A NOTE ON THE MODALITY OF PARABOLIC SUBGROUPS

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Abstract. For an algebraic group R acting morphically on an algebraic variety X the modality of the action, $\operatorname{mod}(R : X)$, is the maximal number of parameters on which a family of R-orbits on X depends upon.

Let G be a simple algebraic group defined over an algebraically closed field K of characteristic 0. Let P be a parabolic subgroup of G. Then P acts on its unipotent radical P_u via conjugation. The modality of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : P_u)$.

Let r and s be the semisimple rank of G and P respectively. We show that there is a quadratic polynomial f with rational coefficients such that the modality of P is at least $f(r-s)$. In particular, the modality of a Borel subgroup B of G grows at least quadratically with r. As a consequence, we obtain a finiteness result for algebraic groups from [8]: there is only a finite number of simple algebraic groups admitting parabolic subgroups of prescribed semisimple rank and prescribed modality. Combining our lower bounds with upper bounds from [6], we can compute the modality of Borel subgroups in some small rank cases. Here $\operatorname{mod} B$ agrees with our lower bounds.

1. Introduction

Let G be a reductive algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Let P be a parabolic subgroup of G. By P_u we denote the unipotent radical of P. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on P_u, $\operatorname{mod}(P : P_u)$, is the maximal number of parameters a family of P-orbits on P_u depends upon. Likewise for $\operatorname{mod}(P : \mathfrak{p}_u)$. Since the characteristic is zero, the exponential mapping is a P-equivariant morphism between $\mathfrak{p}_u$ and P_u. Thus we have $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$. The modality of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : P_u)$.

In [11, Theorem 3.1] it is shown that $\operatorname{mod} P$ is positive provided the nilpotent class of P_u is sufficiently large - at least 5 or 6, depending on the type of P. In the present paper we provide some explicit lower bounds for $\operatorname{mod} P$. These turn out to be quadratic polynomials in the difference of the semisimple ranks of G and P. We observe that the degree of these polynomials is optimal. In particular, the modality of a Borel subgroup of G grows quadratically with the rank of G. As a consequence, we obtain a finiteness result for simple algebraic groups admitting parabolic subgroups of a fixed modality and fixed semisimple rank from [8]. With the aid of these lower bounds for $\operatorname{mod} P$ we also obtain some refinements of the results from [11]. Combining our lower bounds with some upper bounds from [6], we can determine the modality of Borel subgroups in some small rank cases. It turnes out that in these instances $\operatorname{mod} B$ agrees with our lower bounds.

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2. Notation and Recollection

For an algebraic group R we denote its Lie algebra by $\operatorname{Lie} R$ or simply by the corresponding lower case Gothic letter $\mathfrak{r}$. The identity component of R is denoted by R^0 and its unipotent radical by R_u. The Lie algebra of R_u is denoted by $\mathfrak{r}_u$. Throughout the identity element of an algebraic group is denoted by e. For the rank and semisimple rank of R we write $\operatorname{rank} R$ and $\operatorname{rank}_{ss} R$ respectively.

The descending central series $\{C^i R \mid i \geq 0\}$ of R is defined as usual: $C^0 R := R$ and $C^{i+1} R := [C^i R, R]$ for $i \geq 0$. If $C^i R = \{e\}$ for some $i \geq 0$, then R is nilpotent. In that case the *length* of the descending central series of R, denoted by $\ell(R)$, is the smallest integer m such that $C^m R = \{e\}$, i.e., $\ell(R) = m$; equivalently, R is nilpotent of class m.

Recall from [9] that for an algebraic group R acting morphically on an algebraic variety X the *modality* of the action, $\operatorname{mod}(R : X)$, is defined as

$$\operatorname{mod}(R : X) = \max_{Z} \min_{z \in Z} \operatorname{codim}_Z R^0 \cdot z,$$

where Z runs through all irreducible R^0-invariant subvarieties of X. Clearly, $\operatorname{mod}(R : X) \geq \dim X - \dim R$. In case X is an irreducible variety let $K(X)^R$ denote the field of R-invariant rational functions on X. Since $\min_{x \in X} \operatorname{codim}_X R \cdot z = \operatorname{trdeg} K(X)^R$ (e.g. see [10, 2.3]), $\operatorname{mod}(R : X)$ measures the maximal number of parameters on which a family of R-orbits on X depends upon. The modality of the action of R on X is 0 precisely when R admits only a finite number of orbits on X. The notion of modality originates in the work of Arnold [1]; see also Vinberg [13] and [10, 5.2]. We call $\operatorname{mod}(P : P_u)$ the *modality* of P.

Let G be a connected reductive algebraic group, T a maximal torus in G and Ψ the set of roots of G with respect to T. Fix a set of simple roots Π of Ψ and let B be the Borel subgroup of G corresponding to Π. A parabolic subgroup of G containing B is called a standard parabolic subgroup. For a subset J of Π we denote by P_J the standard parabolic subgroup corresponding to J such that $P_\varnothing = B$.

As every parabolic subgroup P of G is conjugate to a standard one, and since $\ell(P_u)$ and $\operatorname{mod} P$ are invariant under conjugation by elements from G, we may if we wish assume that the parabolic subgroups under consideration are standard.

As a general reference for algebraic groups we cite Borel's book [2] and for information on root systems we refer the reader to Bourbaki [3]. The simple roots in a base of a root system of a simple algebraic group are indexed in accordance with *loc. cit.* (Planches I - IX).

We follow [5] by calling a closed connected subgroup H of G *regular* if the normalizer of H in G contains a maximal torus T of G. In that case the root spaces of $\mathfrak{h}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of H with respect to T, $\Psi(H)$, is a subset of Ψ, the root system of G relative to T, and is closed under addition in Ψ. If H is a reductive regular subgroup of G, then $\Psi(H)$ is a subsystem of Ψ. Conversely, by choosing a subsystem $\Psi(H)$ of Ψ, or equivalently, a set of simple roots $\Pi(H)$ for $\Psi(H)$, we can define a connected semisimple regular subgroup H of G whose root system is precisely $\Psi(H)$. Moreover, a set of simple roots $\Pi(H)$ for $\Psi(H)$ can always be chosen such that the Borel subgroup of H relative to $\Pi(H)$ is precisely $H \cap B$, where B is the Borel subgroup of G relative to Π. In particular, $\Psi^+(H) \subseteq \Psi^+$. Likewise, we speak of regular subalgebras.

Let P be a parabolic subgroup of G and let N be a normal subgroup of P contained in P_u. Then N is regular in G, since $T \subset P \subseteq N_G(N)$. Thus $\mathfrak{n} = \operatorname{Lie} N = \oplus_{\beta \in \Psi(N)} \mathfrak{g}_\beta$ and consequently, $N = \prod U_\beta$, where the product is taken over some fixed order in $\Psi(N)$. Further, as $\mathfrak{n}$ is invariant under the adjoint action of P, it is a regular ideal of $\mathfrak{p}$. Define

$$\mu(\mathfrak{n}) := 2 \dim \mathfrak{n} - \dim \mathfrak{p} - \dim [\mathfrak{n}, \mathfrak{n}].$$

The following lower bound appears in [8, Proposition 1].

Lemma 2.1. *Let P and N be as above. Then*

$$\operatorname{mod} P \geq \mu(\mathfrak{n}).$$

Proof. The action of P on $\mathfrak{n}$ induces an action of P on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ and this action factors through P/N, as N acts trivially on this space. Whence we have

$$\begin{aligned} \operatorname{mod} P \geq \operatorname{mod}(P : \mathfrak{n}) &\geq \operatorname{mod}(P : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) = \operatorname{mod}(P/N : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) \\ &\geq \dim \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] - \dim P/N = \mu(\mathfrak{n}), \end{aligned}$$

as desired.

Remark 2.2. If $\operatorname{mod} P > 0$, then there need not be a normal subgroup $N \subseteq P_u$ of P with $\mu(\mathfrak{n}) > 0$. For instance see [9, Remarks 3.5].

3. On the Modality of Borel subgroups

Throughout this section G is a simple algebraic group defined over an algebraically closed field K of arbitrary characteristic and B is a Borel subgroup of G. The principal result of this section is

Theorem 3.1. *Let G be a classical simple algebraic group and B a Borel subgroup of G. Let $r = \operatorname{rank} G$. There exists a quadratic polynomial $f \in \mathbb{Q}[t]$ such that*

$$\operatorname{mod} B \geq f(r).$$

That is, the modality of B grows quadratically with the rank of G. More specifically, depending on the type of G the polynomial f may be taken as in Table 1 below.

Type of G	$f(r)$
A_r	$\frac{1}{12}(r^2 - 4r)$
B_r	$\frac{1}{6}(r^2 - r - 2)$
C_r	$\frac{1}{6}(r^2 - r - 2)$
D_r	$\frac{1}{6}(r^2 - 2r - 5)$

Table 1: uniform lower bounds for the modality of Borel subgroups in classical groups.

Among the bounds in Table 1 the one for type A_r is minimal (for $r \geq 4$). Thus, we may formulate a uniform lower bound for $\operatorname{mod} B$ independent of the type of G:

Corollary 3.2. *Let G be a classical simple algebraic group and B a Borel subgroup of G. Let $r = \operatorname{rank} G$. Then*

$$\operatorname{mod} B \geq \frac{1}{12}(r^2 - 4r).$$

In view of Lemma 2.1, Theorem 3.1 is a consequence of

Proposition 3.3. *Let G be a classical simple algebraic group and B the standard Borel subgroup of G relative to some maximal torus T of G. Set $r = \operatorname{rank} G$. Let $\mathfrak{a}$ be the ideal of $\mathfrak{b}$ generated by the root spaces relative to the simple roots given in column 3 of Table 2 below. Then $\mu(\mathfrak{a})$ is a quadratic polynomial in r. Moreover, $\mu(\mathfrak{a}) \geq f(r)$, where $f(r)$ may be taken as in Table 1 above.*

Proof. By Lemma 2.1 we have $\operatorname{mod} B \geq \mu(\mathfrak{a})$. Thus it suffices to provide the values $\mu(\mathfrak{a})$ for the chosen ideals. In Table 2 below we list the simple roots whose root spaces generate the ideal $\mathfrak{a}$ and list $\mu(\mathfrak{a})$ for each case. The details of the calculations are omitted.

For a fixed classical type we choose for $f(r)$ the polynomial $\mu(\mathfrak{a})$ which is minimal for that type. This yields the lower bounds of Table 1. Whence, Proposition 3.3 and with it Theorem 3.1 follow.

Remarks 3.4. (1) For each classical simple group G, the dimension of B_u for a Borel subgroup B of G (which equals the number of positive roots of G) grows quadratically with the rank of G. Thus the polynomial bounds in Theorem 3.1 are optimal in terms of their degrees. Considering the ratio of $\operatorname{mod} B$ by $\dim B_u$ as r grows for a fixed classical type, we infer from Table 1 that for all classical groups

$$\frac{1}{6} \leq \lim_{r\to\infty} \frac{\operatorname{mod} B}{\dim B_u} \leq 1.$$

The same lower bound can be derived for type A_r from [7] (second part of the proof of Theorem 3.2). Using Theorem 5.3 from [12] in our situation, one can improve the upper bound of this limit to $1/3$ for type A_r.

(2) With the aid of a computer program written by U. Jürgens it was checked for all classical types and $r \leq 40$, that among *all* ideals which are generated by root spaces relative to simple roots the ones in Table 2 yield maximal values for μ.

(3) It follows from work in [6] that if G is of type A_r for $r \leq 9$, B_r or D_r for $r \leq 6$, or C_r for $r \leq 5$, then the bounds given in Table 2 are also upper bounds for $\operatorname{mod} B$. Thus we have $\operatorname{mod} B = \mu(\mathfrak{a})$ in these instances. We list these cases in Table 3 below together with the ideals $\mathfrak{a}$ from Table 2. For G of type A_r, for $r \leq 7$, B_3, B_4, and C_3, the modality of Borel subgroups can also be determined from the information in Table 1 in [4].

At present is not known whether $\operatorname{mod} B$ is a polynomial in r as suggested by these results.

Type of G	r	$\mathfrak{a}$	$\mu(\mathfrak{a})$
A_r	$6n$	$\alpha_n, \alpha_{3n}, \alpha_{5n}$	$\frac{1}{12}(r^2 - 4r)$
	$6n+1$	$\alpha_n, \alpha_{3n+1}, \alpha_{5n+2}$	$\frac{1}{12}(r^2 - 4r + 3)$
	$6n+2$	$\alpha_n, \alpha_{3n+1}, \alpha_{5n+2}$	$\frac{1}{12}(r^2 - 4r + 4)$
	$6n+3$	$\alpha_n, \alpha_{3n+1}, \alpha_{5n+3}$	$\frac{1}{12}(r^2 - 4r + 3)$
	$6n+4$	$\alpha_n, \alpha_{3n+2}, \alpha_{5n+4}$	$\frac{1}{12}(r^2 - 4r)$
	$6n+5$	$\alpha_{n+1}, \alpha_{3n+3}, \alpha_{5n+5}$	$\frac{1}{12}(r^2 - 4r + 7)$
B_r	$6n$	α_n, α_{4n}	$\frac{1}{6}(r^2 - r)$
	$6n+1$	α_n, α_{4n+1}	$\frac{1}{6}(r^2 - r)$
	$6n+2$	$\alpha_{n+1}, \alpha_{4n+2}$	$\frac{1}{6}(r^2 - r - 2)$
	$6n+3$	α_n, α_{4n+2}	$\frac{1}{6}(r^2 - r)$
	$6n+4$	$\alpha_{n+1}, \alpha_{4n+3}$	$\frac{1}{6}(r^2 - r)$
	$6n+5$	$\alpha_{n+1}, \alpha_{4n+4}$	$\frac{1}{6}(r^2 - r - 2)$
C_r	$3n$	α_n, α_{3n}	$\frac{1}{6}(r^2 - r)$
	$3n+1$	α_n, α_{3n+1}	$\frac{1}{6}(r^2 - r)$
	$3n+2$	α_n, α_{3n+2}	$\frac{1}{6}(r^2 - r - 2)$
D_r	$6n$	α_n, α_{4n}	$\frac{1}{6}(r^2 - 2r)$
	$6n+1$	α_n, α_{4n+1}	$\frac{1}{6}(r^2 - 2r - 5)$
	$6n+2$	α_n, α_{4n+1}	$\frac{1}{6}(r^2 - 2r)$
	$6n+3$	α_n, α_{4n+2}	$\frac{1}{6}(r^2 - 2r - 3)$
	$6n+4$	α_n, α_{4n+2}	$\frac{1}{6}(r^2 - 2r - 2)$
	$6n+5$	α_n, α_{4n+3}	$\frac{1}{6}(r^2 - 2r - 3)$

Table 2: lower bounds for the modality of Borel subgroups in classical groups.

Remark 3.5. For exceptional groups G we may equally use Lemma 2.1 in order to obtain lower bounds on the modality of a Borel subgroup B of G. In Table 4 below we list an ideal $\mathfrak{a}$ of $\mathfrak{b}$ in $\mathfrak{b}_u$ together with $\mu(\mathfrak{a})$ in each of the exceptional cases. Among *all* ideals of $\mathfrak{b}$ which are generated by root spaces relative to simple roots, the ones in Table 4 yield maximal values for μ. This was checked using the same computer program which was already applied in the classical situation.

It is known that for G of type G_2, F_4, and E_6, and $\mathfrak{a}$ as in Table 4, $\operatorname{mod} B = \mu(\mathfrak{a})$, cf. [6]. For G_2 this information can also be read off from Table 1 in [4].

Type of G	$\mathfrak{a}$	$\dim \mathfrak{a}$	$\operatorname{mod} B = \mu(\mathfrak{a})$
A_5	$\alpha_1, \alpha_3, \alpha_5$	13	1
A_6	$\alpha_1, \alpha_3, \alpha_5$	18	1
A_7	$\alpha_1, \alpha_4, \alpha_7$	22	2
A_8	$\alpha_1, \alpha_4, \alpha_7$	29	3
A_9	$\alpha_1, \alpha_4, \alpha_8$	35	4
B_3	α_2	7	1
B_4	α_1, α_3	14	2
B_5	α_1, α_4	21	3
B_6	α_1, α_4	29	5
C_3	α_1, α_3	8	1
C_4	α_1, α_4	13	2
C_5	α_1, α_5	19	3
D_4	α_2	9	1
D_5	α_3	15	2
D_6	α_1, α_4	25	4

Table 3: the modality of Borel subgroups in classical groups of small rank.

Type of G	$\mathfrak{a}$	$\dim \mathfrak{a}$	$\mu(\mathfrak{a})$
G_2	α_1	5	1
F_4	α_2	20	4
E_6	α_4	29	5
E_7	α_5	50	10
E_8	α_2	92	20

Table 4: lower bounds for the modality of Borel subgroups in exceptional groups.

Remarks 3.6. (1) For the calculations of the polynomials $\mu(\mathfrak{a})$ in Tables 2 and 4 we assumed that $\operatorname{char} K$ does not divide the structure constants of the Chevalley commutator relations for G. Otherwise, as the dimension of the commutator subalgebras may be smaller, the values $\mu(\mathfrak{a})$ are possibly larger (in fact $\mu(\mathfrak{a})$ is independent of $\operatorname{char} K$ for any of the ideals $\mathfrak{a}$ in these tables). Therefore, the lower bounds for $\operatorname{mod} B$ given in Table 2 and thus the ones in Theorem 3.1 in particular do apply in any characteristic, as claimed.

(2) Strictly speaking, the bounds in this section apply for $\operatorname{mod}(B : \mathfrak{b}_u)$. Replacing $\mathfrak{n}$ by the corresponding normal subgroup N of B, one obtains the same bounds for $\operatorname{mod} B$ *mutatis mutandis*.

4. On the Modality of Parabolic subgroups

We continue with the same notation as in the previous sections. In this chapter we assume that $\operatorname{char} K = 0$. Let H be a reductive regular subgroup of G relative to T. Let P be a parabolic subgroup of G containing T and let $Q = H \cap P$ which is a parabolic subgroup of H. The following result generalizes [9, Corollary 2.6], cf. [9, Remark 2.14].

Proposition 4.1. *Assume that G, P, H, and $Q = H \cap P$ are as above. Then*

$$\operatorname{mod} P \geq \operatorname{mod} Q.$$

Our main result in this section is

Theorem 4.2. *Let G be a classical simple algebraic group and P a parabolic subgroup of G. Let $r = \operatorname{rank} G$ and $s = \operatorname{rank}_{ss} P$. There exists a quadratic polynomial $f \in \mathbb{Q}[t]$ such that*

$$\operatorname{mod} P \geq f(r - s).$$

That is, the modality of P grows at least quadratically with $r - s$, the difference of the semisimple ranks of G and P. Moreover, the polynomial f may be taken from Table 1 above.

Proof. We may assume that P is standard, i.e., $P = P_J$, where $J \subset \Pi$. The special case when J is empty is the one from Theorem 3.1. So we may suppose that J is non-empty. We construct a sequence of simple regular subgroups G_k of G together with parabolic subgroups P_k of G_k for $k \geq 1$ with certain properties. This sequence will be obtained by means of an iteration process whose initial step is defined as follows:

Since J is non-empty, there exists a pair of 'consecutive' simple roots, where precisely one of them is in J. We fix such a pair which is either of the form $\{\alpha_i, \alpha_{i+1}\}$ for some $i < r$, or $\{\alpha_{r-2}, \alpha_r\}$ in case G is of type D_r. For the definition of G_1 we distinguish three cases:

1. For G of type A_r, B_r, or C_r and $i < r - 1$, or D_r and $i < r - 2$, let G_1 be the connected simple regular subgroup of G defined by the set of 'simple' roots

$$\Pi(G_1) := \{\alpha_1, \dots, \alpha_{i-1}, \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_r\}.$$

2. If G is of type C_r and $i = r - 1$, then let G_1 be the connected simple regular subgroup of G defined by

$$\Pi(G_1) := \{\alpha_1, \dots, \alpha_{r-2}, 2\alpha_{r-1} + \alpha_r\}.$$

3. In case G is of type D_r and the chosen pair of consecutive simple roots is either $\{\alpha_{r-2}, \alpha_{r-1}\}$ or $\{\alpha_{r-2}, \alpha_r\}$ we let G_1 be the connected simple regular subgroup of G given by

$$\Pi(G_1) := \{\alpha_1, \dots, \alpha_{r-2} + \alpha_{r-1}, \alpha_{r-2} + \alpha_r\}.$$

In each case the subgroup G_1 is of the same classical type as that of G and $\operatorname{rank} G_1 = \operatorname{rank} G - 1$.

Define $P_1 := P \cap G_1$. As, by construction, only one of the two chosen simple roots is in J, we see that either

$$\operatorname{rank} G_1 - \operatorname{rank}_{ss} P_1 = \operatorname{rank} G - \operatorname{rank}_{ss} P, \tag{4.2.1}$$

or

$$\operatorname{rank} G_1 - \operatorname{rank}_{ss} P_1 = \operatorname{rank} G - \operatorname{rank}_{ss} P + 1. \tag{4.2.2}$$

The latter case only occurs in case 3 above when $\{\alpha_{r-1}, \alpha_r\} \subseteq J$. If P_1 is not a Borel subgroup of G_1, we may repeat the same procedure now with G_1 in place of G etc. Iterating this process defines the desired sequence of simple subgroups G_k of rank $r - k$ and corresponding parabolic subgroups $P_k := G_k \cap P$ for $k \geq 1$. This procedure stops once we have arrived at the standard Borel subgroup of the corresponding simple subgroup. It follows from 4.2.1 and 4.2.2 that the length of our sequence of G_k's is either s or $s - 1$. The later case, illustrated by an example below, occurs precisely when G is of type D_r and $\{\alpha_{r-1}, \alpha_r\} \subseteq J$. In the first case set $H := G_s$ and $Q := P_s$, while in the second we let H be the simple regular subgroup of G_{s-1} corresponding to the usual subsystem of type D_{r-s} and we set $Q := H \cap P$.

Since H is again a classical simple group, Q a Borel subgroup of H, and $\operatorname{rank} H = r - s$, using Proposition 4.1 and Theorem 3.1, we infer that

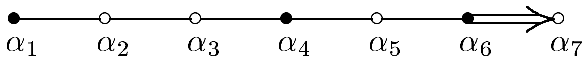
$$\operatorname{mod} P \geq \operatorname{mod} Q \geq f(r - s)$$

for some $f \in \mathbb{Q}[t]$ from Table 1 according to the type of H.

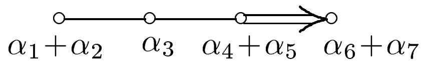
We illustrate the procedure in the proof of Theorem 4.2 with two examples.

Examples 4.3. In our first example G is of type B_7 and P is a standard parabolic subgroup of G of semisimple rank 3. The coloured nodes label the simple roots in the standard Levi subgroup of P. We indicate a regular embedding of G_3 in G. Here $H = G_3$. This example describes the general situation, while the second one shows an instance when 4.2.2 applies.

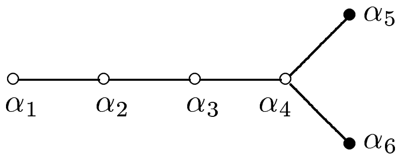
G, P



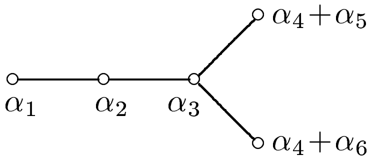
$H = G_3$, $Q = P_3 = G_3 \cap P$



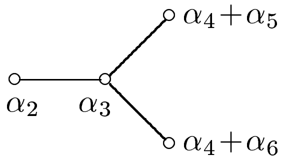
G, P



G_1, $P_1 = G_1 \cap P$



H, Q



Analogous to Corollary 3.2, we may formulate here an explicit uniform bound for $\operatorname{mod} P$ using the polynomial f for type A_r from Table 1:

Corollary 4.4. *Let G be a classical simple algebraic group and P a parabolic subgroup of G. Set $r = \operatorname{rank} G$ and $s = \operatorname{rank}_{ss} P$. Then*

$$\operatorname{mod} P \geq \frac{1}{12}((r-s)^2 - 4(r-s)).$$

From Corollary 4.4 we immediately derive

Corollary 4.5. *Let G be a classical simple algebraic group. Fix non-negative integers s and m. Suppose that G admits a parabolic subgroup P with $\operatorname{rank}_{ss} P = s$ and $\operatorname{mod} P = m$. Then*

$$\operatorname{rank} G \leq 2\sqrt{3m+1} + s + 2.$$

Proof. Solving the inequality in Corollary 4.4 with $m = \operatorname{mod} P$ for r yields the desired bound.

Since there is only a finite number of isomorphism classes of exceptional simple algebraic groups, we readily conclude the following finiteness result from [8].

Corollary 4.6. *There is only a finite number of simple algebraic groups G (up to isomorphism) admitting parabolic subgroups with prescribed semisimple rank and prescribed modality.*

Remarks 4.7. (1) Let G and P be as in Theorem 4.2. As in the case of Borel subgroups, the degree of the lower bound $f(r-s)$ in Theorem 4.1 is optimal, since $\dim P_u$ is a quadratic polynomial in r for fixed s.

(2) If $P = P_J$ is standard, the nilpotent class ℓ of P_u may be determined as the sum of the coefficients of the simple roots in $\Pi \setminus J$ of the highest root in Ψ, cf. [11, Lemma 2.5]. Since 2 is the largest coefficient of a simple root in the highest root for any classical root system, $2(r-s) \geq \ell$. From that and Corollary 4.4 one easily determines a quadratic polynomial in ℓ as a lower bound for $\operatorname{mod} P$.

5. Further consequences

With the aid of Table 2 and Proposition 4.1 we can obtain further refinements of the principal result from [11]. We indicate this with an example below. The following is a reformulation of the part of Theorem 3.1 from [11] which concerns classical groups.

Theorem 5.1. *Let G be a simple classical algebraic group. Let P be a parabolic subgroup of G. Then* $\operatorname{mod} P \geq 1$ *provided one of the following holds:*

(i) *G is of type A_r, B_r, or C_r, and $\ell(P_u) \geq 5$;*
(ii) *G is of type D_r and $\ell(P_u) \geq 6$;*
(iii) *G is of type D_r, $\ell(P_u) = 5$, and $\Theta P = P$, where Θ denotes the graph automorphism of G of order two stemming from interchanging the simple roots α_{r-1} and α_r.*

Thus, we conclude

Corollary 5.2. *Let G and P be as in Theorem 5.1. If $\ell(P_u) \geq 6$, then* $\operatorname{mod} P \geq 1$.

Theorem 5.3. *Let G be a simple classical algebraic group. Let P be a parabolic subgroup of G. Then* $\operatorname{mod} P \geq 2$ *provided one of the following holds:*

(i) *G is of type A_r, B_r, or C_r, and $\ell(P_u) \geq 7$;*
(ii) *G is of type D_r and $\ell(P_u) \geq 8$;*
(iii) *G is of type D_r, $\ell(P_u) = 7$, and $\Theta P = P$, where Θ denotes the graph automorphism of G of order two stemming from interchanging the simple roots α_{r-1} and α_r.*

Proof. With the same methods as the ones applied in the proof of Theorem 4.2 we can reduce any such configuration to a Borel subgroup of a simple regular subgroup of G of type A_7, B_4, C_4 or D_5. The result then follows from Proposition 4.1 and the information in Table 3.

Thus, we have

Corollary 5.4. *Let G and P be as in Theorem 5.3. If $\ell(P_u) \geq 8$, then* $\operatorname{mod} P \geq 2$.

6. Modality for non-algebraically closed fields

Let K be as before an algebraically closed field and let E be a proper infinite subfield. Suppose that G is defined over E and let $G(E)$ denote the group of E-rational points of G. Let P be a parabolic subgroup of G and $P(E)$ the corresponding parabolic subgroup in $G(E)$. The notion of modality also applies to groups and varieties defined over arbitrary infinite fields. We define the modality for $P(E)$ as

$$\operatorname{mod}_E P(E) := \operatorname{mod}_E(P(E) : P(E)_u).$$

Suppose that $\operatorname{mod}_K P = m > 0$. Since G is linear, an m-parameter family of P-orbits on P_u may be realized in terms of matrices in some $GL_m(K)$. Then the same family of matrices also defines an m-parameter-family of $P(E)$-orbits on $P(E)_u$. Thus we have

$$\operatorname{mod}_E P(E) \geq \operatorname{mod}_K P(E).$$

Examples show that we may have a strict inequality here. For instance let $G = SL_2(\mathbb{C})$ and let B be the standard Borel subgroup of G. We have $\operatorname{mod}_{\mathbb{C}} B = 0$, while $\operatorname{mod}_{\mathbb{Q}} B(\mathbb{Q}) > 0$, since $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is infinite.

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