APPROXIMATION OF QUADRATIC FORMS OF INDEPENDENT RANDOM VECTORS BY ACCOMPANYING LAWS

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Abstract. Let $X, X_1, X_2, \dots$ be i.i.d. random vectors taking values in $\mathbf{R}^d$. Assume that $\mathbf{E}\, X = 0$, $\mathbf{E}\,|X|^{8/3} < \infty$ and that X is not concentrated in a proper subspace of $\mathbf{R}^d$. Let $Y, Y_1, Y_2, \dots$ denote i.i.d. random vectors with common distribution which is accompanying to that of X. We compare the distributions of the non-degenerate quadratic forms $Q[S_N]$ and $Q[T_N]$ of the normalized sums $S_N = N^{-1/2}\,(X_1 + \dots + X_N)$ and $T_N = N^{-1/2}\,(Y_1 + \dots + Y_N)$ and prove that

$$\sup_x \Big| \mathbf{P}\,\big\{\, Q[S_N - a] < x \,\big\} - \mathbf{P}\,\big\{\, Q[T_N - a] < x \,\big\} \Big| = O\big((1 + |a|^4)\, N^{-1}\big), \qquad a \in \mathbf{R}^d,$$

provided that $9 \leqslant d \leqslant \infty$. The constant in this bound depends on $\mathbf{E}\,|X|^{8/3}$, on Q and on the covariance operator of X. We also show the optimality of the bound $O\big(N^{-1}\big)$.

1. Introduction

In the limit theory for sums of independent identically distributed random variables and vectors main efforts were devoted to study the rate of approximation in the Central Limit Theorem (CLT) and for problems with stable limit distributions. Compared to the general structure of limit distributions of sums these are still rather special situations. Moreover, in many problems the accuracy of normal approximation is too slow and one needs to enlarge the class of approximating distributions. A natural approximation for the distributions of sums of a large number of i.i.d.

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random vectors is given by infinitely divisible (in particular, accompanying) distributions.

The problem of the estimation of the rate of infinitely divisible approximations for the distribution functions of sums of N i.i.d. one-dimensional random variables as $N \to \infty$ was stated by Kolmogorov (1953). Prohorov (1955) showed that these distribution functions are approximable by the set of infinitely divisible laws in the sense of the uniform distance between distribution functions. Kolmogorov (1956) showed that the rate of this approximation is uniform with respect to the class of all one-dimensional distributions and proved a rate of the form $c\,N^{-1/5}$, where c is an absolute constant. The best possible rate $c\,N^{-2/3}$ was obtained by Arak (1981, 1982). For accompanying approximating distributions, the optimal bound $c\,N^{-1/3}$ was proved by Le Cam (1965). This bound was sharpened up to $c\,N^{-1/2}$ for symmetrically distributed summands by Zaitsev (1981) (see also Ibragimov and Presman (1973)) and up to $c\,N^{-1}$ by Arak (1980) for summands with non-negative characteristic functions, see also Čekanavičius (1989). A more detailed review of one-dimensional results mentioned above and detailed proofs can be found in the monograph of Arak and Zaitsev (1988).

The majority of the corresponding multidimensional results used the uniform distance between distribution functions. Presman (1973) proved an analog of Le Cam's result with a constant depending on the dimension $d < \infty$ only. When the distribution of the summands is symmetric or even has a non-negative characteristic function, the corresponding analogs of the one-dimensional results of Zaitsev (1981) and Arak (1980) were obtained by Zaitsev (1988, 1992). These results can be easily extended to the comparison of distributions on convex polytopes (see Zaitsev (1994)). The constants in these bounds depend on certain characteristics of the polytopes what do not allow to obtain estimates for arbitrary convex sets and balls by passing to the limit.

In spite of the fact that the results described above give the definitive answer to the question about the asymptotic behaviour of the bounds which are uniform with respect to the class of all one-dimensional or d-dimensional distributions, these inequalities do not describe the asymptotics of the rate of infinitely divisible approximation of N-fold convolutions for fixed distribution of summands. The analog of the result of Prohorov (1955) mentioned above for finite dimensional convex sets was obtained by Bakštys and Paulauskas (1986, 1987). For the approximation by accompanying laws they also proved an estimate for the Prohorov distance between suitably normalized vectors in the infinite dimensional case which is of order $O(N^{-1/6})$ in the case of Hilbert space valued vectors. Nagaev (1990) proved that if $4 \leqslant d \leqslant \infty$, the norm of the summands has finite second moment and their distribution is non-degenerate, then the rate of approximation by accompanying laws for balls is $O(N^{-1/2})$. From the recent estimate in the CLT of Bentkus and Götze (1995) it follows that this rate of approximation is $O(N^{-1})$ for arbitrary ellipsoids and hyperboloids when $9 \leqslant d \leqslant \infty$

and the summands have non-degenerate distribution and their norm has a finite fourth moment.

In this paper we investigate the rate of approximation of the distribution of the sum of i.i.d. random vectors by the accompanying laws on the balls, ellipsoids and hyperboloids in spaces of sufficiently large dimension assuming the existence of a finite moment of order 8/3 for the norm of the summands. In particular, we show that this rate of approximation is $O(N^{-1})$ provided $9 \leqslant d \leqslant \infty$.

Let us introduce the necessary notation. Let $\mathbf{R}^d$ denote the d-dimensional space of real vectors $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ defined by $\langle x, y\rangle = x_1 y_1 + \dots + x_d y_d$, $|x|^2 = \langle x, x\rangle$. Since our results are independent of the dimension (provided $d \geqslant 9$), it will be convenient to denote by $\mathbf{R}^\infty$ a real separable Hilbert space. Thus, $\mathbf{R}^\infty$ consists of all real sequences $x = (x_1, x_2, \dots)$ such that $|x|^2 = x_1^2 + x_2^2 + \dots < \infty$.

Let $\mathfrak{F}_d$ be the set of probability distributions defined on the Borel σ-field of subsets of $\mathbf{R}^d$, $\mathfrak{D}_d \subset \mathfrak{F}_d$ the set of infinitely divisible laws, $E_a \in \mathfrak{F}_d$ the distribution concentrated at a point $a \in \mathbf{R}^d$, $E = E_0$; $\mathcal{L}(\zeta) \in \mathfrak{F}_d$ and $\operatorname{cov} \zeta$ the distribution and the covariance operator of a random vector $\zeta \in \mathbf{R}^d$, $\mathbf{I}\{A\}$ the indicator function of an event A. For $H_1, H_2 \in \mathfrak{F}_d$, $x \in \mathbf{R}^d$ we denote the corresponding distribution functions by $H_1(x)$, $H_2(x)$; $\rho(H_1, H_2) \overset{\text{def}}{=} \sup_x \bigl| H_1(x) - H_2(x) \bigr|$. Products $H_1 H_2 = H_1 * H_2$ and powers $H^m = H^{m*}$, $H^0 = E$ of (signed) measures should be understood in the convolution sense. By $c(\cdot), c_1(\cdot), c_2(\cdot), \dots$ we shall denote positive quantities depending on the argument only. The symbols c, c_1, $c_2, \dots$ will denote absolute positive constants. Notice that $c(\cdot)$, c may denote different quantities in different (or even in the same) formulas. By $\lceil \alpha \rceil$ we shall denote the integer part of a number α. The ends of proofs will be denoted by $\square$. Usually we shall assume that all random variables and vectors are independent if the contrary is not clear from the context.

Let $\Xi = \bigl\{\zeta, \zeta_1, \zeta_2, \dots\bigr\}$ be a sequence of i.i.d. random vectors in $\mathbf{R}^d$ with $\mathcal{L}(\zeta) = H$. By $\widetilde{\zeta}$ we shall denote a symmetrization of ζ, for example $\widetilde{\zeta} = \zeta - \zeta_1$. Let Π_λ denote a random variable which is independent of Ξ and has Poisson distribution with parameter $\lambda > 0$. Then it is evident that

$$\mathcal{L}\bigl(\zeta_1 + \dots + \zeta_{\Pi_\lambda}\bigr) = e(\lambda H) \overset{\text{def}}{=} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m H^m}{m!} \in \mathfrak{D}_d \,. \tag{1.1}$$

The distribution $e(\lambda H)$ is often called compound Poisson. It can be easily verified that, for $H_1, H_2 \in \mathfrak{F}_d$ and $\lambda_1, \lambda_2 > 0$,

$$e\bigl(\lambda_1 H_1\bigr)\ e\bigl(\lambda_2 H_2\bigr) = e\bigl((\lambda_1 + \lambda_2) H\bigr), \tag{1.2}$$

where

$$H = \frac{\lambda_1}{\lambda_1 + \lambda_2} H_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} H_2 \in \mathfrak{F}_d \,. \tag{1.3}$$

In particular, for $H \in \mathfrak{F}_d$ and $m = 2, 3, \dots,$

$$\big(e(H)\big)^m = e(mH). \tag{1.4}$$

For $H \in \mathfrak{F}_d$ and $N = 1, 2, \dots,$ the set of accompanying laws is defined by

$$\mathcal{A}_{d,N}(H) = \Big\{ E_{Nb}\, e(NH E_{-b}) : b \in \mathbf{R}^d \Big\} \subset \mathfrak{D}_d. \tag{1.5}$$

When we approximate distributions of sums of i.i.d. summands by accompanying laws, the shift b is usually the mean of truncated summands. In the case when our summands have mean zero, it is natural to consider the accompanying laws "without shift": $D_N(H) = e(NH) \in \mathcal{A}_{d,N}(H)$.

Let us formulate the main result of this paper.

Let $\big\{X, X_1, X_2, \dots\big\}$ and $\big\{Y, Y_1, Y_2, \dots\big\}$ be two sequences of i.i.d. random vectors taking values in $\mathbf{R}^d$ and such that $\mathcal{L}(Y) = e\big(\mathcal{L}(X)\big)$. If $\mathbf{E}\, X = 0$ and $\mathbf{E}\, |X|^2 < \infty$, then the distributions of sums

$$S_N = N^{-1/2}\,(X_1 + \dots + X_N), \qquad T_N = N^{-1/2}\,(Y_1 + \dots + Y_N) \tag{1.6}$$

converge weakly to the distribution of a Gaussian random variable, say η, such that, according to (1.1),

$$\mathbf{E}\, Y = \mathbf{E}\, X = \mathbf{E}\, \eta = 0, \qquad \operatorname{cov} Y = \operatorname{cov} X = \operatorname{cov} \eta. \tag{1.7}$$

We shall assume that η (or, equivalently, X) is not concentrated on a subspace of $\mathbf{R}^d$, i.e., that $\mathbf{P}\big\{\eta \in \mathbf{L}\big\} < 1$ whenever $\mathbf{L} \subset \mathbf{R}^d$ is a proper closed linear subspace of $\mathbf{R}^d$.

According to (1.1), (1.4),

$$\mathcal{L}(T_N) = e\Big(N\,\mathcal{L}\big(X/\sqrt{N}\big)\Big) = \mathcal{L}\big(N^{-1/2}\,(X_1 + \dots + X_{\Pi_N})\big), \tag{1.8}$$

where the random variable Π_N has a Poisson distribution with parameter N and is independent of $\big\{X_1, X_2, \dots\big\}$.

For a linear symmetric and bounded operator $Q : x \mapsto Qx$ mapping $\mathbf{R}^d$ into $\mathbf{R}^d$, define the quadratic form

$$Q[x] = \big\langle Qx, x \big\rangle. \tag{1.9}$$

Assume that the operator Q is non-degenerate, i.e., that $\ker Q = \big\{0\big\}$.

Theorem 1.1. *Assume that* $9 \leqslant d \leqslant \infty$. *Let* $\mathbf{E}\, X = 0$, *and* $\mathbf{E}\, |X|^{8/3} < \infty$. *Then*

$$\sup_x \Big| \mathbf{P}\big\{ Q[S_N - a] < x \big\} - \mathbf{P}\big\{ Q[T_N - a] < x \big\} \Big| = O\big((1+|a|^4)\, N^{-1}\big). \tag{1.10}$$

The constant in this bound depends on $\mathbf{E}\, |X|^{8/3}$, *on* Q *and on the covariance operator of* X *only.*

In the proof of Theorem 1.1 we can assume that $Q^2 = I$, where I is the identity operator, and, hence, that the symmetric operator Q is isometric. This will not restrict the generality, since any symmetric operator Q can be represented as $Q = Q_+ Q_0 Q_+$, where Q_+ is a non-negative operator and Q_0 is a symmetric operator with $Q_0^2 = I$. This can be derived, e.g., from Lusternik and Sobolev (1974), Theorem 1, Section 7.5. Therefore, we may replace $S_N - a$ by $Q_+(S_N - a)$, X by $Q_+ X$, η by $Q_+ \eta$, etc.

An explicit bound of constant in (1.10) is given in Theorem 1.2 below. In Theorem 1.2 we assume that the conditions of Theorem 1.1 are satisfied and $Q^2 = I$. Let $\sigma_1^2 \geqslant \sigma_2^2 \geqslant \dots$ be ordered eigenvalues of the operator $\operatorname{cov} X = \operatorname{cov} \eta$ and $z_1, z_2 \dots$ an orthonormal basis which diagonalizes the operator $\operatorname{cov} \eta$. Define

$$\sigma_9^2(Q) = \sup_{\mathbf{R}^9 \subset \mathbf{R}^d} \min \Big\{ \inf_{e \in \mathbf{R}^9,\, |e|=1} \mathbf{E}\, \langle \eta, e \rangle^2 ,\ \inf_{e \in Q\, \mathbf{R}^9,\, |e|=1} \mathbf{E}\, \langle \eta, e \rangle^2 \Big\}. \tag{1.11}$$

Note that if the operator Q is diagonal in the basis $z_1, z_2 \dots$, then $\sigma_9^2(Q) \geqslant \sigma_9^2$.

Theorem 1.2. *Let the conditions described above be satisfied. Then*

$$\sup_x \Big| \mathbf{P}\big\{ Q[S_N - a] < x \big\} - \mathbf{P}\big\{ Q[T_N - a] < x \big\} \Big| \leqslant \frac{1}{N} \Big(\frac{(\mathbf{E}\, |X|^{8/3})^3}{\sigma^8} + \frac{|a|^4}{\sigma^4} \Big) \exp\Big\{ \frac{c\sigma^2}{\sigma_9^2(Q)} \Big\}, \tag{1.12}$$

where $\sigma^2 = \mathbf{E}\, |X|^2$.

A more exact but less explicit than (1.12) bound is given in (3.31).

Theorems 1.1 and 1.2 are formulated for normalized sums S_N and T_N. It is clear that one can easily derive from them the corresponding statements for non-normalized sums $X_1 + \dots + X_N$ and $Y_1 + \dots + Y_N$.

In the case when $Q = I$ the relation (1.10) turns into

$$\sup_x \Big| \mathbf{P}\big\{ |S_N - a|^2 < x \big\} - \mathbf{P}\big\{ |T_N - a|^2 < x \big\} \Big| = O\big((1+|a|^4)\, N^{-1}\big). \tag{1.13}$$

This bound is non-uniform with respect to a and the constants will depend on the distribution of summands. This is essential since otherwise, varying a or $\mathcal{L}(X)$, one could derive from (1.13) the estimate $\rho\big(\mathcal{L}(S_N),\,\mathcal{L}(T_N)\big)= O\big(N^{-1}\big)$ in dimension one which is evidently wrong because for the one-dimensional symmetric Bernoulli scheme with $\mathbf{P}\big\{X=1\big\} = \mathbf{P}\big\{X=-1\big\} = {}^1\!/_2$ we have $\rho\big(\mathcal{L}(S_N),\,\mathcal{L}(T_N)\big)\geqslant c\,N^{-1/2}$.

On the other hand, the rate of convergence in (1.10), (1.12), (1.13) cannot be faster. Consider, for example, the case when $d<\infty$ is odd, $Q=I$, $a=0$, $\mathbf{P}\big\{X=x\big\}=2^{-d}$ for all $x=(x_1,\dots,x_d)$ such that $x_j=\pm1$, $j=1,\dots,d$.

Denote by $\mathcal{Z}_1=\big\{1,3,5,\dots\big\}$ and $\mathcal{Z}_2=\big\{0,2,4,\dots\big\}$ the sets of odd and even non-negative integers respectively. It is not difficult to check that

$$\mathbf{P}\big\{\,|S_N|^2\in\mathcal{Z}_1/N\,\big\}=1-\,\mathbf{P}\big\{\,|S_N|^2\in\mathcal{Z}_2/N\,\big\}=\begin{cases}1, & \text{for } N \text{ odd,}\\ 0, & \text{for } N \text{ even.}\end{cases}\tag{1.14}$$

From (1.8), we infer

$$\mathbf{P}\big\{\,|T_N|^2\in\mathcal{Z}_k/N\,\big\}=\,\mathbf{P}\big\{\Pi_N\in\mathcal{Z}_k\big\},\qquad k=1,2.\tag{1.15}$$

Using the Local Limit Theorem, see, e.g., Petrov (1975), Theorem 13, Chapter VII, it can be easily verified that

$$\mathbf{P}\big\{\,\Pi_N\in\mathcal{Z}_k\,\big\}\xrightarrow[N\to\infty]{}\frac{1}{2}\,,\qquad k=1,2.\tag{1.16}$$

Choose $c_1(d)$ so that $\mathbf{P}\big\{\,|\eta|^2\leqslant c_1(d)\,\big\}=\dfrac{5}{6}\,.$ By the CLT,

$$\mathbf{P}\,\big\{\,|T_N|^2\leqslant c_1(d)\,\big\}\xrightarrow[N\to\infty]{}\frac{5}{6}\,.\tag{1.17}$$

According to (1.15)–(1.17), for sufficiently large N we have

$$\mathbf{P}\Big\{\,|T_N|^2\in\mathcal{Z}_k/N,\ |T_N|^2\leqslant c_1(d)\,\Big\}\geqslant\frac{1}{4}\,,\qquad k=1,2.\tag{1.18}$$

It is clear that (1.18) implies that

$$\max\Big\{\,\mathbf{P}\,\big\{\,|T_N|^2=l/N\,\big\}:\ l\in\mathcal{Z}_k,\ l\leqslant c_1(d)\,N\Big\}\geqslant\frac{c_2(d)}{N}\,,\qquad k=1,2.\tag{1.19}$$

From (1.14), (1.19) it follows that

$$\sup_x\,\Big|\,\mathbf{P}\,\big\{\,|S_N|^2<x\,\big\}-\,\mathbf{P}\,\big\{\,|T_N|^2<x\,\big\}\Big|\geqslant\frac{c_2(d)}{2\,N}\,.\tag{1.20}$$

For even d a similar example can be easily constructed too. It suffices to add to X in the previous example an independent $(d+1)$-th coordinate taking values ±2 with probability ${}^1\!/_2$ which allows to repeat the arguments above.

If $\mathbf{E}\,|X|^4 < \infty$, the bound of the form $O(N^{-1})$ in (1.10) follows from an estimate in the CLT, contained in the main result of Bentkus and Götze (1995). Theorem 1.1 show that this rate of approximation remains valid given weaker moment assumptions where the Gaussian approximation is essentially slower. Thus, Theorem 1.1 can be considered as an analog of a result of Zaitsev (1996). He proved that if $d = 1$, $\mathbf{E}\,X = 0$, and $\mathbf{E}\,|X|^{3/2} < \infty$, then $\rho\big(\mathcal{L}(S_N),\, \mathcal{L}(T_N)\big) = O(N^{-1/2})$.

For the proof of Theorems 1.1 and 1.2 we use an analog of the well known one-dimensional Esseen bound which estimates the uniform distance $\rho(\cdot,\cdot)$ in terms of characteristic functions (see Lemma 2.6). For sufficiently large values of $|t|$, provided that $O(N^{2/9}) \leqslant |t| \leqslant O(N)$ we estimate the integrals of characteristic functions with the help of recent results of Bentkus and Götze (1994, 1995) (see Lemmas 2.1 and 2.2). For $|t| \leqslant O(N^{2/9})$ we use an expansion of characteristic functions contained in Lemma 2.5. Besides, we use a truncation and the standard techniques of estimation in Hilbert spaces.

2. Auxiliary results

Let us define the following classes of probability distributions which will appear in our arguments. Let $s \leqslant d$ be a natural number and $0 < r \leqslant s^{-1}$, $0 < \delta \leqslant 1/2$. Assume that $P : \mathbf{R}^d \to \mathbf{R}^d$ is a linear operator and $y_1, \dots, y_s \in \mathbf{R}^d$ an orthonormal system of vectors. Let $\varepsilon_1, \varepsilon_2, \dots$ denote i.i.d. symmetric Rademacher random variables.

We say that a distribution $\mathcal{L}(\zeta) \in \mathfrak{F}_d$ belongs to the class $\mathcal{N}(s, r, \delta, P;\, y_1, \dots, y_s)$ if

$$\min\Big\{ \min_{1\leqslant j\leqslant s} \mathbf{P}\,\big\{|\zeta - y_j| \leqslant \delta\big\},\ \min_{1\leqslant j\leqslant s} \mathbf{P}\,\big\{|\zeta - Py_j| \leqslant \delta\big\}\Big\} \geqslant r. \tag{2.1}$$

We say that the distribution of a discrete random vector $W \in \mathbf{R}^d$ belongs to the class $\Gamma(s, \delta;\, y_1, \dots, y_s)$ if W is distributed as $\varepsilon_1 w_1 + \dots + \varepsilon_s w_s$, with some (non-random) $w_1, \dots, w_s \in \mathbf{R}^d$ such that $|w_j - y_j| \leqslant \delta$, for all $j = 1, \dots, s$.

Introduce the function $M_s(v, w)$, $v \in \mathbf{R}$, $w > 0$, by

$$M_s(v,w) = \begin{cases} |v|^{-s/2}, & \text{for } |v| \leqslant \sqrt{w}, \\ \big(|v|/w\big)^{s/2}, & \text{for } |v| \geqslant \sqrt{w}. \end{cases} \tag{2.2}$$

Notice that, for all $v \in \mathbf{R}$, $w > 0$,

$$c_3(s)\,\Big(|v|^{-s/2} + \big(|v|/w\big)^{s/2}\Big) \leqslant M_s(v,w) \leqslant c_4(s)\,\Big(|v|^{-s/2} + \big(|v|/w\big)^{s/2}\Big). \tag{2.3}$$

It is clear that (2.2), (2.3) imply that

$$\begin{aligned} &\text{if}\quad c_5(s)\,|v_0| \leqslant |v| \leqslant c_6(s)\,|v_0|, \quad 0 \leqslant c_7(s)\,w_0 \leqslant w \leqslant c_8(s)\,w_0, \\ &\qquad \text{then}\quad c_9(s)\,M_s(v_0,w_0) \leqslant M_s(v,w) \leqslant c_{10}(s)\,M_s(v_0,w_0). \end{aligned} \tag{2.4}$$

In the proof of Theorem 1.1 we shall use the following Lemmas 2.1–2.6 in which Q denotes the symmetric bounded operator defined in Section 1.

Lemma 2.1 (Bentkus and Götze (1995), Corollary 5.5 and Theorems 6.3, 7.1). *Let s be a natural number, $1 \leqslant s \leqslant d$, $0 < r \leqslant s^{-1}$, $0 < \delta \leqslant 1/2$. Suppose that the operator Q is isometric. Assume that the symmetrization $\widetilde{Z}^*$ of a random vector $Z^* \in \mathbf{R}^d$, satisfies*

$$\mathcal{L}(\widetilde{Z}^*) \in \mathcal{N}(s, r, \delta, Q;\ y_1, \dots, y_s), \tag{2.5}$$

for an orthonormal system of vectors $y_1, \dots, y_s \in \mathbf{R}^d$. Let n be a natural number,

$$\varUpsilon_n^* = Z_1^* + \dots + Z_n^*, \tag{2.6}$$

where $Z_1^*, Z_2^*, \dots$ denote independent copies of Z^*. Let $l = \lceil rn/(4s) \rceil \geqslant 1$, $0 < A \leqslant B$. Then

$$\sup_{z \in \mathbf{R}^d} \int\limits_{A \leqslant |t| \leqslant B} \left| \mathbf{E} \exp\bigl\{ 2itQ[\varUpsilon_n^* - z] \bigr\} \right| \frac{dt}{|t|} \leqslant J + c(s)\,(rn)^{-2} \log \frac{B}{A}, \tag{2.7}$$

where

$$\begin{gathered} J = \sup_{\varLambda} \sup_{z \in \mathbf{R}^d} \int\limits_{A \leqslant |t| \leqslant B} \sqrt{\varphi_\varLambda(z,\, 4tl)}\, \frac{dt}{|t|}, \\ \varphi_\varLambda(z, u) = \left| \mathbf{E} \exp\bigl\{ iul^{-1}\, Q[\varLambda_l + Q\varLambda_l' - z] \bigr\} \right|^2, \end{gathered} \tag{2.8}$$

and the supremum $\sup\limits_{\varLambda}$ is taken over all random vectors $\varLambda_l$ and $\varLambda_l'$ of type

$$\varLambda_l = W_1 + \dots + W_l, \qquad \varLambda_l' = W_1' + \dots + W_l', \tag{2.9}$$

where $W_1, \dots, W_l$, $W_1', \dots, W_l' \in \mathbf{R}^d$ denote independent (generally speaking, non-identically distributed) random vectors with distributions belonging to the class $\Gamma(s, \delta;\ y_1, \dots, y_s)$. ∎

If we additionally assume that $\delta = \dfrac{1}{50\,s}$, then, for all $\gamma \geqslant 0$, $u \in \mathbf{R}$, we have

$$\varphi_\varLambda(z,\, u + \gamma)\ \varphi_\varLambda(z,\, u - \gamma) \leqslant c(s)\, M_s^2(\gamma,\, l) \tag{2.10}$$

and

$$\sup_{z \in \mathbf{R}^d} \left| \mathbf{E} \exp\bigl\{ iun^{-1} Q[\varUpsilon_n^* - z] \bigr\} \right| \leqslant c(s)\, M_s\bigl(ru,\, rn\bigr). \tag{2.11}$$

Lemma 2.2 (Bentkus and Götze (1994), Lemma 6.1). *Assume that a non-negative function $\psi(t)$, $t \geqslant 0$, is continuous and $\psi(0) = 1$. Furthermore, suppose that there exists a number $\lambda \geqslant 1$ such that, for some $s \geqslant 9$, $D \geqslant 1$,*

$$\psi(t)\, \psi(t + \Delta) \leqslant \lambda^2 M_s^2(\Delta,\, D), \qquad \text{for all} \quad t \geqslant 0 \quad \text{and} \quad \Delta \geqslant 0. \tag{2.12}$$

Then

$$\int_{D^{(s-2)/s}}^{D} \sqrt{\psi(t)}\, \frac{dt}{t} \leqslant \frac{c(s)\, \lambda}{D}. \tag{2.13}$$

We shall need also the following auxiliary bound for the convergence rate in the CLT in finite dimensional and Hilbert spaces for expectations of smooth functions.

Lemma 2.3 (Rosenthal's inequality). *Let $\psi_1,\dots,\psi_\rho\in\mathbf{R}^d$ be independent mean zero random vectors. Then*

$$\mathbf{E}\,\Big|\sum_{l=1}^{\rho}\psi_l\Big|^{\beta}\leqslant c(\beta)\,\Big(\sum_{l=1}^{\rho}\mathbf{E}\,|\psi_l|^{\beta}+\Big(\sum_{l=1}^{\rho}\mathbf{E}\,|\psi_l|^2\Big)^{\beta/2}\Big),\qquad 2\leqslant\beta<\infty.$$

Proof. A result of de Acosta (1981) implies

$$\mathbf{E}\,\Big|\sum_{l=1}^{\rho}\psi_l\Big|^{\beta}\leqslant c(\beta)\,\Big(\Big(\mathbf{E}\,\Big|\sum_{l=1}^{\rho}\psi_l\Big|^2\Big)^{\beta/2}+\sum_{l=1}^{\rho}\mathbf{E}\,|\psi_l|^{\beta}+\Big(\sum_{l=1}^{\rho}\mathbf{E}\,|\psi_l|^2\Big)^{\beta/2}\Big).$$

Thus, an application of the obvious identity $\mathbf{E}\,\Big|\sum_{l=1}^{\rho}\psi_l\Big|^2=\sum_{l=1}^{\rho}\mathbf{E}\,|\psi_l|^2$ completes the proof. □

Lemma 2.4. *Let a function $\Psi:\mathbf{R}^d\to\mathbf{R}$ be three times Frechét differentiable. Then there exists an absolute constant c such that*

$$\big|\,\mathbf{E}\,\Psi(S_m)-\mathbf{E}\,\Psi(\eta)\,\big|\leqslant c\,\mathbf{E}\,|X|^{8/3}\,m^{-1/3}\,\big\|\,\Psi^{(2)}\,\big\|_\infty^{1/3}\,\big\|\,\Psi^{(3)}\,\big\|_\infty^{2/3},\tag{2.14}$$

where $\big\|\,\Psi^{(l)}\,\big\|_\infty=\sup_{x\in\mathbf{R}^d}\big\|\,\Psi^{(l)}(x)\,\big\|$, $l=2,3$, and $\big\|\,\Psi^{(l)}(x)\,\big\|$ denotes the norm of the l-th Frechét derivative.

Proof. The result can be proved like Theorem 1.2 in Bentkus et al. (1991). For the sake of completeness we include a sketch of the proof. Without loss of generality we assume that

$$\eta=m^{-1/2}\,(\eta_1+\cdots+\eta_m),\tag{2.15}$$

where $\eta_1,\dots,\eta_m$ are i.i.d. Gaussian random vectors with $\mathcal{L}(\eta_j)=\mathcal{L}(\eta)$. Assume also that the vectors $\big\{X_j,\,\eta_k;\ j,k=1,\dots,m\big\}$ are independent. Denote

$$V_j=m^{-1/2}\,\big(X_1+\cdots+X_{j-1}+\eta_{j+1}+\cdots+\eta_m\big).\tag{2.16}$$

Then

$$\big|\,\mathbf{E}\,\Psi(S_m)-\mathbf{E}\,\Psi(\eta)\,\big|\leqslant\delta_1+\cdots+\delta_m,\tag{2.17}$$

where

$$\delta_j=\Big|\,\mathbf{E}\,\Psi\big(V_j+X_j/\sqrt{m}\big)-\mathbf{E}\,\Psi\big(V_j+\eta_j/\sqrt{m}\big)\Big|,\qquad j=1,\dots,m.\tag{2.18}$$

It is clear that, by the Taylor expansions in powers of $X_j/\sqrt{m}$ and $\eta_j/\sqrt{m}$,

$$\begin{aligned}\Big|\Psi\big(V_j+X_j/\sqrt{m}\big)-\Psi\big(V_j+\eta_j/\sqrt{m}\big)\Big|&\leqslant c\,m^{-1}\,\big(|X_j|^2+|\eta_j|^2\big)\,\big\|\,\Psi^{(2)}\,\big\|_\infty,\\ \Big|\Psi\big(V_j+X_j/\sqrt{m}\big)-\Psi\big(V_j+\eta_j/\sqrt{m}\big)\Big|&\leqslant c\,m^{-3/2}\,\big(|X_j|^3+|\eta_j|^3\big)\,\big\|\,\Psi^{(3)}\,\big\|_\infty.\end{aligned}\tag{2.19}$$

Using Lemma 2.3 and Hölder's inequality, one can easily derive that

$$\mathbf{E}\,|\eta|^{8/3}\leqslant c\,\big(\mathbf{E}\,|\eta|^2\big)^{4/3}=c\,\big(\mathbf{E}\,|X|^2\big)^{4/3}\leqslant c\,\mathbf{E}\,|X|^{8/3},\tag{2.20}$$

Combining the estimates (2.19), and using (2.20), we see that the inequality (2.14) follows from (2.17) and (2.18). □

Lemma 2.5. *Let* $\zeta_1^*, \zeta_2^* \in \mathbf{R}^d$; $\tau_1, \tau_2 \in \mathbf{R}$ *be independent random vectors and variables such that* $\mathbf{E}\,|\zeta_1^*|^4 < \infty$, $\mathbf{E}\,|\zeta_2^*|^4 < \infty$. *Assume that* τ_1, τ_2 *are uniformly distributed on the interval* $[0,1]$. *Let* $\xi \in \mathbf{R}^d$ *and* $t \in \mathbf{R}$ *be non-random. Write*

$$J_0 = \mathbf{E}\,\exp\bigl\{ it\,Q[\xi+\zeta_1^*+\zeta_2^*]\bigr\} - \mathbf{E}\,\exp\bigl\{ it\,Q[\xi+\zeta_1^*]\bigr\} \\ - \mathbf{E}\,\exp\bigl\{ it\,Q[\xi+\zeta_2^*]\bigr\} + \exp\bigl\{ it\,Q[\xi]\bigr\} \tag{2.21}$$

and

$$\begin{aligned} v_1 &= 2\,it\,\langle Q\xi, \zeta_1^*\rangle, & v_2 &= it\,Q[\zeta_1^*], \\ w_1 &= 2\,it\,\langle Q\xi, \zeta_2^*\rangle, & w_2 &= it\,Q[\zeta_2^*], \end{aligned} \qquad u = 2\,it\,\langle Q\,\zeta_1^*, \zeta_2^*\rangle. \tag{2.22}$$

Then we have

$$J_0 = J_1 + J_2, \tag{2.23}$$

where

$$\begin{aligned} J_1 = {} & \mathbf{E}\,v_2\,w_2\,\exp\bigl\{ w_1+\tau_1 w_2\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,\bigl(v_1+v_2+u\bigr)^2 w_2\,(1-\tau_2)\,\exp\bigl\{ w_1+\tau_1 w_2+\tau_2\,(v_1+v_2+u)\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,\bigl(w_1+u\bigr)^2 v_2\,(1-\tau_2)\,\exp\bigl\{ v_1+\tau_2\,(w_1+u)+\tau_1 v_2\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,u^2\,(1-\tau_1)\,\exp\bigl\{ v_1+w_1+\tau_1 u\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,u\,v_1\,w_1\,\exp\{\tau_1 w_1\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,u\,v_1^2\,w_1\,(1-\tau_1)\,\exp\bigl\{ \tau_1 v_1+\tau_2 w_1\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,v_1^2\,w_1^2\,(1-\tau_1)\,(1-\tau_2)\,\exp\bigl\{ \tau_2 v_1+\tau_1 w_1\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} J_2 = {} & \mathbf{E}\,\bigl(v_1+u\bigr)\,w_2\,\exp\bigl\{ w_1+\tau_1 w_2\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,\bigl(w_1+u\bigr)\,v_2\,\exp\bigl\{ v_1+\tau_1 v_2\bigr\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,(u+u\,v_1)\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,u\,w_1\,\exp\{\tau_1 w_1\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,u\,v_1^2\,(1-\tau_1)\,\exp\{\tau_1 v_1\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,w_1\,v_1\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,w_1\,v_1^2\,(1-\tau_1)\,\exp\{\tau_1 v_1\}\,\exp\bigl\{ it\,Q[\xi]\bigr\} \\ & + \mathbf{E}\,v_1\,w_1^2\,(1-\tau_1)\,\exp\{\tau_1 w_1\}\,\exp\bigl\{ it\,Q[\xi]\bigr\}. \end{aligned} \tag{2.25}$$

Proof. The proof is based on the Taylor expansion with remainder term in integral form in powers of v_1, v_2, w_1, w_2, u. For example, using (1.9), we can use the following expansions:

$$\begin{aligned} \mathbf{E}\,\exp\bigl\{ it\,Q[\xi+\zeta_1^*] - it\,Q[\xi]\bigr\} &= \mathbf{E}\,\exp\bigl\{ v_1+v_2\bigr\} \\ &= \mathbf{E}\,\exp\{v_1\} + \mathbf{E}\,v_2\,\exp\bigl\{ v_1+\tau_1 v_2\bigr\}, \end{aligned}$$

$$\begin{aligned}\mathbf{E}\,\exp\bigl\{\,it\,Q[\xi+\zeta_2^*\,]-it\,Q[\xi]\,\bigr\} &= \mathbf{E}\,\exp\bigl\{\,w_1+w_2\,\bigr\}\\ &= \mathbf{E}\,\exp\{w_1\}+\mathbf{E}\,w_2\,\exp\bigl\{\,w_1+\tau_1\,w_2\,\bigr\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,\exp\bigl\{\,it\,Q[\xi+\zeta_1^*+\zeta_2^*\,]-it\,Q[\xi]\,\bigr\} &= \mathbf{E}\,\exp\bigl\{\,v_1+v_2+w_1+w_2+u\,\bigr\}\\ = \mathbf{E}\,\exp\bigl\{\,v_1+v_2+w_1+u\,\bigr\} &+ \mathbf{E}\,w_2\,\exp\bigl\{\,v_1+v_2+w_1+u+\tau_1\,w_2\,\bigr\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,w_2\,\exp\bigl\{\,v_1+v_2+w_1+u+\tau_1\,w_2\,\bigr\} &= \mathbf{E}\,w_2\,\exp\bigl\{\,w_1+\tau_1\,w_2\,\bigr\}\\ &+ \mathbf{E}\,\bigl(v_1+v_2+u\bigr)\,w_2\,\exp\bigl\{\,w_1+\tau_1\,w_2\,\bigr\}\\ &+ \mathbf{E}\,\bigl(v_1+v_2+u\bigr)^2\,w_2\,(1-\tau_2)\,\exp\bigl\{\,w_1+\tau_1\,w_2+\tau_2\,(v_1+v_2+u)\,\bigr\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,\exp\bigl\{\,v_1+v_2+w_1+u\,\bigr\} &= \mathbf{E}\,\exp\bigl\{\,v_1+w_1+u\,\bigr\}\\ &+ \mathbf{E}\,v_2\,\exp\bigl\{\,v_1+w_1+u+\tau_1\,v_2\,\bigr\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,v_2\,\exp\bigl\{\,v_1+w_1+u+\tau_1\,v_2\,\bigr\} &= \mathbf{E}\,v_2\,\exp\bigl\{\,v_1+\tau_1\,v_2\,\bigr\}+\mathbf{E}\,\bigl(w_1+u\bigr)\,v_2\,\exp\bigl\{\,v_1+\tau_1\,v_2\,\bigr\}\\ &+ \mathbf{E}\,\bigl(w_1+u\bigr)^2\,v_2\,(1-\tau_2)\,\exp\bigl\{\,v_1+\tau_2\,(w_1+u)+\tau_1\,v_2\,\bigr\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,\exp\bigl\{\,v_1+w_1+u\,\bigr\} &= \mathbf{E}\,\exp\bigl\{\,v_1+w_1\,\bigr\}\\ &+ \mathbf{E}\,u\,\exp\bigl\{\,v_1+w_1\,\bigr\}+\mathbf{E}\,u^2\,(1-\tau_1)\,\exp\bigl\{\,v_1+w_1+\tau_1\,u\,\bigr\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,u\,\exp\bigl\{\,v_1+w_1\,\bigr\} &= \mathbf{E}\,(u+u\,v_1)\,\exp\{w_1\}+\mathbf{E}\,u\,v_1^2\,(1-\tau_1)\,\exp\bigl\{\,\tau_1\,v_1+w_1\,\bigr\},\\ \mathbf{E}\,(u+u\,v_1)\,\exp\{w_1\} &= \mathbf{E}\,(u+u\,v_1)+\mathbf{E}\,(u+u\,v_1)\,w_1\,\exp\{\tau_1\,w_1\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,u\,v_1^2\,(1-\tau_1)\,\exp\bigl\{\,\tau_1\,v_1+w_1\,\bigr\} &= \mathbf{E}\,u\,v_1^2\,(1-\tau_1)\,\exp\{\tau_1\,v_1\}\\ &+ \mathbf{E}\,u\,v_1^2\,w_1\,(1-\tau_1)\,\exp\bigl\{\,\tau_1\,v_1+\tau_2\,w_1\,\bigr\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,\exp\bigl\{\,v_1+w_1\,\bigr\} &= \mathbf{E}\,\exp\{v_1\}+\mathbf{E}\,w_1\,\exp\{v_1\}+\mathbf{E}\,w_1^2\,(1-\tau_1)\,\exp\bigl\{\,v_1+\tau_1\,w_1\,\bigr\},\\ \mathbf{E}\,\exp\{w_1\} &= 1+\mathbf{E}\,w_1+\mathbf{E}\,w_1^2\,(1-\tau_1)\,\exp\{\tau_1\,w_1\},\\ \mathbf{E}\,w_1\,\exp\{v_1\} &= \mathbf{E}\,w_1+\mathbf{E}\,w_1\,v_1+\mathbf{E}\,w_1\,v_1^2\,(1-\tau_1)\,\exp\{\tau_1\,v_1\},\end{aligned}$$

$$\begin{aligned}\mathbf{E}\,w_1^2\,(1-\tau_1)\,\exp\bigl\{\,v_1+\tau_1\,w_1\,\bigr\} &= \mathbf{E}\,\bigl(w_1^2+v_1\,w_1^2\bigr)\,(1-\tau_1)\,\exp\{\tau_1\,w_1\}\\ &+ \mathbf{E}\,v_1^2\,w_1^2\,(1-\tau_1)\,(1-\tau_2)\,\exp\bigl\{\,\tau_2\,v_1+\tau_1\,w_1\,\bigr\}.\ \square\end{aligned}$$

The following Lemma 2.6 can be easily derived from the results of Le Cam (1965) and Esseen (1968) (see also Petrov (1975), Theorem 1 of Chapter V and Lemma 3 of Chapter III, or Arak and Zaitsev (1988), Theorem 1.2 of Chapter III and Lemma 1.4 of Chapter II)

Lemma 2.6. *Let H_1, H_2 be one-dimensional probability distributions with characteristic functions $h_1(\cdot), h_2(\cdot)$. Then, for any $T>0$,*

$$\rho(H_1,H_2)\leqslant c\int_0^T\Big|\,\frac{h_1(t)-h_2(t)}{t}\,\Big|\,dt+\frac{c}{T}\,\min\Big\{\int_0^T\big|\,h_1(t)\,\big|\,dt,\ \int_0^T\big|\,h_2(t)\,\big|\,dt\Big\}.$$

The expectation $\mathbf{E}_\mu$ with respect to a random vector (or variable) μ we define as the conditional expectation

$$\mathbf{E}_\mu\,f(\mu,\nu,\xi,\dots)=\mathbf{E}\,\big\{\,f(\mu,\nu,\xi\dots)\,\big|\,\nu,\xi,\dots\big\}$$

given all random objects except μ. Below we shall use that if the random vectors $\mu\in\mathbf{R}^d$, ν are independent and $f(\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$ are measurable functions (probably, complex valued), then

$$\big|\,\mathbf{E}\,f(\mu,\nu)\,\big|=\Big|\,\mathbf{E}\,\big(\mathbf{E}_\nu f(\mu,\nu)\big)\Big|\leqslant\mathbf{E}\,\big|\,\mathbf{E}_\nu f(\mu,\nu)\,\big|\leqslant\sup_{z\in\mathbf{R}^d}\,\big|\,\mathbf{E}\,f(z,\nu)\,\big| \tag{2.26}$$

and, analogously,

$$\begin{aligned}\int\big|\,\mathbf{E}\,g(t,\mu,\nu)\,\big|\,dt&\leqslant\int\mathbf{E}\,\big|\,\mathbf{E}_\nu\,g(t,\mu,\nu)\,\big|\,dt=\mathbf{E}\,\int\big|\,\mathbf{E}_\nu\,g(t,\mu,\nu)\,\big|\,dt\\&\leqslant\sup_{z\in\mathbf{R}^d}\int\big|\,\mathbf{E}\,g(t,z,\nu)\,\big|\,dt.\end{aligned} \tag{2.27}$$

3. Proof of Theorem 1.2

Without loss of generality we assume that

$$\sigma^2=\mathbf{E}\,|X|^2=c_{11},\quad\text{and, hence,}\quad\mathbf{E}\,|X|^{8/3}\geqslant c, \tag{3.1}$$

where c_{11} is an absolute positive constant which will be chosen at the end of the proof. Otherwise, one should consider the vectors $\big\{c_{11}\,\sigma^{-1}X_1,\,c_{11}\,\sigma^{-1}X_2,\dots\big\}$ instead of $\big\{X_1,X_2,\dots\big\}$.

Suppose that the distribution

$$H=H_N=\mathcal{L}\big(X/\sqrt{N}\big)\in\mathfrak{F}_d \tag{3.2}$$

is represented in the form

$$H=(1-q)\,U+q\,V, \tag{3.3}$$

where $U = U_N \in \mathfrak{F}_d$, $V = V_N \in \mathfrak{F}_d$,

$$U\Big\{x \in \mathbf{R}^d : |x| \leqslant N^{1/4}\Big\} = V\Big\{x \in \mathbf{R}^d : |x| > N^{1/4}\Big\} = 1 \tag{3.4}$$

and

$$0 \leqslant q = q_N = \mathbf{P}\left\{|X| > N^{3/4}\right\} \leqslant 1. \tag{3.5}$$

Set

$$W = W_N = (1-q)\,U + q\,E. \tag{3.6}$$

Introduce the truncated i.i.d. random vectors

$$X_j^* = X_j\,\mathbf{I}\big\{|X_j| \leqslant N^{3/4}\big\}, \qquad j = 1, 2, \dots, \tag{3.7}$$

with

$$\mathcal{L}\Big(X_j^*/\sqrt{N}\Big) = W, \qquad j = 1, 2, \dots, \tag{3.8}$$

and denote

$$X_j^{**} = X_j - X_j^* = X_j\,\mathbf{I}\big\{|X_j| > N^{3/4}\big\}, \qquad j = 1, 2, \dots. \tag{3.9}$$

Obviously,

$$\mathbf{E}\,X_j^{**} + \mathbf{E}\,X_j^* = \mathbf{E}\,X_j = 0, \qquad j = 1, 2, \dots. \tag{3.10}$$

In order to simplify the notation it is convenient to suppose that the i.i.d. vectors $Y, Y_1, Y_2, \dots$ are defined on the same probability space as $X, X_1, X_2, \dots$ as follows. Assume that the random variable Π and the random vectors $Y_1^*, Y_2^*, \dots;$ $Y_1^{**}, Y_2^{**}, \dots;$ $X, X_1, X_2, \dots$ are mutually independent. Let Π have the Poisson distribution with parameter 1. Denote

$$Y = X_1 + \dots + X_\Pi. \tag{3.11}$$

Assume that $Y_1^*, Y_2^*, \dots;$ $Y_1^{**}, Y_2^{**}, \dots$ have distributions such that

$$\mathcal{L}\Big(Y_j^*/\sqrt{N}\Big) = e(W) = e\big((1-q)\,U\big), \qquad \mathcal{L}\Big(Y_j^{**}/\sqrt{N}\Big) = e(qV), \qquad j = 1, 2, \dots. \tag{3.12}$$

Write

$$Y_j = Y_j^* + Y_j^{**}, \qquad j = 1, 2, \dots, \tag{3.13}$$

$$S_N^* = N^{-1/2}\,(X_1^* + \dots + X_N^*), \qquad T_N^* = N^{-1/2}\,(Y_1^* + \dots + Y_N^*). \tag{3.14}$$

Obviously, $Y_1, Y_2, \dots$ are i.i.d. random vectors with

$$\mathcal{L}\Big(Y_j/\sqrt{N}\Big) = e\big((1-q)\,U\big)\,e(qV) = e(H) = \mathcal{L}\Big(Y/\sqrt{N}\Big), \qquad j = 1, 2, \dots, \tag{3.15}$$

and

$$\begin{aligned}\mathcal{L}(S_N) = H^N, &\qquad \mathcal{L}(T_N) = \big(e(H)\big)^N = e(NH),\\ \mathcal{L}(S_N^*) = W^N, &\qquad \mathcal{L}(T_N^*) = \big(e(W)\big)^N = e(NW)\end{aligned} \tag{3.16}$$

(see (1.1)–(1.3), (1.6), (3.2), (3.3), (3.11)–(3.14)). By construction, the vectors $Y_1, Y_2, \ldots$ are independent of $X, X_1, X_2, \ldots$. Note that (1.1), (3.2), (3.15) automatically imply that

$$\mathcal{L}(Y_j) = \mathcal{L}(Y) = e\big(\mathcal{L}(X)\big), \qquad j = 1, 2, \ldots. \tag{3.17}$$

According to (1.7), (3.17),

$$\mathbf{E}\, Y_j = \mathbf{E}\, X_j = \mathbf{E}\, \eta = 0, \qquad \operatorname{cov} Y_j = \operatorname{cov} X_j = \operatorname{cov} \eta, \qquad j = 1, 2, \ldots. \tag{3.18}$$

Introduce also the symmetrized random vectors

$$\begin{aligned}\widetilde{X}_j = X_j - X_{N+j}, &\qquad \widetilde{Y}_j = Y_j - Y_{N+j},\\ \widetilde{X}_j^* = X_j^* - X_{N+j}^*, &\qquad \widetilde{Y}_j^* = Y_j^* - Y_{N+j}^*,\end{aligned} \qquad j = 1, 2, \ldots. \tag{3.19}$$

It is clear that (3.18) and (3.19) imply that, for $j = 1, 2, \ldots,$

$$\mathbf{E}\,\big(\widetilde{X}_j/\sqrt{2}\big) = \mathbf{E}\,\big(\widetilde{Y}_j/\sqrt{2}\big) = 0, \qquad \operatorname{cov}\big(\widetilde{Y}_j/\sqrt{2}\big) = \operatorname{cov}\big(\widetilde{X}_j/\sqrt{2}\big) = \operatorname{cov} \eta, \tag{3.20}$$

and

$$\mathbf{E}\,\big|\,\widetilde{X}_j/\sqrt{2}\,\big|^{8/3} \leqslant c\, \mathbf{E}\, |X|^{8/3}. \tag{3.21}$$

Besides, using (3.11), (3.15), (3.19) and Lemma 2.3 for fixed values of Π, we see that

$$\begin{aligned}\mathbf{E}\,\big|\,\widetilde{Y}_j/\sqrt{2}\,\big|^{8/3} &\leqslant c\, \mathbf{E}\, |Y|^{8/3} = c\, \mathbf{E}\,\big|\, X_1 + \cdots + X_\Pi\,\big|^{8/3}\\ &\leqslant c\, \mathbf{E}\, \Pi^{4/3}\, \mathbf{E}\, |X|^{8/3} \leqslant c\, \mathbf{E}\, |X|^{8/3}.\end{aligned} \tag{3.22}$$

Using (1.6), (3.5), (3.7), (3.14), and applying the Chebyshev inequality, we get

$$\mathbf{P}\,\big\{\, |S_N| \neq |S_N^*|\,\big\} \leqslant \sum_{j=1}^N \mathbf{P}\,\big\{\, X_j \neq X_j^*\,\big\} = N q \leqslant \frac{\mathbf{E}\, |X|^{8/3}}{N}. \tag{3.23}$$

Using (1.1), (1.6) and (3.12)–(3.14), we obtain

$$\begin{aligned}\mathbf{P}\,\big\{\, Y_j \neq Y_j^*\,\big\} = \mathbf{P}\,\big\{\, Y_j^{**} \neq 0\big\} \leqslant 1 - e^{-q} \leqslant q \leqslant \frac{\mathbf{E}\, |X|^{8/3}}{N^2}, &\qquad j = 1, 2 \ldots,\\ \mathbf{P}\,\big\{\, |T_N| \neq |T_N^*|\,\big\} \leqslant \sum_{j=1}^N \mathbf{P}\,\big\{\, Y_j \neq Y_j^*\,\big\} \leqslant \frac{\mathbf{E}\, |X|^{8/3}}{N}. &\end{aligned} \tag{3.24}$$

Introduce the distribution functions

$$\begin{aligned} F(x) &= \mathbf{P}\big\{Q[S_N-a]<x\big\}, & G(x) &= \mathbf{P}\big\{Q[T_N-a]<x\big\}, \\ F^*(x) &= \mathbf{P}\big\{Q[S_N^*-a]<x\big\}, & G^*(x) &= \mathbf{P}\big\{Q[T_N^*-a]<x\big\}, \end{aligned} \qquad x\in\mathbf{R}, \tag{3.25}$$

and the characteristic functions

$$f^*(t)=\mathbf{E}\,\exp\big\{it\,Q[S_N^*-a]\big\},\qquad g^*(t)=\mathbf{E}\,\exp\big\{it\,Q[T_N^*-a]\big\},\qquad t\in\mathbf{R}. \tag{3.26}$$

From (3.23)–(3.25) it is easy to derive that

$$\begin{aligned} \big|F(x)-F^*(x)\big| &\leqslant \mathbf{P}\big\{|S_N|\neq|S_N^*|\big\}\leqslant\frac{\mathbf{E}\,|X|^{8/3}}{N}, \\ \big|G(x)-G^*(x)\big| &\leqslant \mathbf{P}\big\{|T_N|\neq|T_N^*|\big\}\leqslant\frac{\mathbf{E}\,|X|^{8/3}}{N}, \end{aligned}\qquad \text{for all}\quad x\in\mathbf{R}. \tag{3.27}$$

Fix an orthonormal system of vectors $e_1,\dots,e_9\in\mathbf{R}^d$ and put

$$\delta_0=\frac{1}{900},\qquad p=\min\Big\{\min_{1\leqslant j\leqslant 9}\mathbf{P}\big\{|\eta-e_j|\leqslant\delta_0\big\},\ \min_{1\leqslant j\leqslant 9}\mathbf{P}\big\{|\eta-Q\,e_j|\leqslant\delta_0\big\}\Big\}. \tag{3.28}$$

The number p is positive since the Gaussian vector η is not concentrated in any proper closed linear subspace of $\mathbf{R}^d$ and, therefore, $\mathbf{P}\big\{\eta\in B\big\}>0$, for any ball $B\subset\mathbf{R}^d$ with positive radius. According to the definition of the classes $\mathcal{N}(\cdot)$, we have

$$\mathcal{L}(\eta)\in\mathcal{N}(9,p,\delta_0,Q;\ e_1,\dots,e_9). \tag{3.29}$$

It is clear that

$$p\leqslant c\,\mathbf{E}\,|\eta|^{8/3}\leqslant c\,\mathbf{E}\,|X|^{8/3} \tag{3.30}$$

(see (2.20), (3.28)). If p is fixed, one can consider it as a function of the covariance operator $\operatorname{cov}X=\operatorname{cov}\eta$. The same applies to functions of p.

Below we shall show that

$$\sup_x\big|F(x)-G(x)\big|\leqslant\frac{c\,\big(|a|^4+\big(\mathbf{E}\,|X|^{8/3}\big)^3\big)}{p^4\,N} \tag{3.31}$$

provided that $\mathbf{E}\,|X|^2=c_{11}$, the operator Q is isometric and p is defined in (3.28).

Let c_{12} be a sufficiently large constant which will be specified later. Let m be the minimal natural number such that $m\geqslant c_{12}\big(\mathbf{E}\,|X|^{8/3}\big)^3\big/p^3$. In view of (3.30), we can choose c_{12} to be so large that

$$c_{12}\big(\mathbf{E}\,|X|^{8/3}\big)^3\big/p^3\leqslant m\leqslant 2\,c_{12}\big(\mathbf{E}\,|X|^{8/3}\big)^3\big/p^3. \tag{3.32}$$

Write

$$K = c_{13}\, N/m, \qquad N_0 = pK = c_{13}\, p\, N/m \leqslant p\, N/m, \tag{3.33}$$

$$t_0 = N_0^{2/9}/p, \qquad t_1 = \sqrt{N_0}/p, \qquad t_2 = c_{14}\, N_0^{7/9}/p, \tag{3.34}$$

where $c_{13} \leqslant 1$ will be chosen to be sufficiently small and c_{14} sufficiently large. From (3.32), (3.33) it follows that

$$\frac{c_{15}\, p^4\, N}{2\,\big(\mathbf{E}\,|X|^{8/3}\big)^3} \leqslant N_0 \leqslant \frac{c_{15}\, p^4\, N}{\big(\mathbf{E}\,|X|^{8/3}\big)^3}. \tag{3.35}$$

Without loss of generality (see (3.28), (3.30)–(3.33), (3.35)) we can assume that

$$1 \leqslant m \leqslant N/20, \qquad p N\, m^{-1} \geqslant c_{16}, \qquad N_0 \geqslant c_{17}, \tag{3.36}$$

where $c_{16} = c_{16}(c_{12})$, $c_{17} = c_{17}\big(c_{12},\, c_{13},\, c_{14},\, c_{16}\big)$ will be chosen to be sufficiently large. We have

$$0 < t_0 < t_1 < t_2 < K, \tag{3.37}$$

if c_{17} in (3.36) is large enough. The function $M_9\big(pt, N_0\big)$ attains its minimal value at the point $t = t_1$ and, according to (2.2),

$$M_9\big(pt, N_0\big) = \begin{cases} |pt|^{-9/2}, & \text{for} \quad |t| \leqslant t_1, \\ \big(|pt|/N_0\big)^{9/2}, & \text{for} \quad |t| \geqslant t_1. \end{cases} \tag{3.38}$$

Introduce the integral

$$I(f^*,\, g^*) = \int\limits_{|t| \leqslant t_0} \left| \frac{f^*(t) - g^*(t)}{t} \right| dt. \tag{3.39}$$

Applying Lemma 2.6 with $H_1 = F^*$, $H_2 = G^*$, $T = K$, splitting the integrals and using (3.37) and that $1/K \leqslant 1/|t|$, for $|t| \leqslant K$, we obtain

$$\big| F^*(x) - G^*(x) \big| \leqslant c\, I(f^*,\, g^*) + \sum_{k=1}^{4} \big(I_k(f^*) + I_k(g^*) \big), \tag{3.40}$$

where we denote

$$\begin{aligned} I_1(f^*) &= \frac{1}{K} \int\limits_{|t| \leqslant t_0} \big| f^*(t) \big|\, dt, & I_2(f^*) &= \int\limits_{t_0 \leqslant |t| \leqslant t_1} \big| f^*(t) \big| \, \frac{dt}{|t|}, \\ I_3(f^*) &= \int\limits_{t_1 \leqslant |t| \leqslant t_2} \big| f^*(t) \big| \, \frac{dt}{|t|}, & I_4(f^*) &= \int\limits_{t_2 \leqslant |t| \leqslant K} \big| f^*(t) \big| \, \frac{dt}{|t|}. \end{aligned} \tag{3.41}$$

Below we shall show that each of the integrals $I_k(f^*)$, $k=1,\dots,4$, is bounded from above by c/N_0.

Define the functions

$$\begin{aligned} R_1(t) &= \sup_{b\in\mathbf{R}^d} \Big|\,\mathbf{E}\,\exp\big\{ it^* Q[S^*_{N_1}-b]\big\}\Big|, \\ R_2(t) &= \sup_{b\in\mathbf{R}^d} \Big|\,\mathbf{E}\,\exp\big\{ it^* Q[T^*_{N_1}-b]\big\}\Big|, \end{aligned} \qquad t^* = t\,N_1 N^{-1}, \tag{3.42}$$

where

$$N/20 \leqslant N_1 \overset{\mathrm{def}}{=} \lceil (N-1)/10 \rceil \leqslant N/10 \tag{3.43}$$

(see (3.36)). To estimate the integrals I_1, I_2 and I_3 we note that the sums $S^*_N, S^*_{N_1}$ can be represented as

$$S^*_N = (N_1/N)^{1/2}\, S^*_{N_1} + b_1, \qquad S^*_{N_1} = (2\,m/N_1)^{1/2}\, \varUpsilon^*_n + b_2, \tag{3.44}$$

where the random vectors $\varUpsilon^*_n$ can be written in the form

$$\varUpsilon^*_n = Z^*_1 + \dots + Z^*_n, \qquad n = \lceil N_1/m \rceil; \tag{3.45}$$

$Z^*_1,\dots,Z^*_n$ are independent copies of Z^* defined by

$$Z^* = (2\,m)^{-1/2}\,(X^*_1 + \dots + X^*_m), \tag{3.46}$$

and b_1, b_2 are independent of $S^*_{N_1}, \varUpsilon^*_n$ respectively. Furthermore, introduce the symmetrized random vectors (see (3.19))

$$\begin{aligned} \widetilde Z &= m^{-1/2}\,\big(\widetilde X_1/\sqrt{2} + \dots + \widetilde X_m/\sqrt{2}\big) \qquad \text{and} \\ \widetilde Z^* &= m^{-1/2}\,\big(\widetilde X^*_1/\sqrt{2} + \dots + \widetilde X^*_m/\sqrt{2}\big). \end{aligned} \tag{3.47}$$

According to (3.36), (3.43) and (3.45), we have

$$\frac{N}{40\,m} \leqslant n \leqslant \frac{N}{10\,m}. \tag{3.48}$$

Let $e\in\mathbf{R}^d$, $|e|=1$. Consider a function $L:\mathbf{R}^d\to\mathbf{R}$ with infinitely many bounded derivatives such that

$$0\leqslant L(\cdot)\leqslant 1, \qquad L(x)=1, \text{ for } |x|\leqslant 1, \qquad L(x)=0, \text{ for } |x|\geqslant 2. \tag{3.49}$$

Using (3.21), (3.47)–(3.49) and applying Lemma 2.4, we have

$$\mathbf{P}\big\{\,|\widetilde Z - e|\leqslant 2\,\delta_0\big\} \geqslant \mathbf{E}\,L\Big(\frac{\widetilde Z - e}{\delta_0}\Big) \geqslant \mathbf{E}\,L\Big(\frac{\eta - e}{\delta_0}\Big) - \frac{c\,\mathbf{E}\,|X|^{8/3}}{m^{1/3}\,\delta_0^{8/3}}. \tag{3.50}$$

From (3.49) it follows that

$$\mathbf{E}\,L\Big(\frac{\eta-e}{\delta_0}\Big) \geqslant \mathbf{P}\big\{\,|\eta-e|\leqslant\delta_0\,\big\}. \tag{3.51}$$

It is evident that

$$\mathbf{P}\big\{\,|\widetilde{Z}^*-e|\leqslant 2\,\delta_0\big\} \geqslant \mathbf{P}\big\{\,|\widetilde{Z}-e|\leqslant 2\,\delta_0\big\} - \mathbf{P}\big\{\,\widetilde{Z}\neq\widetilde{Z}^*\,\big\}. \tag{3.52}$$

According to (3.5), (3.7), (3.19), (3.24) and (3.36), we have

$$\mathbf{P}\big\{\,\widetilde{Z}\neq\widetilde{Z}^*\,\big\} \leqslant \sum_{j=1}^{m}\mathbf{P}\big\{\,\widetilde{X}_j\neq\widetilde{X}_j^*\,\big\} \leqslant 2\,m\,q \leqslant \frac{2\,m\,\mathbf{E}\,|X|^{8/3}}{N^2} \leqslant \frac{2\,\mathbf{E}\,|X|^{8/3}}{m^{1/3}}. \tag{3.53}$$

Applying (3.50)–(3.53) for $e=e_j$ and $e=Q\,e_j$, $j=1,\dots,9$, choosing c_{12} in (3.32) to be large enough and using (3.28), we obtain

$$2\,\mathbf{P}\big\{\,|\widetilde{Z}^*-e_j\,|\leqslant 2\,\delta_0\big\} \geqslant 2\,\mathbf{P}\big\{\,|\eta-e_j\,|\leqslant\delta_0\big\} - \frac{c\,\mathbf{E}\,|X|^{8/3}}{m^{1/3}} \geqslant 2p-p=p \tag{3.54}$$

and, analogously,

$$2\,\mathbf{P}\big\{\,|\widetilde{Z}^*-Q\,e_j\,|\leqslant 2\,\delta_0\big\} \geqslant p. \tag{3.55}$$

According to (2.1), the relations (3.54), (3.55) imply that

$$\mathcal{L}(\widetilde{Z}^*)\in\mathcal{N}(9,\,p/2,\,2\,\delta_0,\,Q;\;e_1,\dots,e_9). \tag{3.56}$$

Taking into account (3.28), (3.44), (3.45), (3.56), we see that Lemma 2.1 is applicable with $s=9$, $\delta=2\,\delta_0$, $r=p/2$; $y_j=e_j$, $j=1,\dots,9$.

Conditioning on b_2 (see (2.26)), applying the inequality (2.11) of Lemma 2.1 with $u=2\,t\,m\,N^{-1}n$, and taking into account (2.4), (3.33), (3.42), (3.44), (3.48), we obtain that, for any $b\in\mathbf{R}^d$, $t\in\mathbf{R}$,

$$\begin{aligned}\Big|\,\mathbf{E}\,\exp\big\{\,it^*\,Q[S_{N_1}^*-b]\,\big\}\Big| &= \Big|\,\mathbf{E}\,\exp\big\{\,2\,i\,t\,m\,N^{-1}\,Q\big[\varUpsilon_n^*+(2\,m/N_1)^{-1/2}(b_2-b)\big]\,\big\}\Big|\\ &\leqslant \sup_{z\in\mathbf{R}^d}\Big|\,\mathbf{E}\,\exp\big\{\,2\,i\,t\,m\,N^{-1}\,Q[\varUpsilon_n^*-z]\,\big\}\Big|\\ &\leqslant c\,M_9\big(p\,t\,m\,N^{-1}n,\;pn/2\big)\\ &\leqslant c\,M_9\big(pt,\;N_0\big),\end{aligned} \tag{3.57}$$

Note that in the last inequality we used (2.4) to change constants in the arguments of the function M_9.

Using (2.26) again, we see that the relations (3.26), (3.42), (3.44), (3.57) together imply that, for any $t \in \mathbf{R}$,

$$\big|\, f^*(t) \,\big| \leqslant R_1(t) \leqslant c\, \min\Big\{1,\ M_9\big(pt,\ N_0\big)\Big\}. \tag{3.58}$$

Using (3.34), (3.37), (3.38), (3.41) and (3.58), we conclude

$$I_1(f^*) \leqslant \frac{c}{K} \int_0^{1/p} dt + \frac{c}{K} \int_{1/p}^{\infty} \frac{dt}{(pt)^{9/2}} = \frac{c}{p\,K} = \frac{c}{N_0}\,. \tag{3.59}$$

Similarly,

$$I_2(f^*) \leqslant c \int_{t_0}^{t_1} \frac{1}{(pt)^{9/2}}\ \frac{dt}{t} \leqslant c \int_{p\,t_0}^{\infty} \frac{dt}{t^{1+9/2}} = \frac{c}{N_0}\,, \tag{3.60}$$

and

$$I_3(f^*) \leqslant c \int_0^{t_2} \Big(\frac{pt}{N_0}\Big)^{9/2}\ \frac{dt}{t} \leqslant c\Big(\frac{pt_2}{N_0}\Big)^{9/2} = \frac{c}{N_0}\,. \tag{3.61}$$

Using (2.27), (3.26), (3.33), (3.34), (3.41), (3.44), (3.45), (3.48), (3.56), applying the inequality (2.7) of Lemma 2.1 and changing variables, we obtain

$$I_4(f^*) \leqslant c\, N_0^{-2} \log N_0 + \sup_{\varLambda}\ \sup_{z\in\mathbf{R}^d} \int\limits_{t_2 \leqslant |t| \leqslant K} \sqrt{\varphi_\varLambda(z,\ c\,m\,t\,l\,N^{-1})}\ \frac{dt}{|t|} \tag{3.62}$$

with $\varphi_\varLambda(\cdot\,,\,\cdot)$ defined by (2.8) and

$$l = \lceil r\,n/36 \rceil = \lceil pn/72 \rceil \leqslant c\,N_0 \tag{3.63}$$

(see (3.33), (3.48)). The supremum $\sup_\varLambda$ is taken over all random vectors $\varLambda_l$ and $\varLambda_l'$ of type (2.9), where $W_1, \dots, W_l,\ W_1', \dots, W_l' \in \mathbf{R}^d$ denote independent random vectors with distributions from the class $\Gamma(9,\ 2\,\delta_0;\ e_1, \dots, e_9)$. In addition, we used that, in view of (3.33), (3.48), (3.63), we can choose c_{16}, c_{17} in (3.36) to be so large that

$$l = \lceil pn/72 \rceil \geqslant \lceil pN/(2880\, m) \rceil \geqslant c\,N_0 \geqslant 1. \tag{3.64}$$

Changing variables, using (3.33), (3.34), (3.63), (3.64) and choosing appropriate constants c_{13}, c_{14} in the definitions of t_2 and K, we have

$$\int\limits_{t_2 \leqslant |t| \leqslant K} \sqrt{\varphi_\varLambda(z,\ c\,m\,t\,l\,N^{-1})}\ \frac{dt}{|t|} \leqslant \int\limits_{N_0^{7/9} \leqslant |u| \leqslant N_0} \sqrt{\varphi_\varLambda(z,\ u)}\ \frac{du}{|u|}\,. \tag{3.65}$$

Applying the inequality (2.10) of Lemma 2.1 and using (2.4), (3.63) and (3.64), we obtain, for all $\gamma \geqslant 0,\ u \in \mathbf{R}$,

$$\varphi_\Lambda(z,\, u+\gamma)\ \varphi_\Lambda(z,\, u-\gamma) \leqslant c\, M_9^2(\gamma,\, l) \leqslant c_{18}\, M_9^2(2\gamma,\, N_0). \tag{3.66}$$

Therefore, Lemma 2.2 is applicable with $\psi(t) = \varphi_\Lambda(z,\, t)$ and $\psi(t) = \varphi_\Lambda(z,\, -t)$; $s = 9,\ D = N_0,\ \lambda = \sqrt{c_{18}}$. By Lemma 2.2, we have

$$\int\limits_{N_0^{7/9} \leqslant |u| \leqslant N_0} \sqrt{\varphi_\Lambda(z,\, u)}\ \frac{du}{|u|} \leqslant \frac{c}{N_0}\,. \tag{3.67}$$

Using (3.62), (3.65), (3.67), we see that $I_4(f^*) \leqslant c/N_0$ and, taking into account (3.59)–(3.61),

$$\Big|\, I_1(f^*) + I_2(f^*) + I_3(f^*) + I_4(f^*) \Big| \leqslant \frac{c}{N_0}\,. \tag{3.68}$$

Similarly to (3.58), (3.68), one proves

$$\big|\, g^*(t) \big| \leqslant R_2(t) \leqslant c\, \min\Big\{ 1,\ M_9\big(pt,\, N_0\big) \Big\}, \tag{3.69}$$

$$\Big|\, I_1(g^*) + I_2(g^*) + I_3(g^*) + I_4(g^*) \Big| \leqslant \frac{c}{N_0}\,. \tag{3.70}$$

In order to verify these relations one should use (3.25), (3.26), (3.41), (3.42) and the fact that, according to (1.6), (3.14), (3.15), (3.18), (3.20), (3.22), (3.24), the random vectors $Y, Y_1, Y_2, \ldots;\ Y_1^*, Y_2^*, \ldots;\ T_N, T_N^*$ have all the properties of the random vectors $X, X_1, X_2, \ldots;\ X_1^*, X_2^*, \ldots;\ S_N, S_N^*$, which were used in the proof of (3.58), (3.68).

In order to estimate $I(f^*,\, g^*)$ we write the characteristic functions $f^*(t)$ and $g^*(t)$ in the form

$$\begin{aligned} f^*(t) &= \int_{\mathbf{R}^d} \exp\big\{ itQ[x-a] \big\}\, W^N\{dx\}, \\ g^*(t) &= \int_{\mathbf{R}^d} \exp\big\{ itQ[x-a] \big\}\, \big(e(W)\big)^N\{dx\} \end{aligned} \tag{3.71}$$

(see (3.16), (3.26)). It is clear that

$$W^N - \big(e(W)\big)^N = \big(W - e(W)\big) \sum_{j=1}^{N} W^{j-1} \big(e(W)\big)^{N-j}, \tag{3.72}$$

$$e(W) = \exp\Big\{ W - E \Big\} = E + \sum_{\mu=1}^{\infty} \frac{(W-E)^\mu}{\mu\,!} \tag{3.73}$$

and, hence,

$$W - e(W) = -\sum_{\mu=2}^{\infty} \frac{(W-E)^{\mu}}{\mu\,!} = -(W-E)^2 \sum_{\mu=0}^{\infty} \frac{(W-E)^{\mu}}{(\mu+2)!}$$
$$= (W-E)^2 \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \alpha_{\mu,\nu} W^{\nu}, \tag{3.74}$$

where $\alpha_{\mu,\nu}$ are easily computable coefficients with

$$\sum_{\nu=0}^{\mu} |\alpha_{\mu,\nu}| = \frac{2^{\mu}}{(\mu+2)!}\,, \qquad \mu = 0, 1, \dots . \tag{3.75}$$

According to (3.71), (3.72), (3.74),

$$f^*(t) - g^*(t) = \sum_{j=1}^{N} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\mu} \alpha_{\mu,\nu}\, J(t,j,\nu,N)$$
$$= \sum_{j=1}^{N} \sum_{\mu=0}^{N} \sum_{\nu=0}^{\mu} \alpha_{\mu,\nu}\, J(t,j,\nu,N) \;+\; \sum_{j=1}^{N} \sum_{\mu=N+1}^{\infty} \sum_{\nu=0}^{\mu} \alpha_{\mu,\nu}\, J(t,j,\nu,N)$$
$$= \;\sum\nolimits_1 \alpha_{\mu,\nu}\, J(t,j,\nu,N) \;+\; \sum\nolimits_2 \alpha_{\mu,\nu}\, J(t,j,\nu,N), \tag{3.76}$$

where

$$J(t,j,\nu,N) = \int\limits_{\mathbf{R}^d} \exp\bigl\{ it\, Q[x-a] \bigr\} \Bigl((W-E)^2 W^{j-1+\nu} \bigl(e(W)\bigr)^{N-j} \Bigr)\{dx\}. \tag{3.77}$$

It is easy to see that, for $j = 1, \dots, N$; $\mu = 1, 2, \dots$; $\nu = 0, \dots, \mu$, the integral $J(t,j,\nu,N)$ can be rewritten in the form

$$J(t,j,\nu,N) = \mathbf{E} \exp\bigl\{ it\, Q[\xi + \zeta_1^* + \zeta_2^*] \bigr\} - \mathbf{E} \exp\bigl\{ it\, Q[\xi + \zeta_1^*] \bigr\}$$
$$- \mathbf{E} \exp\bigl\{ it\, Q[\xi + \zeta_2^*] \bigr\} + \mathbf{E} \exp\bigl\{ it\, Q[\xi] \bigr\}, \tag{3.78}$$

where

$$\zeta_k^* = \zeta_k\, \mathbf{I}\bigl\{ |\zeta_k| \leqslant N^{1/4} \bigr\}, \qquad k = 1, 2, \tag{3.79}$$

and the random vectors $\zeta_1,\, \zeta_2 \in \mathbf{R}^d$ with distributions

$$\mathcal{L}(\zeta_1) = \mathcal{L}(\zeta_2) = H = \mathcal{L}\Bigl(X/\sqrt{N} \Bigr) \tag{3.80}$$

form together with $X_1^*, X_2^*, \dots;\ Y_1^*, Y_2^*, \dots$ a system of mutually independent random vectors such that

$$\begin{gathered}\xi = N^{-1/2}\left(X_1^* + \dots + X_{j-1+\nu}^* + Y_1^* + \dots + Y_{N-j}^*\right) - a,\\ \mathcal{L}(\xi) = W^{j-1+\nu}\big(e(W)\big)^{N-j} E_{-a}\end{gathered} \tag{3.81}$$

(see (3.8), (3.12)). Here we used that (3.7), (3.8), (3.79), (3.80) together imply that

$$\mathcal{L}(\zeta_1^*) = \mathcal{L}(\zeta_2^*) = W = \mathcal{L}\Big(X_1^*/\sqrt{N}\Big). \tag{3.82}$$

Furthermore, we introduce random vectors ζ_k^{**} by

$$\zeta_k^{**} = \zeta_k - \zeta_k^* = \zeta_k\,\mathbf{I}\big\{|\zeta_k| > N^{1/4}\big\}, \qquad k = 1, 2. \tag{3.83}$$

According to (3.80) and (3.83),

$$\mathbf{E}\,\zeta_k^{**} + \mathbf{E}\,\zeta_k^* = \mathbf{E}\,\zeta_k = 0, \qquad k = 1, 2. \tag{3.84}$$

For fixed values of ξ, we apply Lemma 2.5 to the right-hand side of (3.78). Taking the expectation with respect to ξ, we see that

$$J(t, j, \nu, N) = J_1 + J_2, \tag{3.85}$$

where J_1, J_2 are defined by (2.24), (2.25) with v_1, v_2, w_1, w_2, u satisfying (2.22) and with τ_1, τ_2 independent of $\big\{\zeta_1^*, \zeta_2^*;\ X_1^*, X_2^*, \dots;\ Y_1^*, Y_2^*, \dots\big\}$ and uniformly distributed on the interval $[0, 1]$.

It can be shown that

$$|J_1| \leqslant c\,(t^2 + t^4)\,\mathbf{E}\,|\zeta_1^*|^2\left(D_1^{(\nu)}\,\mathbf{E}\,|\zeta_1^*|^2 + D_2^{(\nu)}\,\mathbf{E}\,|\zeta_1^*|^4\right)\big(R_1(t) + R_2(t)\big), \tag{3.86}$$

$$|J_2| \leqslant c\,(|t| + |t|^3)\,\mathbf{E}\,|\zeta_1^{**}|\left(\mathbf{E}\,|\zeta_1^{**}| + \mathbf{E}\,|\zeta_1^*|^2 + \mathbf{E}\,|\zeta_1^*|^3\right) D_1^{(\nu)}\big(R_1(t) + R_2(t)\big) \tag{3.87}$$

with

$$\begin{aligned} D_1^{(\nu)} &= 1 + |a|^4 + \sup_{1\leqslant\rho\leqslant N_2}\Big(\mathbf{E}\,\big|X_1^* + \dots + X_\rho^*\big|^4 + \mathbf{E}\,\big|Y_1^* + \dots + Y_\rho^*\big|^4\Big)/N^2,\\ D_2^{(\nu)} &= 1 + |a|^2 + \sup_{1\leqslant\rho\leqslant N_2}\Big(\mathbf{E}\,\big|X_1^* + \dots + X_\rho^*\big|^2 + \mathbf{E}\,\big|Y_1^* + \dots + Y_\rho^*\big|^2\Big)/N,\\ N_2 &= N_2(\nu) = \big\lceil (N + \nu - 1)/10\big\rceil \geqslant N_1\end{aligned} \tag{3.88}$$

(see (3.42), (3.43)). To justify this bounds we note that, according to (3.81), the random variable ξ can be represented in the form

$$\xi = \xi_0 + \xi_1 + \xi_2 + \dots + \xi_{10} + \xi_{11}, \tag{3.89}$$

where $\xi_0, \xi_1, \xi_2, \dots, \xi_{10}, \xi_{11}$ are independent random vectors which are independent of $\tau_1, \tau_2, \zeta_1^*, \zeta_2^*$ and

$$\begin{gathered}\xi_0 = -a; \qquad \mathcal{L}(\xi_1) = \mathcal{L}\big(\varkappa_1^{(1)} + \dots + \varkappa_k^{(1)}\big), \qquad \text{for some} \quad k \leqslant N_2; \\ \mathcal{L}(\xi_l) = \mathcal{L}\big(\varkappa_1^{(l)} + \dots + \varkappa_{N_2}^{(l)}\big), \qquad l = 2, \dots, 11; \\ \mathcal{L}(\varkappa_m^{(l)}) = \mathcal{L}\big(X_1^*/\sqrt{N}\big) \quad \text{or} \quad \mathcal{L}(\varkappa_m^{(l)}) = \mathcal{L}\big(Y_1^*/\sqrt{N}\big), \quad m = 1, \dots, N_2, \quad l = 1, \dots, 11,\end{gathered} \tag{3.90}$$

and, moreover, at least 5 of ξ_l are identically distributed with

$$\mathcal{L}(\xi_l) = \mathcal{L}\Big(N^{-1/2}\left(X_1^* + \dots + X_{N_2}^*\right)\Big) \quad \text{or} \quad \mathcal{L}(\xi_l) = \mathcal{L}\Big(N^{-1/2}\left(Y_1^* + \dots + Y_{N_2}^*\right)\Big). \tag{3.91}$$

To prove (3.86) one should separately estimate each summand on the right-hand side of (2.24). For example, using (2.22), (3.89), we see that

$$\begin{aligned}&\mathbf{E}\, v_1^2 w_1^2 (1-\tau_1)(1-\tau_2) \exp\{\tau_2 v_1 + \tau_1 w_1\} \exp\{i t\, Q[\xi]\} \\ &\qquad = 16\, t^4 \sum_{l_1, l_2, l_3, l_4 = 0}^{11} \mathbf{E}\, A(l_1, l_2, l_3, l_4)\, B,\end{aligned} \tag{3.92}$$

where

$$\begin{aligned}A(l_1, l_2, l_3, l_4) &= \langle Q\xi_{l_1}, \zeta_1^*\rangle \langle Q\xi_{l_2}, \zeta_1^*\rangle \langle Q\xi_{l_3}, \zeta_2^*\rangle \langle Q\xi_{l_4}, \zeta_2^*\rangle (1-\tau_1)(1-\tau_2), \\ B &= \exp\Big\{2it\,\langle Q\xi_0 + \dots + Q\xi_{11},\ \tau_2\zeta_1^* + \tau_1\zeta_2^*\rangle + it\, Q\big[\xi_0 + \dots + \xi_{11}\big]\Big\}.\end{aligned} \tag{3.93}$$

Consider a summand from the right-hand side of (3.92) with fixed l_1, l_2, l_3, l_4. It is clear that at least one of the variables ξ_l, $l = 1, \dots, 11$, say ξ_{l_0}, has the distribution (3.91) and is independent of $\xi_{l_1}, \xi_{l_2}, \xi_{l_3}, \xi_{l_4}$ and, of course, of $\tau_1, \tau_2, \zeta_1^*, \zeta_2^*$. The same independence properties are satisfied for the summands composing ξ_{l_0}. Denote by $\mathbf{E}_0$ the conditional expectation with respect to some N_1 of the N_2 summands composing ξ_{l_0} given all the remaining variables and vectors. According to (2.26), (3.14), (3.42), (3.88), (3.91), (3.93),

$$|\mathbf{E}_0 B| \leqslant R_1(t) + R_2(t) \qquad \text{a.s.} \tag{3.94}$$

Using the relations $\big|\langle x, y\rangle\big| \leqslant |x| \cdot |y|$, (3.82), (3.88), (3.90), (3.93), the isometry property of Q and Hölder's inequality, we have

$$\mathbf{E}\,\big|A(l_1, l_2, l_3, l_4)\big| \leqslant \big(\mathbf{E}\,|\zeta_1^*|^2\big)^2 D_1^{(\nu)}. \tag{3.95}$$

By virtue of (3.94), (3.95),

$$\begin{aligned} \big|\, \mathbf{E}\, A(l_1,\, l_2,\, l_3,\, l_4)\, B \,\big| &= \big|\, \mathbf{E}\, \big(A(l_1,\, l_2,\, l_3,\, l_4)\, \mathbf{E}_0 B\big) \,\big| \\ &\leqslant \mathbf{E}\, \Big(\big|\, A(l_1,\, l_2,\, l_3,\, l_4) \big|\, \big|\, \mathbf{E}_0\, B \,\big|\Big) \\ &\leqslant \mathbf{E}\, \big|\, A(l_1,\, l_2,\, l_3,\, l_4) \big|\, \big(R_1(t) + R_2(t)\big) \\ &\leqslant \big(\mathbf{E}\, |\zeta_1^*|^2\big)^2 D_1^{(\nu)}\, \big(R_1(t) + R_2(t)\big). \end{aligned} \tag{3.96}$$

Therefore, the left-hand side of (3.92) can be estimated in absolute value by the right-hand side of (3.86). The estimation of the remaining 6 summands on the right-hand side of (2.24) can be done in a similar way. Here the elementary inequality $x^y \leqslant 1 + x^z$, for $x > 0$, $0 \leqslant y \leqslant z$, has to be used several times.

The proof of (3.87) almost coincides with the proof of (3.86). A unique difference consists in changing ζ_k^* (involving in the corresponding summand of (2.25) in one place only) by $-\zeta_k^{**}$ (see (2.22), (2.25), (3.84)). Notice that these ζ_k^*, ζ_k^{**} are independent of all other random variables and vectors in the expression of this summand. For example, using (2.22), (3.79), (3.81), (3.83), (3.84), we see that the vectors ζ_1, ζ_1^*, ζ_1^{**} are independent of ξ, ζ_2^*, τ_1 and

$$\begin{aligned} &\mathbf{E}\, v_1\, w_1^2\, (1 - \tau_1)\, \exp\{\tau_1 w_1)\, \exp\big\{\, i t\, Q[\xi]\, \big\} \\ &\quad = -8\, i t^3\, \mathbf{E}\, \big\langle Q\xi, \zeta_1^* \big\rangle \big\langle Q\xi, \zeta_2^* \big\rangle^2 (1 - \tau_1)\, \exp\Big\{2\, i t \tau_1 \big\langle Q\xi, \zeta_2^* \big\rangle\Big\}\, \exp\big\{\, i t\, Q[\xi]\, \big\} \\ &\quad = 8\, i t^3\, \mathbf{E}\, \big\langle Q\xi, \zeta_1^{**} \big\rangle \big\langle Q\xi, \zeta_2^* \big\rangle^2 (1 - \tau_1)\, \exp\Big\{2\, i t \tau_1 \big\langle Q\xi, \zeta_2^* \big\rangle\Big\}\, \exp\big\{\, i t\, Q[\xi]\, \big\}. \end{aligned} \tag{3.97}$$

The further estimation of this quantity is similar to the estimation of the right-hand side of (3.92). One should also use that ζ_1^{**}, ζ_2^{**} are identically distributed and, hence, $\mathbf{E}\, |\zeta_2^{**}| = \mathbf{E}\, |\zeta_1^{**}|$.

Consider the estimation of the right-hand sides of (3.86), (3.87). According to (3.1), (3.9), (3.79), (3.80), (3.83), we have

$$\mathbf{E}\, |\zeta_1^*|^2 \leqslant c N^{-1}, \qquad \mathbf{E}\, |\zeta_1^*|^3 \leqslant c N^{-3/4}, \qquad \mathbf{E}\, |\zeta_1^*|^4 \leqslant N^{-1}\, \mathbf{E}\, |X|^{8/3}, \tag{3.98}$$

$$\mathbf{E}\, |\zeta_1^{**}| \leqslant N^{-5/4}\, \mathbf{E}\, |X_1^{**}|^2 \leqslant c N^{-5/4}. \tag{3.99}$$

Let $y \overset{\text{def}}{=} \mathbf{E}\, X_1^*$. In view of (1.1), (3.8), (3.12), it is easy to verify that

$$\mathbf{E}\, Y_k^* = \mathbf{E}\, X_k^* = y, \qquad \operatorname{cov} Y_k^* = \operatorname{cov} X_k^* = \operatorname{cov} X_1^*, \qquad k = 1, 2 \ldots . \tag{3.100}$$

Introduce the centered random vectors $\overline{X}_k^*$, $\overline{Y}_k^*$ by

$$\overline{X}_k^* = X_k^* - y, \qquad \overline{Y}_k^* = Y_k^* - y, \qquad k = 1, 2 \ldots . \tag{3.101}$$

Using (3.101), we see that, for $\rho = 1, 2, \ldots,$

$$\begin{aligned}
\mathbf{E}\,\big|\,X_1^* + \cdots + X_\rho^*\,\big|^4 &\leqslant c\,\Big(\,\mathbf{E}\,\big|\,\overline{X}_1^* + \cdots + \overline{X}_\rho^*\,\big|^4 + \rho^4\,|y|^4\Big),\\
\mathbf{E}\,\big|\,Y_1^* + \cdots + Y_\rho^*\,\big|^4 &\leqslant c\,\Big(\,\mathbf{E}\,\big|\,\overline{Y}_1^* + \cdots + \overline{Y}_\rho^*\,\big|^4 + \rho^4\,|y|^4\Big),\\
\mathbf{E}\,\big|\,X_1^* + \cdots + X_\rho^*\,\big|^2 &\leqslant c\,\Big(\rho\,\mathbf{E}\,\big|\,\overline{X}_1^*\big|^2 + \rho^2\,|y|^2\Big),\\
\mathbf{E}\,\big|\,Y_1^* + \cdots + Y_\rho^*\,\big|^2 &\leqslant c\,\Big(\rho\,\mathbf{E}\,\big|\,\overline{Y}_1^*\big|^2 + \rho^2\,|y|^2\Big).
\end{aligned} \tag{3.102}$$

Using Lemma 2.3, Hölder's inequality and the relations (3.100), (3.101), we obtain that

$$\begin{aligned}
\mathbf{E}\,\big|\,\overline{X}_1^* + \cdots + \overline{X}_\rho^*\,\big|^4 &\leqslant c\,\Big(\rho\,\mathbf{E}\,\big|\,\overline{X}_1^*\big|^4 + \rho^2\big(\mathbf{E}\,\big|\,\overline{X}_1^*\big|^2\big)^2\Big),\\
&\leqslant c\,\Big(\rho\,\big(\mathbf{E}\,\big|\,X_1^*\big|^4 + |y|^4\big) + \rho^2\,\big(\mathbf{E}\,\big|\,X_1^*\big|^2 + |y|^2\big)^2\Big)\\
&\leqslant c\,\Big(\rho\,\mathbf{E}\,\big|\,X_1^*\big|^4 + \rho^2\big(\mathbf{E}\,\big|\,X_1^*\big|^2\big)^2\Big)
\end{aligned} \tag{3.103}$$

and, analogously,

$$\mathbf{E}\,\big|\,\overline{Y}_1^* + \cdots + \overline{Y}_\rho^*\,\big|^4 \leqslant c\,\Big(\rho\,\mathbf{E}\,\big|\,Y_1^*\big|^4 + \rho^2\big(\mathbf{E}\,\big|\,Y_1^*\big|^2\big)^2\Big). \tag{3.104}$$

According to (3.1), (3.7), (3.100), (3.101),

$$\mathbf{E}\,\big|\,\overline{Y}_1^*\big|^2 = \mathbf{E}\,\big|\,\overline{X}_1^*\big|^2 \leqslant 4\,\mathbf{E}\,\big|\,X_1^*\big|^2 \leqslant c, \qquad \mathbf{E}\,\big|\,X_1^*\big|^4 \leqslant N\,\mathbf{E}\,|X|^{8/3}. \tag{3.105}$$

Recall that Π denotes an independent of $X_1^*, X_2^*, \ldots$ random variable with Poisson distribution of parameter 1. In view of the relations (1.1), (3.8), (3.12), we obtain $\mathcal{L}(Y_1^*) = \mathcal{L}(X_1^* + \cdots + X_\Pi^*)$. Therefore, for $\beta > 0$,

$$\begin{aligned}
\mathbf{E}\,\big|\,Y_1^*\big|^\beta &= \mathbf{E}\,\big|\,X_1^* + \cdots + X_\Pi^*\,\big|^\beta \leqslant \mathbf{E}\,\Pi^\beta\big(|X_1^*|^\beta + \cdots + |X_\Pi^*|^\beta\big)\\
&= \mathbf{E}\,\mathbf{E}\,\Big\{\,\Pi^\beta\big(|X_1^*|^\beta + \cdots + |X_\Pi^*|^\beta\big)\,\big|\,\Pi\Big\}\\
&= \mathbf{E}\,|X_1^*|^\beta\,\mathbf{E}\,\Pi^{\beta+1} = c(\beta)\,\mathbf{E}\,|X_1^*|^\beta.
\end{aligned} \tag{3.106}$$

Using (3.1), (3.9), (3.10), (3.100), we see that

$$|y| = \big|\,\mathbf{E}\,X_1^{**}\big| \leqslant \mathbf{E}\,\big|\,X_1^{**}\big| \leqslant c\,N^{-3/4}. \tag{3.107}$$

From (3.88), (3.102)–(3.107) it follows that

$$\begin{aligned}
D_1^{(\nu)} &\leqslant 1 + |a|^4 + c\,\Big(N_2\,N^{-1}\,\mathbf{E}\,|X|^{8/3} + N_2^2\,N^{-2} + N_2^4\,N^{-5}\Big),\\
D_2^{(\nu)} &\leqslant 1 + |a|^2 + c\,\Big(N_2\,N^{-1} + N_2^2\,N^{-5/2}\Big).
\end{aligned} \tag{3.108}$$

Recall that in the sum $\sum_1$ from (3.76) we have

$$0 \leqslant \nu \leqslant \mu \leqslant N, \qquad N_2 \leqslant N/5 \tag{3.109}$$

(see (3.88)). Therefore, (3.108) and (3.109) together imply

$$D_1^{(\nu)} \leqslant c\left(1+|a|^4+\mathbf{E}\,|X|^{8/3}\right), \qquad D_2^{(\nu)} \leqslant c\left(1+|a|^2\right). \tag{3.110}$$

Using (3.1), (3.58), (3.69), (3.85)–(3.87), (3.98), (3.99) and (3.110), we infer that the integrals $J(t,j,\nu,N)$ from $\sum_1$ can be estimated in absolute value as follows:

$$\left|\,J(t,j,\nu,N)\right| \leqslant c\,C\,N^{-2}\left(|t|+t^4\right)\,\min\left\{1,\,M_9\left(pt,\,N_0\right)\right\}, \tag{3.111}$$

where

$$C=\left(|a|^4+\left(1+|a|^2\right)\mathbf{E}\,|X|^{8/3}\right). \tag{3.112}$$

Analogously, for the sum $\sum_2$ from (3.76) we have

$$0 \leqslant \max\left\{\nu,\,N\right\} \leqslant \mu, \qquad N_2 \leqslant \mu/5 \tag{3.113}$$

(see (3.88)). In this case (3.108) and (3.113) imply

$$D_1^{(\nu)} \leqslant c\mu^4\left(1+|a|^4+\mathbf{E}\,|X|^{8/3}\right), \qquad D_2^{(\nu)} \leqslant c\mu^2\left(1+|a|^2\right). \tag{3.114}$$

Using (3.114) instead of (3.110), we obtain for the integrals $J(t,j,\nu,N)$ from $\sum_2$ the following analog of (3.111):

$$\left|\,J(t,j,\nu,N)\right| \leqslant c\,C\,\mu^4\,N^{-2}\left(|t|+t^4\right)\,\min\left\{1,\,M_9\left(pt,\,N_0\right)\right\}. \tag{3.115}$$

From (3.75), (3.76), (3.111) and (3.115) it follows that, for all $t\in\mathbf{R}$,

$$\left|\,f^*(t)-g^*(t)\right| \leqslant c\,C\,N^{-1}\left(|t|+t^4\right)\,\min\left\{1,\,M_9\left(pt,\,N_0\right)\right\}. \tag{3.116}$$

Using (3.28), (3.37), (3.38), we see that

$$\int_{|t|\leqslant t_0}\left(1+|t|^3\right)\,\min\left\{1,\,M_9\left(pt,\,N_0\right)\right\}dt = 2\int_0^{1/p} + 2\int_{1/p}^{t_0} \leqslant \frac{c}{p^4}\,. \tag{3.117}$$

In view of (3.39), (3.116) and (3.117), we have

$$\left|\,I(f^*,\,g^*)\right| \leqslant \frac{c\,C}{p^4\,N}\,. \tag{3.118}$$

Now the statement of Theorem 1.1 follows from (3.25), (3.27), (3.30), (3.35), (3.40), (3.68), (3.70) and (3.118). Moreover, we have in fact proved the following explicit bound

$$\sup_x \bigl| F(x) - G(x) \bigr| \leqslant \frac{c\,(C + (\mathbf{E}\,|X|^{8/3})^3)}{p^4 N}. \tag{3.119}$$

It is clear that (3.1), (3.112), (3.119) imply (3.31).

Below we show that the vectors $e_1, \dots, e_9$ can be chosen so that

$$p \geqslant \exp\Bigl\{ -\frac{c}{\sigma_9^2(Q)} \Bigr\}. \tag{3.120}$$

Let $e \in \mathbf{R}^d$ and $|e| = 1$. Define $\sigma_e^2 = \mathbf{E}\,\langle \eta, e\rangle^2$. According to (1.11), (3.28), it suffices to verify that

$$\mathbf{P}\bigl\{ |\eta - e| \leqslant \delta_0 \bigr\} \geqslant \exp\Bigl\{ -\frac{c}{\sigma_e^2(Q)} \Bigr\}. \tag{3.121}$$

Introduce the Gaussian vectors $\eta - \langle \eta, e\rangle\, e$ and $\langle \eta, e\rangle\, e$. These vectors are independent, since they are uncorrelated. Consequently,

$$\begin{aligned} \mathbf{P}\Bigl\{ |\eta - e| \leqslant \delta_0 \Bigr\} &\geqslant \mathbf{P}\Bigl\{ \bigl|\eta - \langle \eta, e\rangle\, e\bigr| + \bigl|\langle \eta, e\rangle\, e - e\bigr| \leqslant \delta_0 \Bigr\} \\ &\geqslant \mathbf{P}\Bigl\{ \bigl|\eta - \langle \eta, e\rangle\, e\bigr| \leqslant \delta_0/2 \Bigr\}\, \mathbf{P}\Bigl\{ \bigl|\langle \eta, e\rangle\, e - e\bigr| \leqslant \delta_0/2 \Bigr\} \\ &\stackrel{\text{def}}{=} p_1 \cdot p_2. \end{aligned} \tag{3.122}$$

Recall that $\delta_0 = \dfrac{1}{900}$. Therefore, we have

$$\begin{aligned} p_1 &= 1 - \mathbf{P}\Bigl\{ \bigl|\eta - \langle \eta, e\rangle\, e\bigr|^2 \geqslant \delta_0^2/4 \Bigr\} \geqslant 1 - c\,\mathbf{E}\,\bigl|\eta - \langle \eta, e\rangle\, e\bigr|^2 \\ &\geqslant 1 - c\,\mathbf{E}\,|\eta|^2 = 1 - c\,\mathbf{E}\,|X|^2 \geqslant 1/2 \end{aligned} \tag{3.123}$$

and

$$\sigma_e^2 = \mathbf{E}\,\langle \eta, e\rangle^2 \leqslant \mathbf{E}\,|\eta|^2 = \mathbf{E}\,|X|^2 \leqslant 1 \tag{3.124}$$

provided that c_{11} in (3.1) is chosen to be sufficiently small. It remains to estimate p_2. It is easy to see that

$$p_2 = \mathbf{P}\Bigl\{ \bigl|\eta_0 - \sigma_e^{-1}\bigr| \leqslant \delta_0/2\,\sigma_e \Bigr\}, \tag{3.125}$$

where η_0 is a random variable with standard normal distribution. Taking into account that $\delta_0 = \dfrac{1}{900}$, we see that the relations (3.122)–(3.125) imply (3.121) and, hence, (3.120).

Now the inequality (1.12) follows from (3.1), (3.25), (3.31), (3.120). □

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