Finitistic dimension and Ziegler spectrum [1]

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Abstract: Given a two-sided artinian ring Λ, it is shown that the Ziegler spectrum of Λ forms a test class for certain homological properties of Λ. We discuss the finitistic dimension of Λ, Nunke's condition, and also the relation between the big and the little finitistic dimension.

Let Λ be a two-sided artinian ring. Denote by $\operatorname{Mod}\Lambda$ the category of (right) Λ-modules and by $\operatorname{mod}\Lambda$ the full subcategory of all finitely presented Λ-modules. Given a Λ-module M, we denote by $\operatorname{pd} M$ its projective dimension, and the *finitistic dimension* $\operatorname{Fd}\Lambda$ of Λ is the supremum of the projective dimensions of the Λ-modules with finite projective dimension. The *Ziegler spectrum* $\operatorname{Zsp}\Lambda$ of Λ is by definition the set of isomorphism classes of indecomposable pure-injective Λ-modules.

The aim of this note is to show that the Ziegler spectrum forms a test class for certain homological properties of Λ. Furthermore, the Ziegler spectrum carries a topology and we shall use its compactness to obtain an equivalent formulation of Nunke's condition. In the final part of this note we study the relation between $\operatorname{Fd}\Lambda$ and the little finitistic dimension $\operatorname{fd}\Lambda$ of Λ. Our first result is easily stated as follows.

Theorem 1 $\operatorname{Fd}\Lambda = \sup\{\operatorname{pd} M \mid M \in \operatorname{Zsp}\Lambda \text{ and } \operatorname{pd} M < \infty\}$.

We postpone the proof and first discuss some related homological properties of Λ. Suppose that Λ has a self-duality D between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$. We are interested in *Nunke's condition* [8]:

$$\coprod_{i\geq 0} \operatorname{Ext}^i_\Lambda(D\Lambda, M) \neq 0 \text{ for every } \Lambda\text{-module } M \neq 0.$$

To this end let us define the *Nunke dimension* of Λ as follows:

$$\operatorname{Nd}\Lambda = \inf\{d \in \mathbb{N}_0 \mid \coprod_{i=0}^{d} \operatorname{Ext}^i_\Lambda(D\Lambda, M) \neq 0 \text{ for all } M \neq 0\}.$$

The following relation between Nunke dimension and finitistic dimension is essentially due to Jans [9], see also [4].

Proposition 2 $\operatorname{Nd}\Lambda \leq \operatorname{Fd}\Lambda$.

Proof: We use the fact that the functor

$$\operatorname{Inj}\Lambda \longrightarrow \operatorname{Proj}\Lambda, \qquad X \mapsto \operatorname{Hom}_\Lambda(D\Lambda, X)$$

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is an equivalence between the full subcategories of injective and projective Λ-modules. Suppose now that $\operatorname{Nd}\Lambda \geq n$ and choose a non-zero Λ-module M with $\coprod_{i=0}^{n-1} \operatorname{Ext}^i_\Lambda(D\Lambda, M) = 0$. Take an injective coresolution

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{\psi_0} I_1 \xrightarrow{\psi_1} I_2 \xrightarrow{\psi_2} \dots$$

and put $N = \operatorname{Coker}\operatorname{Hom}_\Lambda(D\Lambda, \psi_{n-1})$. It is easily checked that $\operatorname{pd} N = n$. Thus $\operatorname{Fd}\Lambda \geq n$.

Remark Given any $n \in \mathbb{N}_0$, there is a finite dimensional algebra Λ over a field (with $\operatorname{rad}^2\Lambda = 0$ and $\operatorname{Nd}\Lambda = 2$) such that $\operatorname{Fd}\Lambda - \operatorname{Nd}\Lambda = n$ [6].

Theorem 3 $\operatorname{Nd}\Lambda = \inf\{d \in \mathbb{N}_0 \mid \coprod_{i=0}^{d} \operatorname{Ext}^i_\Lambda(D\Lambda, M) \neq 0$ *for all* $M \in \operatorname{Zsp}\Lambda\}$ *and Nunke's condition holds if and only if* $\operatorname{Nd}\Lambda < \infty$.

Some further definitions are needed in order to give the proof of both theorems. An additive functor $F\colon \operatorname{Mod}\Lambda \to \operatorname{Ab}$ into the category of abelian groups is *finitely presented* if there is a presentation

$$\operatorname{Hom}_\Lambda(Y, \ ) \longrightarrow \operatorname{Hom}_\Lambda(X, \ ) \longrightarrow F \longrightarrow 0$$

with X and Y in $\operatorname{mod}\Lambda$. We denote by $\operatorname{Ker} F$ the full subcategory of Λ-modules M with $F(M) = 0$. A full subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ is *definable* if there is a family F_i, $i \in I$, of finitely presented functors such that $\mathcal{X} = \cap_{i\in I} \operatorname{Ker} F_i$. A definable subcategory is automatically closed under direct limits, products and pure submodules, and it can be shown that this property characterizes a definable subcategory [11]. The notion of a definable subcategory allows us to describe the topology on $\operatorname{Zsp}\Lambda$ which was first introduced by Ziegler in model theoretic terms [12].

Proposition 4 (1) *The assignment* $\mathcal{X} \mapsto \mathcal{X} \cap \operatorname{Zsp}\Lambda$ *defines a bijection between the definable subcategories of* $\operatorname{Mod}\Lambda$ *and the closed subsets of* $\operatorname{Zsp}\Lambda$.

(2) $\operatorname{Zsp}\Lambda$ *is a quasi-compact space.*

Proof: (1) This result is due to Crawley-Boevey [2] and he relies on work of Herzog [5] and Ziegler [12]. We refer to [11] for a proof which uses the localization theory for locally coherent categories developed in [10]. However, the inverse of the assignment $\mathcal{X} \mapsto \mathcal{X} \cap \operatorname{Zsp}\Lambda$ is easily constructed; it sends a closed subset $\mathcal{U}$ of $\operatorname{Zsp}\Lambda$ to $\cap_{F(\mathcal{U})=0} \operatorname{Ker} F$.

(2) See [12] or [10].

The next lemma shows that some relevant subcategories of $\operatorname{Mod}\Lambda$ are definable.

Lemma 5 *Let $n \in \mathbb{N}_0$.*

(1) *The full subcategory $\mathcal{P}_n$ of Λ-modules M with $\operatorname{pd} M \leq n$ is definable.*

(2) *The full subcategory $\mathcal{Q}_n$ of Λ-modules M with $\coprod_{i=0}^{n} \operatorname{Ext}^i_\Lambda(D\Lambda, M) = 0$ is definable.*

Proof: Projective dimension and flat dimension coincide for any module over an artinian ring and therefore $\mathcal{P}_n = \cap_{X \in \operatorname{mod} \Lambda^{\mathrm{op}}} \operatorname{Ker} \operatorname{Tor}^\Lambda_{n+1}(\ , X)$ since $\operatorname{Tor}^\Lambda_{n+1}(M, \ )$ commutes with direct limits and every module can be written as a direct limit of finitely presented modules. Therefore $\mathcal{P}_n$ is definable and $\mathcal{Q}_n$ is definable since $\mathcal{Q}_n = \operatorname{Ker} \coprod_{i=0}^{n} \operatorname{Ext}^i_\Lambda(D\Lambda, \ )$.

An easy consequence of the preceding lemma is the following.

Proposition 6 *Let $\mathcal{U}$ be a subset of $\operatorname{Zsp}\Lambda$. Then $\operatorname{pd} M \leq \sup\{\operatorname{pd} N \mid N \in \mathcal{U}\}$ for every module M in the closure of $\mathcal{U}$.*

Proof of Theorem 1: Suppose that $\mathcal{P}_n \subsetneq \mathcal{P}_{n+1}$. The first part of Proposition 4 then implies that there is $M \in \operatorname{Zsp}\Lambda$ with $M \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n$ since $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ are definable. Thus the assertion follows.

Proof of Theorem 3: The proof of the first part of the assertion is analogous to that of Theorem 1. Suppose now that Nunke's condition holds. Thus $\cap_{n \geq 0} \mathcal{Q}_n = 0$ and the compactness of $\operatorname{Zsp}\Lambda$ then implies the existence of $d \in \mathbb{N}_0$ with $\cap_{n=0}^{d} \mathcal{Q}_n = 0$. We conclude that $\operatorname{Nd}\Lambda < \infty$.

Remark The finitistic dimension of Λ is finite if and only if the full subcategory of all Λ-modules with finite projective dimension is definable. Therefore one might conjecture that $\operatorname{Fd}\Lambda < \infty$ if and only if the modules in the Ziegler spectrum having finite projective dimension form a closed subset of $\operatorname{Zsp}\Lambda$.

The rest of this paper is devoted to studying the relation between the finitistic dimension $\operatorname{Fd}\Lambda$ and the *little finitistic dimension* $\operatorname{fd}\Lambda$ which is the supremum of the projective dimensions of the finitely presented Λ-modules with finite projective dimension. Note that there are examples of finite dimensional algebras where $\operatorname{fd}\Lambda$ is different from $\operatorname{Fd}\Lambda$ [13]. Our results are motivated by recent work of Huisgen-Zimmermann and Smalø who have shown that both dimensions coincide for an interesting class of artin algebras [7].

In order to state the next result we recall that for any subcategory $\mathcal{C}$ of $\operatorname{Mod}\Lambda$ a morphism $M \to M'$ is a *left $\mathcal{C}$-approximation* of M if M' lies in $\mathcal{C}$ and the induced map $\operatorname{Hom}_\Lambda(M', X) \to \operatorname{Hom}_\Lambda(M, X)$ is surjective for every X in $\mathcal{C}$. We denote by $\mathcal{P}_\infty$ the full subcategory of all Λ-modules with finite projective dimension.

Theorem 7 *The following conditions are equivalent.*

(1) *A Λ-module has finite projective dimension if and only if it is a direct limit of finitely presented modules having finite projective dimension.*

(2) *The little finitistic dimension* $\operatorname{fd}\Lambda$ *is finite, and every finitely presented Λ-module has a left $\mathcal{P}_\infty$-approximation which is finitely presented.*
Moreover, in this case $\operatorname{fd}\Lambda = \operatorname{Fd}\Lambda < \infty$.

Examples of rings which satisfy the conditions in this theorem are artin algebras Λ where every finitely presented Λ-modules has a right $\mathcal{P}_\infty \cap \operatorname{mod}\Lambda$-approximation [1, 7]. The proof of Theorem 7 is based on the following lemma.

Lemma 8 *Let $\mathcal{C}$ be any additive subcategory of* $\operatorname{mod}\Lambda$. *Then a Λ-module M is a direct limit of objects in $\mathcal{C}$ iff every morphism $X \to M$ with $X \in \operatorname{mod}\Lambda$ factors through an object in $\mathcal{C}$.*

Proof: See [3, Lemma 4.1].

Proof of Theorem 7: (1) $\Rightarrow$ (2). In order to show that $\operatorname{fd}\Lambda < \infty$ assume there is a family M_i, $i \in \mathbb{N}$, in $\operatorname{mod}\Lambda$ with $\operatorname{pd} M_i < \operatorname{pd} M_{i+1} < \infty$ for all i. Then $\coprod_{i\in\mathbb{N}} M_i$ is a direct limit of finitely presented objects in $\mathcal{P}_\infty$ but $\operatorname{pd}\coprod_{i\in\mathbb{N}} M_i = \infty$. Thus (1) implies $\operatorname{fd}\Lambda = n < \infty$ and therefore $\operatorname{Fd}\Lambda = n$ since $\mathcal{P}_n$ is closed under direct limits by Lemma 5. To show the second part of the assertion we use an idea from [3]. Let M be in $\operatorname{mod}\Lambda$ and choose a representative set $\varphi_i\colon M \to N_i$, $i \in I$, of morphisms in $\operatorname{mod}\Lambda$ such that the domain has finite projective dimension. Then $N = \prod_{i\in I} N_i$ lies in $\mathcal{P}_\infty$ since $\mathcal{P}_\infty = \mathcal{P}_n$ is closed under products by Lemma 5. Thus the induced morphism $\varphi\colon M \to N$ factors through a finitely presented object M' in $\mathcal{P}_\infty$ by the preceding lemma. It is easily checked that the corresponding morphism $M \to M'$ is a left $\mathcal{P}_\infty$-approximation of M.

(2) $\Rightarrow$ (1). It follows immediately from the preceding lemma that every module in $\mathcal{P}_\infty$ is a direct limit of finitely presented modules in $\mathcal{P}_\infty$. Conversely, the projective dimension of a direct limit of finitely presented Λ-modules in $\mathcal{P}_\infty$ is bounded by $\operatorname{fd}\Lambda = n$ since $\mathcal{P}_n$ is closed under direct limits by Lemma 5.

If the ring Λ has the property that every finitely presented Λ-module is pure-injective, for instance if Λ has a self-duality between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{\mathrm{op}}$, then condition (1) in the preceding theorem implies that the finitely presented modules form a dense subset of $\{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M < \infty\}$. We point out that this condition is already sufficient for the equality $\operatorname{fd}\Lambda = \operatorname{Fd}\Lambda$.

Corollary 9 *Let $n \in \mathbb{N}_0$ and suppose that the finitely presented modules $M \in \operatorname{Zsp}\Lambda$ with $\operatorname{pd} M \leq n$ form a dense subset of $\{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M < \infty\}$. Then* $\operatorname{fd}\Lambda = \operatorname{Fd}\Lambda \leq n$.

Proof: Let $\mathcal{U}_n = \{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M \leq n \text{ and } M \in \operatorname{mod}\Lambda\}$ and we may assume that $n \in \mathbb{N}_0$ is the minimal choice with $\{M \in \operatorname{Zsp}\Lambda \mid \operatorname{pd} M < \infty\} \subseteq \overline{\mathcal{U}_n}$. It follows from Proposition 6 and Theorem 1 that $\operatorname{Fd}\Lambda \leq n$ and we obtain therefore $n \leq \operatorname{fd}\Lambda \leq \operatorname{Fd}\Lambda \leq n$. Thus $\operatorname{fd}\Lambda = \operatorname{Fd}\Lambda = n$.

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