random environment, can be tuned by the number of stored patterns. As a matter of fact, if this number equals one the standard Hopfield model (as compared to the generalized Hopfield models to be introduced in the next section) is (by a simple change of variables) equivalent to the classical Curie-Weiss model of a ferromagnet and thus the interactions are purely deterministic. In contrast to that in the region where the number of patterns

roughly equals the number of neurons the model is expected to behave similar to the Sherrington–Kirkpatrick model of a spin glass (which still cannot be described in rigorous mathematical terms). Therefore the standard Hopfield model often is used to try techniques which should illuminate the thermodynamic behavior of spin glasses.

In the last few years the standard Hopfield model in the regime where the patterns are chosen independently and their number is "small" compared to the number of neurons has been well understood. The corresponding analysis has been carried out in a series of papers by Bovier and Gayrard, partially in collaboration with Picco [BGP94], [BGP95a], [BGP95b],[BG95a], [BG95b]. Also a recent result due to Gentz [G95] should be mentioned where a central limit theorem for the overlap parameter in this model is proved.

If the patterns are chosen to be correlated, the picture is less clear. Some investigations based on heuristic arguments claim that correlations among the patterns can only increase the storage capacity of the Hopfield model [Mo92],[TL93]. This may be supported by the fact that in terms of information content a sequence of correlated patterns contains less informations than a sequence of uncorrelated ones. On the other hand – based on a signal-noise-analysis for biased data – some other authors (see e.g. [HK91]) argue that the Hopfield model cannot store any increasing (in N) amount of correlated patterns. The basic idea behind this argument is that in the presence of correlations among the patterns the cross-talk terms between two patterns become non-vanishing and therefore the noise created by these cross-talk terms becomes large compared to the signal coming from a single pattern.

In this paper we will show, that indeed even the standard Hopfield model can store an increasing number of semantically correlated data (which in our case come from a Markov chain) no matter whether the "dynamic" or the "static" (the latter being the one due to Newman) notion of storage is used. The storage capacities we obtain are below those for independent patterns (though similar techniques are employed) and the capcities decrease when the correlation becomes larger.

Moreover, we will show that if one slightly changes the "geometry" of the Hopfield model in an appropriate way, i.e. the scalar product the L^2-norm is taken in, then the Hopfield model is able to store a number of semantically correlated patterns even higher than the bound for the number of independent patterns in the standard Hopfield model.

In this paper we will only consider the case of semantical correlations and analyze storage capacities and appropriate models in this case. The situation with spatially correlated patterns where similar models are supposed to work as well as the thermodynamic properties of the models described below will be investigated in a forthcoming paper.

We organize this paper in the following way: Section 2 will contain the basic setup, especially the type of correlations we consider, the definition of the classical Hopfield model as well as a mathematical description of the "change of geometry" mentioned above. Section 3 deals with the storage capacity of the standard Hopfield model for Markov-dependent sequences of patterns. Here we analyze both the dynamic and

the static notion of storage. Finally in Section 4 we will show that an appropriate choice of the geometry leads to an increased storage capacity for dependent patterns with the dynamic notion of storage. This will show that the class of Hopfield models with a transformed geometry is worth some more investigations. Finally, we will also show that these models are able to store extensively many patterns in the sense of Newman [N88] with a critical storage capacity bounded from below.

2 The Hopfield Model

Throughout the rest of the paper we consider a sequence $(\xi^\nu)_{\nu\in\mathbb{N}}$ of patterns, each of them being itself a sequence of plus and minus ones. So altogether we have an $\mathbb{N}\times\mathbb{N}$ array $((\xi_i^\nu)_{i\in\mathbb{N}})_{\nu\in\mathbb{N}} \in \{-1,+1\}^{\mathbb{N}\times\mathbb{N}}$. The Hamiltonian of the standard Hopfield model on N neurons now is defined as

$$H_N(\sigma) = -\frac{1}{2N}\sum_{i,j=1}^{N}\sum_{\nu=1}^{M(N)}\sigma_i\sigma_j\xi_i^\nu\xi_j^\nu, \tag{1}$$

where $M(N)$ denotes the number of patterns to be stored with N neurons, i.e. an index for the complexity of the neural network on one hand and for the complexity of the random environment $\sum_\nu \xi_i^\nu\xi_j^\nu$ on the other. The idea behind the choice of this Hamiltonian is that it is in a way the "simplest" reasonable function of the so-called overlap vector given by

$$m_N(\sigma) := (m_N^\nu(\sigma))_{\nu\in\mathbb{N}} := (\frac{1}{N}\sum_{i=1}^{N}\xi_i^\nu\sigma_i)_{\nu\in\mathbb{N}}$$

which measures how much a configuration σ agrees with one of the given patterns. More precisely we have

$$H_N(\sigma) = -\frac{N}{2}\sum_{\nu=1}^{M(N)}(m_N^\nu(\sigma))^2 = -\frac{N}{2}||m_N(\sigma)||_2^2$$

where $||\cdot||_2$ denotes the norm in l_2. The Hopfield model itself now can be identified with the Gibbs measure associated with $H_N(\sigma)$, i.e.

$$\varrho_{N,\beta}(\sigma) := \frac{e^{-\beta H_N(\sigma)}}{Z_{N,\beta}}$$

where β denotes the inverse temperature and $Z_{N,\beta}$ is the normalizing constant

$$Z_{N,\beta} := \sum_{\sigma\in\{-1,1\}^N} e^{-\beta H_N(\sigma)}.$$

As already mentioned in the introduction and has been proved in [MPRS87] and [N88], resp., (also see [P95]) this model works well as an associative memory if the patterns are chosen i.i.d., i.e. if $P(\xi_i^\nu = 1) = P(\xi_i^\nu = -1) = 1/2$ independently of all the other ξ_j^μ and if $M(N) < \frac{N}{6\log N}$ and $M(N) \le \alpha N$ for a small α, resp., depending on which definition of storage is used.

Now most of the patterns one would like to consider in realistic situations are not at all independent. Essentially there are two reasonable ways to introduce dependencies among the patterns. One is to consider spatially correlated patterns, i.e. to consider a correlation between ξ_i^ν and ξ_j^ν even if $i \neq j$, but to leave the ξ_i^ν and ξ_j^μ independent for $\mu \neq \nu$. This model may be of interest when storing e.g. images that can be considered to come from a Markov random field. The other type of dependency one may assume is semantical or sequential dependency among the patterns. That means that we consider random variables ξ_i^ν such that ξ_i^ν and ξ_j^μ still are independent if $i \neq j$, but that we may have correlations between ξ_i^ν and ξ_i^μ even if $\mu \neq \nu$. Such sequences may be interesting if non deterministic sequences of patterns have to be learned, and in some sense every type of human behavior is such a sequence.

Due to the objections against the Hopfield model with correlated patterns other models have been suggested, e.g. the so-called "pseudo-inverse learning rule", which basically proposes to consider a Hamiltonian of the form

$$\tilde{H}_N(\sigma) := -\sum_{i,j=1}^{N} \sigma_i \sigma_j J_{ij}$$

with

$$J_{ij} := \frac{1}{N} \sum_{\mu,\nu=1}^{M(N)} \xi_i^\mu (\tilde{C}^{-1})_{\mu,\nu} \xi_j^\nu = J_{ji}$$

where $\tilde{C}^{-1}$ is the inverse of the empirical correlation matrix or overlap matrix $\tilde{C}$ given by $\tilde{C}_{\mu,\nu} := \frac{1}{N} \sum_{i=1}^{N} \xi_i^\nu \xi_i^\mu$.
However, the definition of $\tilde{C}$ shows that $(\tilde{C}^{-1})_{\mu,\nu}$ and thus also J_{ij} depends on all the spins ξ_i^ν (and not only on those with indices i or j). This property – called non-locality by the experts – makes the pseudo-inverse learning rule rather unattractive from physiological constraints, even though even in the case of independent patterns its storage capacity is considerably higher than the storage capacity of the standard Hopfield model (up to $M(N) = N$ for i.i.d. patterns).

However, the following can be learned from the pseudo-inverse-rule: one basic reason, why the Hopfield model works well as an associative memory provided that the patterns are chosen i.i.d. and the number of patterns is not too large is that the randomly chosen patterns behave nearly as if they were orthogonal in the l_2-scalar product (and it is easy to check that the standard Hopfield model can store even N orthogonal patterns), which more exactly means that the overlap is $\tilde{C}_{\mu,\mu} = 1$ for all μ, whereas $\tilde{C}_{\mu,\nu} = \mathcal{O}(N^{-1/2})$ for all $\mu \neq \nu$. Thus for a sufficiently small number of patterns the "noise" created by the $\tilde{C}_{\mu,\nu}$ is negligible with respect to the "signal" $\tilde{C}_{\mu,\mu}$. If the patterns are now sampled dependently such a behavior cannot be expected, different patterns will have a non vanishing overlap, instead. Of course the question is, how much and in which way this influences the storage capacity and as a matter of fact we will show in the next section that small correlation only have a small influence on the storage capacity. But in principle the concept developed above leads to the idea to transform the geometry of the underlying space l_2 in such a way that the patterns are again "nearly orthogonal". Thus we are motivated to study a new class of models to be defined now:

We will speak about a measure $\varrho_{\beta,N,G}$ on $\{-1,1\}^N$ as a generalized Hopfield model with geometry G, if $G = (G_{\mu,\nu})_{\mu,\nu}$ is a symmetric, positive semi-definite $M(N) \times M(N)$ matrix and $\varrho_{\beta,N,G}$ is given by

$$\varrho_{\beta,N,G}(\sigma) := \frac{e^{-\beta H_{N,G}(\sigma)}}{Z_{\beta,N,G}}$$

for $\sigma \in \{-1,1\}^N$, where

$$\begin{aligned} H_{N,G}(\sigma) &:= -\frac{N}{2} < m_N(\sigma), G m_N(\sigma) > \\ &= -\frac{1}{2N} \sum_{i,j=1}^{N} \sum_{\mu,\nu=1}^{M(N)} \xi_i^\mu \xi_j^\nu \sigma_i \sigma_j G_{\mu,\nu} \qquad (2) \\ &=: -\sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j \end{aligned}$$

and $Z_{\beta,N,G}$ again is a normalizing constant to make $\varrho_{\beta,N,G}$ a probability measure:

$$Z_{\beta,N,G} := \sum_{\sigma \in \{-1,1\}^N} e^{-\beta H_{N,G}(\sigma)}.$$

Note that the choice of $G = Id$ (the $M(N) \times M(N)$ identity matrix) reproduces the standard Hopfield model.

We will analyze the storage capacities of some suitably chosen generalized Hopfield models for dependent patterns in Section 4. In Section 3 we will have a closer look at the abilities of the standard Hopfield model when storing correlated patterns. So throughout the rest of the paper we will assume that the correlation among the patterns is purely semantical and comes from a Markov chain. More precisely, we shall assume that the $(\xi_i^\nu)_{i \in \mathbb{N}, \nu \in \mathbb{N}}$ form a Markov chain with initial distribution

$$P(\xi_i^1 = x_i^1,\ i = 1, \ldots, N) = 2^{-N} \quad \text{for all } x_i^1 \in \{-1,1\} \text{ and all } i = 1, \ldots, N. \qquad (3)$$

and transition probabilities

$$\begin{aligned} & P(\xi_i^\nu = x_i^\nu | \xi_j^\mu = x_j^\mu, j = 1, \ldots, N, \mu = 1, \ldots, \nu - 1) \qquad (4) \\ = \ & P(\xi_i^\nu = x_i^\nu | \xi_i^{\nu-1} = x_i^{\nu-1}) = Q(x_i^{\nu-1}, x_i^\nu). \end{aligned}$$

Here Q denotes a symmetric 2×2 matrix with entries

$$Q = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

where $0 < p < 1$ and we may assume that $p \neq 1/2$ since this is the case of independent patterns.

3 The Capacity of the Standard Hopfield Model with Correlated Patterns

In this section we will first give a lower bound on the number of patterns of the form (3) and (4) that can be stored by the standard Hopfield model.

The purpose of the following investigations is at least twofold: On one hand we show that the standard Hopfield model is indeed able to store an increasing number of (weakly) correlated patterns, no matter which notion of storage is used. On the other hand we introduce the basic techniques which also will be important to analyze the storage capacity of the generalized Hopfield models in the next section.

Before doing so let us briefly explain the two different concepts of storage capacity we are dealing with on a technical level.
For the first consider a Hamiltonian of the form

$$\overline{H}(\sigma) = -\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j \tag{5}$$

for some J_{ij} and $\sigma_i \in \{-1, 1\}$. Define the gradient dynamics T on the energy landscape given by $\overline{H}$ via

$$T : \sigma_i \mapsto sgn(\sum_{j=1}^{N} \sigma_j J_{ij})$$

(where sgn is the sign function) and call a configuration $\sigma = (\sigma_i)_{i\leq N}$ stable if it is not changed by the dynamics T , i.e.

$$\sigma_i = sgn(\sum_{j=1}^{N} \sigma_j J_{ij}) \quad \text{for all } i = 1, \ldots, N$$

which means that σ is a local minimum of the Hamiltonian. The storage capacity (and we will often refer to this concept as to the dynamic approach) is defined as the greatest number of patterns $M(N)$ such that the patterns ξ^ν are almost surely stable in the above sense.

The other approach to storage capacity is due to Newman [N88]. Here a pattern ξ^ν is called stable, if it is surrounded by a sufficiently high energy barrier, in other words, if there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\inf_{\sigma\in S_\delta(\xi^\nu)} \overline{H}(\sigma) \geq \overline{H}(\xi^\nu) + \varepsilon N. \tag{6}$$

Here the set $S_\delta(\xi^\nu)$ the infimum is taken over is the Hamming sphere of radius δN centered in ξ^ν. Again we will use the notion of storage capacity for the maximal number $M(N)$ of patterns such that (6) holds true for all ξ^ν almost surely.

Now we are ready to formulate our first results.

Theorem 3.1 *Assume the random patterns ξ^ν fulfill (3) and (4) and $M(N) = \frac{N}{\gamma \log N}$.*
Then for the Hopfield model (1) the following assertions hold true:

1. *If* $\gamma > \frac{3(p^2+(1-p)^2)}{p(1-p)}$

$$P(\liminf_{N\to\infty}(\cap_{\nu=1}^{M(N)} T\xi^\nu = \xi^\nu)) = 1$$

i.e. the patterns are almost surely stable.

2. *If* $\gamma > \frac{2(p^2+(1-p)^2)}{p(1-p)}$

$$P((\cap_{\nu=1}^{M(N)} T\xi^\nu = \xi^\nu)) = 1 - R_N$$

with $\lim_{N\to\infty} R_N = 0$.

Proof: Fix $1 \le \nu \le M(N)$. According to the definition of the dynamics T and the standard Hopfield model the pattern ξ^ν is stable if and only if

$$\xi_i^\nu = sgn(\sum_{j=1}^{N} \xi_j^\nu J_{ij}) = sgn(\sum_{j=1}^{N}\sum_{\mu=1}^{M(N)} \xi_j^\nu \xi_i^\mu \xi_j^\mu)$$

for all $i = 1, \ldots, N$, or, in other words, if

$$\sum_{j=1}^{N}\sum_{\mu=1}^{M(N)} \xi_i^\nu \xi_j^\nu \xi_i^\mu \xi_j^\mu \ge 0$$

for all $i = 1, \ldots, N$ (here we have introduced the convention $sgn(0) = 1$ which does not really influence our calculations since the probability that the above sum equals zero vanishes for large N).
Hence for all $t \ge 0$

$$\begin{aligned} P(\xi^\nu \text{ is not stable }) &\le NP(\sum_{j=1}^{N}\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{M(N)} \xi_1^\nu \xi_j^\nu \xi_1^\mu \xi_j^\mu \le -N) \\ &\le Ne^{-tN} E\left(\exp(-t(\sum_{j=1}^{N}\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{M(N)} \xi_1^\nu \xi_j^\nu \xi_1^\mu \xi_j^\mu))\right) \\ &= Ne^{-tN}\left(E\left(\exp(-t(\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{M(N)} \xi_1^\nu \xi_2^\nu \xi_1^\mu \xi_2^\mu))\right)\right)^N \end{aligned} \tag{7}$$

where the second inequality is an application of the exponential Chebyshev–Markov inequality. Now putting $Y_\mu := \xi_1^\mu \xi_2^\mu$ and calculating the expectation in (7) leads to

$$E(\exp(-t\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{M(N)} \xi_1^\nu \xi_2^\nu \xi_1^\mu \xi_2^\mu)) = E(\exp(-t\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{M(N)} Y_\mu Y_\nu))$$

$$= \sum_{y_\mu=-1,+1} P(Y_1 = y_1)P(Y_2 = y_2|Y_1 = y_1)\cdots P(Y_\nu = y_\nu|Y_{\nu-1} = y_{\nu-1}) \times$$

$$\times \quad P(Y_{\nu+1} = y_{\nu+1}|Y_\nu = y_\nu)\cdots P(Y_M = y_M|Y_{M-1} = y_{M-1}) \exp(-t\sum_{\substack{\mu=1\\ \mu\neq\nu}}^{M(N)} y_\mu y_\nu)$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{\substack{y_\mu=-1,+1,\\ \mu\neq\nu}} R(y_1,y_2)e^{-ty_1}\cdots R(y_{\nu-1},,1)e^{-ty_{\nu-1}} \times \\
&\qquad \times R(1,y_{\nu+1})e^{-ty_{\nu+1}}\cdots R(y_{M-1},y_M)e^{-ty_M} \qquad (8)\\
&+ \frac{1}{2} \sum_{\substack{y_\mu=-1,+1,\\ \mu\neq\nu}} R(y_1,y_2)e^{ty_1}\cdots R(y_{\nu-1},,-1)e^{ty_{\nu-1}} \times \\
&\qquad \times R(-1,y_{\nu+1})e^{ty_{\nu+1}}\cdots R(y_{M-1},y_M)e^{ty_M}
\end{aligned}
$$

where $M := M(N)$ and R is the transition matrix for the Y_μ, i.e.

$$
R = \begin{pmatrix} p^2+(1-p)^2 & 2p(1-p) \\ 2p(1-p) & p^2+(1-p)^2 \end{pmatrix} =: \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}.
$$

Due to the symmetry of R and the Y_μ the two sums on the right hand side in (8) agree such that

$$
\begin{aligned}
& E(\exp(-t\sum_{\substack{\mu=1,\\ \mu\neq\nu}}^{M(N)} Y_\mu Y_\nu)) \\
&= \sum_{\substack{y_\mu=-1,+1,\\ \mu\neq\nu}} R(y_1,y_2)e^{-ty_1}\cdots R(y_{\nu-1},,1)e^{-ty_{\nu-1}} \times \\
&\qquad \times R(1,y_{\nu+1})e^{-ty_{\nu+1}}\cdots R(y_{M-1},y_M)e^{-ty_M} \\
&= \sum_{\substack{y_1=-1,1,\\ y_M=-1,1}} \Pi_L^{\nu-1}(y_1,1)\Pi_R^{M-\nu}(1,y_M)
\end{aligned}
$$

where

$$
\Pi_L := \begin{pmatrix} qe^{-t} & (1-q)e^{-t} \\ (1-q)e^{t} & qe^{t} \end{pmatrix}
$$

and

$$
\Pi_R := \begin{pmatrix} qe^{-t} & (1-q)e^{t} \\ (1-q)e^{-t} & qe^{t} \end{pmatrix}.
$$

Observe that $(\Pi_L)^t = \Pi_R$, implying that Π_L and Π_R have the same eigenvalues. These eigenvalues are

$$
\lambda_1 = q\cosh(t) + \sqrt{1-2q+q^2\cosh^2(t)} \qquad (9)
$$

and

$$
\lambda_2 = q\cosh(t) - \sqrt{1-2q+q^2\cosh^2(t)}.
$$

Now

$$
\begin{aligned}
&\sum_{y_1=-1,1,y_M=-1,1} \Pi_L^{\nu-1}(y_1,1)\Pi_R^{M-\nu}(1,y_M) \\
&= \left(\begin{pmatrix}1\\1\end{pmatrix}\Pi_L^{\nu-1}\begin{pmatrix}1\\0\end{pmatrix}\right) \times \left(\begin{pmatrix}1\\0\end{pmatrix}\Pi_R^{M-\nu}(1,\cdot)\begin{pmatrix}1\\1\end{pmatrix}\right) \leq \lambda_1^{M-1}
\end{aligned}
$$

which can be seen from bringing the matrices Π_L and Π_R into diagonal form and observing that then $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Pi_L^{\nu-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leq \lambda_1^{\nu-1}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Pi_R^{M-\nu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leq \lambda_1^{M-\nu}$. Since the above estimates hold true independently of the choice of ν we obtain

$$\begin{aligned} P(\exists \nu : \xi^\nu \text{ is not stable }) &\leq M(N) P(\xi^1 \text{ is not stable }) \\ &\leq M(N) N e^{-tN} \lambda_1^{(M(N)-1)N}. \end{aligned}$$

Expanding the root in (9) using $\sqrt{(1+x)} \leq 1 + \frac{x}{2}$ yields

$$\lambda_1 \leq q \cosh(t) + (1-q) + \frac{q^2 \sinh^2(t)}{2(1-q)}.$$

Moreover recalling that

$$\cosh(t) = 1 + \frac{t^2}{2} + \mathcal{O}(t^4)$$

leads to

$$\lambda_1 \leq 1 + t^2 \frac{q}{2(1-q)} + \mathcal{O}(t^4) \leq \exp(t^2 \frac{q}{2(1-q)})(1 + \mathcal{O}(t^4)).$$

Hence we arrive at

$$\begin{aligned} & P(\exists \nu : \xi^\nu \text{ is not stable }) \\ \leq\ & M(N) N \exp(-tN + t^2 M(N) N \frac{q}{2(1-q)})(1 + \mathcal{O}(t^4))^{M(N)N} \end{aligned}$$

where the last factor on the right hand side converges to 1, if t goes to zero fast enough (and then e.g. is bounded by 2 for all t small enough). Choosing $t = \frac{1-q}{qM(N)}$ gives

$$P(\exists \nu : \xi^\nu \text{ is not stable }) \leq M(N) N \exp(\frac{1-q}{2q} \frac{N}{M(N)})(1 + \mathcal{O}(t^4))^{M(N)N}.$$

So if $M(N) = \frac{N}{\gamma \log N}$ the last factor on the right hand side can be bounded by $\exp(const. \frac{(\log N)^4}{N^2})$ which is converging to one and therefore the choice of $\gamma > \frac{6q}{1-q} = \frac{3(p^2+(1-p)^2)}{p(1-p)}$ leads to the converging series $\sum \frac{1}{N^\kappa \log N}$ for an $\kappa > 1$ and thus proves part one of the theorem by the Borel–Cantelli Lemma. The choice of $\gamma > \frac{4q}{1-q} = \frac{2(p^2+(1-p)^2)}{p(1-p)}$ yields

$$P(\exists \nu : \xi^\nu \text{ is not stable }) \to 0$$

and therefore part two of the theorem. □

Theorem 3.1 in other words states that the patterns are fixed points of the gradient dynamics and hence are recognized if one starts with them. But just recalling patterns if they are presented without errors can hardly be called an associative memory. What one would like to have is that even if a pattern is corrupted by a certain percentage of noise the gradient dynamics is able to retrieve this pattern in spite of the noise. The following theorem shows that also noised patterns can be successfully reconstructed.

Theorem 3.2 *Let $r \in [0, \frac{1}{2})$ and for each $\nu = 1, \ldots, M(N)$ let $\tilde{\xi^\nu}$ be an element of the Hamming sphere of radius rN centered at ξ^ν. Assume the random patterns ξ^ν fulfill (3) and (4) and $M(N) = (1-2r)^2 \frac{N}{\gamma \log N}$.*
Then for the Hopfield model (1) the following assertions hold true:

1. *If $\gamma > \frac{3(p^2+(1-p)^2)}{p(1-p)}$*
$$P(\liminf_{N\to\infty}(\cap_{\nu=1}^{M(N)} T\tilde{\xi^\nu} = \xi^\nu)) = 1$$
i.e. the noised patterns are almost surely attracted.

2. *If $\gamma > \frac{2(p^2+(1-p)^2)}{p(1-p)}$*
$$P((\cap_{\nu=1}^{M(N)} T\tilde{\xi^\nu} = \xi^\nu)) = 1 - R_N$$
with $\lim_{N\to\infty} R_N = 0$.

Proof: Following the lines of the proof of Theorem 3.1 we obtain for any $\nu = 1, \ldots, M(N)$ and any $t \geq 0$ that

$$P((T\tilde{\xi^\nu})_i = -\xi_i^\nu) \leq \exp(-tN(1-2r) + t^2 M(N) N \frac{q}{2(1-q)})(1 + \mathcal{O}(t^4))^{M(N)N}$$

and hence

$$P(\cup_{\nu=1}^{M(N)} (T\tilde{\xi^\nu})_i = -\xi_i^\nu) \leq$$
$$M(N)N \exp(-tN(1-2r) + t^2 M(N) N \frac{q}{2(1-q)})(1 + \mathcal{O}(t^4))^{M(N)N}.$$

Optimizing in t and applying the Borel–Cantelli Lemma again proves the theorem. □

Remark 3.3 *The above estimates can probably be improved in some respects. Note however, that our bound on the storage capacity of the standard Hopfield model is a decreasing function in the correlation of the patterns (which in Theorems 3.1 and 3.2 appears as a factor $\frac{2p(1-p)}{p^2+(1-p)^2} \leq 1$).*

We now check, whether the above results are just artifacts of our notion of storage capacity or if the standard Hopfield model can even store "extensively many" patterns (i.e. the number of patterns grows proportional to N) provided that Newman's concept of storage is used. The latter is indeed the case.

Theorem 3.4 *Suppose that the random patterns fulfill (3) and (4). There exists an $\alpha_c > 0$ (depending on p) such that if $M(N) \leq \alpha_c N$, then there are $\varepsilon > 0$ and $0 < \delta < 1/2$ such that for the standard Hopfield model (1)*

$$P\left(\liminf_{N\to\infty}(\cap_{\nu=1}^{M(N)} \cap_{\sigma\in S_\delta(\xi^\nu)} (H_N(\sigma) \geq H_N(\xi^\nu) + \varepsilon N))\right) = 1$$

where $S_\delta(\xi^\nu)$ is the Hamming sphere of radius δN centered in ξ^ν.

Proof: Again the proof uses large deviation estimates. These can be carried out by a centering of the variables together with the idea of replacing the ξ_i^μ by appropriate Gaussian random variables which is a standard idea in the framework of the Hopfield model with independent patterns (see e.g [N88] or [BG92]) – but however, in our case technically quite involved.
Let us set

$$h_N(\sigma,\delta) := \inf_{\sigma' \in S_\delta(\sigma)} H_N(\sigma').$$

Then

$$\begin{aligned} & P\left(\{\cap_{\nu=1}^{M(N)}(h_N(\xi^\nu,\delta) \geq H_N(\xi^\nu)+\varepsilon N)\}^c\right) \\ = \; & P\left(\cup_{\nu=1}^{M(N)} \cup_{J:|J|=\delta N} H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \\ \leq \; & \sum_{J:|J|=\delta N} \sum_{\nu=1}^{M(N)} P\left(H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \end{aligned}$$

where ξ_J^ν denotes a configuration differing from ξ^ν exactly in the coordinates J and for convenience we have chosen δN to be an integer.
Estimating the probability on the right hand side of the above inequality again with the help of the exponential Chebyshev–Markov inequality we obtain for a fixed $1 \leq \nu \leq M(N)$ and all $t \geq 0$

$$\begin{aligned} & P\left(H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \\ = \; & P\left(-\frac{1}{2N}\sum_{\mu=1}^{M(N)}\sum_{i,j=1}^{N}(\xi_{J,i}^\nu \xi_{J,j}^\nu - \xi_i^\nu \xi_j^\nu))\xi_i^\mu \xi_j^\mu \leq \varepsilon N\right) \\ = \; & P\left(\frac{1}{N}\sum_{\mu=1}^{M(N)}\sum_{\substack{i\in J, j\notin J \\ i\notin J, j\in J}} \xi_i^\nu \xi_j^\nu \xi_i^\mu \xi_j^\mu \leq \varepsilon N\right) \\ = \; & P\left(\frac{2}{N}\sum_{\mu=1}^{M(N)}\sum_{i\in J, j\notin J} \xi_i^\nu \xi_j^\nu \xi_i^\mu \xi_j^\mu \leq \varepsilon N\right) \\ = \; & P\left(\frac{1}{N}\sum_{\mu\neq\nu}\sum_{i\in J, j\notin J} \xi_i^\nu \xi_j^\nu \xi_i^\mu \xi_j^\mu \leq (\varepsilon/2 - \delta(1-\delta))N\right) \\ \leq \; & e^{t\varepsilon' N} E\left(\exp\left(-\frac{t}{N}\sum_{\mu\neq\nu}\sum_{i\in J, j\notin J} \xi_i^\nu \xi_j^\nu \xi_i^\mu \xi_j^\mu\right)\right) \end{aligned}$$

where we have set $\varepsilon' = \varepsilon/2 - \delta(1-\delta)$ and we are again left with estimating the expectation of an exponential.

To this end let us assume that $\xi_i^\nu = 1$ for all $i = 1, \ldots, N$ (this can be done without loss of generality since the initial situation is completely symmetric). Moreover we will center the variables ξ_i^μ conditioned on their past. Thus we put

$$\overline{\xi_i^\mu} = \xi_i^\mu - (2p-1)\xi_i^{\mu-1}. \tag{10}$$

for $i = 1, \ldots, N$ and $\mu > \nu$. Observe that

$$\xi_i^\mu \xi_j^\mu = \overline{\xi_i^\mu}\,\overline{\xi_j^\mu} + (2p-1)(\xi_i^{\mu-1}\overline{\xi_j^\mu} + \overline{\xi_i^\mu}\xi_j^{\mu-1}) + (2p-1)^2 \xi_i^{\mu-1}\xi_j^{\mu-1}$$

Hence

$$\begin{aligned}
\sum_{i\in J, j\notin J}\sum_{\mu>\nu}^{M} \xi_i^\mu \xi_j^\mu &= \sum_{i\in J, j\notin J}\sum_{\mu>\nu}^{M}\sum_{n=0}^{M-\mu} (2p-1)^{2n}\overline{\xi_i^\mu}\,\overline{\xi_j^\mu} \\
&+ \sum_{i\in J, j\notin J}\sum_{\mu>\nu}^{M}\sum_{n=0}^{M-\mu} (2p-1)^{2n+1}(\xi_i^{\mu-1}\overline{\xi_j^\mu} + \overline{\xi_i^\mu}\xi_j^{\mu-1}) \\
&+ \sum_{i\in J, j\notin J}\sum_{n=0}^{M-\nu-1} (2p-1)^{2n}.
\end{aligned}$$

Centering also the variables in the second sum on the right hand side above gives

$$\begin{aligned}
&\sum_{i\in J, j\notin J}\sum_{\mu>\nu}^{M} \xi_i^\mu \xi_j^\mu \\
&= \sum_{i\in J, j\notin J}\left(\sum_{\mu_1,\mu_2>\nu}^{M} a_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}} + \sum_{\mu>\nu}^{M} a_{\mu,\nu}(\overline{\xi_i^\mu} + \overline{\xi_j^\mu}) + \sum_{n=0}^{M-\nu-1}(2p-1)^{2n}\right),
\end{aligned}$$

where

$$a_{\mu_1,\mu_2} := \sum_{n=0}^{M-\max\{\mu_1,\mu_2\}} (2p-1)^{2n+|\mu_1-\mu_2|} \tag{11}$$

for $\mu_1, \mu_2 \geq \nu$, $(\mu_1, \mu_2) \neq (\nu, \nu)$. Note that $a_{\mu_1,\mu_2} = a_{\mu_2,\mu_1}$.

For the summands with an index $\mu < \nu$ in $\sum_{i\in J, j\notin J}\sum_{\mu\neq\nu} \xi_i^\mu \xi_j^\mu$ we observe that the Markov chains $(\xi_i^\mu)_{\mu<\nu}$ and $(\xi_i^\mu)_{\mu>\nu}$, $i = 1, \ldots, N$ conditioned on $\xi_i^\nu = 1$ are independent and that reversing the chains $(\xi_i^\mu)_{\mu<\nu}$ $(i = 1, \ldots, N)$ does not change their distribution. So applying the same transformation as above to the reversed Markov chains $(\xi_i^\mu)_{\mu<\nu}$ $(i = 1, \ldots, N)$ yields

$$\begin{aligned}
&E(\exp(-\frac{t}{N}\sum_{i\in J, j\notin J}\sum_{\mu\neq\nu} \xi_i^\mu \xi_j^\mu)) = \exp(-\frac{t}{N}\sum_{i\in J, j\notin J}(\sum_{n=0}^{M-\nu-1}(2p-1)^{2n} + \sum_{n=0}^{\nu-1}(2p-1)^{2n})) \\
\times\ & E\left(\exp\left(-\frac{t}{N}\sum_{i\in J, j\notin J}\left(\sum_{\mu>\nu} a_{\mu,\nu}(\overline{\xi_i^\mu} + \overline{\xi_j^\mu}) + \sum_{\mu<\nu}\tilde{a}_{\mu,\nu}(\overline{\xi_i^\mu} + \overline{\xi_j^\mu}) + \right.\right.\right. \\
&\left.\left.\left. + \sum_{\mu_1,\mu_2>\nu} a_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}} + \sum_{\mu_1,\mu_2<\nu}\tilde{a}_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}}\right)\right)\right),
\end{aligned}$$

where

$$\tilde{a}_{\mu_1,\mu_2} := \sum_{n=0}^{\nu-1-\min\{\nu-\mu_1,\nu-\mu_2\}} (2p-1)^{2n+|\mu_1-\mu_2|}. \tag{12}$$

Using the independence of the initial part and the tail part of the Markov chains mentioned above together with Hölder's inequality to split up the moment generating function of the linear part from the moment generating function of the genuine quadratic form we obtain for all $\lambda > 1$

$$
\begin{aligned}
& E\left(\exp\left(-\frac{t}{N}\sum_{\mu\neq\nu}\sum_{i\in J, j\notin J}\xi_i^\nu\xi_j^\nu\xi_i^\mu\xi_j^\mu\right)\right) \\
\leq\ & \exp(-tN\delta(1-\delta)(\sum_{n=0}^{M-\nu-1}(2p-1)^{2n}+\sum_{n=0}^{\nu-1}(2p-1)^{2n})) \qquad (13)\\
\times\ & \left(E\left(\exp\left(-\frac{t}{N}\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}a_{\mu,\nu}\sum_{i\in J,j\notin J}(\overline{\xi_i^\mu}+\overline{\xi_j^\mu})\right)\right)\right)^{\frac{\lambda-1}{\lambda}}\times \\
\times\ & \left(E\left(\exp\left(-\frac{t}{N}\frac{\lambda}{\lambda-1}\sum_{\mu<\nu}\tilde{a}_{\mu,\nu}\sum_{i\in J,j\notin J}(\overline{\xi_i^\mu}+\overline{\xi_j^\mu})\right)\right)\right)^{\frac{\lambda-1}{\lambda}} \\
\times\ & \left(E\left(\exp\left(-\frac{t}{N}\lambda\sum_{i\in J,j\notin J}\sum_{\mu_1,\mu_2>\nu}a_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}}\right)\right)\right)^{\frac{1}{\lambda}} \\
\times\ & \left(E\left(\exp\left(-\frac{t}{N}\lambda\sum_{i\in J,j\notin J}\sum_{\mu_1,\mu_2<\nu}\tilde{a}_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}}\right)\right)\right)^{\frac{1}{\lambda}}.
\end{aligned}
$$

Now we estimate the factors on the right hand side of (13). First of all note that for M large enough (which is possible, since M is growing with N)

$$
(\sum_{n=0}^{M-\nu-1}(2p-1)^{2n}+\sum_{n=0}^{\nu-1}(2p-1)^{2n}))\geq\frac{1}{C'(1-(2p-1)^2)}.
$$

for any $C' > 1$

To treat the other terms let us agree on the following notation: With $E_I^{I'}$ (where $I \subset \{1,\ldots,N\}$ and $I' \subset \{1,\ldots,M\}$) we denote the integration with respect to those random variables ξ_i^μ with $i \in I$ and $\mu \in I'$. Especially, if we drop the upper or lower indices we will usually mean the expectation with respect to all the random variables occuring in the argument of the integral. By the independence of the coordinate processes and the identical distribution of the ξ_i^μ we obtain for the moment generating function of the linear part

$$
E\left(\exp\left(-\frac{t}{N}\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}a_{\mu,\nu}\sum_{i\in J,j\notin J}(\overline{\xi_i^\mu}+\overline{\xi_j^\mu})\right)\right)
$$

$$= \left[E\left(\exp(-\frac{t}{N}\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}a_{\mu,\nu}\overline{\xi_1^\mu})\right)\right]^{\delta(1-\delta)N^2}$$

Moreover

$$\begin{aligned}
& E\left(\exp\left(-t\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}a_{\mu,\nu}\overline{\xi_1^\mu}\right)\right) \\
= \; & E^{\nu<\mu\leq M-1}\left(\exp\left(-t\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}^{M-1}a_{\mu,\nu}\overline{\xi_1^\mu}\right)\right)E^M\left(\exp(-t\frac{\lambda}{\lambda-1}a_{M,\nu}\overline{\xi_1^M})\right) \\
= \; & E^{\nu<\mu\leq M-1}\left(\exp\left(-t\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}^{M-1}a_{\mu,\nu}\overline{\xi_1^\mu}\right)\right)\times \\
\times \; & \left(p\exp(-2t\frac{\lambda}{\lambda-1}a_{M,\nu}(1-p)\xi_1^{M-1})+(1-p)\exp(2t\frac{\lambda}{\lambda-1}a_{M,\nu}p\xi_1^{M-1}\right) \\
\leq \; & E^{\nu<\mu\leq M-1}\left(\exp\left(-t\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}^{M-1}a_{\mu,\nu}\overline{\xi_1^\mu}\right)\right)\cosh(t\frac{\lambda}{\lambda-1}a_{M,\nu}(1+|2p-1|)\xi_1^{M-1}) \\
\leq \; & E^{\nu<\mu\leq M-1}\left(\exp\left(-t\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}^{M-1}a_{\mu,\nu}\overline{\xi_1^\mu}\right)\right)\exp(\frac{1}{2}t^2(\frac{\lambda}{\lambda-1})^2a_{M,\nu}^2(1+|2p-1|)^2)
\end{aligned}$$

where we have used $|\xi_1^{M-1}|=1$,

$$p\exp(-2(1-p)t)+(1-p)\exp(2pt)\leq\cosh((1+|2p-1|)t)$$

for all $0<p<1$ and all $t\in\mathbb{R}$ and finally

$$\cosh(x)\leq\exp(x^2/2).$$

Iterating these estimates gives

$$\begin{aligned}
& E\left(\exp\left(-t\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}a_{\mu,\nu}\overline{\xi_1^\mu}\right)\right) \\
\leq \; & \exp(\frac{1}{2}t^2(\frac{\lambda}{\lambda-1})^2(1+|2p-1|)^2\sum_{\mu>\nu}^{M}a_{\mu,\nu}^2) \\
\leq \; & \exp\left(\frac{1}{2}t^2(\frac{\lambda}{\lambda-1})^2(1+|2p-1|)^2\frac{1}{(1-(2p-1)^2)^3}\right).
\end{aligned}$$

So altogether (replacing t by $\frac{t}{N}$) we arrive at

$$\begin{aligned}
& E\left(\exp\left(-\frac{t}{N}\frac{\lambda}{\lambda-1}\sum_{\mu>\nu}a_{\mu,\nu}\sum_{i\in J,j\notin J}(\overline{\xi_i^\mu}+\overline{\xi_j^\mu})\right)\right) \\
\leq \; & \exp\left(\frac{1}{2}t^2\delta(1-\delta)(\frac{\lambda}{\lambda-1})^2(1+|2p-1|)^2\frac{1}{(1-(2p-1)^2)^3}\right)
\end{aligned}$$

Thus applying the same techniques to the second linear term on the right hand side of (13) we obtain

$$\begin{aligned}
& \left(E\left(\exp\left(-\frac{t}{N}\frac{\lambda}{\lambda-1}\sum_{\mu>\nu} a_{\mu,\nu} \sum_{i\in J, j\notin J} (\overline{\xi_i^\mu} + \overline{\xi_j^\mu}) \right)\right)\right)^{\frac{\lambda-1}{\lambda}} \times \\
\times \quad & \left(E\left(\exp\left(-\frac{t}{N}\frac{\lambda}{\lambda-1}\sum_{\mu<\nu} \tilde{a}_{\mu,\nu} \sum_{i\in J, j\notin J} (\overline{\xi_i^\mu} + \overline{\xi_j^\mu}) \right)\right)\right)^{\frac{\lambda-1}{\lambda}} \\
\leq \quad & \exp\left(t^2\delta(1-\delta)(\frac{\lambda}{\lambda-1})(1+|2p-1|)^2 \frac{1}{(1-(2p-1)^2)^3} \right)
\end{aligned}$$

We will see that due to our final choice of t this factor will have a negligible contribution to the final estimate.

The moment generating function of the quadratic form is treated similarly using the independence of the ξ_i^μ for different i to replace them by Gaussian random variables:

$$\begin{aligned}
& E\left(\exp\left(-\frac{t}{N}\lambda \sum_{i\in J, j\notin J} \sum_{\mu_1,\mu_2>\nu} a_{\mu_1,\mu_2} \overline{\xi_i^{\mu_1}}\, \overline{\xi_j^{\mu_2}} \right)\right) \\
= \quad & E^{\nu<\mu_1,\mu_2\leq M-1} E_{J^c}^M \left(\exp\left(-\frac{t}{N}\lambda \sum_{i\in J, j\notin J} \sum_{\mu_2>\nu}^{M} \sum_{\mu_1>\nu}^{M-1} a_{\mu_1,\mu_2} \overline{\xi_i^{\mu_1}}\, \overline{\xi_j^{\mu_2}} \right)\right. \\
& \left. E_J^M \left(\exp(-\frac{t}{N}\lambda \sum_{i\in J, j\notin J} \overline{\xi_i^M} \sum_{\mu_2=\nu+1}^{M} a_{M,\mu_2} \overline{\xi_j^{\mu_2}}) \right)\right) \\
= \quad & E^{\nu<\mu_1,\mu_2\leq M-1} E_{J^c}^M \left(\exp\left(-\frac{t}{N}\lambda \sum_{i\in J, j\notin J} \sum_{\mu_2>\nu}^{M} \sum_{\mu_1>\nu}^{M-1} a_{\mu_1,\mu_2} \overline{\xi_i^{\mu_1}}\, \overline{\xi_j^{\mu_2}} \right)\right. \\
& \left. \prod_{i\in J} E_{\{i\}}^M \left(\exp(-\frac{t}{N}\lambda \overline{\xi_i^M} \sum_{j\notin J} \sum_{\mu_2=\nu+1}^{M} a_{M,\mu_2} \overline{\xi_j^{\mu_2}}) \right)\right) \\
\leq \quad & E^{\nu<\mu_1,\mu_2\leq M-1} E_{J^c}^M \left(\exp\left(-\frac{t}{N}\lambda \sum_{i\in J, j\notin J} \sum_{\mu_2>\nu}^{M} \sum_{\mu_1>\nu}^{M-1} a_{\mu_1,\mu_2} \overline{\xi_i^{\mu_1}}\, \overline{\xi_j^{\mu_2}} \right)\right) \times \\
\times \quad & \prod_{i\in J} \exp\left(\frac{1}{2}\frac{t^2}{N^2}\lambda^2(1+|2p-1|)^2 (\sum_{j\notin J} \sum_{\mu_2=\nu+1}^{M} a_{M,\mu_2} \overline{\xi_j^{\mu_2}})^2 \right) \\
= \quad & E^{\nu<\mu_1,\mu_2\leq M-1} E_{J^c}^M \left(\exp\left(-\frac{t}{N}\lambda \sum_{i\in J, j\notin J} \sum_{\mu_2>\nu}^{M} \sum_{\mu_1>\nu}^{M-1} a_{\mu_1,\mu_2} \overline{\xi_i^{\mu_1}}\, \overline{\xi_j^{\mu_2}} \right)\right) \times \\
\times \quad & \prod_{i\in J} E_{z_i^M} \exp\left(z_i^M \frac{t}{N}\lambda(1+|2p-1|) \sum_{i\in J, j\notin J} \sum_{\mu_2=\nu+1}^{M} a_{M,\mu_2} \overline{\xi_j^{\mu_2}} \right)
\end{aligned}$$

$$= E^{\nu<\mu_1,\mu_2\leq M-1} E^M_{J^c}\left(\exp\left(-\frac{t}{N}\lambda\sum_{i\in J,j\notin J}\sum_{\mu_2>\nu}^{M}\sum_{\mu_1>\nu}^{M-1} a_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}}\right)\right)\times$$

$$\times\; E_{z_J^M}\exp\left(\frac{t}{N}\lambda(1+|2p-1|)\sum_{i\in J,j\notin J}\sum_{\mu_2=\nu+1}^{M} a_{M,\mu_2} z_i^M\overline{\xi_j^{\mu_2}}\right)$$

where z_i^M are Gaussian random variables with expectation 0 and identity covariance matrix independent of the ξ_i^μ, $E_{z_i^M}$ denotes the expectation with respect to z_i^M, and finally $E_{z_J^M}$ denotes the expectation with respect to the vector $(z_i^M)_{i\in J}$. Here we have used the well known identity

$$\exp(\frac{1}{2}x^2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(xy-\frac{1}{2}y^2)dy.$$

Interchanging the order of integration and using the above technique on every ξ_i^μ we are now able to consecutively replace all the variables ξ_i^μ by Gaussian random variables z_i^μ with expectation zero and identity covariance matrix. This leads to

$$E\left(\exp\left(-\frac{t}{N}\lambda\sum_{i\in J,j\notin J}\sum_{\mu_1,\mu_2>\nu} a_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}}\right)\right)$$

$$\leq\; E_z\left(\exp\left(\frac{t}{N}\lambda(1+|2p-1|)^2\sum_{i\in J,j\notin J}\sum_{\mu_1,\mu_2>\nu} a_{\mu_1,\mu_2} z_i^{\mu_1} z j^{\mu_2}\right)\right)$$

$$\leq\; E_z\left(\exp\left(t\lambda(1+|2p-1|)^2\sqrt{\delta(1-\delta)}\sum_{\mu_1,\mu_2>\nu} a_{\mu_1,\mu_2} z^{\mu_1} z^{\mu_2}\right)\right)$$

$$=\; E_z\left(\exp\frac{1}{2}\left(t\lambda(1+|2p-1|)^2\sqrt{\delta(1-\delta)}\langle z,\hat{A}z\rangle\right)\right)$$

where (by normalizing) the $(z^\mu)_{\mu=\nu+1,\ldots,M}$ are now Gaussian random variables with expectation 0 and identity covariance matrix, z denotes the vector of the z_i^μ and E_z is integration with respect to z. Finally $\hat{A}$ is an $2(M-\nu)\times 2(M-\nu)$-matrix with entries

$$\hat{A} = \left(\begin{array}{c|c} 0 & A \\ \hline A & 0 \end{array}\right)$$

and the $(M-\nu)\times(M-\nu)$-matrix A is given by

$$A = (A_{\mu_1,\mu_2}) = (a_{\mu_1-\nu,\mu_2-\nu}).$$

.

Observe that the above integral only exists if t is small enough (i.e. if $Id - t\lambda(1+|2p-1|)^2\sqrt{\delta(1-\delta)}\hat{A}$ is positive definite) and in this case it equals the

inverse of the square-root of the determinant of $Id - t\lambda(1+|2p-1|)^2\sqrt{\delta(1-\delta)}\hat{A}$. On the other hand

$$\begin{aligned}\det(Id - t\lambda(1+|2p-1|)\sqrt{(\delta(1-\delta)}\hat{A}) &= \prod_{i=1}^{2(M-\nu)} \varrho_i \\ &= \prod_{i=1}^{2(M-\nu)} (1 - t\lambda(1+|2p-1|)^2\sqrt{\delta(1-\delta)}\alpha_i)\end{aligned}$$

where the ϱ_i are the eigenvalues of $Id - t\lambda(1+|2p-1|)^2\sqrt{\delta(1-\delta)}\hat{A}$ and the α_i are the eigenvalues of $\hat{A}$. Moreover note that $\hat{A}$ has a symmetric spectrum, i.e. if α_i is an eigenvalue of $\hat{A}$ then so is $-\alpha_i$. Thus

$$\begin{aligned}\det(Id - t\lambda(1+|2p-1|)\sqrt{(\delta(1-\delta)}\hat{A}) &= \prod_{i=1}^{M-\nu} (1 - t^2\lambda^2(1+|2p-1|)^4\delta(1-\delta)\alpha_i^2) \\ &\geq (1 - t^2\lambda^2(1+|2p-1|)^4\delta(1-\delta)\alpha_{\max}^2)^{M-\nu}\end{aligned}$$

where the product is taken over all non-negative eigenvalues and $\alpha_{\max}$ denotes the maximal eigenvalue of $\hat{A}$. Bounding the maximal eigenvalue by Gershgorin's theorem, i.e.

$$\alpha_{\max} \leq \max_{\mu_1} \sum_{\mu_2} |a_{\mu_1,\mu_2}| \leq \frac{1}{1-(2p-1)^2}\frac{2}{1-|2p-1|}$$

we arrive at

$$\begin{aligned}& E_z\left(\exp\left(\frac{t}{N}\lambda(1+|2p-1|)\sqrt{\delta(1-\delta)}\sum_{\mu_1,\mu_2>\nu} a_{\mu_1,\mu_2}z^{\mu_1}z^{\mu_2}\right)\right) \\ \leq\ & \left(\frac{1}{\sqrt{1 - t^2\lambda^2(1+|2p-1|)^2\delta(1-\delta)(\frac{1}{1-(2p-1)^2}\frac{2}{1-|2p-1|})^2}}\right)^{M-\nu}.\end{aligned}$$

where we have assumed that t is so small that the latter quantity is real.

Thus repeating the estimate for the moment generating function of the second quadratic form and setting $M = \alpha N$

$$\begin{aligned}& P\left(H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \\ \leq\ & \inf_{t^*\geq t\geq 0} \exp\left(t\varepsilon' N - tN\delta(1-\delta)\frac{1}{C'(1-(2p-1)^2)}\right) \\ \times\ & \exp\left(-\log\left(1 - t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{4(1+|2p-1|)^2}{(1-|2p-1|)^2}\right)\frac{M-\nu}{2}\right) \\ \times\ & \exp\left(-\log\left(1 - t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{4(1+|2p-1|)^2}{(1-|2p-1|)^2}\right)\frac{\nu}{2}\right) \\ \times\ & \exp\left(t^2\delta(1-\delta)(\frac{\lambda}{\lambda-1})(1+|2p-1|)^2\frac{1}{(1-(2p-1)^2)^3}\right) \\ =\ & \inf_{t^*\geq t\geq 0} \exp\left(t\varepsilon' N - tN\delta(1-\delta)\frac{1}{C'(1-(2p-1)^2)}\right)\end{aligned}$$

$$\begin{aligned}
&\times \quad \exp\left(-\log\left(1-t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{4(1+|2p-1|)^2}{(1-|2p-1|)^2}\right)\frac{M}{2}\right)\\
&\times \quad \exp\left(t^2\delta(1-\delta)(\frac{\lambda}{\lambda-1})(1+|2p-1|)^2\frac{1}{(1-(2p-1)^2)^3}\right)
\end{aligned}$$

where $t^* = \frac{(1-(2p-1)^2)(1-|2p-1|)}{2\lambda(1+|2p-1|)}\sqrt{\frac{1}{\delta(1-\delta)}}$.
Finally by Stirlings formula and the above estimate

$$\begin{aligned}
&\sum_{J:|J|=\delta N}\sum_{\nu=1}^{M(N)} P\left(H_N(\xi_J^\nu)-H_N(\xi^\nu)\le\varepsilon N\right)\\
\le\ & M(N)\binom{N}{\delta N}\exp\left(t\varepsilon' N - tN\delta(1-\delta)\frac{1}{C'(1-(2p-1)^2)}\right)\times\\
\times\ & \exp\left(-\log(1-t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{4(1+|2p-1|)^2}{(1-|2p-1|)^2})\frac{\alpha}{2}N\right)\\
\times\ & \exp\left(t^2\delta(1-\delta)(\frac{\lambda}{\lambda-1})(1+|2p-1|)^2\frac{1}{(1-(2p-1)^2)^3}\right)\\
\le\ & \alpha N \inf_{t^*\ge t\ge 0}\exp\left((-\delta\log\delta-(1-\delta)\log(1-\delta))N\right)\times\\
\times\ & \exp\left(t\varepsilon' N - tN\delta(1-\delta)\frac{1}{C'(1-(2p-1)^2)}\right)\\
\times\ & \exp\left(-\log(1-t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{4(1+|2p-1|)^2}{(1-|2p-1|)^2})\frac{\alpha}{2}N\right)\\
\times\ & \exp\left(t^2\delta(1-\delta)(\frac{\lambda}{\lambda-1})(1+|2p-1|)^2\frac{1}{(1-(2p-1)^2)^3}\right)
\end{aligned}$$

and we have to find an admissible t (i.e. $0\le t\le t^*$) and values of δ and α such that the above exponent becomes negative. To this end first of all note that for all admissible t

$$\exp\left(t^2\delta(1-\delta)(\frac{\lambda}{\lambda-1})(1+|2p-1|)^2\frac{1}{(1-(2p-1)^2)^3}\right)=\mathcal{O}(1)$$

and therefore this term does not influence the convergence.

Moreover if $t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{4(1+|2p-1|)^2}{(1-|2p-1|)}\le 3/4$

$$\begin{aligned}
&\frac{1}{\sqrt{1-t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{4(1+|2p-1|)^2}{(1-|2p-1|)^2}}}\\
\le\ & \exp\left(4t^2\lambda^2\frac{\delta(1-\delta)}{(1-(2p-1)^2)^2}\frac{(1+|2p-1|)^2}{(1-|2p-1|)^2}\right).
\end{aligned}$$

and hence the essential argument of the above exponent becomes

$$(-\delta\log\delta-(1-\delta)\log(1-\delta))N+t\varepsilon' N-tN\delta(1-\delta)\frac{1}{C'(1-(2p-1)^2)}$$

$$
\begin{aligned}
&- \log(1 - t^2\lambda^2 \frac{\delta(1-\delta)}{(1-(2p-1)^2)^2} \frac{4(1+|2p-1|)^2}{(1-|2p-1|)^2}) \\
&\leq (-\delta\log\delta - (1-\delta)\log(1-\delta))N + t\varepsilon' N - tN\delta(1-\delta)\frac{1}{C'(1-(2p-1)^2)} \\
&+ 4t^2\lambda^2 \frac{\delta(1-\delta)}{(1-(2p-1)^2)^2} \frac{(1+|2p-1|)^2}{(1-|2p-1|)^2}\alpha N
\end{aligned}
$$

if

$$
t \leq t^{**} := \frac{(1-(2p-1)^2)(1-|2p-1|)}{4\lambda(1+|2p-1|)} \sqrt{\frac{3}{\delta(1-\delta)}}.
$$

Choosing ε sufficiently small the exponent is minimized by a t which is close to

$$
t_{\min} = \frac{1}{\alpha} \frac{1}{8\lambda^2(1+|2p-1|)^2}((1-(2p-1)^2) + \frac{1}{C'})(1-(2p-1)^2)(1-|2p-1|)^2.
$$

Observe that $t_{\min} \leq t^{**}$ if

$$
\alpha \geq \sqrt{\delta(1-\delta)} \frac{1}{\sqrt{3}\lambda(1+|2p-1|)}(1-(2p-1)^2 + \frac{1}{C'})(1-|2p-1|). \tag{14}
$$

On the other hand inserting $t_{\min}$ into the essential part of the exponent and choosing ε sufficiently small gives

$$
\begin{aligned}
&(-\delta\log\delta - (1-\delta)\log(1-\delta))N + t_{\min}\varepsilon' N - t_{\min}N\delta(1-\delta)\frac{1}{C'(1-(2p-1)^2)} \\
&+ t_{\min}^2\lambda^2 \frac{\delta(1-\delta)}{1-(2p-1)^2} \frac{2(1+|2p-1|)^2}{1-|2p-1|}\alpha N \qquad (15) \\
&\leq (-\delta\log\delta - (1-\delta)\log(1-\delta))N - \gamma \frac{\delta(1-\delta)(1-|2p-1|)^2(1-(2p-1)^2+\frac{1}{C'})^2}{16\lambda^2(1+|2p-1|)^2}\frac{1}{\alpha}
\end{aligned}
$$

with $\gamma < 1$ and close to 1 (as ε becomes small). The right hand side of this inequality becomes negative when δ and α become small appropriately. To check whether this can be done in agreement with (14) we insert

$$
\alpha = \sqrt{\delta(1-\delta)} \frac{1}{\sqrt{3}\lambda(1+|2p-1|)}(1-(2p-1)^2 + \frac{1}{C'})(1-|2p-1|)
$$

into the right hand side of (15) and obtain

$$
\left(-\frac{\sqrt{3}\gamma(1-(2p-1)^2+\frac{1}{C'}}{16\lambda(1+|2p-1|)}(1-|2p-1|)\sqrt{\delta(1-\delta)} - \delta\log\delta - (1-\delta)\log(1-\delta) \right) N. \tag{16}
$$

As it is quickly checked that for each positive constant C there is an interval $[0,r]$ (depending on C, of course) such that

$$
C\sqrt{\delta(1-\delta)} \geq -\delta\log\delta - (1-\delta)\log(1-\delta)
$$

for all $\delta \in [0,r]$, the above exponent becomes negative if we choose δ small enough and e.g. α as the right hand side of (14). This completes the proof of the theorem. □

Remark 3.5 *Observe that the bound on the moment generating function in (16) as well as the bound on α in (15) depends on p mainly via the factor $(1-|2p-1|)$ (the other terms containing p are bounded from above and away from 0) which converges to zero for p close to one or close to zero and therefore can only deteriorate the bounds for δ or α (allowing smaller α's or δ's only) for large correlations. However, this might either indicate that the Hopfield model has problems to store patterns with large correlations or it just shows that our estimates get worse for large p (which is probably true). A rigorous analysis about how the storage capacity behaves for $p \to 1$ or $p \to 0$ seems quite difficult to obtain, since it requires e.g. bounds on the moment generating function that are uniform in p.*

4 On the Capacity of a Class of Generalized Hopfield Models

The previous section may have supported the idea that some essential features of the standard Hopfield model have to be changed to achieve agreement with the information theoretical picture that a reasonable neural network should be able to store more correlated data than independent ones. We will follow this strategy in this section by analyzing the storage capacity of a class of generalized Hopfield models as introduced in Section 2 (2). Of course, the properties of these models depend sensitively on the choice of the geometry G. Led by the picture behind the pseudo-inverse rule which is idea that the standard Hopfield model works the better the "more orthogonal" the patterns are, we choose the geometry G in such a way that the correlated patterns are "most orthogonal" in average, i.e. we choose G as the inverse of the covariance matrix C given by

$$C_{\mu\nu} = E(\xi_1^\nu \xi_1^\mu). \tag{17}$$

With this setting we are able to prove the following theorem:

Theorem 4.1 *Assume the random patterns ξ^ν fulfill (3) and (4) and $M(N) = \frac{N}{\gamma \log N}$.*
Then for the generalized Hopfield model (2) with geometry $G = C^{-1}$, where the matrix C is given by (17) the following assertions hold true

1. *If $\gamma > 6\left(\frac{1-(2p-1)^2}{1+(2p-1)^2}\right)^2$*

$$P(\liminf_{N\to\infty}(\cap_{\nu=1}^{M(N)} T\xi^\nu = \xi^\nu)) = 1$$

i.e. the patterns are almost surely stable.

2. *If $\gamma > 4\left(\frac{1-(2p-1)^2}{1+(2p-1)^2}\right)^2$*

$$P((\cap_{\nu=1}^{M(N)} T\xi^\nu = \xi^\nu)) = 1 - R_N$$

with $\lim_{N\to\infty} R_N = 0$.

Proof: Basically the ideas of the proof are the same as those of the proof of Theorem 3.1. Just due to the more difficult geometry the calculation of the moment generating function becomes more involved.
Fix $1 \leq \nu \leq M(N)$. Then ξ^ν is stable if and only if $T(\xi_i^\nu) = \xi_i^\nu$ for all $1 \leq i \leq N$. Thus

$$P(\xi^\nu \text{ is not stable }) \leq NP(T\xi_1^\nu = -\xi_1^\nu) = NP(\sum_{j=1}^{N} \sum_{\mu_1,\mu_2} \xi_1^\nu \xi_j^\nu \xi_1^{\mu_1} \xi_j^{\mu_2} G_{\mu_1,\mu_2} < 0).$$

Note that our choice of the patterns, i.e. (3) and (4), implies that

$$C = \begin{pmatrix} 1 & a & \cdots & \cdots & a^{M-1} \\ a & 1 & a & \cdots & a^{M-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a^{M-1} & \cdots & \cdots & a & 1 \end{pmatrix}$$

with $a = 2p - 1$ and thus

$$G = C^{-1} = \frac{1}{1-a^2} \begin{pmatrix} 1 & -a & 0 & \cdots & \cdots & 0 \\ -a & 1+a^2 & -a & 0 & \cdots & 0 \\ 0 & -a & 1+a^2 & -a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -a & 1+a^2 & -a \\ 0 & \cdots & \cdots & 0 & -a & 1 \end{pmatrix}.$$

Hence again applying the exponential Chebyshev–Markov inequality we obtain for all $t \geq 0$ and for $1 < \nu < M$

$$\begin{aligned} P(\xi^\nu \text{ is not stable }) &\leq NP(\sum_{j=1}^{N} \sum_{(\mu_1,\mu_2)\neq(\nu,\nu)} \xi_1^\nu \xi_j^\nu \xi_1^{\mu_1} \xi_j^{\mu_2} G_{\mu_1,\mu_2} \leq -NG_{\nu\nu}) \\ &\leq Ne^{-tN\frac{1+a^2}{1-a^2}} \left(E \exp(-t \sum_{(\mu_1,\mu_2)\neq(\nu,\nu)} \xi_1^\nu \xi_2^\nu \xi_1^{\mu_1} \xi_2^{\mu_2} G_{\mu_1,\mu_2}) \right)^N \end{aligned}$$

and we are led to calculating the integral

$$E \exp(-t \sum_{(\mu_1,\mu_2)\neq(\nu,\nu)} \xi_1^\nu \xi_2^\nu \xi_1^{\mu_1} \xi_2^{\mu_2} G_{\mu_1,\mu_2}).$$

Due to the structure of the matrix G this computation can be carried out similarly to the computation of the moment generating function in the proof of Theorem 3.1. To simplify notation we may without loss of generality assume that $\xi_1^\nu = \xi_2^\nu = 1$. Then

$$E \exp \ (\ -t \sum_{(\mu_1,\mu_2)\neq(\nu,\nu)} \xi_1^\nu \xi_2^\nu \xi_1^{\mu_1} \xi_2^{\mu_2} G_{\mu_1,\mu_2})$$

$$
\begin{aligned}
= \sum_{\substack{x_1^\mu=-1,+1 \\ x_1^\mu=-1,+1}} \quad & P((\xi_1^1,\xi_2^1)=(x_1^1,x_2^1))P((\xi_1^2,\xi_2^2)=(x_1^2,x_2^2)|(\xi_1^1,\xi_2^1)=(x_1^1,x_2^1))\times \\
& \exp(-t(x_1^1x_2^1G_{11}+x_1^1x_2^2G_{12}+x_1^2x_2^1G_{21}))\times \\
& P((\xi_1^3,\xi_2^3)=(x_1^3,x_2^3)|(\xi_1^2,\xi_2^2)=(x_1^2,x_2^2))\times \\
& \exp(-t(x_1^2x_2^2G_{22}+x_1^2x_2^3G_{23}+x_1^3x_2^2G_{32}))\times \\
& \cdots P((\xi_1^\nu,\xi_2^\nu)=(1,1)|(\xi_1^{\nu-1},\xi_2^{\nu-1})=(x_1^{\nu-1},x_2^{\nu-1}))\times \\
& \exp(-t(x_1^{\nu-1}x_2^{\nu-1}G_{\nu-1,\nu-1}+x_1^{\nu-1}x_2^{\nu}G_{\nu-1,\nu}+x_1^{\nu}x_2^{\nu-1}G_{\nu,\nu-1}))\times \\
& P((\xi_1^{\nu+1},\xi_2^{\nu+1})=(x_1^{\nu+1},x_2^{\nu+1})|(\xi_1^{\nu},\xi_2^{\nu})=(1,1))\times \\
& \exp(-t(x_1^{\nu+1}x_2^{\nu+1}G_{\nu+1,\nu+1}+x_1^{\nu+1}x_2^{\nu}G_{\nu+1,\nu}+x_1^{\nu}x_2^{\nu+1}G_{\nu,\nu+1}))\times \\
& \cdots P((\xi_1^M,\xi_2^M)=(x_1^M,x_2^M)|(\xi_1^{M-1},\xi_2^{M-1})=(x_1^{M-1},x_2^{M-1}))\times \\
& \exp(-t(x_1^Mx_2^MG_{M,M}+x_1^Mx_2^{M-1}G_{M,M-1}+x_1^{M-1}x_2^MG_{M-1,M}))\times \\
= \sum_{\substack{(x_1^1,x_2^1) \\ (x_1^M,x_2^M)}} \quad & \Gamma_L^{\nu-1}((x_1^1,x_2^1),(1,1))\Gamma_R^{M-\nu}((1,1),(x_1^M,x_2^M))\times \qquad (18) \\
& \exp(t(G_{22}-G_{11})(x_1^1x_2^1+x_1^Mx_2^M))
\end{aligned}
$$

where the lines (and columns) of the matrices Γ_L and Γ_R are indexed by $(1,1),(1,-1),(-1,1)(-1,-1)$ and the matrices themselves are given by

$$
\Gamma_L = \begin{pmatrix}
p^2e^{-t(G_{22}+2G_{21})} & p(1-p)e^{-tG_{22}} & p(1-p)e^{-tG_{22}} & (1-p)^2e^{-t(G_{22}-2G_{21})} \\
p(1-p)e^{tG_{22}} & p^2e^{t(G_{22}+2G_{21})} & (1-p)^2e^{t(G_{22}-2G_{21})} & p(1-p)e^{tG_{22}} \\
p(1-p)e^{tG_{22}} & (1-p)^2e^{t(G_{22}-2G_{21})} & p^2e^{t(G_{22}+2G_{21})} & p(1-p)e^{tG_{22}} \\
(1-p)^2e^{-t(G_{22}-2G_{21})} & p(1-p)e^{-tG_{22}} & p(1-p)e^{-tG_{22}} & p^2e^{-t(G_{22}+2G_{21})}
\end{pmatrix}
$$

and

$$
\Gamma_R = \begin{pmatrix}
p^2e^{-t(G_{22}+2G_{21})} & p(1-p)e^{tG_{22}} & p(1-p)e^{tG_{22}} & (1-p)^2e^{-t(G_{22}-2G_{21})} \\
p(1-p)e^{-tG_{22}} & p^2e^{t(G_{22}+2G_{21})} & (1-p)^2e^{t(G_{22}-2G_{21})} & p(1-p)e^{-tG_{22}} \\
p(1-p)e^{-tG_{22}} & (1-p)^2e^{t(G_{22}-2G_{21})} & p^2e^{t(G_{22}+2G_{21})} & p(1-p)e^{-tG_{22}} \\
(1-p)^2e^{-t(G_{22}-2G_{21})} & p(1-p)e^{tG_{22}} & p(1-p)e^{tG_{22}} & p^2e^{-t(G_{22}+2G_{21})}
\end{pmatrix}
$$

which is due to the symmetry of the matrix G and the fact that almost all of its diagonal elements are equal (indeed the exponential term in (18) stems from the fact that for G_{11} and $G_{M,M}$ mild corrections have to be carried out). Observe that $(\Gamma_L)^t = \Gamma_R$ such that Γ_L and Γ_R have the same eigenvalues. In order to calculate these eigenvalues we express p and G_{22} and G_{21} in terms of the covariance $a = 2p-1$. The characteristic polynomial of the matrix Γ_L factorizes into two quadratic polynomials which are

$$
\begin{aligned}
P(x) &= x^2 + \frac{x}{2}\left((a-1)^2\cosh(t\frac{a+1}{a-1}) - (a+1)^2\cosh(t\frac{a-1}{a+1})\right) \\
&- \frac{1}{8}(a^2-1)^2\cosh\left(\frac{4at}{a^2-1}\right) + \frac{1}{8}(a^4+6a^2+1)
\end{aligned}
$$

and

$$\begin{aligned} Q(x) &= x^2 - \frac{x}{2}\left((a-1)^2\cosh(t\frac{a+1}{a-1}) + (a+1)^2\cosh(t\frac{a-1}{a+1})\right) \\ &+ \frac{1}{8}(a^2-1)^2\cosh\left(\frac{4at}{a^2-1}\right) - \frac{1}{8}(a^4-10a^2+1). \end{aligned}$$

Hence the eigenvalues are

$$\begin{aligned} \lambda_1 &= \frac{1}{4}\left((a+1)^2\cosh(t\frac{a-1}{a+1}) - (a-1)^2\cosh(t\frac{a+1}{a-1})\right) \\ &+ \frac{1}{4}\left((a+1)^2\sinh(t\frac{a-1}{a+1}) - (a-1)^2\sinh(t\frac{a+1}{a-1})\right),, \end{aligned}$$

$$\begin{aligned} \lambda_2 &= \frac{1}{4}\left((a+1)^2\cosh(t\frac{a-1}{a+1}) - (a-1)^2\cosh(t\frac{a+1}{a-1})\right) \\ &- \frac{1}{4}\left((a+1)^2\sinh(t\frac{a-1}{a+1}) - (a-1)^2\sinh(t\frac{a+1}{a-1})\right), \end{aligned}$$

$$\begin{aligned} \lambda_3 &= \frac{1}{4}(a-1)^2\cosh\left(t\frac{a+1}{a-1}\right) + \frac{1}{4}(a+1)^2\cosh\left(t\frac{a-1}{a+1}\right) \\ &+ \frac{1}{8}\left(2(a-1)^4\cosh(\frac{2t(a+1)^2}{a^2-1}) + 2(a+1)^4\cosh(\frac{2t(a-1)^2}{a^2-1})\right. \\ &- \left. 4(a^2-1)^2\cosh(\frac{4ta}{a^2-1}) + 4(a^2-1)^2\cosh(\frac{2t(a^2+1)}{a^2-1}) + 12a^4 - 56a^2 + 12\right)^{\frac{1}{2}} \end{aligned} \tag{19}$$

and finally

$$\begin{aligned} \lambda_4 &= \frac{1}{4}(a-1)^2\cosh\left(t\frac{a+1}{a-1}\right) + \frac{1}{4}(a+1)^2\cosh\left(t\frac{a-1}{a+1}\right) \\ &- \frac{1}{8}\left(2(a-1)^4\cosh(\frac{2t(a+1)^2}{a^2-1}) + 2(a+1)^4\cosh(\frac{2t(a-1)^2}{a^2-1})\right. \\ &- \left. 4(a^2-1)^2\cosh(\frac{4ta}{a^2-1}) + 4(a^2-1)^2\cosh(\frac{2t(a^2+1)}{a^2-1}) + 12a^4 - 56a^2 + 12\right)^{\frac{1}{2}}. \end{aligned}$$

Next observe that for each t λ_3 is the largest of these eigenvalues. To see why, note that using $2\cosh(x)\cosh(y) = \cosh(x+y) + \cosh(x-y)$ and $2\sinh(x)\sinh(y) = \cosh(x+y) - \cosh(x-y)$ the discriminant D_P of P can be calculated as

$$\begin{aligned} D_P &= \frac{1}{16}\left(\frac{(a-1)^4}{2}\left(\cosh(2t\frac{a+1}{a-1}) - 1\right) + \frac{(a+1)^4}{2}\left(\cosh(2t\frac{a-1}{a+1}) - 1\right)\right. \\ &- \left.(a^2-1)^2\left(\cosh(2t\frac{a^2+1}{a^2-1} - \cosh(\frac{4at}{a^2-1})\right)\right) \\ &= \frac{1}{16}\left((a-1)^2\sinh(t\frac{a+1}{a-1}) - (a+1)^2\sinh(t\frac{a-1}{a+1})\right)^2 \geq 0 \end{aligned}$$

the last inequality showing that indeed the eigenvalues λ_1 and λ_2 of Γ_L are real. Moreover for the discriminant D_Q of Q we obtain

$$\begin{aligned} D_Q &= D_P + \frac{1}{8}(a^2-1)^2\left(\cosh(2t\frac{a^2+1}{a^2-1}) + \cosh(\frac{4at}{a^2-1})\right) \\ &- \frac{1}{4}(a^2-1)\cosh(\frac{4at}{a^2-1}) + \frac{1}{4}(a^2-1)^2 \\ &= D_P + \frac{1}{4}(a^2-1)^2\sinh(t\frac{a+1}{a-1})\sinh(t\frac{a-1}{a+1}) + \frac{1}{4}(a^2-1)^2 \geq D_P \end{aligned}$$

since $\sinh(xy)\sinh(\frac{x}{y}) \geq 0$ for all $x, y \in \mathbb{R}, y \neq 0$.
Now $D_Q \geq D_P$ immediately implies that the largest zero of Q – which is λ_3 – also is the largest eigenvalue of Γ_L. (For t in a neighborhood of 0 this can also be seen from the fact that for $t = 0$ the matrix Γ_L is stochastic and irreducible and $\lambda_3 = 1$).

To analyze the behavior of λ_3 we again apply the approximations $\cosh(x) = 1 + \frac{x^2}{2} + \mathcal{O}(x^4)$ and $\sqrt{1+x} \leq 1 + \frac{x}{2}$ to obtain

$$\begin{aligned} \lambda_3 &= \frac{1}{2}(a^2+1) + \frac{1}{4}t^2(a^2+1) + \mathcal{O}(t^4) + \frac{1}{8}\sqrt{16(a^2-1)^2 + 16t^2(a^2-1)^2 + \mathcal{O}(t^4)} \\ &\leq \frac{1}{2}(a^2+1) + \frac{1}{4}t^2(a^2+1) + \frac{1}{2}(1-a^2)(1+\frac{t^2}{2}) + \mathcal{O}(t^4) \\ &= 1 + \frac{t^2}{2} + \mathcal{O}(t^4) = \exp(t^2/2)(1+\mathcal{O}(t^4)) \end{aligned}$$

which is exactly the same asymptotics as in the standard Hopfield model with independent patterns. Since the $\mathcal{O}(t^4)$ will cancel in our further calculations in the very same way as in the proof of Theorem 3.1 we will just drop it here. Just as in Section 3 we may now estimate the moment generating function

$$\begin{aligned} E\exp(-t\sum_{(\mu_1,\mu_2)\neq(\nu,\nu)}\xi_1^\nu\xi_2^\nu\xi_1^{\mu_1}\xi_2^{\mu_2}G_{\mu_1,\mu_2}) &\leq \lambda_3^M\cosh(2t(G_{22}-G_{11})) \\ &\leq \exp(\frac{t^2}{2}(M + 4\frac{a^2}{1-a^2})). \end{aligned}$$

Since the expressions for $\nu = 1, M$ are not much different (and their contribution to the following probability is negligible, anyway), we obtain for any $t \geq 0$

$$\begin{aligned} P(\exists\nu : \xi^\nu \text{ is not stable}) &\leq NM(N)e^{-tN\frac{1+a^2}{1-a^2}}\exp(N\frac{t^2}{2}(M + 4\frac{a^2}{1-a^2})) \\ &= \exp(N\frac{t^2}{2}(M + 4\frac{a^2}{1-a^2}) - Nt\frac{1+a^2}{1-a^2}) \end{aligned}$$

Minimizing the exponent with respect to t leads to

$$t_{\min} = \frac{1}{(M + 4\frac{a^2}{1-a^2})}\frac{1+a^2}{1-a^2}$$

which implies

$$P(\exists\nu : \xi^\nu \text{ is not stable}) \leq NM(N)\exp\left(\frac{1}{2(M + 4\frac{a^2}{1-a^2})}(\frac{1+a^2}{1-a^2})^2\right).$$

Just as before choosing $M(N) = \frac{N}{\gamma \log N}$ with $\gamma > 6(\frac{1-a^2}{1+a^2})^2$ leads to

$$P(\liminf_{N\to\infty}(\cap_{\nu=1}^{M(N)} T\xi^\nu = \xi^\nu)) = 1$$

i.e. part one of the theorem via the Borel–Cantelli Lemma. If only

$$P(\exists \nu : \xi^\nu \text{ is not stable}) \to 0$$

is required, $\gamma > 4(\frac{1-a^2}{1+a^2})^2$ is enough. □

Of course, there is again a version of the theorem dealing with the case of noised patterns, i.e. the attraction of Hamming spheres centered in a stored pattern to its center by the gradient dynamics.
This is the content of

Theorem 4.2 *Let $r \in [0, \frac{1}{2})$ and for each $\nu = 1, \ldots, M(N)$ let $\tilde{\xi^\nu}$ be an element of the Hamming sphere of radius rN centered at ξ^ν. Assume the random patterns ξ^ν fulfill (3) and (4) and $M(N) = (1-2r)^2 \frac{N}{\gamma \log N}$.*
Then for the generalized Hopfield model (2) the following assertions hold true

1. *If $\gamma > 6(\frac{1-a^2}{1+a^2})^2$*
$$P(\liminf_{N\to\infty}(\cap_{\nu=1}^{M(N)} T\tilde{\xi^\nu} = \xi^\nu)) = 1$$
i.e. the noised patterns are almost surely attracted.

2. *If $\gamma \geq 4(\frac{1-a^2}{1+a^2})^2$*
$$P((\cap_{\nu=1}^{M(N)} T\tilde{\xi^\nu} = \xi^\nu)) = 1 - R_N$$
with $\lim_{N\to\infty} R_N = 0$.

Proof: The proof is just a combination of the arguments leading to Theorems 3.2 and 4.1. □

Remark 4.3 *We would like to emphasize that Theorems 4.1 and 4.2 indeed state that there is a version of the Hopfield model that can store a number of correlated patterns that is larger than the lower bound for the number of independent patterns that can be stored in the standard Hopfield model. The version of the Hopfield model to be used depends on the kind of correlation.*

Finally, we will analyze the abilities of the generalized Hopfield models provided Newman's concept of storage is used. We again are able to prove that the generalized Hopfield models are able to store extensively many patterns. Even more we can show that there exists an α_c not depending on p such that if $M \leq \alpha_c N$ the generalized Hopfield models described above can store almost surely M patterns. This especially means that for strong correlations (p close to one or close to zero) these models behave much better that the standard Hopfield model. Also observe that the moment generating function of these models strongly resembles the moment generating function of the standard Hopfield model with independent patterns which may support the idea that they are the "legitimate" variant of the Hopfield model in the case of correlated patterns.

More formally we prove

Theorem 4.4 *Suppose that the random patterns fulfill (3) and (4).There is an* $\alpha_c > 0$ *not depending on* p *such that if* $M(N) \leq \alpha_c N$, *then there are* $\varepsilon > 0$ *and* $0 < \delta < 1/2$ *such that for the generalized Hopfield model (2) with geometry* $G = C^{-1}$ *(where again* C *is the covariance matrix of the* ξ^μ*)*

$$P\left(\liminf_{N\to\infty}(\cap_{\nu=1}^{M(N)} \cap_{\sigma\in S_\delta(\xi^\nu)} (H_N(\sigma) \geq H_N(\xi^\nu) + \varepsilon N))\right) = 1$$

where $S_\delta(\xi^\nu)$ *is the Hamming sphere of radius* δN *centered in* ξ^ν.

Proof: Following the ideas of the proof Theorem 3.4 and using the notations introduced therein we obtain

$$\begin{aligned} & P\left(\{\cap_{\nu=1}^{M(N)}(h_N(\xi^\nu,\delta) \geq H_N(\xi^\nu) + \varepsilon N)\}^c\right) \\ = \;& P\left(\cup_{\nu=1}^{M(N)} \cup_{J:|J|=\delta N} H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \\ \leq \;& \sum_{J:|J|=\delta N} \sum_{\nu=1}^{M(N)} P\left(H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \end{aligned}$$

where again ξ_J^ν denotes a configuration differing from ξ^ν exactly in the coordinates J and δN is an integer. For fixed $2 \leq \nu \leq M(N) - 1$ we obtain for all $t \geq 0$

$$\begin{aligned} & P\left(H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \\ = \;& P\left(-\frac{1}{2N}\sum_{\mu_1,\mu_2=1}^{M(N)}\sum_{i,j=1}^{N}(\xi_{J,i}^\nu\xi_{J,j}^\nu - \xi_i^\nu\xi_j^\nu))\xi_i^{\mu_1}\xi_j^{\mu_2}G_{\mu_1,\mu_2} \leq \varepsilon N\right) \\ = \;& P\left(\frac{1}{N}\sum_{\mu_1,\mu_2=1}^{M(N)}\sum_{\substack{i\in J, j\notin J \\ i\notin J, j\in J}}\xi_i^\nu\xi_j^\nu\xi_i^{\mu_1}\xi_j^{\mu_2}G_{\mu_1,\mu_2} \leq \varepsilon N\right) \\ = \;& P\left(\frac{1}{N}\sum_{(\mu_1,\mu_2)\neq(\nu,\nu)}\sum_{i\in J, j\notin J}\xi_i^\nu\xi_j^\nu\xi_i^\mu\xi_j^\mu \leq (\varepsilon/2 - \frac{1+a^2}{1-a^2}\delta(1-\delta))N\right) \\ \leq \;& e^{t\varepsilon'' N}E\left(\exp\left(-\frac{t}{N}\sum_{(\mu_1,\mu_2)\neq(\nu,\nu)}\sum_{i\in J, j\notin J}\xi_i^\nu\xi_j^\nu\xi_i^{\mu_1}\xi_j^{\mu_2}G_{\mu_1,\mu_2}\right)\right) \end{aligned}$$

where we have set $\varepsilon'' = \varepsilon/2 - \frac{1+a^2}{1-a^2}\delta(1-\delta)$.

To estimate the moment generating function we again choose $\xi_i^\nu = 1$ for all $i = 1,\ldots,N$ and center the variables ξ_i^μ. Expressing the coefficients G_{μ_1,μ_2} as well as the centering in terms of a we obtain

$$\begin{aligned} & E(\exp(-\frac{t}{N}\sum_{i\in J, j\notin J}\sum_{(\mu_1,\mu_2)\neq(\nu,\nu)}\xi_i^{\mu_1}\xi_j^{\mu_2}G_{\mu_1,\mu_2})) = \exp\left(-tN\delta(1-\delta)\frac{2}{1-a^2}\right) \\ \times \;& E\left(\exp\left(-\frac{t}{N}\sum_{i\in J, j\notin J}\left(\sum_{\mu>\nu}b_{\mu,1}(\overline{\xi_i^\mu} + \overline{\xi_j^\mu}) + \sum_{\mu<\nu}\tilde{b}_{\mu,1}(\overline{\xi_i^\mu} + \overline{\xi_j^\mu}) + \right.\right.\right. \\ + \;& \left.\left.\left.\sum_{\mu_1,\mu_2>\nu}b_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}} + \sum_{\mu_1,\mu_2<\nu}\tilde{b}_{\mu_1,\mu_2}\overline{\xi_i^{\mu_1}}\,\overline{\xi_j^{\mu_2}}\right)\right)\right), \end{aligned}$$

where

$$\begin{aligned} b_{\mu,\mu} &:= \frac{1}{1-a^2}\left[(1+a^2)\sum_{n=0}^{M-\mu-1}a^{2n}+a^{2M-2\mu}-2a^2\sum_{n=0}^{M-\mu-1}a^{2n}\right] \\ &= \frac{1}{1-a^2}\left[\frac{1-a^{2M-2\mu}}{1-a^2}(1-a^2)+a^{2M-2\mu}\right]=\frac{1}{1-a^2} \end{aligned}$$

for all $\mu > \nu$ while for $\mu_1 \neq \mu_2$

$$\begin{aligned} b_{\mu_1,\mu_2} := \Bigg(&\frac{1+a^2}{1-a^2}\sum_{n=0}^{M-1-\max\mu_1,\mu_2}a^{2n}+\frac{1}{1-a^2}a^{2(M-\max\mu_1,\mu_2)} \\ &-\frac{1}{1-a^2}\sum_{n=0}^{M-\max\mu_1,\mu_2}a^{2n}-\frac{a^2}{1-a^2}\sum_{n=0}^{M-1-\max\mu_1,\mu_2}a^{2n}\Bigg)a^{|\mu_1-\mu_2|}=0 \end{aligned}$$

and analogously

$$\tilde{b}_{\mu_1,\mu_2} := \begin{cases} \frac{1}{1-a^2} & \text{if } \mu_1=\mu_2 \\ 0 & \text{if } \mu_1 \neq \mu_2. \end{cases}$$

So more explicitly we obtain

$$\begin{aligned} & E(\exp(-\frac{t}{N}\sum_{i\in J,j\notin J}\sum_{(\mu_1,\mu_2)\neq(\nu,\nu)}\xi_i^{\mu_1}\xi_j^{\mu_2}G_{\mu_1,\mu_2})) \\ \leq\ & \exp\left(-tN\delta(1-\delta)\frac{2}{1-a^2}\right) \\ \times\ & E\left(\exp\left(-\frac{t}{N}\sum_{i\in J,j\notin J}\left(\sum_{\mu\neq\nu}\overline{\xi_i^\mu}\,\overline{\xi_j^\mu}\right)\right)\right) \end{aligned}$$

Just as in the proof of Theorem 3.4 we replace the $\overline{\xi_i^\mu}$ by i.i.d. Gaussian random variables with variance one and end up with

$$\begin{aligned} & P\left(H_N(\xi_J^\nu)-H_N(\xi^\nu)\leq\varepsilon N\right) \\ \leq\ & \exp\left(-tN\delta(1-\delta)\frac{2}{1-a^2}\right) \\ \times\ & \left(\frac{1}{1-t^2\lambda^2(1+|2p-1|)^4\delta(1-\delta)(\frac{1}{1-a^2})^2}\right)^{\frac{M}{2}} \end{aligned}$$

which is due to the fact that the matrix of the coefficients (b_{μ_1,μ_2}) is $\frac{1}{1-a^2}$ times the identity matrix and hence all its eigenvalues are equal to $\frac{1}{1-a^2}$. Thus for all $C' > 1$ choosing ε'' and t small enough we obtain

$$\begin{aligned} & P\left(H_N(\xi_J^\nu)-H_N(\xi^\nu)\leq\varepsilon N\right) \\ \leq\ & \exp(t\varepsilon''N+tN\delta(1-\delta)\frac{2}{1-a^2}) \\ \times\ & \exp(t^2\lambda^2(1+|a|)^4\delta(1-\delta)(\frac{1}{1-a^2})^2\alpha N) \\ \leq\ & \exp\left(-\frac{(3+a^2)tN\delta(1-\delta)}{C'(1-a^2)}+t^2\lambda^2(1+|a|)^4\delta(1-\delta)(\frac{1}{1-a^2})^2\alpha N\right) \end{aligned}$$

the above exponent is minimized by

$$t_{\min} := \frac{(3+a^2)(1-a^2)}{2C'(1+|a|)^4\alpha}$$

giving

$$\begin{aligned} & P\left(H_N(\xi_J^\nu) - H_N(\xi^\nu) \leq \varepsilon N\right) \\ \leq \quad & \exp\left(-N\delta(1-\delta)\frac{(3+a^2)^2}{4\alpha(1+|a|)^4}\right) \end{aligned} \tag{20}$$

provided we have chosen α in such a way that $t_{\min}$ is admissible.

The rest of the proof can be carried out along the lines of the proof of Theorem 3.4 □

Remarks 4.5

- *Observe that the reason why α_c can be chosen independently of p is that the exponent in (20) is bounded from above and away from zero for all p.*

- *We are quite convinced that results in the spirit the previous sections also hold true, if the correlation of the patterns does not come from a Markov chain (for example the results of Section 3 should at least also work in the case of exponentially decreasing correlations). On the other hand in all other cases the geometry matrix G is much more difficult to treat and moreover there is no simple way to calculate the moment generating function.*

Acknowledgment: I am grateful to Anton Bovier for his hospitality at the WIAS and for many helpful remarks leading to the final form of the proof of Theorem 3.4. I would like to thank Holger Knöpfel for many useful discussions on several estimates.

References

[AGS87] D.J. Amit, G. Gutfreund, H. Sompolinsky; Statistical mechanics of neural networks near saturation; Ann. Phys. 173, 30-67 (1987)

[BG92] A. Bovier, V. Gayrard; Rigorous bounds on the storage capacity of the dilute Hopfield model; J. Stat. Phys. 69, 597-627 (1992)

[BG95a] A.Bovier, V.Gayrard; An almost sure large deviation principle for the Hopfield model, WIAS-preprint 146, to appear in Prob. Th. Rel. Fields (1996)

[BG95b] A.Bovier, V.Gayrard; The retrieval phase of the Hopfield model, WIAS-preprint 161, to appear in Ann. Prob. (1996)

[BGP94] A.Bovier, V.Gayrard, P.Picco; Gibbs states for the Hopfield model in the regime of perfect memory; Prob. Th. Rel. Fields 100, 329-363 (1994)

[BGP95a] A.Bovier, V.Gayrard, P.Picco; Large deviation principles for the Hopfield model and the Kac-Hopfield model; Prob. Th. Rel. Fields 101, 511-546 (1995)

[BGP95b] A.Bovier, V.Gayrard, P.Picco; Gibbs states for the Hopfield model with extensively many patterns; J. Stat. Phys. 79, 395-414 (1995)

[FP77] L.A. Pastur, A.L. Figotin; Exactly soluble model of a spin-glas; Sov. J. of Low Temprature Phys. 3(6), 378-383 (1977)

[G95] B. Gentz; A central limit theorem for the overlap parameter in the Hopfield model; Preprint; Universität Zürich (1995)

[HK91] L. van Hemmen, R. Kühn, Collective phenomena in neural networks; in Models of neural networks, E. Domany, L. v. Hemmen, R. Schulte (eds.), Springer, Berlin (1991)

[Ho82] J.J. Hopfield; Neural networks and physical systems with emergent collective computational abilities; Proc. nat. Acad. Sci. USA 79, 2554-2558 (1982)

[L94] D. Loukianova; Capacité de mémoire dans le modèle de Hopfield; C.R. Acad. Sci. Paris 318, 157-160 (1994)

[MPRV87] R. McEliece, E. Posner, E. Rodemich. S. Venkatesh; The capacity of the Hopfield associative memory; IEEE Trans. inf. Th. 33, 461-482 (1987)

[Mo92] R. Monasson; Properties of neurla networks storing spatially correlated patterns; J. Phys. A: Math. Gen. 335, 3701-3720 (1992)

[N88] C. Newman; Memory capacity in neural networks; Neural Networks 1, 223-238 (1988)

[P95] D. Petritis; Thermodynamic formalism of neural computing; Preprint, Université de Rennes I (1995)

[T95] M. Talagrand; Résultats rigoureux pour le modèle de Hopfield, C.R. Acad. Sci. 321, I, 309-312,(1995)

[TL93] W. Tarkowski, M. Lewenstein; Storage of sets of correlated data in neural network memories, J. Phys. A: Math. Gen. 26, 2453- 2469 (1993)