Approximation of arbitrary Dirichlet processes by Markov chains

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Abstract

We prove that any Hunt process on a Hausdorff topological space associated with a Dirichlet form can be approximated by a Markov chain in a canonical way. This also gives a new and "more explicit" proof for the existence of Hunt processes associated with strictly quasi-regular Dirichlet forms on general state spaces.

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1 Introduction

In the last few years the theory of *Dirichlet forms* on general (topological) state spaces has been used to construct and analyze a number of fundamental processes on infinite-dimensional "manifold-like" state spaces which so far could not be constructed by other means. Among these some of the most important are: solutions to infinite-dimensional stochastic differential equations with very singular drifts such as the stochastic quantization of (infinite volume) *time zero* and *space-time quantum fields* in Euclidean field theory (see e.g. [AR91], [RZ92], [AMR94]); diffusions on loop spaces (see [DR92], [ALR93], [ElM96]); a class of interacting Fleming-Viot processes (see [ORS95, in particular Subsections 5.2, 5.3] and [OR94]); infinite particle systems with very singular interactions (see e.g. [Os95], [Y95]); stochastic dynamics associated with Gibbs states (see e.g. [AKoR95]). We also refer to the survey article [R96]. All these processes are diffusions (i.e., have continuous sample paths almost surely) and all except for the one in [ALR93] are conservative, hence, in particular, they are Hunt processes.

In [AMR95a] (see also [MR92, Chap. V, Sect. 2]) Dirichlet forms (not necessarily symmetric) on general state spaces which are associated with Hunt processes have been characterized completely through an analytic property which is checkable in examples and is called *strict quasi-regularity*. However, the construction of the Hunt process was quite indirect using "Kolmogorov's scheme" and a number of (partly rather technical) tools from potential theory and the general theory of Markov processes. It has been an open question for quite some time whether the method of constructing Markov processes based on the Yoshida approximation (for the transition semigroup) and tightness arguments (cf. the beautiful exposition in [EK86]) can be extended to this case. This would be desirable, since, in addition, this would yield an approximation of the Hunt processes by Markov chains in a canonical way and would make the limit process more explicit and provide another tool for its analysis.

We recall, however, that this approximation method was, so far, only developed under some additional assumptions on the state space (i.e., it was assumed to be a locally compact separable metric space) and on the underlying transition semigroup (e.g. it was assumed to be Feller). It should be emphasized that these conditions are not even fulfilled in the classical case of regular Dirichlet forms on locally compact sepaerable metric state spacees for which the existence of an associated Hunt process was first established by M. Fukushima in his famous work [F71] (see also [Si74], [F80], [FOT94] and [Ca-Me75], [Le77] for the non-symmetric case).

The purpose of this paper is to prove that the above Markov chain approximation scheme can be extended to any Hunt process associated with a Dirichlet form on a general state space. The proof is divided into several steps and carried out in Sections 3 and 4 below (see Theorems 3.2, 3.3, 4.3, 4.4). Section 2 contains some background material and the necessary terminology resp. definitions.

Finally, we want to emphasize that we also gain a new proof for the existence of an associated Hunt process for the most general class of Dirichlet forms possible (namely those which are strictly quasi-regular). This proof is also new for the classical case in [F71]. As usual in Dirichlet form theory, the price we pay for this generality is that we only get the approximation of the path space measures P_x for quasi-every point x in the state space. However, if we just want the approximation result and assume that the limit process is already given, we can modify our method to obtain an approximation for *each* P_x. The details are contained in the forthcoming paper [MRZ96].

2 Preliminaries

In this section we recall some necessary notions and known facts concerning *quasi-regular* and *strictly quasi-regular Dirichlet forms*. For details we refer to [MR92].

Let E be a Hausdorff space such that its Borel σ-algebra $\mathcal{B}(E)$ is generated by $C(E)$ (:= the set of all continuous functions on E). Let m be a σ-finite (positive) measure on $(E, \mathcal{B}(E))$ where $\mathcal{B}(E)$ is the Borel σ-algebra of E. Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(E, m)$ with associated semigroup $(T_t)_{t\geq 0}$, resolvent $(G_\alpha)_{\alpha>0}$, and co-associated semigroup $(\hat{T}_t)_{t\geq 0}$, and resolvent $(\hat{G}_\alpha)_{\alpha>0}$, respectively.

Define for a closed set $F \subset E$,

$$D(\mathcal{E})_F := \{u \in D(\mathcal{E}) | u = 0 \quad m\text{-a.e. on } F^c\}$$

where $F^c := E \setminus F$.

Definition 2.1 An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$-*nest* if $\bigcup_{k\geq 1} D(\mathcal{E})_{F_k}$ is dense in $D(\mathcal{E})$ (w.r.t. the norm $\|\cdot\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} := (\tilde{\mathcal{E}}(\cdot,\cdot)+(\cdot,\cdot)_{L^2(E;m)})^{\frac{1}{2}}$, where $\tilde{\mathcal{E}}(u,v) := \frac{1}{2}[\mathcal{E}(u,v)+\mathcal{E}(v,u)]$).

A subset $N \subset E$ is called $\mathcal{E}$-*exceptional* if $N \subset \bigcap_{k\geq 1} F_k^c$ for some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$. We say that a property of points in E holds $\mathcal{E}-$*quasi-everywhere* (abbreviated $\mathcal{E}$-*q.e.*), if the property holds outside some $\mathcal{E}$-exceptional set.

Given an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ we define

$$
\begin{aligned}
C(\{F_k\}) \;:=\; & \{f : A \to \mathbb{R} \,|\, \cup_{k=1} F_k \subset A \subset E,\ f_{|F_k} \text{ is continuous} \\
& \text{for every } k \in \mathbb{N}\}. \qquad (2.1)
\end{aligned}
$$

An $\mathcal{E}$-q.e. defined function f on E is called $\mathcal{E}$-*quasi-continuous* if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f \in C(\{F_k\})$.

Definition 2.2 A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called *quasi-regular* if:

(i) There exits an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets.

(ii) There exits an $\tilde{\mathcal{E}}^{\frac{1}{2}}$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous m-versions.

(iii) There exists $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

It is known that the above quasi-regularity condition characterizes all the Dirichlet forms which are associated to a pair of Borel right processes [MR92, Ch. IV]. Though not really necessary (see [AMR95b]), for convenience we shall make use of the well-developed capacity theory of quasi-regular Dirichlet forms below.

Let $h \in D(\mathcal{E})$ be a 1-excessive function (w.r.t. $(T_t)_{t\geq 0}$, i.e., $e^{-t}T_t h \leq h\ \forall t \geq 0$). Then the 1-reduced function h_U of h on an open set U is the unique function in $D(\mathcal{E})$ satisfying

$$
\begin{aligned}
&\text{(i)} \quad h_U = h \quad m\text{-a.e. on } U \\
&\text{(ii)} \quad \mathcal{E}_1(h_U, w) = 0 \quad \text{for all } w \in D(\mathcal{E})_{U^c}. \qquad (2.2)
\end{aligned}
$$

The 1-coreduced function $\hat{h}_U$ for a 1-coexcessive function $\hat{h}$ is defined correspondingly with the two entries of $\mathcal{E}$ interchanged. Given a 1-excessive function h in $D(\mathcal{E})$ and a 1-coexcessive function $g \in D(\mathcal{E})$, define for an open set $U \subset E$

$$\mathrm{Cap}_{h,g}(U) := \mathcal{E}_1(h_U, \hat{g}_U), \tag{2.3}$$

and for arbitrary $A \subset E$

$$\mathrm{Cap}_{h,g}(A) := \inf\{\mathrm{Cap}_{h,g}(U) | A \subset U,\ U \text{ open}\}. \tag{2.4}$$

For our purpose, another capacity is also needed. Let S (resp. $\hat{S}$) denote the family of all 1-excessive (resp. 1-coexcessive) functions in $D(\mathcal{E})$. Let $h \in S$ and $g \in \hat{S}$. We define for an open set $U \subset E$

$$\begin{aligned} \mathrm{Cap}_{1,g}(U) &:= \sup\{\mathrm{Cap}_{u,g}(U) | u \in S,\ u \leq 1\} \\ \mathrm{Cap}_{h,1}(U) &:= \sup\{\mathrm{Cap}_{h,u}(U) | u \in \hat{S},\ u \leq 1\} \end{aligned} \tag{2.5}$$

and for arbitrary $A \subset E$

$$\begin{aligned} \mathrm{Cap}_{1,g}(A) &:= \inf\{\mathrm{Cap}_{1,g}(U) | A \subset U \subset E,\ U \text{ open}\} \\ \mathrm{Cap}_{h,1}(A) &:= \inf\{\mathrm{Cap}_{h,1}(U) | A \subset U \subset E,\ U \text{ open}\}\ . \end{aligned} \tag{2.6}$$

It has been shown in [MR92, III.2 and V.2] that $\mathrm{Cap}_{h,g}$, $\mathrm{Cap}_{h,1}$, and $\mathrm{Cap}_{1,g}$ are all countably subadditive Choquet capacities.

Here is a description of $\mathcal{E}$-nests in terms of capacities:

Proposition 2.3 ([MR92, III. 2.11]) *Let $h = G_1\varphi$, $g = \hat{G}_1\hat{\varphi}$ for some $\varphi,\ \hat{\varphi} \in L^2(E;m)$, $\varphi,\ \hat{\varphi} > 0$. Then an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subets of E is an $\mathcal{E}$-nest if and only if $\lim_{k\to\infty} \mathrm{Cap}_{h,g}(F_k^c) = 0$.*

In what follows we adjoin an extra point Δ (which serves as the "cemetery" for Markov processes) to E and write E_Δ for $E \cup \{\Delta\}$. If E is locally compact, then Δ can be considered either as an isolated point of E_Δ, or as a point "at infinity" of E_Δ with the topology of one point compactification. We select one of the above two topologies and fix it. If E is not locally compact then we simply consider Δ as an isolated point of E_Δ. $\mathcal{B}(E_\Delta)$ denotes the Borel σ-algebra of E_Δ. Any function on E is considered as a function on E_Δ by putting $f(\Delta) = 0$. m is extended to $(E_\Delta, \mathcal{B}(E_\Delta))$ by setting $m(\{\Delta\}) = 0$. For a subset F of E, we still write F^c for $E \setminus F$, while the complement of

F in E_Δ will be explicitly denoted by $E_\Delta \setminus F$. Given an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed sets of E, we define

$$\begin{aligned} C_\infty(\{F_k\}) \;:=\; & \Big\{ f : A \to \mathbb{R} \,|\, \bigcup_{k\geq 1} F_k \subset A \subset E,\ f_{|F_k\cup\{\Delta\}} \text{ is} \\ & \text{continuous for every } k \in \mathbb{N}\Big\}\ . \end{aligned}$$

Note that if Δ is an isolated point of E_Δ, then $C_\infty(\{F_k\})$ coincides with $C(\{F_k\})$.

Definition 2.4 An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed sets of E is called a *strict $\mathcal{E}$-nest* if

$$\mathrm{Cap}_{1,g}(F_k^c) \downarrow 0 \ \text{ as } k \to \infty \text{ for some } g = \hat{G}_1\varphi,\ \varphi \in L^2(E;m),\ \varphi > 0\ .$$

It has been shown that the above definition is independent of the particular choice of φ (cf. [MR92, V.2.5]).

The concepts of *strictly $\mathcal{E}$-exceptional sets* and *strictly $\mathcal{E}$-quasi-everywhere* (*strictly $\mathcal{E}$-q.e.*) are now defined correspondingly, but with "strict $\mathcal{E}$-nests" replacing "$\mathcal{E}$-nests".

A strictly $\mathcal{E}$-q.e. defined function f is called *strictly $\mathcal{E}$-quasi-continuous* if there exists a strict $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $f \in C_\infty(\{F_k\})$.

Definition 2.5 *A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is called* strictly quasi-regular *if:*

(i) There exits a strict $\mathcal{E}$-nest $(E_k)_{k\in\mathbb{N}}$ such that $E_k \cup \{\Delta\}$ is compact in E_Δ for each k.

(ii) There exits an $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$-dense subset of $D(\mathcal{E})$ whose elements have strictly $\mathcal{E}$-quasi-continuous m-versions.

(iii) There exist $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, having strictly $\mathcal{E}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and a strictly $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \mathbb{N}\}$ separates the points of $E_\Delta \setminus N$.

It is well-known that a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is strictly quasi-regular if and only if it is associated with a Hunt process (see [MR92, V.2] for details). It is also well-known that if $1 \in D(\mathcal{E})$ and Δ is an isolated point of E_Δ, then $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular if and only if it is strictly quasi-regular (cf. [MR92, Proof of V.2.15]). For the rest of this section we assume that our Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is strictly quasi-regular. Then, in particular, each $u \in D(\mathcal{E})$ has a strictly $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ (cf. [MR92, V.2.22 (iii)]).

Let $(E_k)_{k\in\mathbb{N}}$ be a strict $\mathcal{E}$-nest specified in Definition 2.5 (i). Set: $Y_1 := \bigcup_{k\in\mathbb{N}} E_k$.

Proposition 2.6 ([MR92, V. 2.23]) *Let $\alpha > 0$. There exits a kernel $\widetilde{R}_\alpha(z,\cdot)$ from $(E, \mathcal{B}(E))$ to $(Y_1, \mathcal{B}(Y_1))$ satisfying*

(i) $\widetilde{R}_\alpha f(z)$ is a strictly $\mathcal{E}$-quasi-continuous version of $G_\alpha f$ for each $f \in L^2(Y_1; m)$.

(ii) $\alpha\widetilde{R}_\alpha(z, Y_1) \leq 1$, for all $z \in E$.

The kernel is strictly $\mathcal{E}$-q.e. unique in the sense that if K is another kernel satisfying (i), then $K(z,\cdot) = \widetilde{R}_\alpha(z,\cdot)$ for strictly $\mathcal{E}$-q.e. $z \in E$.

Let $\mathbb{Q}_+$, $\mathbb{Q}_+^*$ denote the non-negative respectively the strictly positive rational numbers. Adapting the argument of [MR92, IV. 3.4, 3.10 and 3.11] to the strictly quasi-regular case, we have the following results.

Lemma 2.7 *There exits a countable family J_0 of bounded strictly $\mathcal{E}$-quasi-continuous 1-excessive functions in $D(\mathcal{E})$ and a Borel set $Y \subset Y_1$ (Y_1 as specified in Proposition 2.6) satisfying:*

(i) If $u, v \in J_0$, $\alpha, c_1, c_2 \in \mathbb{Q}_+^*$, then
$\widetilde{R}_\alpha u$, $u \wedge v$, $u \wedge 1$, $(u+1)\wedge v$, $c_1 u + c_2 v$ are all in J_0.

(ii) $N := E \setminus Y$ is strictly $\mathcal{E}$-exceptional and
$\widetilde{R}_\alpha(z, N) = 0$, for all $z \in Y$, $\alpha \in \mathbb{Q}_+^*$.

(iii) J_0 separates the points of $Y_\Delta := Y \bigcup \{\Delta\}$.

(iv) If $u \in J_0$, $x \in Y$, then
$$\beta\widetilde{R}_{1+\beta}u(x) \leq u(x) \quad \text{for all } \beta \in \mathbb{Q}_+^*,$$
$$\widetilde{R}_\alpha u(x) - \widetilde{R}_\beta u(x) = (\beta - \alpha)\widetilde{R}_\alpha \widetilde{R}_\beta u(x), \text{ for all } \alpha, \beta \in \mathbb{Q}_+^*,$$
$$\lim_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}_+^*}} \alpha\widetilde{R}_\alpha u(x) = u(x).$$

We now define for $\alpha \in \mathbb{Q}_+^*$, $A \in \mathcal{B}(Y_\Delta)$ $\quad (\mathcal{B}(Y_\Delta) := \mathcal{B}(E_\Delta \bigcap Y_\Delta))$

$$R_\alpha(x, A) = \begin{cases} \widetilde{R}_\alpha(x, A \cap Y) + (\frac{1}{\alpha} - \widetilde{R}_\alpha(x, Y)) I_A(\Delta), & \text{if } x \in Y \\ \dfrac{1}{\alpha} I_A(\Delta), & \text{if } x = \Delta \end{cases} \tag{2.7}$$

and set

$$J := \{u + cI_{Y_\Delta} | u \in J_0,\ c \in \mathbb{Q}_+\}\,. \tag{2.8}$$

Note that by our convention $u(\Delta) = 0$ for all $u \in J_0$. Hence the following lemma is clear.

Lemma 2.8 *Let $(R_\alpha)_{\alpha \in \mathbb{Q}_+^*}$ and J be defined by* (2.7) *and* (2.8) *respectively. Then the properties (i) and (iv) of Lemma 2.7 remain true with J_0, Y, and $\widetilde{R}_\alpha$ replaced by J, Y_Δ and R_α respectively.*

3 Compound Poisson processes associated with $\mathcal{E}^\beta$ and their weak limit

Throughout this section $(\mathcal{E}, D(\mathcal{E}))$ is a strictly quasi-regular Dirichlet form. Let J, Y_Δ and $(R_\alpha)_{\alpha \in \mathbb{Q}_+^*}$ be as in Lemma 2.8.

For a fixed $\beta \in \mathbb{Q}_+^*$, let $\{Y^\beta(k),\ k = 0, 1, \ldots\}$ be a Markov chain in Y_Δ with some initial distribution ν and the transition function βR_β, and let $(\Pi_t^\beta)_{t \geq 0}$ be a Poisson process with parameter β, i.e.,

$$P\left(\Pi_t^\beta = k\right) = e^{-\beta t} \frac{(\beta t)^k}{k!}\ .$$

Assume that $(\Pi_t^\beta)_{t \geq 0}$ is independent of $\{Y^\beta(k),\ k = 0, 1, \ldots\}$ and define

$$X_t^\beta = Y^\beta(\Pi_t^\beta), \quad t \geq 0\ , \tag{3.1}$$

then $(X_t^\beta)_{t \geq 0}$ is a strong Markov process in Y_Δ. Let $\mathcal{B}_b(Y_\Delta)$ denote the set of all bounded Borel functions on Y_Δ and define

$$P_t^\beta f := e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} (\beta R_\beta)^k f \quad ,\ f \in \mathcal{B}_b(Y_\Delta)\ . \tag{3.2}$$

It is known that $(P_t^\beta)_{t \geq 0}$ is the transition semigroup of $(X_t^\beta)_{t \geq 0}$, i.e., for all $f \in \mathcal{B}_b(Y_\Delta)$, $t, s \geq 0$, we have

$$E\left[f(X_t^\beta) | \sigma(X_{s'}^\beta : s' \leq s)\right] = \left(P_t^\beta f\right)\left(X_s^\beta\right) \tag{3.3}$$

(see [EK86, IV. 2]). Note that $(P_t^\beta)_{t\geq 0}$ is a strongly continuous contraction semigroup on the Banach space $(\mathcal{B}_b(Y_\Delta), \|\cdot\|_\infty)$ ($\|f\|_\infty := \sup_{x\in Y_\Delta} |f(x)|$), and the corresponding generator is given by

$$
\begin{aligned}
L^\beta u(x) = \beta(\beta R_\beta u(x) - u(x)) &= \beta \int_{Y_\Delta} (u(y) - u(x)) \beta R_\beta(x, dy), \\
\text{for all } u \in \mathcal{B}_b(Y_\Delta)\ . & \qquad\qquad (3.4)
\end{aligned}
$$

On the other hand, if we define

$$
\mathcal{E}^\beta(u,v) := \beta(u - \beta G_\beta u, v), \text{ for } u, v \in L^2(E;m) \tag{3.5}
$$

(recall that $(G_\beta)_{\beta>0}$ is the resolvent of $(\mathcal{E}, D(\mathcal{E}))$, then $\mathcal{E}^\beta$ is a Dirichlet form on $L^2(E;m)$ and the associated semigroup is given by

$$
T_t^\beta f = e^{-\beta t} \sum_{j=0}^\infty \frac{(\beta t)^j}{j!} (\beta G_\beta)^j\ f \qquad \forall\ f \in L^2(E;m)\ . \tag{3.6}
$$

Comparing (3.6) with (3.3), we see that (X_t^β) is a process associated with $\mathcal{E}^\beta$. More precisely, let $\Omega_{E_\Delta} := D_{E_\Delta}[0,\infty)$ be the space of all cadlag functions from $[0,\infty)$ to E_Δ, equipped with the Prohorov metric (see [EK86, Chap. III]). Let $(X_t)_{t\geq 0}$ be the coordinate process on Ω_{E_Δ}. Let P_x^β be the law of (X_t^β) on Ω_{E_Δ} with initial distribution δ_x for $x \in Y_\Delta$; and for $x \in E\setminus Y_\Delta$ let P_x^β be the Dirac measure on Ω_{E_Δ} such that $P_x^\beta\{X_t = x \text{ for all } t \geq 0\} = 1$. Finally, let $(\mathcal{F}_t^\beta)_{t\geq 0}$ be the natural filtration of $(X_t)_{t\geq 0}$ (completed w.r.t. $(P_x^\beta)_{x\in E}$, cf. e.g. [MR92, IV. 1].) Then we have:

Proposition 3.1 $I\!M^\beta := (\Omega_{E_\Delta}, (X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (P_x^\beta)_{x\in E_\Delta})$ *is a Hunt process* associated with $\mathcal{E}^\beta$, *i.e., for all* $t > 0$ *and any (m-version of)* $u \in L^2(E;m)$, $x \mapsto \int u(X_t^\beta) dP_x^\beta$ *is an m-version of* $T_t^\beta u$.

Indeed, the fact that $I\!M^\beta$ is a Hunt process can be checked by a routine argument following [G75] (see e.g. Section 4 below). The association of $\mathcal{E}^\beta$ and $I\!M^\beta$ is an easy consequence of (3.2), (3.3) and (3.6).

Remark For a general Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ (not necessarily quasi-regular) on $L^2(E;m)$, and for an arbitrary $\beta > 0$ one can always construct a kernel βR_β on $(E_\Delta, \mathcal{B}(E_\Delta))$ sucht that $R_\beta f$ is an m-version of $G_\beta f$ for $f \in L^2(E;m)$. Hence one can always construct a Hunt process $I\!M^\beta$ associated with $\mathcal{E}^\beta$ as above.

Now we want to prove the uniform tightness of (X_t^β), $\beta \in \mathbb{Q}_+^*$. To this end we need to embed Y_Δ into another space $\overline{E}$ which is quasi-homeomorphic to E_Δ in the sense of [CMR94].

Let J be as specified in Lemma 2.8 and let $J = \{u_n | n \in \mathbb{N}\}$. We set $g_n = R_1 u_n,\ n \in \mathbb{N}$. Define for $x, y \in Y_\Delta$

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} |g_n(x) - g_n(y)| \wedge 1 \ . \tag{3.7}$$

It is easy to check from Lemmas 2.7, 2.8 that $\{g_n | n \in \mathbb{N}\}$ separates the points of Y_Δ. Hence, ρ is a metric on Y_Δ. Let $\overline{E}$ be the completion of Y_Δ with respect to ρ. Then $\overline{E}$ is a compact metric space. We extend the kernel $(R_\alpha)_{\alpha \in \mathbb{Q}_+^*}$ to the space $\overline{E}$ by setting for $\alpha \in \mathbb{Q}_+^*$, $A \in \mathcal{B}(\overline{E})$,

$$R_\alpha(x, A) = \begin{cases} R_\alpha(x, A \cap Y_\Delta)\ , & x \in Y_\Delta\ , \\ \dfrac{1}{\alpha}\ , & x \in \overline{E} \setminus Y_\Delta \end{cases} \tag{3.8}$$

Let $(X_t^\beta)_{t \geq 0}$ be defined by (3.1). Then (X_t^β) can be regarded as a cadlag process with state space $\overline{E}$. We use the same notation as before: P_x^β denotes the law of (X_t^β) in $D_{\overline{E}}[0,\infty)$ with initial distribution δ_x. Each g_n in (3.7) is uniformly continuous w.r.t. ρ and hence extends uniquely to a continuous function on $\overline{E}$ which we denote again by g_n.

Theorem 3.2 *$\{P_x^\beta | \beta \in \mathbb{Q}_+^*\}$ is tight on $D_{\overline{E}}[0,\infty)$ for any $x \in \overline{E}$.*

Proof. (cf. [EK86, Proof of Theorem 2.5 in Chap. IV]) Since $\{g_n | n \in \mathbb{N}\}$ separates the points of $\overline{E}$, we can apply [EK86, Ch. III, Theorem 9.1] (due to the same arguments as in the proof of [EK86, Ch. III, Corollary 9.2]) to prove the assertion once we have shown that for any finite collection $\{g_k | 1 \leq k \leq N\}$, the laws of $(g_1(X^\beta), g_2(X^\beta), \ldots, g_N(X^\beta))_{\beta \in \mathbb{Q}_+^*}$ are uniformly tight on $\Omega_{\mathbb{R}^N}$. Here (X_t^β) is constructed in the manner of (3.1) but with βR_β extended by (3.8) and with initial distribution δ_x. Since $g_i \in D(L^\beta)$, it follows that for $1 \leq i \leq N$

$$g_i(X_t^\beta) - g_i(X_0^\beta) = M_t^{\beta,i} + \int_0^t L^\beta g_i(X_s^\beta) ds\ ,\ t \geq 0\ , \tag{3.9}$$

where $(M_t^{\beta,i})_{t \geq 0}$ is an $(\mathcal{F}_t^\beta)$-martingale. Note that by (3.8), (3.4) and Lemma 2.8 $L^\beta g_i(x) = I_{Y_\Delta} \beta R_\beta (g_i - u_i)(x)$. Therefore for any $T > 0$, using the contraction property of βR_β we have for all $1 \le i \le N$

$$
\begin{aligned}
&\sup_{\beta\in\mathbb{Q}_+^*} \|L^\beta(g_i)\|_\infty = \sup_{\beta\in\mathbb{Q}_+^*} \|I_{Y_\Delta}\beta R_\beta(g_i-u_i)\|_\infty \\
&\le \|I_{Y_\Delta}(g_i-u_i)\|_\infty < +\infty \ . \qquad (3.10)
\end{aligned}
$$

This together with Theorem 9.4 in [EK86, Chap. III] gives the (uniform) tightness of the laws of $(g_1(X_{t_1}^\beta), g_2(X_{t_2}^\beta), \dots, g_N(X_{t_N}^\beta))_{\beta\in\mathbb{Q}_+^*}$ and the proof is completed. $\square$

Below for a Borel subset $S \subset Y$, we shall write S_Δ for $S \cup \{\Delta\}$. Except otherwise stated the topology of S_Δ is always meant to be the one induced by the metric ρ. Note that the ρ-topology and the original topology generate the same Borel σ-algebra on S_Δ.

The rest of this section is devoted to the following key theorem.

Theorem 3.3 *There exists a Borel subset $Z \subset Y$ and a Borel subset $\Omega \subset D_{\overline{E}}[0,\infty)$ with the following properties:*

(i) *$E \setminus Z$ is strictly $\mathcal{E}$-exceptional.*

(ii) $R_\alpha(x, \overline{E} \setminus Z_\Delta) = 0$, $\forall x \in Z_\Delta$, $\alpha \in \mathbb{Q}_+^*$.

(iii) *If $\omega \in \Omega$, then $\omega_t, \omega_{t-} \in Z_\Delta$ for all $t \ge 0$. Moreover, each $\omega \in \Omega$ is cadlag in the original topology of Y_Δ and $\omega_{t-}^0 = \omega_{t-}$ for all $t > 0$, where ω_{t-}^0 denotes the left limit in the original topology.*

(iv) *If $x \in Z_\Delta$ and P_x is a weak limit of some sequence $(P_x^{\beta_j})_{j\in\mathbb{N}}$ with $\beta_j \in \mathbb{Q}_+^*$, $\beta_j \uparrow \infty$, then $P_x[\Omega] = 1$.*

The proof of Theorem 3.3 relies on several lemmas which may be of their own interest.

Lemma 3.4 *There exists a strictly $\mathcal{E}$-quasi-continuous 1-excessive function $h \in D(\mathcal{E})$ such that $0 < h \le 1$ pointwise on Y, where Y is as speciefied in Lemma 2.7.*

Proof. Let $\{h_j | j \ge 1\}$ be the countable family J_0 specified in Lemma 2.7. Since J_0 separates the points of Y_Δ and by our convention $h_j(\Delta) = 0$ for all j, for each $x \in Y$ there exists at least one h_j such that $h_j(x) > 0$. We now define

$$
h := \sum_{j\ge1} 2^{-j}(1 + \|h_j\|_\infty + \|h_j\|_{\tilde{\mathcal{E}}_1^{\frac12}})^{-1} h_j
$$

Then h is as desired. $\square$

We now fix a function $\varphi \in L^1(E;m) \cup L^2(E;m)$, $\varphi > 0$. Set $g = \hat{G}_1\varphi$. It is known (cf. [MR92, V.2.4]) that for every open set $U \subset E$, there exists a function $e_U \in L^\infty(E;m)$ such that

$$\mathrm{Cap}_{1,g}(U) = \int_E e_U \, \varphi \, dm \ . \tag{3.11}$$

Lemma 3.5 *Let $U_n \subset E$, $n \geq 1$ be a decreasing sequence of open sets. If $\mathrm{Cap}_{1,g}(U_n) \longrightarrow 0$, as $n \to \infty$, then we can find m-versions e_n of e_{U_n} such that*

(i) $e_n \geq 1$, *strictly $\mathcal{E}$-q.e. on U_n for $n \geq 1$.*

(ii) $\alpha \tilde{R}_{\alpha+1}(e_n) \leq e_n$, *strictly $\mathcal{E}$-q.e. for $\alpha \in \mathbb{Q}_+^*$, $n \geq 1$.*

(iii) $e_n \downarrow 0$, *strictly $\mathcal{E}$-q.e. as $n \to \infty$.*

Proof. Let h be the 1-excessive function specified in Lemma 3.4. In what follows for simplicity we identify a function in $D(\mathcal{E})$ with (one of) its strictly $\mathcal{E}$-quasi-continuous m-version(s). Let $U \subset E$ open. We first prove that there exists an m-version $e_U \in L^\infty(E;m)$ satisfying (3.11) such that

$$e_U \geq 1 \ , \ \text{strictly } \mathcal{E}\text{-q.e. on } U \tag{3.12}$$

$$\alpha \tilde{R}_{\alpha+1}(e_U) \leq e_U \ , \ \text{strictly } \mathcal{E}\text{-q.e.}, \ \alpha \in \mathbb{Q}_+^* \tag{3.13}$$

and $e_U \wedge h$ is strictly $\mathcal{E}$-quasi-continuous. Set $S_U := \{u_U | u \in S, \ u \leq 1\}$, where S is as defined in Section 2.

According to the proof of Lemma 2.4 in [MR92, Ch. V] and since S is upper directed (cf. [MR92, p. 155]), we can take $f_n \in S$, $f_n \leq 1$, $n \in \mathbb{N}$, $(f_n)_U \uparrow$ such that e_U can be choosen as

$$e_U = \lim_{n \to \infty} (f_n \vee u_n)_U \ , \tag{3.14}$$

where $u_n = (nh) \wedge 1$. We claim that e_U is the desired function. Since $(f_n \vee u_n)_U \geq u_n$ strictly $\mathcal{E}$-q.e. on U and $u_n \uparrow 1$, (3.12) follows.

Since $(f_n \vee u_n)_U$ is 1-excessive and strictly $\mathcal{E}$-quasi-continuous, we have

$$\alpha \tilde{R}_{\alpha+1}((f_n \vee u_n)_U) \leq (f_n \vee u_n)_U \ , \text{strictly } \mathcal{E}\text{-q.e.}$$

Letting $n \to \infty$, we get (3.13)

Since $(f_n \vee u_n)_U \wedge h(x)$ is 1-excessive, it follows from [MR92, III. 1.2(iii)] that

$$\mathcal{E}_1\left((f_n \vee u_n)_U \wedge h\ ,\ (f_n \vee u_n)_U \wedge h\right) \le K^2\ \mathcal{E}_1(h,h)\ ,$$

where K is the constant from the sector condition satisfied by $(\mathcal{E}, D(\mathcal{E}))$ (see [MR92, I.(2.3)]). Since $(f_n \vee u_n)_U \wedge h \uparrow e_U \wedge h$ in $L^2(E;m)$, we can apply [MR92, I. 2.12 and III. 3.5] to conclude that $e_U \wedge h$ is strictly $\mathcal{E}$-quasi-continuous and $e_U \wedge h \in D(\mathcal{E})$ with

$$\mathcal{E}_1(e_U \wedge h, e_U \wedge h) \le K^2 \mathcal{E}_1(h,h)\ . \tag{3.15}$$

Now we can easily complete the proof of the lemma as follows.

For $n \in \mathbb{N}$ let e_n be the m-version of e_{U_n} satisfying (3.11)–(3.13), constructed above with U replaced by U_n. By (3.11) and the fact that $\mathrm{Cap}_{1,g}(U_n) \downarrow 0$, we have that $e_n \downarrow 0$ m-a.e. and hence, $e_n \wedge h \downarrow 0$ m-a.e.

On the other hand, from (3.15) we have that

$$\sup_n \mathcal{E}_1(e_n \wedge h\,,\, e_n \wedge h) \le C\ \mathcal{E}_1(h,h)\ .$$

Thus again by [MR92, III. 3.5], the Cesaro mean $\omega_n = \frac{1}{n}\sum_{j=1}^n e_{n_j} \wedge h$ of some subsequence $(e_{n_j} \wedge h)_{j\ge 1}$ converges to zero strictly $\mathcal{E}$-quasi-uniformly. But $(e_n \wedge h)_{n\in\mathbb{N}}$ is strictly $\mathcal{E}$-q.e. decreasing, thus $e_n \wedge h \downarrow 0$ strictly $\mathcal{E}$-q.e. Hence, $e_n \downarrow 0$ strictly $\mathcal{E}$-q.e. This completes the proof. □

Lemma 3.6 *In the situation of Lemma 3.5 there exists $S \in \mathcal{B}(E)$, $S \subset Y$ such that $E \setminus S$ is strictly $\mathcal{E}$-exceptional and the following holds:*

(i) $\tilde{R}_\alpha(x, Y - S) = 0$, *for all* $x \in S, \alpha \in \mathbb{Q}_+^*$.

(ii) $e_n(x) \ge 1$, *for* $x \in S \cap U_n$, $n \ge 1$.

(iii) $\alpha \tilde{R}_{\alpha+1}(e_n)(x) \le e_n(x)$, *for all* $x \in S, \alpha \in \mathbb{Q}_+^*, n \ge 1$.

(iv) $e_n(x) \downarrow 0$, *for all* $x \in S$.

Proof. The proof is just a modification of the proof of IV. 3.11 in [MR92]. Indeed, by Lemma 3.5, there exists a Borel set $S_1 \subset Y$ such that assertions (ii) – (iv) hold pointwise on S_1 and $Y \setminus S_1$ is strictly $\mathcal{E}$-exceptional. Thus we can find a Borel set $S_2 \subset S_1$ such that $\tilde{R}_\alpha(x, Y - S_1) = 0$ for all $x \in S_2$, $\alpha \in \mathbb{Q}_+^*$

and $E \setminus S_2$ is strictly $\mathcal{E}$-exceptional. Repeating this argument, we get a decreasing sequence $(S_n)_{n\geq 1}$ such that $E \setminus S_n$ is strictly $\mathcal{E}$-exceptional and $\tilde{R}_\alpha(x, Y - S_n) = 0$ for all $x \in S_{n+1}$, $\alpha \in \mathbb{Q}_+^*$. Clearly, $S := \cap_{n\geq 1} S_n$ is the desired set. $\square$

Lemma 3.7 *Let $S \in \mathcal{B}(E)$, $S \subset Y$ such that Lemma 3.6 (i) holds. Then*

$$P_x^\beta[X_t \in S_\Delta,\ X_{t-} \in S_\Delta,\ \forall t \geq 0] = 1 \quad \forall x \in S_\Delta\ . \tag{3.16}$$

Proof. Lemma 3.6 (i) implies that $(\beta R_\beta)^n(x, \overline{E} \setminus S_\Delta) = 0$, $\forall x \in S_\Delta$, $\beta \in \mathbb{Q}_+^*$, $n \geq 1$. Therefore, if $Y^\beta(k)$, $k = 1, 2, \ldots$ is a Markov chain starting from some $x \in S_\Delta$ with transition function βR_β, then

$$P[Y^\beta(k) \in \overline{E} \setminus S_\Delta \text{ for some } k] = 0\ .$$

Clearly, this implies

$$P[Y^\beta(\Pi_t^\beta) \in \overline{E} \setminus S_\Delta \text{ for some } t \geq 0] = 0\ ,$$

which in turn implies (3.16). $\square$

Recall that by our convention any function f on E is extended to E_Δ by setting $f(\Delta) = 0$. In the next two lemmas we consider the situation of Lemma 3.6.

Lemma 3.8 *Let $\beta \in \mathbb{Q}_+^*$, $\beta \geq 2$. $n \geq 1$. Then e_n is a (P_t^β)-2-excessive function on S_Δ, i.e., $e^{-2t} P_t^\beta e_n(x) \leq e_n(x)$ and $\lim_{t\to 0} e^{-2t} P_t^\beta e_n(x) = e_n(x)$ $\forall x \in S_\Delta$.*

Proof. By Lemma 3.6 (i) and (iii), (2.7), and induction it is easy to see that $((\beta - 1)R_\beta)^k(e_n)(x) \leq e_n(x)$, for all $x \in S_\Delta$. Hence, for $x \in S_\Delta$

$$\begin{aligned} P_t^\beta e_n(x) &= e^{-\beta t} \sum_{k=0}^\infty \frac{(\beta t)^k}{k!} (\beta R_\beta)^k e_n(x) \\ &= e^{-\beta t} \sum_{k=0}^\infty \frac{(\beta t)^k}{k!} \left(\frac{\beta}{\beta - 1}\right)^k ((\beta - 1)R_\beta)^k e_n(x) \\ &\leq e^{-\beta t} \sum_{k=0}^\infty \frac{((\frac{\beta^2}{\beta-1})t)^k}{k!} e_n(x) = e^{(1+\frac{1}{\beta-1})t} e_n(x)\ . \end{aligned}$$

This gives

$$e^{-2t} P_t^\beta e_n(x) \le e_n(x) \ , \ x \in S_\Delta \ .$$

But $\lim_{t\to 0} e^{-2t} P_t^\beta f(x) = f(x)$ holds for all $x \in \overline{E}$ and $f \in \mathcal{B}_b(\overline{E})$ and the proof is completed. $\square$

Define for $n \in \mathbb{N}$ the stopping time,

$$\tau_n := \inf\{t \ge 0 | X_t \in U_n\} \ .$$

Lemma 3.9 *Let $\beta \in \mathbb{Q}_+^*$, $\beta \ge 2$, and $M^\beta := (D_{\overline{E}}[0,\infty),\ (X_t)_{t\ge 0},\ (P_x^\beta)_{x\in\overline{E}})$ be the canonical realization of the Markov process (X_t^β). Then*

$$E_x^\beta[e^{-2\tau_n}] \le e_n(x), \quad x \in S_\Delta \,, \tag{3.17}$$

where E_x^β denotes expectation w.r.t. P_x^β.

Proof. Since by (3.16) S_Δ is an invariant set of M^β, the restriction $M^\beta_{S_\Delta}$ of M^β to S_Δ is still a Hunt process. Applying Lemma 3.8 we get

$$E_x^\beta[e^{-2\tau_n} e_n(X_{\tau_n})] \le e_n(x), \quad x \in S_\Delta \ . \tag{3.18}$$

Since obviously for every $x \in S_\Delta$

$$X_{\tau_n} \in U_n \quad P_x^\beta - a.s. \text{ on } \{\tau_n < \infty\} \ ,$$

we have by Lemma 3.6 (ii) that for all $x \in S_\Delta$

$$e^{-2\tau_n} \le e^{-2\tau_n} e_n(X_{\tau_n}) \quad P_x^\beta - a.s.$$

Therefore, (3.17) follows from (3.18). $\square$

Proof of Theorem 3.3 Take a strict $\mathcal{E}$-nest $(F_k^{(1)})_{k\in\mathbb{N}}$ such that $J_0 \subset C_\infty(\{F_k^{(1)}\})$, $F_k^{(1)} \cup \{\Delta\}$ is compact, and $\bigcup_{k\ge 1} F_k^{(1)} \subset Y$. Let $U_k := E \setminus F_k^{(1)}$ and $\tau_k := \inf\{t \ge 0 \mid X_t \in U_k\}$. We can find a subset $S^{(1)} \in \mathcal{B}(E)$ satisfying Lemma 3.6 (i)–(iv). Without loss of generality we may assume that $S^{(1)} \subset \bigcup_{k\ge 1} F_k^{(1)}$. Fix any $T > 0$, $\beta \in \mathbb{Q}_+^*$, $\beta \ge 2$, $k \in \mathbb{N}$, and $x \in S_\Delta^{(1)}$. By Lemma 3.9,

$$\begin{aligned} P_x^\beta[\tau_k < T] &\le E_x^\beta[e^{-2\tau_k}] e^{2T} \\ &\le e^{2T} e_k(x) \ . \end{aligned} \tag{3.19}$$

Since the trace topology of $\overline{E}$ on $F_k^{(1)} \cup \{\Delta\}$ is the same as the original one, $B_k^T := \{\omega \in D_{\overline{E}}[0,\infty) | \omega(t) \in F_k^{(1)} \cup \{\Delta\},$ for all $t < T\}$ is a closed subset of $D_{\overline{E}}[0,\infty)$. Thus, if P_x is a weak limit of some sequence $(P_x^{\beta_j})_{j\in\mathbb{N}}$ with $\beta_j \uparrow \infty$, $\beta_j \in \mathbb{Q}_+^*$, then

$$\begin{aligned} P_x[B_k^T] &\geq \overline{\lim_{j\to\infty}} P_x^{\beta_j}[B_k^T] \geq \overline{\lim_{j\to\infty}} P_x^{\beta_j}[\tau_k \geq T] \\ &= \overline{\lim_{j\to\infty}} (1 - P_x^{\beta_j}[\tau_k < T]) \geq 1 - e^{2T} e_k(x) \ . \end{aligned} \tag{3.20}$$

By Lemma 3.6 (iv) it follows that

$$P_x[\bigcup_k B_k^T] \geq \overline{\lim_{k\to\infty}} P_x[B_k^T] \geq \overline{\lim_{k\to\infty}} (1 - e^{2T} e_k(x)) = 1 \ .$$

Let $\Omega_1 := \bigcap_{N\geq 1} \bigcup_{k\geq 1} B_k^N$. Then $P_x[\Omega_1] = 1$ for $x \in S^{(1)}$ and Ω_1 satisfies Theorem 3.3 (iii) with Z_Δ replaced by $\bigcup_{k\geq 1} F_k^{(1)} \cup \{\Delta\}$. We now take another strict $\mathcal{E}$-nest $(F_k^{(2)})_{k\geq 1}$ such that $F_k^{(2)} \subset F_k^{(1)}$ for each k and $\bigcup_{k\geq 1} F_k^{(2)} \subset S^{(1)}$. Repeating the above argument we get $S^{(2)} \subset \bigcup_{k\geq 1} F_k^{(2)}$ and $\Omega_2 \subset \Omega_1$, satisfying the same property as above. By continuing this procedure we obtain the following sequences of objects: strict $\mathcal{E}$-nests $(F_k^{(n)})_{k\geq 1}$, Borel sets $S^{(n)} \subset E$ such that Theorem 3.3 (ii) holds with Z_Δ replaced by $S_\Delta^{(n)}$ and

$$Y \supset \bigcup_{k\geq 1} F_k^{(1)} \supset S^{(1)} \supset \ldots \supset \bigcup_{k\geq 1} F_k^{(n)} \supset S^{(n)} \supset \ldots \ ,$$

and finally Borel sets $\Omega_n \subset D_{\overline{E}}[0,\infty)$ such that

$$D_{\overline{E}}[0,\infty) \supset \Omega_1 \supset \ldots \supset \Omega_n \supset \ldots \ .$$

Ω_n satisfies Theorem 3.3 (iii) with Z_Δ replaced by $\bigcup_{k\geq 1} F_k^{(n)} \cup \{\Delta\}$,and satisfies Theorem 3.3 (iv) with Z_Δ replaced by $S_\Delta^{(n)}$. We now define $\Omega := \bigcap_{n\geq 1} \Omega_n$, $Z := \bigcap_{n\geq 1} S^{(n)} = \bigcap_{n\geq 1} (\bigcup_k F_k^{(n)})$. Then Z and Ω satisfy Theorem 3.3 (i) - (iv). $\square$

4 Hunt processes associated with $(\mathcal{E}, D(\mathcal{E}))$

All the assumptions and notations are the same as in the previous section. Let $\{P_x^\beta | \beta \in \mathbb{Q}_+^*\}$ be as specified in Theorem 3.2.

Lemma 4.1 *If we define for $\alpha \in \mathbb{Q}_+^*$, $\beta \in \mathbb{Q}_+^*$,*

$$R_\alpha^\beta f(x) := E_x^\beta \left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt \right] \ , \quad f \in \mathcal{B}_b(\overline{E}),\ x \in \overline{E}.$$

Then

$$R_\alpha^\beta f = \left(\frac{\beta}{\alpha+\beta} \right)^2 R_{\frac{\alpha\beta}{\alpha+\beta}} f + \frac{1}{\alpha+\beta} f \ . \tag{4.1}$$

Proof. Note that (3.2) and (3.4) hold also in the space $\mathcal{B}_b(\overline{E})$. Therefore, if R_α^β is given by (4.1), then one can directly check that

$$\left(R_\alpha^\beta (\alpha - L^\beta) f \right) = \left((\alpha - L^\beta) R_\alpha^\beta \right) = f \ ,$$

proving the lemma. $\square$

Lemma 4.2 *Let $x \in \overline{E}$ and let P_x be a weak limit of a subsequence $(P_x^{\beta_j})_{j\geq 1}$ with $\beta_j \uparrow \infty$, $\beta_j \in \mathbb{Q}_+^*$. Define the kernel*

$$P_t(x, f) := P_t f(x) := E_x[f(X_t)] \ \forall f \in \mathcal{B}_b(\overline{E}). \tag{4.2}$$

Then

$$\int_0^\infty e^{-\alpha t} P_t f(x)\, dt = R_\alpha f(x) \ \forall f \in \mathcal{B}_b(\overline{E}),\ \alpha \in \mathbb{Q}_+^* \ . \tag{4.3}$$

In particular, the kernels P_t, $t \geq 0$, are independent of the subsequence $(P_x^{\beta_j})_{j\geq 1}$.

Proof. Since $P_x^{\beta_j} \longrightarrow P_x$ weakly in $D_{\overline{E}}[0,\infty)$, we have by [EK86, Chap. III, Lemma 7.7 and Theorem 7.8]

$$E_x^{\beta_j} \left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt \right] = \int_0^\infty e^{-\alpha t} E_x^{\beta_j} [f(X_t)]\, dt \underset{j\to\infty}{\longrightarrow} \int_0^\infty e^{-\alpha t} P_t f(x)\, dt$$

for any $f \in C_b(\overline{E})$ and $\alpha > 0$. But by Lemma 4.1

$$\begin{aligned} E_x^{\beta_j} \left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt \right] &= R_\alpha^{\beta_j} f(x) = \left(\frac{\beta_j}{\alpha+\beta_j} \right)^2 R_{\frac{\alpha\beta_j}{\alpha+\beta_j}} f(x) + \frac{1}{\alpha+\beta_j} f(x) \\ &\underset{j\to\infty}{\longrightarrow} R_\alpha f(x) \end{aligned} \tag{4.4}$$

for all $f \in \mathcal{B}_b(\overline{E})$ (due to the resolvent equation). Hence, (4.3) holds for all $f \in C_b(\overline{E})$. The usual monotone class argument implies that (4.3) holds also for all $f \in \mathcal{B}_b(\overline{E})$. The last assertion is derived from (4.3) by the right continuity of $P_t f(x)$ in t for $f \in C_b(\overline{E})$ and the uniqueness of the Laplace transform. $\square$

Let Z be as specified in Theorem 3.3.

Theorem 4.3 *For every $x \in Z_\Delta$ the uniformly tight family $\{P_x^\beta | \beta \in \mathbb{Q}_+^*\}$ has a unique limit P_x for $\beta \uparrow \infty$. The process $(D_{\overline{E}}[0,\infty), (X_t)_{t\geq 0}, (P_x)_{x\in Z_\Delta})$ is a Markov process with the transition semigroup $(P_t)_{t\geq 0}$ determined by (4.3). Moreover, $P_x[X_t \in Z_\Delta,\ X_{t-} \in Z_\Delta$ for all $t \geq 0] = 1$ for all $x \in Z_\Delta$.*

Proof. The last assertion follows from Theorem 3.3. In view of the last assertion and Lemma 4.2, we only need to show that if $x \in Z_\Delta$ and P_x is a weak limit for some sequence $(P_x^{\beta_j})_{j\geq 1}$ for $\beta_j \uparrow \infty$, then

$$\begin{aligned} &E_x\Big[f_1(X_{t_1}) f_2(X_{t_1+t_2}) \ldots f_n(X_{t_1+t_2+\ldots+t_n})\Big] \\ &= P_{t_1}[f_1\, P_{t_2}[f_2 \ldots P_{t_n}[f_n] \ldots](x) \end{aligned}$$

for any $n \geq 1$, $t_1, t_2, \ldots, t_n \geq 0$ and $f_1, f_2, \ldots, f_n \in \mathcal{B}_b(Z_\Delta)$.

By induction, the above formula trivially follows from

$$\begin{aligned} &E_x\Big[f_1(X_{t_1}) f_2(X_{t_1+t_2}) \ldots f_n(X_{t_1+t_2+\ldots+t_n})\Big] \\ &= E_x\Big[f_1(X_{t_1}) f_2(X_{t_1+t_2}) \ldots f_{n-1}(X_{t_1+t_2+\ldots+t_{n-1}})\, P_{t_n}(f_n)(X_{t_1+t_2+\ldots+t_{n-1}})\Big] \end{aligned} \tag{4.5}$$

So, it remains to prove (4.5). To this end, we assume first that $f_1, f_2, \ldots, f_{n-1} \in C_b(Z_\Delta)$ and $f_n = R_\alpha R_1 u$ for some $\alpha \in \mathbb{Q}_+^*$, $\alpha > 1$, $u \in J$ (cf. (2.8)). In this case

$$P_t^\beta f_n = e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} (\beta R_\beta)^k R_\alpha R_1 u \in C_b(Z_\Delta \times [0,T]) \tag{4.6}$$

for $\beta \in \mathbb{Q}_+^*$ and any $T > 0$, and for $\beta_1,\ \beta_2 \in \mathbb{Q}_+^*$

$$\begin{aligned} P_t^{\beta_1} f_n - P_t^{\beta_2} f_n &= \int_0^t \frac{d}{ds}(P_s^{\beta_1} P_{t-s}^{\beta_2} f_n) ds \\ & \int_0^t P_s^{\beta_1}\, (P_{t-s}^{\beta_2})(L^{\beta_1} - L^{\beta_2}) f_n \, ds \ . \end{aligned} \tag{4.7}$$

By a simple calculation

$$(L^{\beta_1} - L^{\beta_2}) f_n \ = \ R_{\beta_1} w - R_{\beta_2} w \ , \tag{4.8}$$

where $w := R_1(\alpha R_\alpha u - u) - (\alpha R_\alpha u - u)$. Consequently, by (4.7)

$$\begin{aligned}\sup_{t\le T}\sup_x |P_t^{\beta_1} f_n(x) - P_t^{\beta_2} f_n(x)| &\le T \sup_x |L^{\beta_1} f_n(x) - L^{\beta_2} f_n(x)| \\ &\le T\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)\|w\|_\infty \ . \end{aligned}\tag{4.9}$$

This together with (4.4) shows that

$$\sup_{t\le T}\sup_x |P_t^{\beta} f_n(x) - P_t f_n(x)| \longrightarrow 0 \tag{4.10}$$

as $\beta \to +\infty$.

In particular, $P_t f_n(x)$ is jointly continuous in (t,x). Since $P_x^{\beta_j} \to P_x$ weakly, by [EK86, Chap. III, Lemma 7.7 and Theorem 7.8] and Lebesgue's dominated convergence theorem we have that

$$\begin{aligned}&\lim_{j\to+\infty} E_x^{\beta_j}\Bigg[\int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} f_1(X_{t_1}) \dots f_{n-1}(X_{t_1+\dots+t_{n-1}}) \\ &\qquad P_{t_n}(f_n)(X_{t_1+\dots+t_{n-1}})\, dt_1 \, \dots dt_n\Bigg] \\ &= E_x\Bigg[\int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} f_1(X_{t_1}) \dots f_{n-1}(X_{t_1+\dots+t_{n-1}}) \\ &\qquad P_{t_n}(f_n)(X_{t_1+\dots+t_{n-1}})\, dt_1 \dots dt_n\Bigg]\end{aligned}\tag{4.11}$$

Because of (4.10), it follows that

$$\begin{aligned}&\lim_{j\to+\infty} E_x^{\beta_j}\Bigg[\int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} f_1(X_{t_1}) \dots f_{n-1}(X_{t_1+\dots+t_{n-1}}) \\ &\qquad P_{t_n}(f_n)(X_{t_1+\dots+t_{n-1}})\, dt_1 \dots dt_n\Bigg] \\ &= \lim_{j\to+\infty} E_x^{\beta_j}\Bigg[\int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} f_1(X_{t_1}) \dots f_{n-1}(X_{t_1+\dots+t_{n-1}}) \\ &\qquad P_{t_n}^{\beta_j}(f_n)(X_{t_1+\dots+t_{n-1}})\, dt_1 \dots dt_n\Bigg]\end{aligned}$$

$$= \lim_{j\to+\infty} E_x^{\beta_j}\left[\int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} f_1(X_{t_1}) \dots f_n(X_{t_1+\dots+t_n})\, dt_1 \dots dt_n\right]$$

$$= E_x\left[\int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} f_1(X_{t_1}) \dots f_n(X_{t_1+\dots+t_n})\, dt_1 \dots dt_n\right], \tag{4.12}$$

where we used the Markov property of $P_x^{\beta_j}$ in the second to last step. (4.11) and (4.12) yield

$$\begin{aligned}
&\int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} E_x\left[f_1(X_{t_1}) \dots f_n(X_{t_1+\dots+t_n})\right] dt_1 \dots dt_n \\
&= \int_0^\infty \dots \int_0^\infty e^{-\alpha_1 t_1 - \dots - \alpha_n t_n} E_x\Big[f_1(X_{t_1}) \dots f_{n-1}(X_{t_1+\dots+t_{n-1}}) \\
&\qquad P_{t_n}(f_n)(X_{t_1+\dots+t_{n-1}})\Big]\, dt_1 \dots dt_n\ . \qquad (4.13)
\end{aligned}$$

Since the above integrands are right continuous, (4.13) implies (4.5) for such $f_1, \dots, f_n$. Applying the monotone convergence theorem in connection with Lemma 2.8 twice and also the usual monotone class argument, we conclude that (4.5) holds for all $f_1, \dots, f_n \in \mathcal{B}_b(Z_\Delta)$. □

In what follows let $(P_x)_{x \in Z_\Delta}$ be as in Theorem 4.3. Let Ω be specified by Theorem 3.3. Since $P_x(\Omega) = 1$ for all $x \in Z_\Delta$, we may restrict P_x and the coordinate process $(X_t)_{t \geq 0}$ to Ω. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of $(X_t)_{t \geq 0}$.

Theorem 4.4 $\mathbb{M} := (\Omega, (X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, (P_x)_{x \in Z_\Delta})$ *is a Hunt process with respect to both the ρ-topology and the original topology.*

Proof. The ρ-topology and the original topology generate the same Borel sets. Hence, by virtue of Theorem 3.3 (iii), $\mathbb{M}$ is a Hunt process in the original topology if and only if so is it in the ρ-topology. Thus we discuss only the ρ-topology case. In this case $R_\alpha f$ is uniformly continuous on Z_Δ for each $\alpha \in \mathbb{Q}_+^*$ and $f \in J$. Therefore, if we set

$$\begin{aligned}
\mathcal{H} := \{f \in \mathcal{B}_b(Z_\Delta)\ :\ & P_x[t \mapsto R_\alpha f(X_t) \text{ is right continuous on } [0,\infty)] = 1 \\
& \forall \alpha \in \mathbb{Q}_+^*,\ \forall x \in Z_\Delta\}\ ,
\end{aligned}$$

then $\mathcal{H} \supset J$. Using [DeMe82, Theorem VI. 18] one can easily check that $\mathcal{H}$ is a linear space such that $f_n \in \mathcal{H}$, $f_n \uparrow f$ bounded, implies $f \in \mathcal{H}$. Therefore, by a monotone class argument we see that $\mathcal{H}$ contains $\mathcal{B}_b(Z_\Delta)$. Now, the strong Markov property of $\mathbb{M}$ follows from [Sh88, (7.4)]. It remains to show the quasi-left-continuity of $(X_t)_{t\geq 0}$. To this end let $(\tau_n)_{n\geq 1}$ be an increasing sequence of $(\mathcal{F}_t)$-stopping times with limit τ. Assume τ is bounded. Define $V^{(\omega)} := \lim_{n\to\infty} X_{\tau_n}(\omega)$. Then following the argument in the proof of [MR92, Ch. IV. 3.21] one can show that

$$E_x[g(V)R_\alpha f(X_\tau)] = E_x[g(V)R_\alpha f(V)]$$

for all $g \in C_b(Z_\Delta)$ and $f \in J$. Consequently, using twice the monotone class argument we obtain that

$$E_x[h(V, X_\tau)] = E_x[h(V, V)]$$

for all $\mathcal{B}(Z_\Delta \times Z_\Delta)$–measurable bounded functions h. This fact is then enough to derive the quasi-left-continuity of $(X_t)_{t\geq 0}$. (See e.g. the argument in the proof of [MR92, Ch. IV. 3.21] for details.) Thus, the proof of Theorem 4.4 is complete. $\square$

Final remarks 4.5 (i) Let $\mathbb{M}$ be the trivial extension of $\mathbb{M}$ to E_Δ (i.e., each point $x \in E_\Delta \setminus S_\Delta$ is a trap for $\overline{\mathbb{M}}$; cf. [MR92, IV. 3.23]). Then one can easily derive that $\overline{\mathbb{M}}$ is a Hunt process which is unique up to the usual equivalence (cf. e.g. [MR92, IV. 6.3]), and by (4.3) $\overline{\mathbb{M}}$ is associated with $(\mathcal{E}, D(\mathcal{E}))$.
(ii) This paper has been written in the framework of Dirichlet forms in order to be able to refer to [MR92]. But the results easily extend to the more general case of Semi-Dirichlet forms as defined in [OMR95].
(iii) If $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular but not strictly quasi-regular, then one can not expect that (X_t^β) converges weakly to a process associated with $(\mathcal{E}, D(\mathcal{E}))$ because in this case the sample paths of the associated process of $(\mathcal{E}, D(\mathcal{E}))$ may fail to be in $D_{E_\Delta}[0,\infty)$ ($X_{\rho-}$ may not exist or may not be in E_Δ). Nevertheless, one can always make use of the *local compactification method* to obtain a regular Dirichlet form $(\mathcal{E}^\#, D(\mathcal{E}^\#))$ which is quasi-homeomorphic to $(\mathcal{E}, D(\mathcal{E}))$ (cf. [MR92, Chap. VI], [AMR92], [CMR94]). Thus the result of this paper applies to $(\mathcal{E}^\#, D(\mathcal{E}^\#))$ since any regular Dirichlet form is strictly quasi-regular ([MR92, V. 2.12]). In particular, the approach given in this paper gives a new way to construct the associated process of a quasi-regular Dirichlet form. This new construction is completely different from those described in [F80], [Si74], [MR92] respectively, and is in a sense "more explicit" because the approximating Markov chains are "more explicit".

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