A Topological Study of the Fractional Fibonacci Groups

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Abstract

We consider the fractional generalization of the Fibonacci groups $F(2, 2n)$, which is naturally from a topological point of views. The topological properties of the arising three-dimensional manifolds are studied. In particular, these manifolds are described in terms of the Dehn surgery, as the two-fold branched coverings of the three-dimensional sphere and as the cyclic coverings of the three-dimensional sphere, branched over the two-bridge knots.

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1 The Fractional Fibonacci Groups

The present article is devoted to some generalization of the Fibonacci groups and studying of geometric and topological properties of the corresponding three-dimensional closed orientable manifolds.

The Fibonacci groups $F(2, m)$ were introduced by J. Conway [5] and have the following presentations:

$$F(2, m) = \langle x_1, x_2, \ldots, x_m \mid x_i \, x_{i+1} = x_{i+2}, \quad i = 1, \ldots, m \rangle, \tag{1}$$

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where the indices are taken $mod\, m$. The nomenclature follows from the inductive definition of the Fibonacci numbers with initials $a_1 = a_2 = 1$:

$$a_i + a_{i+1} = a_{i+2}. \tag{2}$$

The studying of the Fibonacci groups was inspired by the Conway's question: whether or not the groups $F(2,m)$ are finite? The answer was given due to [6], [9], [18], and [21] (a nice survey can be found in [22]) : the group $F(2,m)$ is finite precisely when $m = 1, 2, 3, 4, 5, 7$.

In [12] D. L. Johnson introduced the following generalization of the Fibonacci groups:

$$F(r,m) = \langle x_1, x_2, \ldots, x_m \mid x_i \cdots x_{i+r-1} = x_{i+r}, \quad i = 1, \ldots, m\rangle, \tag{3}$$

for a given integer $r \geq 2$, where the indices are taken $mod\, m$. The properties of $F(r,m)$ were studied in [13] (see [22] for a survey, where the knowledge about the groups $F(r,m)$, $2 \leq r \leq 10$ and $2 \leq n \leq 10$, is summarized).

Another generalization of the Fibonacci groups were introduced by C. Maclachlan [14]. For an integer $k \geq 1$ he considered a group

$$F^k(2,m) = \langle x_1, x_2, \ldots, x_m \mid x_i\, x_{i+1}^k = x_{i+2}, \quad i = 1, \ldots, m\rangle, \tag{4}$$

where the indices are taken $mod\, m$. One can see the correspondence between this generalization of the Fibonacci group and the following generalization $\{a_i^k\}$ of the Fibonacci numbers :

$$a_i^k + k\, a_{i+1}^k = a_{i+2}^k, \tag{5}$$

where the inductive law depends on a given integer parameter k.

It is natural to generalize the above inductive law to the fractional parameter case. More exactly, let us consider a "fractional" analogue $\{a_i^{k/l}\}$ of the Fibonacci numbers with the following inductive law:

$$a_i^{k/l} + \frac{k}{l}\, a_{i+1}^{k/l} = a_{i+2}^{k/l} \tag{6}$$

for a fractional number $\frac{k}{l}$. By the same arguments, it is natural to consider groups with the following presentations:

$$F^{k/l}(2,m) = \langle x_1, x_2, \ldots, x_m \mid x_i^l\, x_{i+1}^k = x_{i+2}^l, \quad i = 1, \ldots, m\rangle \tag{7}$$

for coprime integers k and l, and an integer $m \geq 3$, where the indices are taken $mod\, m$. A group $F^{k/l}(2,m)$ is said to be *a Fractional Fibonacci group.*

One of the aims of the present article is to demonstrate that the extension of the Fibonacci groups $F(2,m)$ to the Fractional Fibonacci groups $F^{k/l}(2,m)$ is naturally from a topological point of views.

We recall that a new stage in the investigation of the Fibonacci groups was started with a remarkable work of H. Helling, A. C. Kim and J. Mennicke [10] (see also [16]), where they showed that the groups $F(2,2n)$ arise as the fundamental groups of three-dimensional closed orientable manifolds. More exactly, the group $F(2,2n)$, $n \geq 4$, is isomorphic to the fundamental group of a three-dimensional closed orientable hyperbolic manifold. The group $F(2,6)$ is isomorphic to the fundamental group of a three-dimensional closed orientable Euclidean Hantzche–Wendt manifold studied in [25]. The group $F(2,4)$ is finite, and isomorphic to the fundamental group of the lens space $L(5,2)$ which is a three-dimensional orientable spherical manifold. Thus, the group $F(2,2n)$, $n \geq 2$, is realizable as a co-compact discrete group of isometries acting without fixed points on the space X_n, where $X_2 = \mathcal{S}^3$ is the spherical three-dimensional space, $X_3 = \mathbb{E}^3$ is the Euclidean three-dimensional space, and for $n \geq 4$ $X_n = \mathbb{H}^3$ is the Lobachevsky three-dimensional space.

A three-dimensional manifold $M_n = X_n / F(2,2n)$, $n \geq 2$, uniformized by a Fibonacci group, is referred to as *a Fibonacci manifold*.

We recall the following topological properties of the Fibonacci manifolds M_n:

(1) it was shown in [11] that the manifold M_n can be obtained as the n–fold cyclic covering of the three-dimensional sphere, branched over the figure-eight knot;

(2) it was shown in [3] that the manifold M_n can be obtained by Dehn surgeries with parameters $(1,1)$ and $(-1,1)$ on the components of the chain of $2n$ circles in $\mathcal{S}^3$;

(3) it was shown in [15] (see also [24]) that the manifold M_n can be obtained as the 2–fold covering of the three-dimensional sphere, branched over the Turk's head link Th_n, that is, the closure of the 3-string braid $(\sigma_1 \sigma_2^{-1})^n$.

A three-dimensional closed orientable manifold $M_n^{k/l}$ with the fundamental group $\pi_1(M_n^{k/l}) = F^{k/l}(2,2n)$ is said to be *a Fractional Fibonacci manifold*. In particular, for $k/l = 1$ we turn to the Fibonacci manifold, and the case when $l = 1$ and k is integer, corresponds to the manifolds from [14].

We will show that the Fractional Fibonacci manifolds $M_n^{k/l}$ have the following topological properties which are direct generalizations of properties (1) – (3):

(1*) the manifold $M_n^{1/l}$ can be obtained as the n–fold cyclic covering of the three-dimensional sphere, branched over the two-bridge knot $K(2l+\frac{1}{2l})$ (see Corollary 3.1);

(2*) the manifold $M_n^{k/l}$ can be obtained by Dehn surgeries with parameters (k,l) and $(-k,l)$ on the components of the chain of $2n$ circles in $\mathcal{S}^3$ (see Theorem 2.1);

(3*) the manifold $M_n^{k/l}$ can be obtained as the 2–fold covering of the three-dimensional sphere, branched over the link $Th_n^{k/l}$, that is the closure of the rational 3-string braid $(\sigma_1^{k/l} \sigma_2^{-k/l})^n$ (see Theorem 3.1).

Moreover, the Fractional Fibonacci manifolds $M_n^{k/l}$ are hyperbolic for all but a finite number of pairs of coprime integers k and l for each integer $n \geq 3$ (see Corollary 2.1).

2 The Fractional Fibonacci Manifolds

In this section we will construct the family of Fractional Fibonacci manifolds $M_n^{k/l}$. Our approach generalizes the studing of A. Cavicchioli and F. Spaggiari for describing of the Fibonacci manifolds M_n in terms of the Dehn surgery [3].

For any integer $n \geq 2$ we can consider the link L_{2n} with $2n$ components, each of which is unknotted and linked with exactly two adjacent components. The diagram of the link L_8 is pictured in Fig. 1.

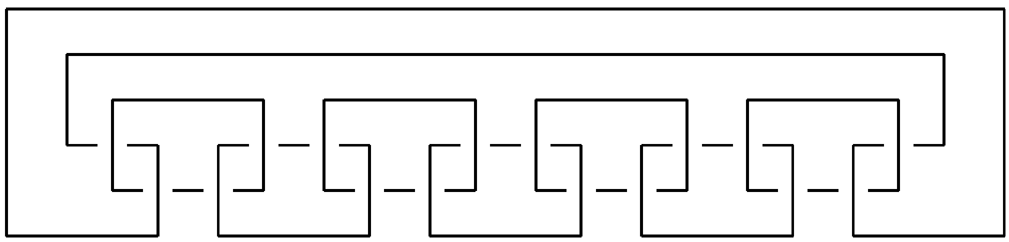

Fig. 1. The link L_8.

It was shown by W. Thurston ([23], Section 6.8.7) that for $n \geq 3$ the link L_{2n} is hyperbolic and the hyperbolic volume of the complement $\mathcal{S}^3 \setminus L_{2n}$ is given by the formula

$$vol\left(\mathcal{S}^3 \setminus L_{2n}\right) = 8\,n\left[\Lambda\left(\frac{\pi}{4} + \frac{\pi}{2n}\right) + \Lambda\left(\frac{\pi}{4} - \frac{\pi}{2n}\right)\right], \tag{8}$$

where $\Lambda(x)$ is the Lobachevsky function [23] :

$$\Lambda(x) = -\int_0^x \ln |\, 2\sin\theta \,| \; d\theta. \tag{9}$$

We use the Wirtinger algorithm ([2], p. 31) for calculating of the fundamental group $\pi_1(\mathcal{S}^3 \setminus L_{2n})$ of the manifold $\mathcal{S}^3 \setminus L_{2n}$. Denoting the generators by x_i and y_i, $i = 1, \ldots, 2n$, in the correspondence to Fig. 2.,

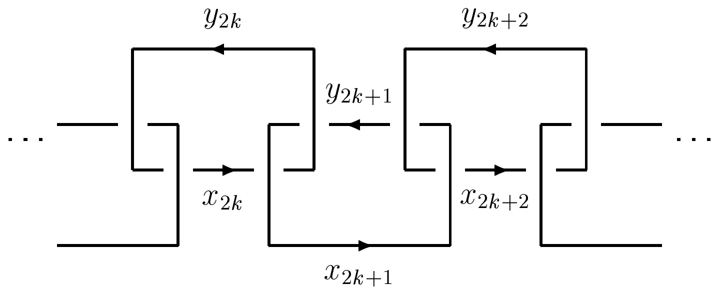

Fig. 2. The generators of $\pi_1(\mathcal{S}^3 \setminus L_{2n})$.

we will get $4n$ relations of the following four types:

$$x_{2k+2}\, x_{2k+1} \;=\; x_{2k+1}\, y_{2k+2}, \tag{10}$$

$$y_{2k+1}\, y_{2k+2} \;=\; y_{2k+2}\, x_{2k+1}, \tag{11}$$

$$x_{2k+1}\, y_{2k} \;=\; x_{2k}\, x_{2k+1}, \tag{12}$$

$$y_{2k}\, x_{2k+1} \;=\; y_{2k+1}\, y_{2k}, \tag{13}$$

where $k = 1, \ldots, n$, and the indices are taken $mod\, 2n$. Therefore, we get the following presentation for the fundamental group:

$$\begin{aligned} \pi_1(\mathcal{S}^3 \setminus L_{2n}) = \quad \langle x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n} \mid \quad & x_{2k+2}\, x_{2k+1} \;=\; x_{2k+1}\, y_{2k+2}, \\ & y_{2k+1}\, y_{2k+2} \;=\; y_{2k+2}\, x_{2k+1}, \\ & x_{2k+1}\, y_{2k} \;=\; x_{2k}\, x_{2k+1}, \\ & y_{2k}\, x_{2k+1} \;=\; y_{2k+1}\, y_{2k}, \\ & k = 1, \ldots, n \rangle. \end{aligned} \tag{14}$$

For an i-th component, $i = 1, \ldots, 2n$, of the link L_{2n} we will consider the meridian m_i and the longitude l_i in the group $\pi_1(\mathcal{S}^3 \setminus L_{2n})$. Suppose that

$$m_{2k+1} \;=\; x_{2k+1}, \tag{15}$$

$$l_{2k+1} \;=\; y_{2k+2}^{-1}\, y_{2k}, \tag{16}$$

and

$$m_{2k} \;=\; y_{2k}, \tag{17}$$

$$l_{2k} \;=\; x_{2k-1}^{-1}\, x_{2k+1}, \tag{18}$$

where $k = 1, \ldots, n$ and all indices are taken $mod\, 2n$. From equalities (16) and (17) we get

$$l_{2k+1} \;=\; m_{2k+2}^{-1}\, m_{2k}. \tag{19}$$

Similarly, from (15) and (18) we get

$$l_{2k} \;=\; m_{2k-1}^{-1}\, m_{2k+1}. \tag{20}$$

Let us express the elements x_i and y_i from (14) in terms of the meridians and longitudes. Directly from (15) and (17), we have:

$$x_{2k+1} \;=\; m_{2k+1} \tag{21}$$

and

$$y_{2k} \;=\; m_{2k}. \tag{22}$$

From (12), using (21) and (22), we get

$$x_{2k} \;=\; x_{2k+1}\, y_{2k}\, x_{2k+1}^{-1} \;=\; m_{2k+1}\, m_{2k}\, m_{2k+1}^{-1}. \tag{23}$$

Similarly, from (13) we get

$$y_{2k+1} \;=\; y_{2k}\, x_{2k+1}\, y_{2k}^{-1} \;=\; m_{2k}\, m_{2k+1}\, m_{2k}^{-1}. \tag{24}$$

Substituting (21), (22) and (23) into (10), we get

$$m_{2k+3}\, m_{2k+2}\, m_{2k+3}^{-1}\, m_{2k+1} \;=\; m_{2k+1}\, m_{2k+2},$$

what by (20) is equivalent to the equality

$$m_{2k+3}\, m_{2k+2}\, l_{2k+2}^{-1} \;=\; m_{2k+1}\, m_{2k+2}.$$

Whence,

$$m_{2k+3} \;=\; m_{2k+1}\, \left(m_{2k+2}\, l_{2k+2}\, m_{2k+2}^{-1}\right). \tag{25}$$

Substituting (21), (22) and (23) into (11), we get

$$m_{2k}\, m_{2k+1}\, m_{2k}^{-1}\, m_{2k+2} \;=\; m_{2k+2}\, m_{2k+1},$$

what by (19) is equivalent to the equality

$$m_{2k}\, m_{2k+1}\, l_{2k+1}^{-1} \;=\; m_{2k+2}\, m_{2k+1}.$$

Whence,

$$m_{2k} \;=\; m_{2k+2}\, \left(m_{2k+1}\, l_{2k+1}\, m_{2k+1}^{-1}\right). \tag{26}$$

Therefore, using (21) – (26), from the presentations (14) we get

$$\begin{aligned} \pi_1(\mathcal{S}^3 \setminus L_{2n}) \;=\; \langle m_1, \ldots, m_{2n}, l_1, \ldots, l_{2n} \;\mid\; & m_{2k+1}\, \left(m_{2k+2}\, l_{2k+2}\, m_{2k+2}^{-1}\right) \;=\; m_{2k+3}, \\ & m_{2k}\, \left(m_{2k+1}\, l_{2k+1}^{-1}\, m_{2k+1}^{-1}\right) \;=\; m_{2k+2}, \\ & k = 1, \ldots, n\rangle. \end{aligned} \tag{27}$$

Let us consider a closed manifold $M(p_1/q_1, \ldots, p_{2n}/q_{2n})$ obtained by Dehn surgeries on the cusps of the manifold $\mathcal{S}^3 \setminus L_{2n}$ with surgery parameters (p_i, q_i), $i = 1, \ldots, 2n$, where p_i and q_i are coprime integers.

Then, from (27), we have the following presentations for the fundamental group $\pi_1(M(p_1/q_1, \ldots, p_{2n}/q_{2n}))$:

$$\begin{aligned} \langle m_1, \ldots, m_{2n}, l_1, \ldots, l_{2n} \mid \quad & m_{2k+1} \left(m_{2k+2}\, l_{2k+2}\, m_{2k+2}^{-1}\right) = m_{2k+3}, \\ & m_{2k} \left(m_{2k+1}\, l_{2k+1}^{-1}\, m_{2k+1}^{-1}\right) = m_{2k+2}, \\ & m_i^{p_i}\, l_i^{q_i} = 1,\ k = 1, \ldots, n,\ i = 1, \ldots 2n\rangle. \end{aligned} \tag{28}$$

In the following lemma we will give the another presentations for the fundamental group $\pi_1(M(p_1/q_1, \ldots, p_{2n}/q_{2n}))$, which was remarked by M. Takahashi [20]. Because the proof was omitted in [20], for reader's convenience we will give it here.

Lemma 2.1 ([20]) *In the above notations the following group presentation holds:*

$$\begin{aligned} \pi_1(M(p_1/q_1, \ldots, p_{2n}/q_{2n})) = \langle a_1, \ldots, a_{2n} \mid \quad & a_{2k+1}^{q_{2k+1}}\, a_{2k+2}^{-p_{2k+2}} = a_{2k+3}^{q_{2k+3}}, \\ & a_{2k}^{q_{2k}}\, a_{2k+1}^{p_{2k+1}} = a_{2k+2}^{q_{2k+2}}, k = 1, \ldots, n\rangle. \end{aligned} \tag{29}$$

Proof. Because p_i and q_i are coprime integers, there exist coprime integers s_i and t_i such that

$$q_i\, s_i - p_i\, t_i = 1. \tag{30}$$

For $i = 1, \ldots, 2n$ we consider elements

$$a_i = m_i^{s_i}\, l_i^{t_i}. \tag{31}$$

Then, from (28) and (30), we have

$$a_i^{q_i} = m_i^{q_i s_i}\, l_i^{q_i t_i} = m_i\, (m_i^{p_i}\, l_i^{q_i})^{t_i} = m_i, \tag{32}$$

and

$$a_i^{-p_i} = m_i^{-p_i s_i}\, l_i^{-p_i t_i} = l_i\, (m_i^{p_i}\, l_i^{q_i})^{-s_i} = l_i. \tag{33}$$

Substituting (32) and (33) into (25) and (26), we get

$$a_{2k+3}^{q_{2k+3}} = a_{2k+1}^{q_{2k+1}} \left(a_{2k+2}^{q_{2k+2}}\, a_{2k+2}^{-p_{2k+2}}\, a_{2k+2}^{-q_{2k+2}}\right) = a_{2k+1}^{q_{2k+1}}\, a_{2k+2}^{-p_{2k+2}}, \tag{34}$$

and

$$a_{2k}^{q_{2k}} = a_{2k+2}^{q_{2k+2}} \left(a_{2k+1}^{q_{2k+1}}\, a_{2k+1}^{-p_{2k+1}}\, a_{2k+1}^{-q_{2k+1}}\right) = a_{2k+2}^{q_{2k+2}}\, a_{2k+1}^{-p_{2k+1}},$$

whence,

$$a_{2k}^{q_{2k}}\, a_{2k+1}^{q_{2k+1}} = a_{2k+2}^{q_{2k+2}}. \tag{35}$$

The statement of the lemma holds from (34) and (35). □

In virtue of lemma 2.1, there is the following connection between the manifolds $M(p_1/q_1, \ldots, p_{2n}/q_{2n})$ and the Fibonacci groups $F(2,2n)$, which was remarked in [3]. Let us suppose that $q_i = 1$ and $p_i = (-1)^{i+1}$ for $i = 1, \ldots, 2n$. Then the group

$$\pi_1 \left(M(1,-1,\ldots,1,-1) \right) = \langle a_1, \ldots, a_{2n} \mid a_i \, a_{i+1} = a_{i+2}, \, i = 1, \ldots, 2n \rangle,$$

where the indices are taken $mod\, 2n$, is the Fibonacci group $F(2,2n)$, and the manifold $M_n = M(1,-1,\ldots,1,-1)$ is the Fibonacci manifold. Whence, we have that the Fibonacci manifold M_n can be obtained by Dehn surgeries with parameters $(1,1)$ and $(-1,1)$ on the components of the link L_{2n}.

The similar property takes place for the Fractional Fibonacci manifolds $M_n^{k/l}$.

Theorem 2.1 *For any coprime integers k and l, and for any integer $n \geq 2$ the Fractional Fibonacci manifold $M_n^{k/l}$ can be obtained by Dehn surgeries with parameters (k,l) and $(-k,l)$ on the components of the link L_{2n}.*

Proof. Suppose that $q_i = l$ and $p_i = (-1)^{i+1} k$, for $i = 1, \ldots, 2n$. Then, in virtue of lemma 2.1, for the manifold $M(k/l, -k/l, \ldots, k/l, -k/l)$ we get

$$\pi_1(M(k/l,-k/l,\ldots,k/l,-k/l)) = \langle a_1, \ldots, a_{2n} \mid a_i^l \, a_{i+1}^k = a_{i+2}^l, \, i = 1, \ldots, 2n \rangle,$$

where all indices are taken $mod\, 2n$, that is the Fractional Fibonacci group $F^{k/l}(2,2n)$. Therefore, $M(k/l,-k/l,\ldots,k/l,-k/l)$ is the Fractional Fibonacci manifold. □

In particular, theorem 2.1 gives a new surgery description of the series of the manifolds M_n^k with the fundamental groups

$$\pi_1 \left(M_n^k \right) = \langle a_1, \ldots, a_{2n} \mid a_i \, a_{i+1}^k = a_{i+2}, i = 1, \ldots, 2n \rangle,$$

introduced by C. Maclachlan [14].

In virtue of theorem 2.1 we can investigate geometric structures on the Fractional Fibonacci manifolds. Because these manifolds can be obtained by Dehn surgeries on the hyperbolic links L_{2n}, according to the Thurston–Jørgensen theory of a hyperbolic surgery [23], we get the following statement.

Corollary 2.1 *For each integer $n \geq 3$ for all but a finite number of pairs of coprime integers k and l the Fractional Fibonacci manifolds $M_n^{k/l}$ are hyperbolic manifolds.*

Considering these manifolds fundamental groups, we get

Corollary 2.2 *For each integer $n \geq 3$ for all but a finite number of pairs of coprime integers k and l the Fractional Fibonacci groups*

$$F^{k/l}(2,2n) = \langle a_1, \ldots, a_{2n} \mid a_i^l \, a_{i+1}^k = a_{i+2}^l, \; i = 1, \ldots, 2n \rangle$$

are hyperbolic groups.

3 The Covering Properties of the Manifolds

In this section we study some topological properties of the Fractional Fibonacci manifolds $M_n^{k/l}$. We will demonstrate that these properties can be considered as a direct generalization of the corresponding properties of the Fibonacci manifolds M_n.

First of all, we define a series of links. We recall [1] that any link can be obtained as a closure of some braid. Let p and q be coprime integers, then by $\sigma_i^{p/q}$ we denote the rational p/q–tangle whose incoming arcs are i–th and $(i+1)$–th strings. For an integer $n \geq 1$ we denote by $Th_n^{p/q}$ the n-periodic link, that is, the closure of the rational 3-strings braid $(\sigma_1^{p/q}\sigma_2^{-p/q})^n$. The diagram for the case $n = 3$, $p/q = 3/2$ is presented in Fig. 3.

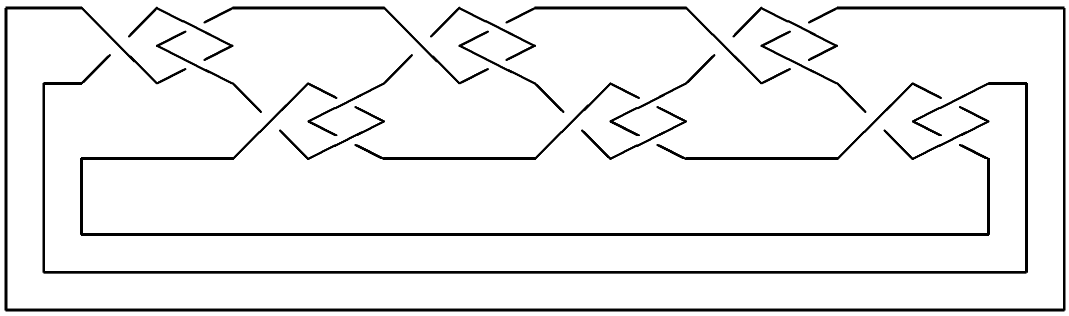

Fig. 3. The link $Th_3^{3/2}$

In particular, for the case $p/q = 1$ we get the Turk's head link Th_n ([7], p. 126), that is, the closure of the 3-strings braid $(\sigma_1\sigma_2^{-1})^n$.

It was shown in [15] that the Fibonacci manifolds M_n, $n \geq 2$, can be obtained as the two-fold coverings of the three-dimensional sphere, branched over the Turk's head links Th_n. The following theorem demonstrates that the Fractional Fibonacci manifolds $M_n^{k/l}$ have analogous property.

Theorem 3.1 *For any coprime integers k and l, and for any integer $n \geq 2$, the Fractional Fibonacci manifold $M_n^{k/l}$ can be obtained as the two-fold covering of the three-dimensional sphere, branched over the link $Th_n^{k/l}$.*

Proof. By theorem 2.1, a manifold $M_n^{k/l}$ can be obtained by Dehn surgeries with parameters (k, l) and $(-k, l)$ on the components of the link L_{2n} and according to the above notations $M_n^{k/l} = M(k/l, -k/l, \ldots, k/l, -k/l)$. It is obvious from Fig. 1 that the link L_{2n} has a strongly-invertible involution τ whose axe intersects each component of the link in two points. Therefore, the manifold $M_n^{k/l}$ is obtained by Dehn surgeries on the strongly-invertible link, according to the terminology of [17]. Thus, in virtue of the Montesinos theorem [17], the manifold $M_n^{k/l}$ can be obtained as the two-fold covering of the three-dimensional sphere, branched over some link. Moreover, the Montesinos algorithm

[17] admits to describe this link. Applying the Montesinos algorithm to our case, we get the n-periodic link whose fragment is pictured in Fig. 4, where boxes labelled by "k/l" and "$-k/l$" denote corresponding rational tangles.

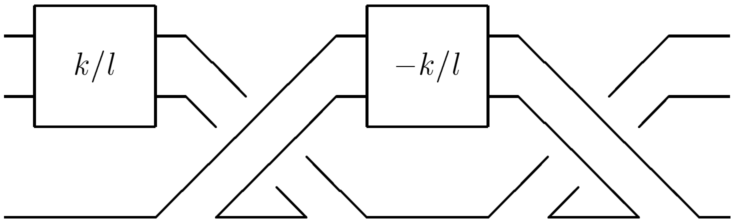

Fig. 4.

Using the Reidemeister moves, one can see that the n-periodic link with the fragment pictured in Fig. 4 is equivalent to the n-periodic link whose fragment is pictured in Fig. 5.

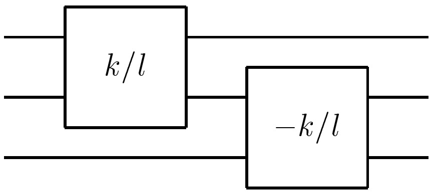

Fig. 5.

Therefore, we get the link $Th_n^{k/l}$, that is the closure of the rational 3-strings braid $(\sigma_1^{k/l}\sigma_2^{-k/l})^n$. $\square$

Let us study a particular case of the link $Th_n^{k/l}$ when $k/l = 1/l$, where l is integer, and $n = 2$. In this case our links $Th_2^{1/l}$, obviously, are knots and they have a simple structure.

Lemma 3.1 *For any integer l the link $Th_2^{1/l}$ is the two-bridge knot $K(2l + \frac{1}{l})$.*

Proof. Let us consider the diagram of the knot $Th_2^{1/l}$ pictured in Fig. 6.

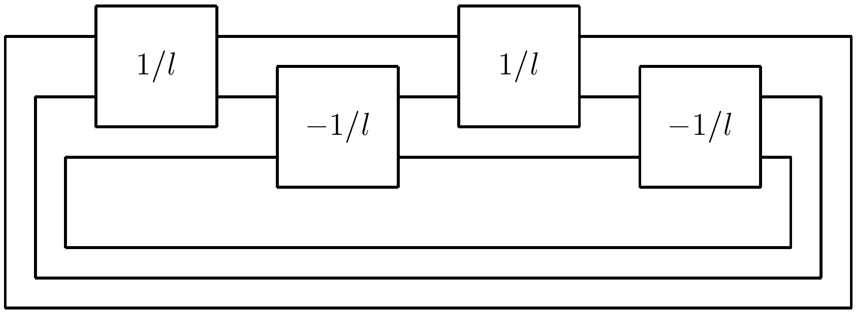

Fig. 6.

Using properties of rational tangles [4], [17], we will get that the diagram in Fig. 6 is equivalent to the diagram in Fig. 7, and next, to the diagram in Fig. 8.

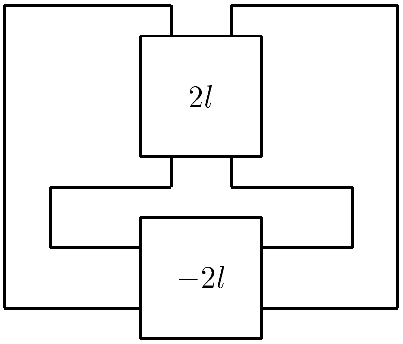

Fig. 7.

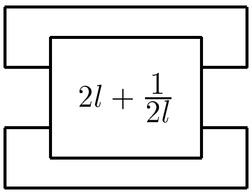

Fig. 8.

Thus, the knot $Th_2^{1/l}$ is equivalent to the closure of the rational $(2l + \frac{1}{2l})$–tangle (see Fig. 8), that is the 2-bridge knot $K(2l + \frac{1}{2l})$. $\square$

We remark that according to lemma 3.1 the knot Th_2^1 is the figure-eight knot $K(5/2) = 4_1$, and $Th_2^{1/2}$ is the knot $K(4 + \frac{1}{4}) = 8_1$ in notations of [19].

Let us consider for an integer l a link $\mathcal{L}^{1/l}$, whose diagram is pictured in Fig. 9.

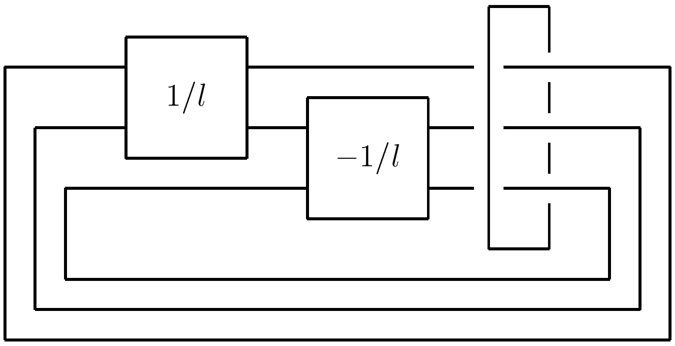

Fig. 9. The link $\mathcal{L}^{1/l}$.

Lemma 3.2 *For any integer l the link $\mathcal{L}^{1/l}$ is a two-component link, and its components are equivalent.*

Proof. It is obvious from Fig. 9 that $\mathcal{L}^{1/l}$ is a two-component link, and both its components are unknotted. The sequence of pictures in Fig. 9 – Fig. 12, each of which is obtained from the previous by Reidemeister moves,

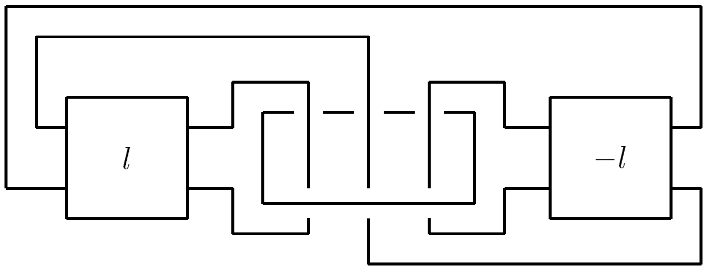

Fig. 10.

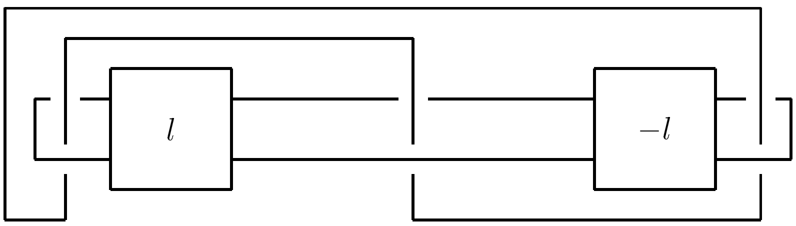

Fig. 11.

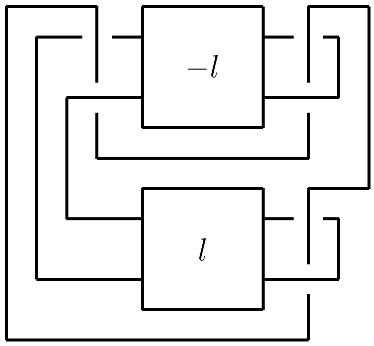

Fig. 12.

shows that components of the link $\mathcal{L}^{1/l}$ are equivalent. $\square$

Let us consider orbifolds whose singular sets are above considered links. Let us denote by $Th_n^{1/l}(2)$ the orbifold whose underlying space is the three-dimensional sphere and whose singular set is the link $Th_n^{1/l}$ with branch index 2. Similarly, let $Th_2^{1/l}(n)$ be the orbifold whose underlying space is the three-dimensional sphere and whose singular set is the link $Th_2^{1/l}$ (that is the 2-bridge knot $K(2l + \frac{1}{2l})$ in virtue of lemma 3.1) with branch index n. Also, let us consider the orbifold $\mathcal{L}^{1/l}(2, n)$ whose underlying space is the three-dimensional sphere and whose singular set is the 2-component link $\mathcal{L}^{1/l}$ with branch indices 2 and n on its components (which are equivalent in virtue of lemma 3.2). Then we have the following

Theorem 3.2 *In the above notations the following covering diagram is commutative:*

$$\begin{tikzcd} & M_n^{1/l} \arrow[dl, "2"'] \arrow[dr, "n"] & \\ Th_n^{1/l}(2) \arrow[dr, "n"'] & & Th_2^{1/l}(n) \arrow[dl, "2"] \\ & \mathcal{L}^{1/l}(2, n) & \end{tikzcd}$$

Proof. Let us consider the orbifold $\mathcal{L}^{1/l}(2, n)$. The diagram of the link was presented in Fig. 9. If we replace the boxes labelled "$1/l$" and "$-1/l$" by the corresponding tangles, we will get the diagram, pictured in Fig. 13.

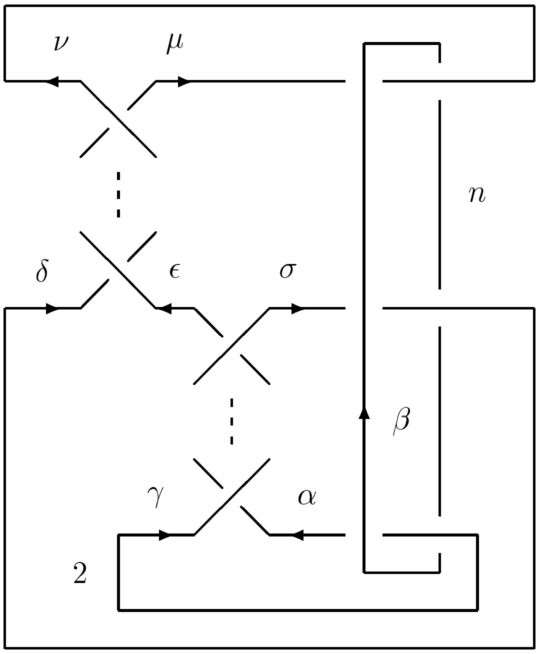

Fig. 13.

Let us consider the loops α, β, γ, δ, ϵ, σ, ν and μ, corresponding to notations in Fig. 13. For finding of the presentation of the fundamental group of the link $\mathcal{L}^{1/l}$ we will use the following remark. Let us consider a $(k,1)$–tangle for an integer k, with loops a and b, corresponding to incoming arcs, and with loops a_k and b_k, corresponding to outcoming arcs (see Fig. 14 for $k<0$).

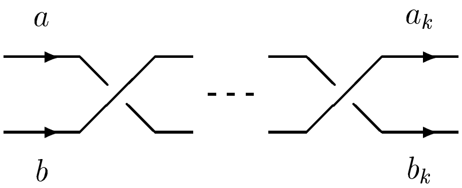

Fig. 14.

Then a_k and b_k can be expressed throught a and b by elementary formulae which hold directly by the induction, based on the Wirtinger algorithm.

Lemma 3.3 *(i) If $k<0$, then $b_k = g_k(a,b)$, $a_k = g_{k-1}(a,b)$, where*

$$g_k(a,b) \,=\, x_1^{-1}\, x_2^{-1} \,\cdots\, x_k^{-1}\, x_{k+1}\, x_k \,\cdots\, x_2\, x_1, \tag{36}$$

and $x_i = \begin{cases} b & i = 1 \mod 2, \\ a & i = 0 \mod 2; \end{cases}$

(ii) If $k > 0$, *then* $a_k = f_k(a,b)$, $b_k = f_{k-1}(a,b)$, *where*

$$f_k(a,b) = y_1\, y_2 \cdots y_k\, y_{k+1}\, y_k^{-1} \cdots y_2^{-1}\, y_1^{-1}, \tag{37}$$

and $y_i = \begin{cases} a & i = 1 \mod 2, \\ b & i = 0 \mod 2. \end{cases}$

Using lemma 3.3, and from Fig. 13 we get

$$\epsilon = f_l(\gamma, \alpha), \qquad \sigma = f_{l-1}(\gamma, \alpha),$$

$$\nu = g_{l-1}(\delta, \epsilon), \qquad \sigma = f_l(\delta, \epsilon).$$

Because

$$\gamma = \beta^{-1}\, \alpha^{-1}\, \beta, \qquad \delta = \beta^{-1}\, \sigma\, \beta,$$

we get

$$\nu = g_{l-1}(\beta^{-1}\, f_{l-1}(\beta^{-1}\alpha^{-1}\beta, \alpha)\, \beta, f_l(\beta^{-1}\alpha^{-1}\beta, \alpha)) \tag{38}$$

and

$$\mu = g_l(\beta^{-1}\, f_{l-1}(\beta^{-1}\alpha^{-1}\beta, \alpha)\, \beta, f_l(\beta^{-1}\alpha^{-1}\beta, \alpha)) \tag{39}$$

Now consider the word $w_l(\alpha, \beta) = \beta^{-1}\, \mu\, \beta\, \nu$. From (38) and (39) we get

$$\begin{aligned} w_l(\alpha,\beta) = \;& \beta^{-1}\, g_l(\beta^{-1}\, f_{l-1}(\beta^{-1}\alpha^{-1}\beta, \alpha)\, \beta, f_l(\beta^{-1}\alpha^{-1}\beta, \alpha)) \\ & \beta\, g_{l-1}(\beta^{-1}\, f_{l-1}(\beta^{-1}\alpha^{-1}\beta, \alpha)\, \beta, f_l(\beta^{-1}\alpha^{-1}\beta, \alpha)) \end{aligned} \tag{40}$$

Therefore, we get the following presentation for the link group:

$$\pi_1\left(\mathcal{S}^3 \setminus \mathcal{L}^{1/l}\right) = \langle\, \alpha,\, \beta \mid w_l(\alpha, \beta) = 1\, \rangle, \tag{41}$$

where $w_l(\alpha, \beta)$ is given by the formula (40).

Let us consider the orbifold $\mathcal{L}^{1/l}(2,n)$. Its singular set is the link $\mathcal{L}^{1/l}$ and branch indices 2 and n correspond to components, noted in Fig. 13. According to the general properties of the fundamental group of an orbifold [8], from (41) we get

$$\pi_1\left(\mathcal{L}^{1/l}(2,n)\right) = \langle\, \tilde{\alpha},\, \tilde{\beta} \mid \tilde{\alpha}^2 = \tilde{\beta}^n = 1,\, w_l(\tilde{\alpha}, \tilde{\beta}) = 1\, \rangle, \tag{42}$$

where generators $\tilde{\alpha}$ and $\tilde{\beta}$ canonically correspond to α and β from Fig. 13.

Let us consider the group

$$\mathbb{Z}_2 \oplus \mathbb{Z}_n = \langle\, a \mid a^2 = 1\, \rangle \oplus \langle\, b \mid b^n = 1\, \rangle$$

and the epimorphism

$$\theta\,:\,\pi_1(\mathcal{L}^{1/l}(2,n))\;\longrightarrow\;\mathbb{Z}_2\,\oplus\,\mathbb{Z}_n,$$

defined by setting $\theta(\tilde{\alpha})\,=\,a$ and $\theta(\tilde{\beta})\,=\,b$.

By the virtue of theorem 3.1 the Fractional Fibonacci manifold $M_n^{1/l}$ is the two-fold covering of the three-dimensional sphere, branched over the link $Th_n^{1/l}$. In other words, $M_n^{1/l}$ is the two-fold covering of the orbifold $Th_n^{1/l}(2)$. According to the definition of the link $\mathcal{L}^{1/l}$, the orbifold $Th_n^{1/l}(2)$ is the n-fold cyclic covering of the orbifold $\mathcal{L}^{1/l}(2,n)$. Hence, we get the following sequence of coverings:

$$M_n^{1/l}\;\stackrel{2}{\longrightarrow}\;Th_n^{1/l}(2)\;\stackrel{n}{\longrightarrow}\;\mathcal{L}^{1/l}(2,n),\tag{43}$$

which induces the sequence of group embeddings

$$F^{1/l}(2,2n)\,\lhd\,\pi_1(Th_n^{1/l}(2))\,\lhd\,\pi_1(\mathcal{L}^{1/l}(2,n)),\tag{44}$$

where $\mid\pi_1(\mathcal{L}^{1/l}(2,n))\,:\,\pi_1(Th_n^{1/l}(2))\mid=2$ and $\mid\pi_1(Th_n^{1/l}(2))\,:\,F^{1/l}(2,2n)\;\mid=\,n$.

By the construction of the n-fold covering $Th_n^{1/l}(2)\;\longrightarrow\;\mathcal{L}^{1/l}(2,n)$, the loop $\tilde{\beta}$ in $\pi_1(\mathcal{L}^{1/l}(2,n))$ lifts to a trivial loop in $\pi_1(Th_n^{1/l}(2))$, and the loop $\tilde{\alpha}$ in $\pi_1(\mathcal{L}^{1/l}(2,n))$ lifts to a loop in $\pi_1(Th_n^{1/l}(2))$ which generates a group of order 2. Hence,

$$\pi_1(Th_n^{1/l}(2))\;=\;\theta^{-1}(\langle a\;\mid\;a^2\;=\;1\,\rangle)\;=\;\theta^{-1}(\mathbb{Z}_2).$$

For the $2n$-fold covering $M_n^{1/l}\;\longrightarrow\;\mathcal{L}^{1/l}(2,n)$ both loops $\tilde{\alpha}$ and $\tilde{\beta}$ in $\pi_1(\mathcal{L}^{1/l}(2,n))$ lift to trivial loops in $F^{1/l}(2,2n)$, hence $F^{1/l}(2,2n)\;=\;Ker\,\theta$.

Let $\Gamma(l,n)$ be the subgroup of $\pi_1(\mathcal{L}^{1/l}(2,n))$ given by

$$\Gamma(l,n)\;=\;\theta^{-1}(\langle b\;\mid\;b^n\;=\;1\,\rangle)\;=\;\theta^{-1}(\mathbb{Z}_n).$$

Thus we have a sequence of normal subgroups

$$F^{1/l}(2,2n)\,\lhd\,\Gamma(l,n)\,\lhd\,\pi_1(\mathcal{L}^{1/l}(2,n)),\tag{45}$$

where $\mid\pi_1(\mathcal{L}^{1/l}(2,n))\,:\,\Gamma(l,n)\mid=\,n$ and $\mid\,\Gamma(l,n)\,:\,F^{1/l}(2,2n)\;\mid=\,2$.

In the two-fold covering, corresponding to the embedding $\Gamma(l,n)\lhd\pi_1(\mathcal{L}^{1/l}(2,n))$, the loop $\tilde{\alpha}$ lifts to a trivial loop, and the loop $\tilde{\beta}$ lifts to a loop which generate a cyclic group of order n. Therefore, this two-fold covering is branched over one of components of the link $\mathcal{L}^{1/l}$. In virtue of lemma 3.2 its components are equivalent. Whence, using lemma 3.1 we get that this covering space is the orbifold $Th_2^{1/l}(n)$ which singular set is the two-bridge knot $K(2l+\frac{1}{2l})$. Considering the embedding $F^{1/l}(2,2n)\,\lhd\,\Gamma(l,n)$ we get that the manifold $M_n^{1/l}$ is the n-fold cyclic covering of the orbifold $Th_2^{1/l}(n)$. Hence, from embeddings (44) and (45) we get that the covering diagram from the statement of the theorem is commutative. □

We recall that the particular case $l=1$ of the theorem 3.2 for the Fibonacci manifolds was considered in [24].

Corollary 3.1 *For any integers l and n the Fractional Fibonacci manifold $M_n^{1/l}$ is the n-fold cyclic covering of the three-dimensional sphere, branched over the two-bridge knot $K(2l+\frac{1}{2l})$.*

We remark in particularly that if $l=1$ then M_n is a Fibonacci manifold and the knot $K(5/2)$ is the figure-eight knot.

Because the two-fold coverings of the three-dimensional sphere, branched over a two-bridge knot, are well-studied ([2], p. 183), we get the following statement for the case $n=2$.

Corollary 3.2 *For any integer l the Fractional Fibonacci manifold $M_2^{1/l}$ is the lens space $L(4l^2+1, 2l)$.*

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