# The Existence of Exactly $m$-coloured Complete Subgraphs

Alan Stacey
University of Cambridge

Peter Weidl
SFB 343, Fakultät für Mathematik,
Universität Bielefeld

June 13, 1996

## 1. Introduction

The classical result of Ramsey [6] for colourings of infinite graphs can be stated in the following way.

**Theorem 1.** *Let $c \geq 1$ be a positive integer and suppose that we have a $c$-colouring of the edges of the complete graph whose vertex set is $\mathbb{N}$, i.e. a function*

$$\Delta : \mathbb{N}^{(2)} \to \{1, \ldots, c\}.$$

*Then there is an infinite complete subgraph all of whose edges have the same colour.* □

This theorem has inspired very many generalizations and further questions. Much of this work concerns the existence of particular monochromatic structures, but one might also take some $m > 1$ and search for substructures where exactly $m$ colours are used. In [2] M. Erickson formulated such a problem in the form of a proposition $P(c, m)$, for positive integers $c, m$:

**P(c,m):** *If the edges of a countable infinite complete graph $K_\omega$ are exactly $c$-coloured, then there exists a countable infinite complete subgraph $H$ of $K_\omega$ whose edges are exactly $m$-coloured.*

Of course, a graph $G$ is said to be *exactly $x$-coloured* ($x \in \mathbb{N}$) if the colouring map from $E(G)$ to $\{1, \ldots, x\}$ is surjective. As usual we denote the vertex set of a graph $V(G)$ and we denote the edge set $E(G)$.

The case $m = 1$ of $P(c, m)$ is, of course, just Ramsey's Theorem, and $P(c, m)$ trivially holds in the case $c = m$. Erickson [2] observed that a fairly straightforward application of Ramsey's Theorem also enables one to show that $P(c, m)$ is true in the case $m = 2$ (provided $c \geq 2$; of course $P(c, m)$ is false if $c < m$). Erickson [2] found counterexamples to $P(c, m)$ in many other cases and he conjectured that the only cases for which $P(c, m)$ holds are those just described. In other words, he conjectured that $P(c, m)$ is true if and only if

$$\begin{aligned} &(1) \quad m = 1, \quad \text{or} \\ &(2) \quad m = 2 \text{ and } c \geq 2, \quad \text{or} \\ &(3) \quad c = m \geq 3. \end{aligned}$$

It is not too difficult to produce various families of counterexamples to $P(c, m)$ each family sustaining the conjecture of Erickson for a significant range of values for $c$ and $m$. The principal difficulty in proving the conjecture seems to be that one family of counterexamples will cover values with a certain property, and another family will cover values with some quite unrelated property. Even if we put together several such families of counterexamples, there always seem to be some parameter values which are not covered.

We are not able to give a complete solution to the conjecture. However, we do obtain counterexamples to $P(c, m)$ which are significantly different from those obtained before; with the aid of random methods we are then able to extend these counterexamples to show that for each $m \geq 3$, $P(c, m)$ is false for all sufficiently large $c$.

In Section 2 of this paper we will describe our main new method for constructing counterexamples. This will enable us to prove the following result.

**Theorem 2.** *Suppose that $c > m \geq 3$ are positive integers. Let $n$, $p$, $k$ and $q$ be the unique natural numbers such that*

$$c = \binom{n}{2} + 2 - p \quad \text{and} \quad 0 \leq p \leq n - 2,$$

$$m = \binom{k}{2} + 2 - q \quad \text{and} \quad 0 \leq q \leq k - 2.$$

*Then, if $p \not\equiv q$ mod 6, or $p = 0$, or $q = 0$, then $P(c, m)$ is false.* □

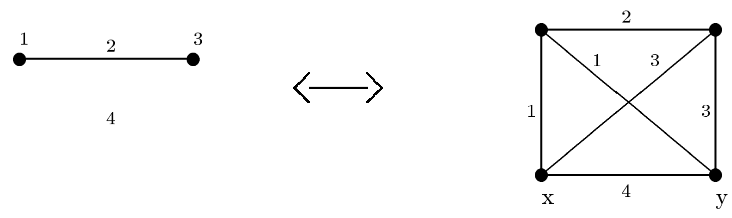
As we indicate at the beginning of Section 4, it is possible to extend our methods constructively to cover certain other cases. In Section 3, we turn our attention to random colourings. These colourings are closely related to the constructive colourings of Section 2 and we are almost able to fill in the gaps in Theorem 2. We prove the following:

**Theorem 3.** *Suppose that $m \geq 3$ is a positive integer. Then there exists an integer $C(m)$ such that $P(c, m)$ is false for all $c \geq C(m)$.* □

Before we proceed with the specific counterexamples of Section 2, we describe a very general way of giving colourings, which essentially reduces the problem of colouring an infinite graph to the colouring of finite graphs. We take a complete subgraph of $K_\omega$ with $n$ vertices, $K_n$, and we colour the edges *and vertices* of this $K_n$ using all the colours $1, \ldots, c-1$, and perhaps also using the colour $c$. We use this colouring to induce an exact $c$-colouring of $K_\omega$ as follows. All the edges of $K_n$ receive their assigned colour. All edges of the form $vw$, where $v \in V(K_n)$ and $w \in V(K_\omega) \setminus V(K_n)$ receive the colour of $v$. All the edges of $K_\omega - K_n$ receive the colour $c$. Now we see that any infinite complete subgraph of $K_\omega$ must use the colour $c$, and it also uses the colours of the edges and vertices of whatever complete subgraph of $K_n$ it contains. Actually, it is not too hard to see that if there is a counterexample to $P(c, m)$ then there must be one that is induced by a finite graph colouring in this way.

In order to illustrate this we give an example: (We will use numbers to represent colours in all figures we give.)

Example 1 (m=3,c=4,n=2)




for all $x, y \in V(K_\omega) \setminus V(K_2)$ with $x \neq y$

We can easily check by testing that the colouring in example 1 is a counterexample to $P(4, 3)$. But how can we check colourings systematically to see whether they are counterexamples or not? We shall restrict our attention to colourings with a special structure, for which we are able to answer this question.

## 2. The construction of counterexamples

This section implicitly includes a proof of Theorem 2.

We shall describe two closely-related general strategies for obtaining colourings, which we shall call type I and type II. Recall that in order to induce an exact $c$-colouring on the edges of $K_\omega$, we colour the edges and vertices of some $K_n$ with either $c-1$ or $c$ colours.

**Type I:** We choose a minimal $n \in \mathbb{N}$ such that $\binom{n}{2} + 2 \geq c$. So we have

$$c = \binom{n}{2} + 2 - p$$

where $0 \leq p \leq n-2$. We then colour the vertices of $K_n$ with colour 1, and use $\binom{n}{2} - p$ colours for the edges of $K_n$. Of these colours, a total of $p$ will be used twice and the remaining colours will only be used on one edge. The final colour, colour $c$, is of course used to colour all those edges which are not incident with any vertex of $K_n$, as described at the end of the introduction.

In order to specify the colouring completely, we must say how the $p$ *pairs* of edges — where both edges of the pair have the same colour — are arranged. Before we turn to this matter in detail, we describe the other type of colouring which we use.

**Type II:** We write $c$ in the form

$$c = \binom{n'}{2} + 1 - p'$$

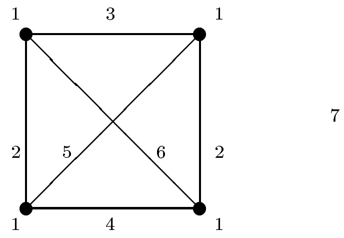
with $0 \leq p' \leq n'-2$ (so, unless $p = 0$ above, then $n' = n$ and $p' = p-1$). We then colour the vertices of $K_{n'}$ with colour $c$ (the same colour used for the edges not incident with $K_{n'}$) and use the remaining $\binom{n'}{2} - p'$ colours to colour the edges of $K_{n'}$. Similarly to type I colourings, we will have $p'$ *pairs* of edges, with both edges of each pair having the same colour, and all the remaining edges having different colours.

It may seem that the difference between type I and type II colourings is so slight that it is of very little interest. As we shall see, however, this very small flexibility is crucial.

Before we explain in detail how we arrange the pairs in such a way as to give useful colourings, we give a simple example of a type I colouring:

Example 2

(m=4,c=7,n=4)




Example 2 shows a counterexample to $P(7,4)$. We have $7 = \binom{4}{2} + 2 - 1$ which signifies that there is one pair in $K_4$ $(p = 1)$.

In a more general sense we can make the following observations. Since the 'structure' of the colouring of $K_n$ is given by, for example, type I, we can only find $m$ colours by using this structure; so, for instance, if we take $k$ vertices from $V(K_n)$ we get at most $\binom{k}{2} + 2$ colours. In general we will need to consider how many of the $p$ pairs particular subgraphs can contain; with this in mind we say that a set of vertices, $V$, *spans* a pair if both edges of that pair are contained in the complete subgraph induced by $V$.
Throughout our discussion of type I colourings we will write $m$ in the form

$$m = \binom{k}{2} + 2 - q,$$

with $0 \leq q \leq k - 2$. We can see that if we have a type I colouring and we choose an infinite complete subgraph containing only $k-1$ vertices of $K_n$, then we will not have as many as $m$ colours. If we can arrange our pairs in such a way that any group of $k+1$ vertices spans no more than $k - 1 + q$ pairs, then we ensure that any infinite complete subgraph containing $k+1$ vertices of $K_n$ contains more than $m$ colours. Then the only way that we can obtain exactly $m$ colours will be to use $k$ vertices, so our task will be to arrange the pairs in such a way that no group of $k$ vertices spans precisely $q$ pairs.

Similar observations can be made for a type II colouring of $K_{n'}$, which we address in more detail in due course. For now, however, we turn to the details of how we arrange the pairs in our type I colourings.

We describe four configurations for the pairs and show, in each case, for which values of $p$ and $q$ the configuration provides a counterexample to $P(c, m)$. There is a central idea running through all these configurations: a group of either two or three pairs can be arranged in such a way that they are all spanned by a particular set of four vertices, but none of the pairs is spanned by any three of those vertices. So, for instance, if $p \equiv 0 \,(\text{mod } 3)$ (which, henceforth, we denote simply $p \equiv 0\,(3)$) then we can arrange all the $p$ pairs in groups of 3; if $q \not\equiv 0(3)$ then no subgraph can contain exactly $q$ pairs.

From the preceding comments, one can see that which arrangement of pairs we use depends on the values of $p$ and $q$ modulo 6. The values of $n$ and $k$ are not important but we must bear in mind that $p$ can be as large as $n-2$ and in all cases we wish to arrange the $p$ pairs in such a way that no group of $k+1$ vertices contains more than $k-1+q$ pairs.
Each description of a configuration starts with a figure.

Configuration 1:

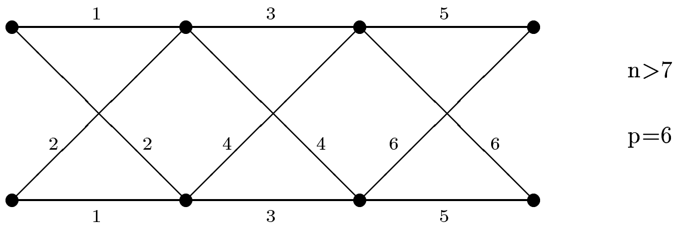



Figure 1

It is easy to see how to generalize this special configuration for $p = 6$ to represent all values for $p$ with $p \equiv 0$ or 2 or $4\,(6)$. If $n$ is even then we can get as many as $n-2$ pairs. If $n$ is odd then, since $p$ is even, we need at most $n-3$ pairs ($p \leq n-2$ and hence $\leq n-3$), which we can indeed construct. So each $p \equiv 0$ or 2 or $4\,(6)$ with $p \leq n-2$ can be represented by configuration 1.
If $q$ is odd then no complete subgraph contains exactly $q$ pairs. We also observe that any set of $k+1$ vertices spans at most $k-1$ pairs and therefore has too many colours. Hence we have counterexamples to $P(c, m)$ for all values $p$ and $q$ with $p \equiv 0, 2$ or $4\,(6)$ and $q \equiv 1, 3$ or $5\,(6)$.

Configuration 2:

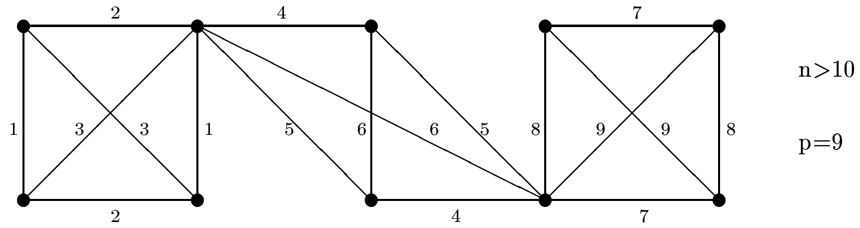



Figure 2

We use copies of $K_4$ to produce 3 pairs in such a way that any set of vertices spans either 0 pairs or all 3 pairs. Then we link the $K_4$ graphs in a chain as

shown (see Figure 2). This configuration enables us to represent all values of $p$ with $p \equiv 0$ or $3\,(6)$. For that we have to check whether $n$ vertices are enough to produce exactly $p = n-2$ pairs. If $n \equiv 1\,(3)$ we can construct as many as $n-1$ pairs in the above way; for $n \equiv 2\,(3)$ we get as many as $n-2$ pairs so again this is enough. If $n \equiv 0\,(3)$ then we can only obtain $n-3$ pairs; since, however, we are assuming $p \equiv 0\,(3)$ and $p \leq n-2$ we see that in fact $p \leq n-3$, so the construction is fine.
Now we claim that these type I configuration 2 colourings produce counterexamples to $P(c,m)$ for $p$ and $q$ with $p \equiv 0$ or $3\,(6)$ and $q \equiv 1, 2, 4$ or $5\,(6)$. Indeed we see that in any complete subgraph the number of pairs is divisible by three so cannot be equal to $q$. Furthermore, if we take $k+1$ vertices we obtain too many colours because we get at most $k$ pairs which is no more than the limit discussed above of $k-1+q$ (using the fact that in these cases, $q \geq 1$).

Configuration 3:

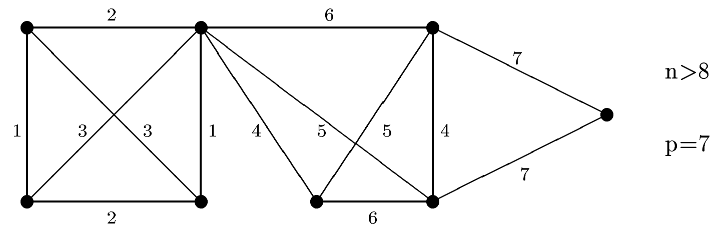



Figure 3

This is the same as for configuration 2 except that we add another pair to reach other congruence classes for $p$.
As a consequence we can only have complete subgraphs with $3r$ or $3r+1$ ($r \in \mathbb{N}_0$) pairs. Hence we cannot obtain exactly $q$ pairs for $q \equiv 2$ or $5\,(6)$. As in previous cases, it is straightforward to check we can always fit in $p$ pairs in such a way that any complete subgraph with $k+1$ vertices has too many colours. Therefore, this configuration supplies us with type I colourings which are counterexamples to $P(c,m)$ for $p \equiv 1$ or $4\,(6)$ and $q \equiv 2$ or $5\,(6)$.

Configuration 4:

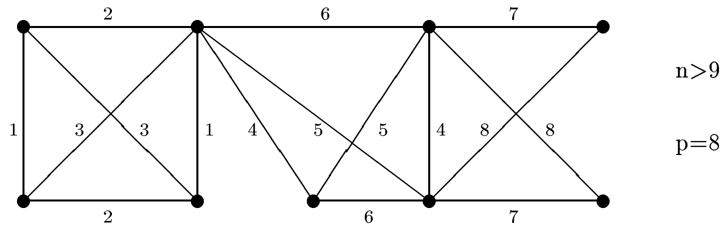



Figure 4

Now we use configuration 2 to construct $3r$ ($r \in \mathbb{N}_0$) pairs. At the end we add 2 pairs by using 2 vertices which are not used so far for building pairs and two vertices which belong to pairs (see Figure 4). With the help of configuration 4 we represent all values for $p \equiv 2$ or $5\,(6)$.

It is again easy to check that $p = n - 2$ pairs can be constructed: since $p \equiv 2\,(3)$ then $n \equiv 1\,(3)$. We would get $n - 1$ pairs by configuration 2 by using $n$ vertices. If we only use $n - 2$ vertices we can get $n - 4$ pairs. Now we add another two pairs with the help of the 'free' vertices. We conclude that $n - 2$ pairs are constructable in a configuration 4 colouring.

Much as before we can see that configuration 4 allows us to find counterexamples in all cases with $p \equiv 2$ or $5$ $(6)$ and $q \equiv 1$ or $4\,(6)$.

The following table summarizes all results which we get by a type I colouring of $K_n$ using configurations 1-4. The numbers in the table stand for representatives of the equivalence classes modulo 6. Recall that we always have $q, p \geq 0$. A cross in the table means that this case is covered by one of the configurations above.

**TABLE I:**

| q \ p | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | | | | | | |
| 1 | × | | × | × | × | × |
| 2 | × | × | | × | × | |
| 3 | × | | × | | × | |
| 4 | × | | × | × | | × |
| 5 | × | × | × | × | × | |

We now turn to the subject of type II colourings. Recall that for a type II colouring we express $c$ in the form

$$c = \binom{n'}{2} + 1 - p'$$

with $0 \leq p' \leq n' - 2$. Let us also express $m$ in a similar way:

$$m = \binom{k'}{2} + 1 - q'$$

with $0 \leq q' \leq k' - 2$. Of course, if $p > 0$ and $q > 0$, then we simply have $n' = n$, $k' = k$, $p' = p - 1$ and $q' = q - 1$.

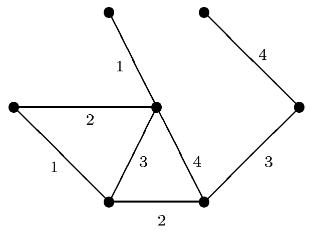
Once again we must arrange pairs in a certain way, and we can do this with configurations 1-4. It is not hard to see what new cases this enables us to cover. For example, by precisely the same argument as that used for type I colourings, a type II colouring using configuration 1 will cover those cases where $p' \equiv 0, 2$ or $4\,(6)$ and $q' \equiv 1, 3$ or $5\,(6)$. Provided $p > 0$ and $q > 0$ then this is equivalent to the condition that $p \equiv 1, 3$ or $5\,(6)$ and $q \equiv 0, 2$ or $4\,(6)$. More generally, if the values of $p'$ and $q'$ modulo 6 (rather than $p$ and $q$) correspond to a cross in table I, then that case is covered by a type II colouring using configurations 1-4. It is a simple matter to check that all the cases where $p \not\equiv q\,(6)$, $p > 0$ and $q > 0$ are covered by either a type I or type II colouring.

We note, furthermore, that if $p < q$ then we have a trivial counterexample: however we arrange $p$ pairs then no subgraph can possibly have $q$ pairs; it is easy to check that we can arrange $p$ pairs in such a way that no set of $k+1$ vertices spans more than $k-1+q$ pairs, so we obtain a counterexample to $P(c, m)$. In particular, the case $p = 0, q > 0$ is easily covered. So except for the case $q = 0$, we have covered all cases where $p \not\equiv q\,(6)$.

We now complete the proof of Theorem 2 by constructively giving counterexamples which cover any case where $q = 0$ (and $c > m \geq 3$). We do this not only for completeness, but also to indicate some new techniques for covering cases where $p \equiv q\,(6)$. Note that — in contrast to our other results and all the counterexamples given in [2] — here we are proving, for certain values of $m$, that $P(c, m)$ is false *for all* $c > m$.

We will use a type II colouring in order to prove the case $q = 0$ for all $p \geq 0$. The corresponding formulas for $c$ and $m$ are: $c = \binom{n}{2} + 1 - (p-1)$ and $m = \binom{k+1}{2} + 1 - (k-1)\big(= \binom{k}{2} + 2 - 0\big)$, except when $p = 0$ in which case we have $c = \binom{n+1}{2} + 1 - (n-1)$.
Our aim is to generalize the distribution of the pairs as given in the following example for $n = 7$ where $p > 0$.

Example 3: $(n = 7, p = 5)$




We will consider the cases $p = 0$ and $p > 0$ separately. Recall that in order to give a type II colouring, we need only specify a colouring, $\Delta$, of the edges of a finite complete graph.

Case 1: $p = 0$.
Let $K_{n+1}$ have vertex set $V_1 = \{x_1, \ldots, x_{n+1}\}$; we define an edge-colouring by $\Delta(x_1 x_i) = i - 1$ for $2 \le i \le n$, $\Delta(x_i x_{i+1}) = i - 2$ for $3 \le i \le n$ and $\Delta(x_2 x_{n+1}) = n - 1$. There are $c - 1 - (n-1)$ colours left to colour $\binom{n+1}{2} - 2(n-1) = c - 1 - (n-1)$ edges in $K_{n+1}$. So every remaining edge receives a different colour. Observe that we produce $n - 1$ pairs.

Case 2: $p > 0$.
Let $V_2 = \{x_1, \ldots, x_{p+2}\}$ be a set of $p + 2$ vertices taken from the vertex set of $K_n$ (recall that $p + 2 \le n$). We define an edge-colouring of $K_n$ in which the vertices of $V_2$ span $p - 1$ pairs: let $\Delta(x_1 x_i) = i - 1$ for $2 \le i \le p$ and $\Delta(x_i x_{i+1}) = i - 2$ for $3 \le i \le p + 1$. Once again every remaining edge in $K_n$ receives a different colour.

For both colourings it is easily seen that if we pick exactly $k+1$ vertices then we can obtain at most $k-2$ pairs, and hence will have too many colours. We will certainly have too few colours if we only have $k$ vertices so we do indeed have the required counterexamples to $P(c, m)$ for $q = 0$ with $p \ge 0$.

## 3. Random counterexamples

In this section we shall use random graph techniques to extend the counterexamples of the previous section and thereby prove Theorem 3 which was stated in the introduction. Before getting down to the details, we describe the general approach.

In the previous section we expressed $c$ and $m$ in the following way:

$$c = \binom{n}{2} + 2 - p \quad \text{with} \quad 0 \le p \le n - 2,$$

$$m = \binom{k}{2} + 2 - q \quad \text{with} \quad 0 \le q \le k - 2.$$

We were able to prove that $P(c, m)$ is false except when $p \equiv q\,(6)$; we were also able to prove it to be false when $p = 0$ or $q = 0$.
Now consider alternative representations for $c$:

$$c = \binom{n+1}{2} + 2 - (p + n) = \binom{n+2}{2} + 2 - (p + 2n + 1).$$

If $p \equiv q\,(6)$ then either $p+n$ or $p+2n+1$ (or both) is not congruent to $q$ modulo 6. This suggests that we might be able to find some counterexamples by working with $K_{n+1}$ or $K_{n+2}$ and including $p+n$ or $p+2n+1$ pairs respectively. Indeed, this turns out to be a fruitful approach but there is an obvious difficulty to overcome. The configurations given in the previous section are only valid when the number of pairs does not exceed $n-2$. Once we have included more pairs, it is rather difficult to arrange them in such a way that no group of $k+1$ vertices spans more than $k+q-1$ pairs, which was an important requirement.

In this section we shall show how to use a certain random arrangement to place a large number of pairs (we need up to about $3n$) in a useful way, with no group of $k+1$ vertices spanning too many pairs. For a fixed $m$, this random arrangement will provide counterexamples for all sufficiently large $c$. The detailed calculations concerning the random placement of pairs are contained in the proof of Lemma 4. In order to motivate the lemma, recall that our general strategy is to place pairs mostly in groups of two or three (in fact, we shall see that in this section it will be sufficient just to use groups of two). Each such group is specified primarily by the selection of a group of four vertices, so we are really concerned with the selection of a number of groups of four vertices from the graph. As well as requiring that no group of $k+1$ vertices should contain too many such groups, we also wish to ensure that no two such groups have more than one vertex in common: if they have two vertices in common then we may wish the corresponding edge to be a member of two pairs at the same time! With these considerations in mind, we now state and prove our main lemma.

**Lemma 4.** *Suppose $\alpha \geq 0$, $l \in \mathbb{N}$. Then $\exists N\big(= N(\alpha, l)\big) \in \mathbb{N}$ such that if $n \geq N$ and $t = \lceil \alpha n \rceil$, then we can find sets $A_1, \ldots, A_t \in [n]^{(4)}$ such that*

$$i \neq j \Rightarrow |A_i \cap A_j| \leq 1 \tag{3.0}$$

*and, for all $V \in [n]^{(l)}$*

$$\#\{1 \leq i \leq t : A_i \subset V\} \leq l/3 - 1. \tag{3.1}$$

*Proof.* We select a random collection of 4-sets by choosing each member of $[n]^{(4)}$ independently with probability

$$p := \frac{24(\alpha+1)}{(n-1)(n-2)(n-3)}.$$

We denote this random collection by $\mathcal{A}$; so the size of $\mathcal{A}$ is binomially distributed and has mean exactly $(\alpha+1)n$ ($p$ was chosen for this reason). We will show that if $n$ is large then there is a positive probability that $\mathcal{A}$ is such that the desired collection can be obtained from $\mathcal{A}$ by discarding a few of its members.

To begin our calculations, we observe that with large probability, $\mathcal{A}$ has cardinality at least $(\alpha+1/2)n$. All we need to know in fact is that

$$\mathbb{P}\big(|\mathcal{A}| \leq (\alpha+1/2)n\big) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.2}$$

Since $|\mathcal{A}|$ has mean $(\alpha+1)n$ and standard deviation less than $\sqrt{(\alpha+1)n}$ then (3.2) follows from as simple a tool as Chebychev's inequality. Of course, using a normal approximation we can see that we can expect the probability in (3.2) to be exponentially small and a little work with probability generating functions enables one to obtain an exact exponential bound. The application of Theorem 7(i) in [1] (p.13) gives a better exponential bound; for $\alpha \leq 5$ (which will comfortably hold in our application of the Lemma) it gives

$$\mathbb{P}\big(\big||\mathcal{A}| - (\alpha+1)n\big| \geq (1/2)n\big) \leq 2e^{-n/12(\alpha+1)^2}\sqrt{(\alpha+1)/n}.$$

With condition (3.0) in mind, we now turn to the question of how many (unordered) pairs of distinct elements of $\mathcal{A}$ intersect in two or more points. We denote the collection of all such pairs by $\mathcal{X}$, so

$$\mathcal{X} = \big\{\{A,B\} \in \mathcal{A}^{(2)} : |A \cap B| \geq 2\big\}.$$

Given any 4-set, there are less than $\binom{4}{2}\binom{n-2}{2} = 3(n-2)(n-3)$ other 4-sets which intersect it in two or more points. So we can bound the total number of unordered pairs of elements of $[n]^{(4)}$ which intersect in two or more points:

$$\begin{aligned}\Big|\Big\{\{A,B\} \in \Big([n]^{(4)}\Big)^{(2)} : |A \cap B| \geq 2\Big\}\Big| \quad &\leq \quad 3(n-2)(n-3)\binom{n}{4}\Big/2 \\ &= \quad \frac{n(n-1)(n-2)^2(n-3)^2}{16}.\end{aligned}$$

Each of the unordered pairs just counted has probability $p^2$ of being in $\mathcal{X}$. Hence

$$\begin{aligned}\mathbb{E}\big(|\mathcal{X}|\big) \quad &\leq \quad \frac{n(n-1)(n-2)^2(n-3)^2}{16}p^2 \\ &= \quad 36\frac{n}{n-1}(\alpha+1)^2.\end{aligned} \tag{3.3}$$

One has to remove at most $|\mathcal{X}|$ sets from $\mathcal{A}$ to obtain a collection satisfying condition (3.0). We would like to have to remove no more than $(1/2)n$ 4-sets;

a simple application of (3.3) gives

$$\mathbb{P}\big(|\mathcal{X}| \geq (1/2)n\big) \leq \frac{72(\alpha+1)^2}{n-1}. \tag{3.4}$$

One should be able to greatly improve on (3.4) with some work, but this crude bound will suffice for our purposes.

For our final calculation we need to bound the probability that there is some $l$-set which contains $l/3$ or more of the members of $\mathcal{A}$. Suppose that $V \in [n]^{(l)}$. Given a collection of $\lceil l/3 \rceil$ members of $V^{(4)}$ the probability that they are all in $\mathcal{A}$ is of course $p^{\lceil l/3 \rceil}$. By simply counting the number of ways of choosing $\lceil l/3 \rceil$ members of $V^{(4)}$ we see that

$$\begin{aligned}
\mathbb{P}\big(|V^{(4)} \cap \mathcal{A}| \geq \lceil l/3 \rceil\big) &\leq \binom{\binom{l}{4}}{\lceil l/3 \rceil} \left(\frac{24(\alpha+1)}{(n-1)(n-2)(n-3)}\right)^{\lceil l/3 \rceil} \\
&\leq \frac{\gamma}{(n-3)^l},
\end{aligned}$$

where $\gamma$ depends on $l$ and $\alpha$ but not on $n$.

Now, given such a $V \in [n]^{(l)}$, the event $\mathcal{S}_V := \{\mathcal{A} : |V^{(4)} \cap \mathcal{A}| < \lceil l/3 \rceil\}$ is a monotone decreasing event (or a down-set) in the usual sense that it is downward closed: if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{A}_2 \in \mathcal{S}_V$, then $\mathcal{A}_1 \in \mathcal{S}_V$. By Harris's Inequality ([4]; see also [5]) any two decreasing events are positively correlated: the probability of their intersection is at least as great as the product of their probabilities. Combining this result with the preceding bound we have

$$\begin{aligned}
\mathbb{P}\big(\forall V \in [n]^{(l)}, |V^{(4)} \cap \mathcal{A}| < \lceil l/3 \rceil\big) &= \mathbb{P}\left(\bigcap_{V \in [n]^{(l)}} \mathcal{S}_V\right) \\
&\geq \left(1 - \frac{\gamma}{(n-3)^l}\right)^{\binom{n}{l}} \\
&\to e^{-\gamma/l!} \text{ as } n \to \infty \qquad (3.5)
\end{aligned}$$

Equations (3.2), (3.4) and (3.5) show that, provided $n$ is sufficiently large, then there is a positive probability that the the random set $\mathcal{A}$ satisfies the three conditions:

(i) $|\mathcal{A}| > (\alpha + 1/2)n$,

(ii) $|\mathcal{X}| \leq (1/2)n$,

(iii) $\forall V \in [n]^{(l)}, |V^{(4)} \cap \mathcal{A}| < \lceil l/3 \rceil$.

Hence, for sufficiently large $n$, a set $\mathcal{A}$ satisfying (i)-(iii) exists. However, condition (ii) ensures that by removing no more than $(1/2)n$ members of $\mathcal{A}$ we obtain a collection satisfying condition (3.0). Condition (i) ensures that the collection still has at least $\alpha n$ members, so by removing further sets if necessary we have $t = \lceil \alpha n \rceil$ sets. Condition (iii) we still hold after removal of some sets, so (3.1) holds and we have the required collection of 4-sets. □

Having proved Lemma 4, we are now in a position to move straight to a proof of our principal result.

*Proof of Theorem 3.* Let $m \geq 3$ be given. We wish to show that for sufficiently large $c$, the statement $P(c, m)$ is false. As usual we write $m$ in the form

$$m = \binom{k}{2} + 2 - q \quad \text{with} \quad 0 \leq q \leq k - 2.$$

If $q = 0$ then we know that $P(c, m)$ is false for all $c > m$ (by Theorem 2). So we may assume that $q \geq 1$ (and hence $k \geq 3$).

Let $N = N(3/2, k+1)$ in the sense of the statement of Lemma 4 (i.e. $\alpha = 3/2$ and $l = k + 1$). We shall show that if $c \geq \binom{N-2}{2} + 2$ (and $c > m$) then there is a counterexample to $P(c, m)$. Given such a $c$ we express it as

$$c = \binom{n}{2} + 2 - p \quad \text{with} \quad 0 \leq p \leq n - 2.$$

In the light of Theorem 2 we may assume $p \geq 1$ and that $p \equiv q\,(6)$. We note that the lower bound on $c$ ensures that $n \geq N - 1$.
All of the necessary counterexamples will be obtained by arranging pairs in a manner rather similar to Configuration 1 of Section 2, which was used in the case when $p$ is even and $q$ is odd. We can write $m$ in the form

$$m = \binom{k}{2} + \epsilon - q^*,$$

with $\epsilon = 1$ or $2$ chosen in such a way that $q^*$ is odd. Let us now write $c$ as

$$c = \binom{n}{2} + \epsilon - \tilde{p},$$

where (using the fact that $\tilde{p} \equiv q^*\,(6)$) $1 \leq \tilde{p} \leq n - 2$.
Now, since $\tilde{p} + 2n + 1$ is even, we may rewrite $c$ as

$$c = \binom{n^*}{2} + \epsilon - p^* \quad \text{with} \quad p^* \text{ even},$$

by taking $n^* = n + 2$ and $p^* = \tilde{p} + n + (n + 1)$ (if $n$ were odd, we could also take $n^* = n + 1$ and $p^* = \tilde{p} + n$).

If $\epsilon = 2$ then we will use a type I colouring and if $\epsilon = 1$ we use a type II colouring. In either case we shall take advantage of the fact that $p^*$ is even and $q^*$ is odd to give an exact $c$-colouring in which the pairs are arranged in groups of two, as follows.

We consider the graph $K_{n^*}$ with vertex set $[n^*]$. Since $n^* \geq N(3/2, k+1)$ and $p^* \leq 3n^*$ (and $p^*$ is even) we can apply Lemma 4 to obtain sets $A_1, \ldots, A_{p^*/2}$ satisfying (3.0) and (3.1) (with $l = k + 1$). To each set $A_i$ we associate two pairs of edges in an obvious way: we arrange the vertices of $A_i$ in an arbitrary order, $v_1, v_2, v_3, v_4$ and take $\{v_1v_2, v_3v_4\}$ as one pair and $\{v_1v_4, v_2v_3\}$ as the other. Property (3.0) from the statement of Lemma 4 ensures that no edge is part of two such pairs. As before, the two edges of any pair both receive the same colour, but apart from that any two edges in $K_{n^*}$ get distinct colours, and all the edge colours are taken from the set $\{1, \ldots, c-1\}$. All the vertices receive the same colour, and this colour is different from the colours used on any of the edges. If $\epsilon = 2$ this is colour 1 (this means we have a type I colouring) and if $\epsilon = 1$ the vertices get colour $c$ (a type II colouring). This colouring of the edges and vertices of $K_{n^*}$ induces a colouring of the edges of an infinite complete graph in the usual way (described at the end of the introduction).

It is now a straightforward matter to check that we have a counterexample to $P(c, m)$. If we take a complete infinite subgraph which contains $k - 1$ vertices of $K_{n^*}$ then we cannot possibly have as many as $m$ colours. If we have $k+1$ vertices of $K_{n^*}$ then condition (3.1) ensures they span strictly less than $(k + 1)/3$ of the 4-sets $A_i$, so we have less than $2(k + 1)/3$ pairs and hence too many colours. Finally if we take exactly $k$ vertices then we need $q^*$ pairs, which is impossible since $q^*$ is odd and any set of vertices must span an even number of pairs. □

We close this section with a couple of remarks about the proof. It is easy to see that the conclusion of Lemma 4 cannot hold for values of $n$ which are not much larger than $l$. Therefore, even if we could prove the Lemma constructively it would not be possible to extend it to find counterexamples for all the open cases of $P(c, m)$: some new idea is required. The proof of Lemma 4 could be simplified a little (essentially avoiding the use of the positive correlations result) to give a slightly weaker conclusion, with $l/3 - 1$ replaced by $l/3$. This would still be sufficient to prove Theorem 3, but we have chosen to give as strong a result as possible in the (perhaps vain) hope that it lays a better groundwork for further developments.

## 4. Related problems and remarks

The constructions of Section 2 left open all the cases when $p \equiv q\,(6)$ — except when $q = 0$ — and Section 3 only filled the gaps in the cases when $c$ is large (relative to $m$). It is possible to construct counterexamples to cover some further special cases, but we are not able to completely cover even one of the six diagonal cases ($p \equiv q\,(6)$) of table I. We remark, without giving details, that we have counterexamples for the cases where $p = q$, for $q \in \{1, 2, 3, k-2, k-3, k-4, k-5\}$ and any value of $p$, and when $n-p < k-q$.

There are a number of interesting variations of $P(c, m)$. Perhaps the most natural is to ask if we can find some complete subgraph, *not necessarily infinite*, which is exactly $m$-coloured. In the Ramsey case — when $m = 1$ — this is, of course, a rather dull question, but for $m > 1$ it becomes non-trivial. To be precise, we formulate the following proposition F(n,c,m):

**F(n,c,m):** *For every exact c-colouring of the edges of $K_n$, there exists a complete subgraph of $K_n$ whose edges are exactly m-coloured.*

To be of any interest at all, $n$ must satisfy the condition $\binom{n}{2} \geq c$ (otherwise there is no exact $c$-colouring on $E(K_n)$ so $F(n, c, m)$ is trivially true for all $m$). The most natural case is probably when $n$ is taken to be arbitrarily large, or equivalently (via the usual compactness arguments) $n$ is replaced by $\omega$.
We make a few observations about the proposition $F(n, c, m)$. It is obviously true for $c = m$ and for $m = 1$. In the case $m = 2$ we can show that for each $c \geq 2$ and for all $n \geq R(c + 1, c)$ (where $R(c + 1, c)$ denotes the Ramsey number for finding a monochromatic $K_{c+1}$ and using $c$ colours) $F(n, c, 2)$ is true. In order to see this we simply have to observe that there exists a monochromatic $K_{c+1}$ in $K_n$ and if we pick a vertex $x \in V(K_n) \setminus V(K_{c+1})$ then by the pigeonhole principle there are two edges from $x$ to $K_{c+1}$ with the same colour. So we find a subgraph, a triangle, coloured with two colours.
For the case $m = 3$ we give a proof that $F(n, c, m)$ holds if $c \geq 3$.

*Proof.*
Case 1: Suppose that there exists a vertex $v \in V(K_n)$ such that there are 3 edges incident to $v$ with different colours. Then it is clear we either obtain a triangle (one of whose vertices is $v$) all of whose edges have distinct colours, or we have a $K_4$ coloured with 3 colours.
Case 2: For each vertex $v \in V(K_n)$ there is either one colour or two colours on the edges incident to $v$. We label the vertices of $K_n$ corresponding to the incident colours, so each vertex receives either one or two colours. We split into two subcases:
1. Suppose there is a vertex, $x$ say, all of whose incident edges have the same

colour, 1 say. From this it follows that every other vertex is incident with an edge of colour 1, so can only receive one further colour. Since $c \geq 3$ we can find two edges with colours 2 and 3; the two vertices incident with the first edge must be coloured "12" and those incident with the second edge must be coloured "13". Therefore the graph spanned by these four vertices is exactly 3-coloured.

2. Every vertex has a label "xy" with $x \neq y$. Pick a vertex, $u$, which is coloured, w.l.o.g. "12", and an edge, $vw$, coloured with colour 3. If the edges $uv$ and $uw$ have different colours (necessarily 1 and 2) then we have an exactly 3-coloured triangle. Otherwise we may assume they both have colour 1. Then we pick a vertex, $x$, such that $ux$ has colour 2, and we see that the graph spanned by $u, v, w$ and $x$ is exactly 3-coloured. $\square$

It is not hard to see that $F(n, c, m)$ and, indeed, $F(\omega, c, m)$, are false in a lot of cases, and the counterexamples of section 2 can be useful here too. However, there are many more gaps than there were in Section 2, due to the simple fact that our spare colour — colour $c$ of Section 2 — which was used on all the edges not incident with the vertices of $K_n$, is not guaranteed to appear in *finite* subgraphs. We do not feel we are close to proving anything analogous to Theorem 3 for proposition $F(n, c, m)$.

Canonical Ramsey theory (see e.g. [3], pp.111-116) provides results about colourings when infinitely many colours are used, and one can also investigate a version of proposition $P(c, m)$ in this case.

**I(m)**: *For every colouring of the edges of $K_\omega$ with infinitely many colours there exists a finite complete subgraph of $K_\omega$ whose edges are exactly $m$-coloured.*

Since we can colour all edges of $K_\omega$ with different colours we know that $I(m)$ is false unless $m$ is of the form $\binom{k}{2}$ for some $k$. The cases $k = 1$ and $k = 2$ are trivial, and one can check that the proof of proposition $F(n, c, m)$ for $m = 3$ can also be used to prove $I(3)$; note that although we can guarantee some complete subgraph with exactly three colours, we cannot guarantee a triangle with three colours. We are unable to determine whether $I(\binom{k}{2})$ holds in other cases, although the methods of Canonical Ramsey Theory may enable a proof to be found that it does. More generally it may be possible to demonstrate that if there are not arbitrarily large complete subgraphs in which every edge has a different colour, then one can draw some strong conclusions about the colouring.

## Acknowledgement

The authors would like to thank "Sonderforschungsbereich 343", University of Bielefeld, for its support.

## References

[1] B. Bollobás, *Random Graphs*, Academic Press, London (1985) *xvi*+447pp.

[2] M. Erickson, A conjecture concerning Ramsey's theorem, in *Discrete Mathematics* **126** (1994) 395-398.

[3] R.L. Graham, B.L. Rothschild and J.H. Spencer, *Ramsey Theory*, Wiley, New York (1980) *ix*+174pp.

[4] T.E. Harris, A lower bound for the critical probability in a certain percolation process, *Proc. Camb. Phil. Soc.* **56** (1960) 13-20.

[5] T.M. Liggett, *Interacting Particle Systems,* Springer-Verlag, New York (1985) *xiii*+ 488pp.

[6] F.P. Ramsey, On a problem of formal logic, in *Proc. London Math. Soc.* **48** (1930) 264-286.