DIFFUSIONS ON PATH AND LOOP SPACES: EXISTENCE, FINITE DIMENSIONAL APPROXIMATION AND HÖLDER CONTINUITY

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1. Introduction and results

In his paper [K95], T. Kazumi gave a construction for the Ornstein–Uhlenbeck process on the path space over a compact Riemannian manifold by finite dimensional approximation. He also derived regularity properties of the sample paths. A major ingredient in his proof is an estimate for the Ornstein–Uhlenbeck operator (resp. its finite–dimensional projections) applied to certain distance functions. This estimate, which is used to prove tightness of the laws of the approximating finite–dimensional diffusions, seems to rely heavily on the special situation considered. In particular, it is not obvious how to carry over the result to Ornstein–Uhlenbeck operators on *loop* spaces. In fact, in contrast to first order objects, second order objects on path spaces over manifolds are rather complicated.

On the other hand, there is a well–known method for proving tightness and Hölder–continuity based on the Lyons–Zheng decomposition (cf. [LZhe88], [LZ94]), which only involves estimations of first order derivatives. This method has been successfully applied to approximate Hilbert–space– and $C([0,1],\mathbb{R}^m)$–valued diffusions, cf. [RZ95], [LRZ95].

In this paper, we describe two approaches to existence, finite-dimensional approximation, and Hölder continuity of symmetric diffusions on path and loop spaces over manifolds, cf. Theorems 1 and 2 below. Both apply not only to Ornstein–Uhlenbeck processes, but also to more general diffusions, provided an easy to check condition holds (cf. (A 1), (A 2) resp. (A 3) below, see also the examples in Section 5). Our first approach (Theorem 1) is somehow similar to Kazumi's proof for the Ornstein–Uhlenbeck operator on path space, but it is based on estimates following from the Lyons–Zheng decomposition. The point of view taken here is to look at path–space–valued processes as manifold–valued two–parameter processes. We obtain both existence/approximation and Hölder continuity results. If one is not interested in Hölder continuity but only existence and finite dimensional approximation, it can be more convenient to view path–space–valued processes as infinite dimensional rather than two–parameter processes. This is carried out in our second approach, leading to a stronger approximation result (Theorem 2), which contains one of the main results from [LRZ95] as a special case. Our method of proof here is partially motivated by an idea of K. Kuwae and T. Uemura [KuwU95].

To state our results in detail, let M be a connected submanifold of $\mathbb{R}^m$, $m \in \mathbb{N}$. We assume that M carries a Riemannian metric g such that

$$K^{-1}|v| \le g(v,v)^{1/2} \le K|v| \quad \text{for all } v \in TM \tag{1.1}$$

for some constant $K \in [1,\infty)$. This is always the case if (M,g) is a compact Riemannian manifold embedded into $\mathbb{R}^m$ by Whitney's theorem. Let d be the corresponding distance function. We assume that M is *complete* w.r.t. d.
The space $C([0,1],M)$ of continuous paths on M with supremum distance

$$d_\infty(\omega,\omega') = \sup_{s\in[0,1]} d(\omega(s),\omega'(s))$$

may be viewed as an infinite–dimensional manifold. Suppose Ω is a subset of type

$$\Omega = \{\omega \in C([0,1],M);\ \varphi(\omega(0),\omega(1)) = 0\}$$

where $\varphi \in C^\infty(M\times M,\mathbb{R}^\ell), \ell \in \mathbb{N}$, is a fixed map.
Ω may be the space of paths resp. loops on M starting at a fixed point, or the space of all paths resp. loops. We fix a probability measure μ on Ω such that supp $\mu = \Omega$. Let

$$\begin{aligned}\mathcal{F}C_b^\infty(\Omega) := \{F:\Omega\to\mathbb{R};\ F(\omega) = f(\omega(s_1),\dots,\omega(s_n)) \quad \text{for some}\\ n\in\mathbb{N},\ s_1,\dots,s_n\in[0,1] \text{ and } f\in C_b^\infty(M^n)\}\end{aligned}$$

be the cylinder functions on Ω. Here $C_b^\infty(M^n)$ denotes the set of all smooth functions on M^n having bounded derivatives of any order.
Suppose we are given a quadratic form $\mathcal{E}$ on $\mathcal{F}C_b^\infty(\Omega)$ of type

$$\mathcal{E}(F,G) = \int \Gamma(F,G)\ d\mu\ ,$$

where $\Gamma : \mathcal{F}C_b^\infty(\Omega)\times\mathcal{F}C_b^\infty(\Omega) \to L^1(\Omega,\mu)$ is a symmetric bilinear operator satisfying $\Gamma(F,F)\ge 0$ μ–a.e., and the product rule

$$\Gamma(FG,H) = F\cdot\Gamma(G,H) + G\cdot\Gamma(F,H) \quad \text{for all } F,G,H\in\mathcal{F}C_b^\infty(\Omega)\ . \tag{1.2}$$

We also write $\mathcal{E}(F)$ and $\Gamma(F)$ instead of $\mathcal{E}(F,F)$ and $\Gamma(F,F)$. The main examples we have in mind are those where

$$\Gamma(F,G) = \langle DF, DG\rangle \tag{1.3}$$

for some kind of gradient D on Ω taking values in a suitable "generalized tangent bundle" to Ω, and some suitable "generalized metric" $\langle\cdot,\cdot\rangle$ on the tangent bundle. We refer to the examples in Section 5 below and to the extensive literature for the definitions of well–suited gradients for the various path and loop spaces, cf. e.g. [MalMal90], [DR92], [ALeR92], [FMal93]. Since the choice of the "metric" is left open, (1.3) includes also the cases where $\Gamma(F,G) = \langle DF, ADG\rangle$ for a nice, fibrewise linear symmetric operator A on the "tangent bundle".

We assume that $(\mathcal{E}, FC_b^\infty)$ is a closable form on $L^2(\Omega,\mu)$. This has been verified for the standard gradients by proving integration by parts identities, cf. the references given above, [ElLi95] and [EnSt95]. The closure is denoted by $(\mathcal{E}, D(\mathcal{E}))$. It is a positive closed quadratic form, and hence associated with a negative definite self–adjoint linear operator $(L, D(L))$ on $L^2(\Omega,\mu)$. Note that in the situation of (1.3), L is equal to $-D^*\overline{D}$, where $\overline{D}$ denotes the L^2–closure of the gradient and D^* its adjoint. In particular, if D is one of the standard gradients on the various path resp. loop spaces, then L is the corresponding Ornstein–Uhlenbeck operator.

We say that an Ω–valued stochastic process $((X_t)_{t\geq 0}, P)$ is a *stationary diffusion generated by* L, if

$$\begin{aligned} &E\left[F_0(X_0)F_1(X_{t_1})F_2\left(X_{t_1+t_2}\right)\cdots F_\ell\left(X_{t_1+\cdots+t_l}\right)\right] \\ &= \int F_0\, e^{t_1L}\left(F_1e^{t_2L}\left(F_2\cdots e^{t_\ell L}F_\ell\right)\cdots\right)\, d\mu \end{aligned}$$

for all $\ell\in\mathbb{N}$, $t_0,\dots,t_\ell\in[0,\infty)$, and all bounded functions $F_0,\dots,F_\ell:\Omega\to\mathbb{R}$. Here E denotes expectation w.r.t. P.

We want to give conditions on Γ (resp. D) and μ that are easy to check in examples providing the following:

1) **Existence** of a stationary diffusion generated by L.
2) **Convergence** to this diffusion of diffusion processes living on suitable finite dimensional manifolds (that is on the spaces of piecewise minimal geodesics on M or piecewise linear paths in $\mathbb{R}^m$ with base–points on M, adapted to fixed partitions of $[0,1]$).
3) **Hölder–continuity** of the sample paths w.r.t. d_∞.

Weak conditions for existence have already been given in [RSchm95] using a general machinery for constructing diffusions with non–locally compact state space (cf. [MR92]). However, in our concrete situation it seems more natural and useful for applications to obtain the process as a limit of finite dimensional diffusions. This is what we do here. The price we have to pay is that we cannot start the diffusion from quasi–every path $\omega\in\Omega$ but only with the invariant measure μ.

We first introduce the approximating objects: Let E denote the Banach space $C([0,1],\mathbb{R}^m)$. We may regard Ω as a submanifold of E. For $n\in\mathbb{N}$ let $\mathcal{D}_n := \{i\cdot 2^{-n}; 0\leq i\leq 2^n\}$ be the n–th dyadic partition of $[0,1]$ and $\Pi_n: E\to(\mathbb{R}^m)^{2^n+1}$, $\Pi_n(\omega) = (\omega(s))_{s\in\mathcal{D}_n}$ the corresponding projection map on path space. Let $\Omega_n := \Pi_n(\Omega)$. Ω_n is a closed submanifold of M^{2^n+1}.

To plug Ω_n back into path space we define

$$i_n : M^{2^n+1}\subset(\mathbb{R}^m)^{2^n+1}\to E, \qquad i_n(x_0,\dots,x_{2^n}) = \omega,$$

where $\omega(i\cdot 2^{-n}) = x_i$ $(0\leq i\leq 2^n)$, and ω is given by piecewise linear interpolation inbetween. Then i_n is a continuous embedding.

Unfortunately, there is no analogous way of embedding M^{2^n+1} (resp. Ω_n) continuously into $C([0,1],M)$ (resp. Ω) in general. However, if $r(x)$ denotes the injectivity radius of a point $x \in M$, then we may at least embed the open subset

$$U_n = \left\{ (x_0, \dots, x_{2^n}) \in M^{2^n+1},\ d\left(x_i, x_{i+1}\right) < r\left(x_i\right)\ \forall\, 0 \le i < 2^n \right\} \tag{1.4}$$

continuously into $C([0,1],M)$ in a natural way. In fact, for $x, y \in M$ such that $d(x,y) < r(x)$ let $\gamma_{x,y} : [0, 2^{-n}] \to M$ be the unique minimal geodesic joining x and y parametrized proportional to arc length. It is well known that $\gamma_{x,y}$ depends continuously on x and y. Hence a continuous embedding j_n of U_n into $C([0,1],M)$ is given by $j_n(x_0, \dots, x_{2^n}) = \omega$, where $\omega(i \cdot 2^{-n} + s) = \gamma_{x_i, x_{i+1}}(s)$ for all $0 \le i < 2^n$ and $s \in [0, 2^{-n}]$. Obviously, $j_n(\Omega_n \cap U_n) \subseteq \Omega$.

We now show how to obtain a diffusion process on Ω_n from the quadratic form $(\mathcal{E}, D(\mathcal{E}))$ (resp. its finite dimensional projections): Let $\mu_n := \mu \circ \Pi_n^{-1}$. For restrictions f, g of functions in $C_b^\infty(M^{2^n+1})$ to Ω_n we define

$$\mathcal{E}^{(n)}(f,g) \;:=\; \mathcal{E}(f \circ \Pi_n, g \circ \Pi_n)\ . \tag{1.5}$$

Using that $(\mathcal{E}, \mathcal{F}C_b^\infty(\Omega))$ is closable on $L^2(\Omega, \mu)$, one immediately verifies that $\mathcal{E}^{(n)}$ is a closable positive quadratic form on $L^2(\Omega_n, \mu_n)$.

We denote the closure by $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$, and the associated negative–definite self–adjoint operator on $L^2(\Omega_n, \mu_n)$ by $(L^{(n)}, D(L^{(n)}))$. We stress that in many of the concrete examples we are interested in, $L^{(n)}$ can be identified explicitly as a certain elliptic differential operator in divergence form on Ω_n (see the examples in Section 5 below).

Under the assumptions we impose below (cf. (A 1) or (A 3)), one can show that there is a stationary Ω_n–valued diffusion process $((X_t^{(n)}), P^{(n)})$ generated by $L^{(n)}$. In general, this follows from Fukushima's method for constructing diffusions associated with regular local Dirichlet forms on locally compact separable metric spaces, cf. Appendix B. In concrete applications, where $L^{(n)}$ is a nice elliptic differential operator, the diffusion can be obtained more directly, e.g. by solving the corresponding stochastic differential equation.

Let $\widehat{E} := C([0,\infty) \times [0,1], \mathbb{R}^m) \cong C([0,\infty), E)$, and $\widehat{\Omega} := \{\hat\omega \in \widehat{E};\ \hat\omega(t,\cdot) \in \Omega\ \forall t \ge 0\}$ $(\cong C([0,\infty), \Omega))$. The process $i_n(X_t^{(n)})$, $t \ge 0$, is under $P^{(n)}$ an E–valued diffusion. We denote its law on $\widehat{E}$ by P_n.

For $\alpha \in (0, \frac{1}{2})$ let $\widehat{\Omega}_\alpha$ denote the locally Hölder–continuous functions of order α in $\widehat{\Omega}$, i.e. all $\hat\omega \in \widehat{\Omega}$ such that

$$\sup \left\{ \frac{d(\hat\omega(t,s), \hat\omega(t',s'))}{|(t,s) - (t',s')|^\alpha}\ ;\ t, t' \in [0,T],\ s, s' \in [0,1],\ (t,s) \neq (t',s') \right\} < \infty$$

for all $T > 0$. For $\hat\omega \in \widehat{\Omega}_\alpha$, the Ω–valued function $t \mapsto \hat\omega(t,\cdot)$ is locally Hölder–continuous of order β w.r.t. d_∞ for all $\beta \in (0,\alpha)$. More generally, it is $C_\beta([0,1],M)$–valued and locally Hölder–continuous of oder γ w.r.t. the distance function in $C_\beta([0,1],M)$ whenever $\beta, \gamma > 0$ such that $\beta + \gamma < \alpha$, see Appendix A. Here

$C_\beta([0,1],M)$ denotes the functions in $C([0,1],M)$ that are Hölder–continuous of order β.

For $\hat\omega \in \widehat{E}$, $t \in [0,\infty)$ and $s \in [0,1]$ let $X_{t,s}(\hat\omega) := \hat\omega(t,s)$ and $X_t(\hat\omega) := X_{t,\cdot}(\hat\omega) \in E$. We fix a constant $p > 4$. Our basic assumptions are:

(A 1) There is a constant $c_\Gamma \in (0,\infty)$ such that

$$\int \Gamma(F)^{p/2}\, d\mu \;\le\; c_\Gamma \cdot \sup_{x\in M} |df|^p_{T^*_x M}\ ,$$

whenever $F(\omega) = f(\omega(s))$ for some $s \in [0,1]$ and $f \in C_b^\infty(M)$.

(A 2) There is a constant $c_\mu \in (0,\infty)$ such that

$$\int d(\omega(s),\omega(s'))^p\ \mu(d\omega) \;\le\; c_\mu \cdot |s-s'|^{p/2}$$

for all $s, s' \in [0,1]$.

Note that (A 2) is the usual condition for proving that the initial distribution μ is supported by Hölder–continuous paths of order α for all $\alpha \in (0, \frac{1}{2} - \frac{1}{p})$.

We can now state our first main result.

Theorem 1. *Suppose (A 1) and (A 2) hold. Then the sequence $(P_n)_{n\in\mathbb{N}}$ converges weakly to a probability measure P on $\widehat{E}$ such that $P[\widehat{\Omega}_\alpha] = 1$ for all $\alpha \in (0, \frac{1}{2} - \frac{2}{p})$. The a.s. Ω–valued process $((X_t)_{t\ge 0}, P)$ is a stationary diffusion generated by L.*

In particular, *there is* a stationary diffusion generated by L, and it has a.s. *locally Hölder–continuous paths of order γ w.r.t. d_∞* for each $\gamma \in (0, \frac{1}{2} - \frac{2}{p})$. It even has locally Hölder–continuous paths of order γ in $C_\beta([0,1],M)$ whenever $\gamma + \beta < \frac{1}{2} - \frac{2}{p}$. The proof of Theorem 1 will be given in Section 3 below.

We can also approximate $((X_t)_{t\ge 0}, P)$ by Ω–valued processes. Define stopping times $\tau_n := \inf\,\{t \ge 0\,;\, X_t^{(n)} \notin U_n\}$, $n \in \mathbb{N}$ (cf. (1.4) for the definition of U_n). The process $j_n(X_t^{(n)})$, $0 \le t < \tau_n$, is under $P^{(n)}$ an Ω–valued diffusion with life–time $\tau_n \in [0,\infty]$. For $T > 0$ let Q_n denote the restriction of the law of $((j_n(X_t^{(n)})), P^{(n)})$ to the space of all $C([0,1],M)$–valued continuous paths with life–time greater than T. We regard the measures Q_n, $n \in \mathbb{N}$, as sub–probability measures on $C([0,T]\times[0,1],M)$. As a consequence of Theorem 1 we obtain:

Corollary. Let P be as in Theorem 1. Then $(Q_n)_{n\in\mathbb{N}}$ converges to P weakly on $C([0,T]\times[0,1],M)$ for all $T > 0$.

For the proof see Section 3 below.

If we are just interested in existence and finite dimensional approximation but not Hölder–continuity, then the assumptions of Theorem 1 may be weakened. We replace (A 1) and (A 2) by:

(A 3) There is a function $A_\Gamma \in \bigcup_{q>1} L^q(\Omega,\mu)$ such that

$$\Gamma(F) \;\le\; A_\Gamma \cdot \sup_{x\in M} |df|^2_{T^*_x M} \qquad \mu\text{–a.e. },$$

whenever $F(\omega) = f(\omega(s))$ for some $s \in [0,1]$ and $f \in C_b^\infty(M)$.

Hence we partially relax condition (A 1), and we drop the assumption on μ completely. (A 3) is similar to a condition for existence obtained in [RSchm95] by completely different means. By considering diffusions on path space as infinite–dimensional rather than two–parameter processes we can show:

Theorem 2. *Suppose (A 3) holds. Then the assertions of Theorem 1 hold apart from the Hölder–continuity.*

The proof will be given in Section 4 below. In Section 5 we apply our results to Ornstein–Uhlenbeck and more general diffusion operators on path and loop spaces of compact manifolds and $\mathbb{R}^m$.

2. Basic estimates

The bilinear operator $\Gamma : \mathcal{F}C_b^\infty(\Omega) \times \mathcal{F}C_b^\infty(\Omega) \to L^1(\Omega,\mu)$ has a unique continuous extension to $D(\mathcal{E})$. In particular, for $n \in \mathbb{N}$ and $f \in D(\mathcal{E}^{(n)})$, $f \circ \Pi_n$ is in $D(\mathcal{E})$ as can be seen easily. Hence $\Gamma(f \circ \Pi_n)$ is well–defined. We denote the expectations w.r.t. $P^{(n)}$ and P_n by $E^{(n)}$ resp. E_n. All our proofs are based on the following lemma (cf. e.g. [LZ94] for a similar estimate):

Lemma 1. *For all $p > 2$ there is a universal constant $B_p \in (0,\infty)$ such that*

$$E^{(n)}\left[\left|f(X_t^{(n)}) - f(X_{t'}^{(n)})\right|^p\right] \le B_p \cdot |t-t'|^{p/2} \cdot \int \Gamma(f \circ \Pi_n)^{p/2}\, d\mu$$

for all $t, t' \ge 0$, all $n \in \mathbb{N}$, and all functions $f \in D(\mathcal{E}^{(n)})$.

Proof. Fix $p > 2$, $T > 0$ and $n \in \mathbb{N}$. As shown in Appendix B, $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ is a regular Dirichlet form with local property. The associated diffusion $((X_t^{(n)}), P^{(n)})$ is conservative. Hence, by [LZhe88], [LZ94], for any function $f \in D(\mathcal{E}^{(n)})$ we have the decomposition

$$f(X_t^{(n)}) - f(X_0^{(n)}) = \tfrac{1}{2}\, M_t^{[f]} - \tfrac{1}{2}(\overline{M}_T^{[f]} - \overline{M}_{T-t}^{[f]}), \quad 0 \le t \le T, \;\; P^{(n)}\text{–a.s.},$$

where $(M_t^{[f]})_{0\le t\le T}$ and $(\overline{M}_t^{[f]})_{0\le t\le T}$ are martingales w.r.t. the filtrations $\mathcal{F}_t := \sigma(X_\tau^{(n)};\ 0 \le \tau \le t)$ resp. $\overline{\mathcal{F}}_t := \sigma(X_{T-\tau}^{(n)};\ 0 \le \tau \le t)$. Moreover, it is easy to check that $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ admits a carré du champ operator $\Gamma^{(n)}$ which is determined by

$$\Gamma^{(n)}(f,g) \circ \Pi_n \;=\; E_\mu\left[\,\Gamma(f \circ \Pi_n, g \circ \Pi_n)\,|\,\Pi_n\,\right] \qquad \mu\text{–a.e.},$$

cf. [BH91, Ch. V, Prop. 1.1.5] for details. Here E_μ denotes conditional expectation w.r.t. μ. Hence

$$\begin{aligned} \langle M^{[f]} \rangle \;&=\; \int_0^\bullet \Gamma^{(n)}(f)(X_t^{(n)})\, dt, \qquad \text{and} \\ \langle \overline{M}^{[f]} \rangle \;&=\; \int_0^\bullet \Gamma^{(n)}(f)(X_{T-t}^{(n)})\, dt \qquad P^{(n)}\text{–a.s.} \end{aligned}$$

We obtain

$$
\begin{aligned}
&E^{(n)}\left[\left|f(X_t^{(n)})-f(X_{t'}^{(n)})\right|^p\right]^{1/p}\\
&\quad\le \frac{1}{2}\,E^{(n)}\left[\left|M_t^{[f]}-M_{t'}^{[f]}\right|^p\right]^{1/p}+\frac{1}{2}\,E^{(n)}\left[\left|\overline{M}_{T-t}^{[f]}-\overline{M}_{T-t'}^{[f]}\right|^p\right]^{1/p}\\
&\quad\le \frac{1}{2}\,B_p\cdot\left(E^{(n)}\left[\left(\int_{t'}^{t}\Gamma^{(n)}(f)\left(X_\tau^{(n)}\right)\,d\tau\right)^{p/2}\right]^{1/p}\right.\\
&\qquad\qquad\left.+E^{(n)}\left[\left(\int_{T-t}^{T-t'}\Gamma^{(n)}(f)\left(X_\tau^{(n)}\right)\,d\tau\right)^{p/2}\right]^{1/p}\right)\\
&\quad\le \frac{1}{2}\,B_p\cdot(t-t')^{\frac{1}{2}-\frac{1}{p}}\cdot\left[\left(\int_{t'}^{t}E^{(n)}\left[\Gamma^{(n)}(f)\left(X_\tau^{(n)}\right)^{p/2}\right]d\tau\right)^{1/p}\right.\\
&\qquad\qquad\left.+\left(\int_{T-t}^{T-t'}E^{(n)}\left[\Gamma^{(n)}(f)\left(X_\tau^{(n)}\right)^{p/2}\right]d\tau\right)^{1/p}\right]\\
&\quad= B_p\cdot(t-t')^{\frac{1}{2}}\,\left(\int\Gamma^{(n)}(f)^{p/2}\,d\mu_n\right)^{1/p}\;\le\;B_p\cdot(t-t')^{\frac{1}{2}}\,\left(\int\Gamma(f\circ\Pi_n)^{p/2}\,d\mu\right)^{1/p}
\end{aligned}
$$

for all $0\le t'\le t\le T$, where $B_p\in(0,\infty)$ is the constant from the Burkholder–Gundy inequality. Here we used that $(X_t^{(n)})$ is stationary w.r.t. $P^{(n)}$. Since B_p is independent of T, the assertion follows. □

As a direct consequence of Lemma 1 we note:

Lemma 2. *Suppose (A 1) holds. Then there is a constant $c_0\in(0,\infty)$ such that*

$$E_n\left[|X_{t,s}-X_{t',s}|^p\right]^{1/p}\le c_0\cdot|t-t'|^{1/2}$$

for all $n\in\mathbb{N}$, $s\in[0,1]$ and $t,t'\in[0,\infty)$.

Proof. Fix $n\in\mathbb{N}$. For $\hat{x}=(x_0,\dots,x_{2^n})\in M^{2^n+1}\subset\mathbb{R}^{m\cdot(2^n+1)}$, $1\le j\le m$, and $0\le i\le 2^n$ let $p_i^{(j)}(\hat{x}):=x_i^{(j)}$ (j–th coordinate of x_i in $\mathbb{R}^m$). We fix functions $\psi_\ell\in C_b^\infty(\mathbb{R})$, $\ell\in\mathbb{N}$, such that $\psi_\ell(r)=r$ for all $r\in[-\ell,\ell]$ and $|\psi_\ell'|\le 1$. Then, because of (1.1), $\psi_\ell\circ p_i^{(j)}\in C_b^\infty\left(M^{2^n+1}\right)\subset D\left(\mathcal{E}^{(n)}\right)$ for all i,j,ℓ as above. Hence, by Lemma 1,

$$
\begin{aligned}
(2.1)\qquad &E^{(n)}\left[\left|(X_t^{(n)})_i-(X_{t'}^{(n)})_i\right|^p\right]^{1/2}\\
&\le \sum\nolimits_{j=1}^{m}E^{(n)}\left[\left|p_i^{(j)}(X_t^{(n)})-p_i^{(j)}(X_{t'}^{(n)})\right|^p\right]^{1/p}\\
&\le \sum\nolimits_{j=1}^{m}\varliminf_{\ell\to\infty}E^{(n)}\left[\left|\psi_\ell\circ p_i^{(j)}(X_t^{(n)})-\psi_\ell\circ p_i^{(j)}(X_{t'}^{(n)})\right|^p\right]^{1/p}\\
&\le |t-t'|^{1/2}\cdot B_p^{1/p}\cdot\sum\nolimits_{j=1}^{m}\varliminf_{\ell\to\infty}\left(\int\Gamma\left(\psi_\ell\circ p_i^{(j)}\circ\Pi_n\right)^{p/2}\,d\mu\right)^{1/p}
\end{aligned}
$$

for all $t, t' \in [0,\infty)$ and $0 \le i \le 2^n$. Here $(X_t^{(n)})_i\,(\in M \subset \mathbb{R}^m)$ is the i–th component of $X_t^{(n)}$.
But $\psi_\ell \circ p_i^{(j)} \circ \Pi_n(\omega) = \psi_\ell\left(\omega^{(j)}\,(i\cdot 2^{-n})\right)$ for all $\omega \in \Omega$, $\omega^{(j)}$ denoting the j–th coordinate in $\mathbb{R}^m$. Hence, by Assumption (A 1) and (1.1),

$$\left(\int \Gamma\left(\psi_\ell \circ p_i^{(j)} \circ \Pi_n\right)^{p/2}\, d\mu\right)^{1/p} \le c_\Gamma^{1/p}\cdot \|\psi_\ell'\|_\infty \cdot K \le c_\Gamma^{1/p}\cdot K$$

for all i, j, ℓ as above. Consequently, the right hand side of (2.1) is dominated by $c_0 \cdot |t-t'|^{1/2}$ where $c_0 := B_p^{1/p}\, m\, c_\Gamma^{1/p} K$ is a finite constant.
Now fix $s \in [0,1]$, $s = (i+\alpha)\cdot 2^{-n}$, where $0 \le i < 2^n$ and $\alpha \in [0,1]$. Then $X_{t,s} = (1-\alpha) X_{t,i\cdot 2^{-n}} + \alpha\, X_{t,(i+1)\cdot 2^{-n}}$ P_n–a.s. for all $t \ge 0$. Hence

$$\begin{aligned}
&E_n\left[|X_{t,s} - X_{t',s}|^p\right]^{1/p}\\
&\quad\le (1-\alpha)\; E_n\left[\left|X_{t,i\cdot 2^{-n}} - X_{t',i\cdot 2^{-n}}\right|^p\right]^{1/p} + \alpha\, E_n\left[\left|X_{t,(i+1)\cdot 2^{-n}} - X_{t',(i+1)\cdot 2^{-n}}\right|^p\right]^{1/p}\\
&\quad= (1-\alpha)\; E^{(n)}\left[\left|(X_t^{(n)})_i - (X_{t'}^{(n)})_i\right|^p\right]^{1/p} + \alpha\, E^{(n)}\left[\left|(X_t^{(n)})_{i+1} - (X_{t'}^{(n)})_{i+1}\right|^p\right]^{1/p}\\
&\quad\le c_0 \cdot |t-t'|^{1/2} \qquad\qquad \text{for all } t, t' \ge 0\ .
\end{aligned}$$

□

If in addition to (A 1) we assume (A 2), then we obtain the following estimates that are the key ingredients in our first tightness proof for $\{P_n; n \in \mathbb{N}\}$ resp. in the proof of Hölder–continuity (cf. Section 3).

Lemma 3. *If (A 1) and (A 2) hold, then there is a constant $c_1 \in (0,\infty)$ such that*

$$E_n\left[|X_{t,s} - X_{t',s'}|^p\right]^{1/p} \le c_1\left|(t,s) - (t',s')\right|^{1/2}$$

for all $n \in \mathbb{N}$, $t, t' \in [0,\infty)$ and $s, s' \in [0,1]$.

Proof. Let $n \in \mathbb{N}$ and $t \in [0,\infty)$. First, note that for all $0 \le i,\ j \le 2^n$,

$$\begin{aligned}
E_n\left[\left|X_{t,i\cdot 2^{-n}} - X_{t,j\cdot 2^{-n}}\right|^p\right] &= E^{(n)}\left[\left|(X_t^{(n)})_i - (X_t^{(n)})_j\right|^p\right]\\
&= \int |x_i - x_j|^p\, \mu_n(d\hat{x}) = \int \left|\omega(i\cdot 2^{-n}) - \omega(j\cdot 2^{-n})\right|^p \mu(d\omega)\\
&\le K^p \cdot \int d\left(\omega(i\cdot 2^{-n}), \omega(j\cdot 2^{-n})\right)^p \mu(d\omega)\\
&\le c_\mu \cdot K^p \cdot \left|i\cdot 2^{-n} - j\cdot 2^{-n}\right|^{p/2}\ .
\end{aligned}$$

Here we used that by (1.1) the Euclidean distance of two points on $M \subset \mathbb{R}^m$ can be estimated by K times their distance on M, and applied Assumption (A 2).
Next, we remark that the obtained estimate

$$E_n\left[|X_{t,s} - X_{t,s'}|^p\right]^{1/p} \le c_\mu^{1/p}\cdot K \cdot |s-s'|^{1/2} \qquad \forall\, t \in [0,\infty)\,,\ n \in \mathbb{N}, \tag{2.2}$$

carries over from $s, s' \in \mathcal{D}_n$ to any $s, s' \in [0,1]$. In fact: Fix $s \in [0,1]$, $s = (i+\alpha)2^{-n}$ for some $0 \le i < 2^n$ and $\alpha \in [0,1]$. If s' is a real in the same dyadic interval, i.e. $s' = (i+\alpha')\cdot 2^{-n}$ for some $\alpha' \in [0,1]$, then $X_{t,s} = (1-\alpha)\cdot X_{t,i\cdot 2^{-n}} + \alpha \cdot X_{t,(i+1)\cdot 2^{-n}}$ and $X_{t,s'} = (1-\alpha')\cdot X_{t,i\cdot 2^{-n}} + \alpha' \cdot X_{t,(i+1)\cdot 2^{-n}}$ P_n–a.s., hence

$$\begin{aligned} E_n\left[|X_{t,s} - X_{t,s'}|^p\right]^{1/p} &= |\alpha - \alpha'| \cdot E_n\left[\left|X_{t,(i+1)\cdot 2^{-n}} - X_{t,i\cdot 2^{-n}}\right|^p\right]^{1/p} \\ &\le |\alpha - \alpha'|^{\frac{1}{2}} \cdot c_\mu^{1/p} \cdot K \cdot 2^{-\frac{n}{2}} = c_\mu^{1/p} \cdot K \cdot |s - s'|^{\frac{1}{2}} \ . \end{aligned}$$

On the other hand, for $s' \in \mathcal{D}_n$ we have

$$\begin{aligned} E_n\left[|X_{t,s} - X_{t,s'}|^p\right]^{1/p} \ &= \ E_n\left[\left|(1-\alpha)\cdot X_{t,i\cdot 2^{-n}} + \alpha \cdot X_{t,(i+1)2^{-n}} - X_{t,s'}\right|^p\right]^{1/p} \\ &\le (1-\alpha)\cdot E_n\left[\left|X_{t,i\cdot 2^{-n}} - X_{t,s'}\right|^p\right]^{1/p} + \alpha \cdot E_n\left[\left|X_{t,(i+1)\cdot 2^{-n}} - X_{t,s'}\right|^p\right]^{1/p} \\ &\le c_\mu^{1/p} \cdot K \cdot \left((1-\alpha)\left|i\cdot 2^{-n} - s'\right|^{1/2} + \alpha\left|(i+1)\cdot 2^{-n} - s'\right|^{1/2}\right) \\ &\le c_\mu^{1/p} \cdot K \cdot \left|(1-\alpha)(i\cdot 2^{-n} - s') + \alpha((i+1)\cdot 2^{-n} - s')\right|^{1/2} = c_\mu^{1/p} K \left|s - s'\right|^{1/2} \ , \end{aligned}$$

since $i\cdot 2^{-n} - s'$ and $(i+1)\cdot 2^{-n} - s'$ are either both positive or negative, and $x \mapsto x^{1/2}$ is concave on $[0,\infty)$ resp. $(-\infty, 0]$. Applying the same argument once more with roles of s and s' interchanged, we see that (2.2) is also true, if s and s' are *not* in the same dyadic interval.

Now we obtain the desired estimates by combining (2.2) and Lemma 2:

$$\begin{aligned} &E_n\left[|X_{t,s} - X_{t',s'}|^p\right]^{1/p} \le E_n\left[|X_{t,s} - X_{t,s'}|^p\right]^{1/p} + E_n\left[|X_{t,s'} - X_{t',s'}|^p\right]^{1/p} \\ &\quad \le c_\mu^{1/p} \cdot K \cdot |s - s'|^{1/2} + c_0 \cdot |t - t'|^{1/2} \le c_1 \cdot |(t,s) - (t',s')|^{1/2}, \\ &E_n\left[|X_{t,s} - X_{t',s} - X_{t,s'} + X_{t',s'}|^p\right]^{1/p} \\ &\quad \le E_n\left[(|X_{t,s} - X_{t',s}| + |X_{t,s'} - X_{t',s'}|)^p\right]^{1/2p} \cdot E_n\left[(|X_{t,s} - X_{t,s'}| + |X_{t',s} - X_{t',s'}|)^p\right]^{1/2p} \\ &\quad \le \left(2\cdot c_0|t - t'|^{\frac{1}{2}}\right)^{\frac{1}{2}} \cdot \left(2c_\mu^{1/p} \cdot K \cdot |s - s'|^{\frac{1}{2}}\right)^{\frac{1}{2}} \ = \ c_2 \cdot |t - t'|^{\frac{1}{4}} \cdot |s - s'|^{\frac{1}{4}} \end{aligned}$$

for all $n \in \mathbb{N}$, $t, t' \in [0,\infty)$ and $s, s' \in [0,1]$, where $c_1 := c_\mu^{1/p} K + c_0$ and $c_2 := 2\cdot\left(c_0 c_\mu^{1/p} K\right)^{1/2}$ are finite constants. □

3. Proof of Theorem 1

Suppose (A 1) and (A 2) hold. We now prove Theorem 1 in two steps:

Step 1: $\{P_n; n \in \mathbb{N}\}$ is tight. Moreover, $P[\widehat{\Omega}_\alpha] = 1$ for any $\alpha \in \left(0, \frac{1}{2} - \frac{2}{p}\right)$, and any accumulation point P of $\{P_n; n \in \mathbb{N}\}$.

Step 2: If P is an accumulation point of $\{P_n; n \in \mathbb{N}\}$, then the ($P$–a.s. Ω–valued) process $(X_t)_{t\ge 0}$ is under P a stationary diffusion generated by L. In particular, there is at most one accumulation point.

By Step 1, every subsequence of $(P_n)_{n\in\mathbb{N}}$ has an accumulation point. By Step 2, all these accumulation points coincide, hence $(P_n)_{n\in\mathbb{N}}$ converges weakly. The weak limit has the properties claimed in Theorem 1.

Proof of Step 1. Fix $\alpha \in \left(0, \frac{1}{2} - \frac{2}{p}\right)$ and $T \in (0,\infty)$. For the moment, the P_n are viewed as probability measures on $\widehat{E}^T := C([0,T]\times[0,1],\mathbb{R}^m)$. By a standard argument of type Kolmogorov–Čentsov (see e.g. [W84, Cor. 1.2]), Lemma 3 implies that for each $n \in \mathbb{N}$ there is a random variable $Y_n : \widehat{E}^T \to (0,\infty)$ such that

$$|X_{t,s} - X_{t',s'}| \le Y_n \cdot |(t,s)-(t',s')|^{\alpha} \qquad \forall\, t,t' \in [0,T] \;\; \forall\, s,s' \in [0,1] \;\; P_n\text{–a.s.},$$

and $\sup_{n\in\mathbb{N}} E[Y_n^p] < \infty$. In particular, if

$$\widehat{E}^T_{\alpha,k} := \{\hat\omega \in \widehat{E}^T; |\hat\omega(t,s) - \hat\omega(t',s')| \le k\cdot|(t,s)-(t',s')|^{\alpha} \;\forall t,t' \in [0,T] \;\forall s,s' \in [0,1]\}$$

$(k \in \mathbb{N})$, then $\sup_{k\in\mathbb{N}} \inf_{n\in\mathbb{N}} P_n\left[\widehat{E}^T_{\alpha,k}\right] = 1$. We even obtain

$$\sup_{k\in\mathbb{N}} \inf_{n\in\mathbb{N}} P_n\left[\widetilde{E}^T_{\alpha,k}\right] = 1, \tag{3.1}$$

where $\widetilde{E}^T_{\alpha,k} := \widehat{E}^T_{\alpha,k} \cap \{|X_{0,0}| \le k\}$, since

$$\begin{aligned} P_n\left[|X_{0,0}| \le k\right] &= P^{(n)}\left[\left|(X_0^{(n)})_0\right| \le k\right] = \mu_n\left(\{\hat x \in \Omega_n; |x_0| \le k\}\right) \\ &= \mu\left(\{\omega \in \Omega; |\omega(0)| \le k\}\right) \qquad \text{for all } n \in \mathbb{N}\,. \end{aligned}$$

By the theorem of Arzelà–Ascoli, $\widetilde{E}^T_{\alpha,k}$ is a compact subset of $\widehat{E}^T$ for all $k \in \mathbb{N}$. Thus $\{P_n; n \in \mathbb{N}\}$ is tight on $\widehat{E}^T$ for all $T \in (0,\infty)$, hence tight on $\widehat{E}$.
To prove the second part of the claim, we fix an accumulation point of $\{P_n; n \in \mathbb{N}\}$. By (3.1), $\sup_{k\in\mathbb{N}} P[\widetilde{E}^T_{\alpha,k}] = 1$ for all $T \in (0,\infty)$. Hence P–a.e. function in $\widehat{E}$ is locally Hölder–continuous of order α w.r.t the Euclidean distance in $\mathbb{R}^m$. Moreover, $P[\widehat{\Omega}] = 1$. In fact, for any dyadic rational $s \in [0,1]$ and all $t \in [0,\infty)$, $X_{t,s}$ is in M P_n–a.s. for all $n \in \mathbb{N}$ large enough, hence

$$P\left[X_{t,s} \in M\right] \;\ge\; \varlimsup_{n\to\infty} P_n\left[X_{t,s} \in M\right] \;=\; 1\,.$$

Thus, X_t is in $C([0,1],M)$ for all $t \in [0,\infty)$ P–a.s. Similarly,

$$P\left[X_t \in \Omega\right] \;=\; P\left[\varphi\left(X_{t,0}, X_{t,1}\right) = 0\right] \;\ge\; \varlimsup_{n\to\infty} P_n\left[\varphi\left(X_{t,0}, X_{t,1}\right) = 0\right] \;=\; 1\,,$$

since the law of $(X_{t,0}, X_{t,1})$ under P_n is the law of the first and last component of $i_n(X_t^{(n)})$ under $P^{(n)}$, and $X_t^{(n)}$ is in Ω_n for all $n \in \mathbb{N}$. Hence $(X_t)_{t\ge 0}$ is P–a.s. Ω–valued, i.e. $P[\widehat{\Omega}] = 1$.

Finally, we conclude that $P[\widehat{\Omega}_\alpha] = 1$, because a function in $\widehat{\Omega}$ is locally Hölder–continuous of order α w.r.t. the Riemannian distance in M if (and only if) the same is true w.r.t. the Euclidean distance in $\mathbb{R}^m$ (cp. the proof in Appendix A). $\square$

Proof of Step 2. This is almost completely analogous to [RZ95, Sect. 3]), so we only give a sketch. For $\alpha \in (0,\infty)$ let $G_\alpha := (\alpha - L)^{-1}$, and $G_\alpha^{(n)} := \left(\alpha - L^{(n)}\right)^{-1}$ ($n \in \mathbb{N}$) be the α–resolvents of L resp. $L^{(n)}$. The form domain $D(\mathcal{E})$ is a Hilbert space w.r.t. the inner product $\mathcal{E}_\alpha(F,\tilde{F}) := \mathcal{E}(F,\tilde{F}) + \alpha \int F\tilde{F}\,d\mu$. The set $A_n := \{f \circ \Pi_n ; f \in D\left(\mathcal{E}^{(n)}\right)\}$ is a closed subspace. A short computation shows that for $f \in L^2(\Omega_n, \mu_n)$, $(G_\alpha^{(n)} f) \circ \Pi_n$ is the orthogonal projection in $(D(\mathcal{E}), \mathcal{E}_\alpha)$ of $G_\alpha(f \circ \Pi_n)$ onto A_n. Since the smooth cylinder functions are in $\bigcup_{n\in\mathbb{N}} A_n$, the union is dense in $D(\mathcal{E})$. Thus $(G_\alpha^{(n)} f_n) \circ \Pi_n \to G_\alpha F$ whenever $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega_n, \mu_n)$ such that $f_n \circ \Pi_n \to F$ in $L^2(\Omega, \mu)$.

Now fix $\ell \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_\ell \in (0,\infty)$ and bounded continuous functions $F_0, F_1, \ldots, F_\ell : E \to \mathbb{R}$. For $\omega \in E$ and $n \in \mathbb{N}$, $i_n \circ \Pi_n(\omega)$ is the linear interpolation of ω w.r.t. the n–th dyadic partition of $[0,1]$. Hence $F_j \circ i_n \circ \Pi_n \to F_j$ pointwise for all $1 \le j \le \ell$. Denoting expectation w.r.t. P by E we obtain:

$$
\begin{aligned}
&\int_{(0,\infty)^\ell} e^{-(\alpha_1 t_1 + \cdots + \alpha_\ell t_\ell)}\, E\left[F_0(X_0) F_1\left(X_{t_1}\right) \cdots F_\ell\left(X_{t_1+\cdots+t_\ell}\right)\right] \; dt_1 \ldots dt_\ell \\
&= \lim_{n\to\infty} \int_{(0,\infty)^\ell} e^{-(\alpha_1 t_1 + \cdots + \alpha_\ell t_\ell)}\, E^{(n)}[F_0(i_n(X_0^{(n)})) \cdots F_\ell(i_n(X^{(n)}_{t_1+\cdots+t_\ell}))]\, dt_1 \ldots dt_\ell \\
&= \lim_{n\to\infty} \int F_0 \circ i_n \;\; G^{(n)}_{\alpha_1} \left(F_1 \circ i_n \;\; G^{(n)}_{\alpha_2} \left(F_2 \circ i_n \;\; \cdots \;\; G^{(n)}_{\alpha_\ell} \left(F_\ell \circ i_n\right)\right) \ldots\right) \, d\mu_n \\
&= \int F_0 \cdot G_{\alpha_1}\left(F_1 \cdot G_{\alpha_2}\left(F_2 \cdots G_{\alpha_\ell} F_\ell\right) \cdots\right) \; d\mu \\
&= \int_{(0,\infty)^\ell} e^{-(\alpha_1 t_1 + \cdots + \alpha_\ell t_\ell)} \int F_0 \cdot e^{t_1 L}\left(F_1 \cdot e^{t_2 L}\left(F_2 \cdots e^{t_\ell L} F_\ell\right) \cdots\right) d\mu \;\; dt_1 \ldots dt_\ell \,.
\end{aligned}
$$

Here we used in the third step that $\mu_n = \mu \circ \Pi_n^{-1}$' and applied the resolvent convergence shown above using induction on ℓ. By the uniqueness of the Laplace transform, P is uniquely determined, and $((X_t), P)$ is the stationary diffusion generated by L. $\square$

Finally, we prove the corollary to Theorem 1.

Proof of the corollary. Fix $T > 0$ and $n \in \mathbb{N}$. Under $P^{(n)}$, the process $X_t^{(n)}$ ($= \Pi_n(i_n(X_t^{(n)}))$), $t \in [0,T]$, has the same distribution as the process $\Pi_n(X_t)$ under P_n. Hence Q_n (resp. its extension to $\widehat{E}^T$) is the law of Z^n under P_n, where $Z^n : \widehat{E} \to \widehat{E}^T \cup \{\Delta\}$ is the random variable given by $Z^n = (Z_t^n)_{t\in[0,T]}$, $Z_t^n := j_n(\Pi_n(X_t))$, if $\Pi_n(X_t) \in U_n \cap \Omega_n$ for all $t \in [0,T]$, $Z^n := \Delta$ else, Δ denoting a cemetery point.

We want to show that $(Q_n)_{n\in\mathbb{N}}$ converges weakly to P on $\widehat{E}^T$. This implies weak convergence on $C([0,T]\times[0,1],M)$, because $Q_n[\widehat{E}^T\setminus C([0,T]\times[0,1],M)]=0$ for all $n\in\mathbb{N}$, and the topology on $C([0,T]\times[0,1],M)$ is the trace topology induced from $\widehat{E}^T$. Since $(P_n)_{n\in\mathbb{N}}$ converges weakly to P on $\widehat{E}^T$, it is enough to show

$$\lim_{n\to\infty} P_n[Z^n=\Delta] \;=\; 0 \quad \text{and} \tag{3.2}$$

$$\lim_{n\to\infty} P_n[\{\hat\omega\in\widehat{E}; Z^n(\hat\omega)\neq\Delta \text{ and } \|Z^n(\hat\omega)-\hat\omega\|_\infty>\varepsilon\}]=0 \;\text{ for all } \varepsilon>0\;. \tag{3.3}$$

Fix $N\in\mathbb{N}$. Let $B_N:=\{\hat\omega\in\widehat{\Omega};\ \sup\limits_{t\in[0,T]}\ \sup\limits_{s\in[0,1]}|X_{t,s}(\hat\omega)|\leq N\}$, and let $r>0$ be the minimum of the injectivity radii of points in $\{x\in M;|x|\leq N\}$. Obviously, $P_n[\,\Pi_n(X_t)\in\Omega_n\ \forall\, t\in[0,T]\,]=1$ for all $n\in\mathbb{N}$. Since the i–th component of $\Pi_n(X_t)$ is $X_{t,i\cdot 2^{-n}}$, we obtain by definition of U_n:

$$\begin{aligned}
&P_n\left[\{Z^n=\Delta\}\cap B_N\right] \\
&\leq P_n[\{\hat\omega\in B_N;\ \sup_{t\in[0,T]}\ \sup_{0\leq i<2^n} d\left(\hat\omega(t,i2^{-n}),\hat\omega(t,(i+1)2^{-n})\right)\geq r\}] \\
&\leq P_n[\sup_{t\in[0,T]}\ \sup_{0\leq i<2^n}|X_{t,i2^n}-X_{t,(i+1)2^{-n}}|\geq\delta]
\end{aligned}\tag{3.4}$$

for all $n\in\mathbb{N}$, where $\delta>0$ is a constant independent of n. Here we used that points $p,q\in M$ such that $|p|,\ |q|\leq N$ and $d(p,q)\geq r$ cannot have arbitrary small Euclidean distance. Because of (3.1), the right hand side of (3.4) converges to 0 for $n\to\infty$. Moreover, again by (3.1), $\sup\limits_{n\in\mathbb{N}} P_n[\widehat{E}\setminus B_N]$ becomes arbitrary small if $N\in\mathbb{N}$ is chosen large enough. This proves (3.2).

Now fix $n\in\mathbb{N}$. Note that for P_n–a.e. $\hat\omega\in\{Z^n\neq\Delta\}$,

$$i_n\left(\Pi_n\left(\hat\omega_{t,\cdot}\right)\right) \;=\; \hat\omega_{t,\cdot} \qquad \text{for all } t\in[0,T]\;, \tag{3.5}$$

i.e. $\hat\omega_{t,\cdot}$ is piecewise linear. Fix an $\hat\omega\in\{Z^n\neq\Delta\}$ such that (3.5) holds, and let $t\in[0,T]$. Then $Z^n_t(\hat\omega)=\left(Z^n_{t,s}(\hat\omega)\right)_{s\in[0,1]}$ is the piecewise minimal geodesic interpolation of $\hat\omega(t,\cdot)$ w.r.t. the partition $\mathcal{D}_n$. Hence for $s\in[i2^{-n},(i+1)2^{-n}]$, $0\leq i<2^n$, we have

$$\begin{aligned}
|Z^n_{t,s}(\hat\omega)-\hat\omega(t,i2^{-n})| \;&\leq\; K\cdot d\left(Z^n_{t,s}(\hat\omega),\hat\omega(t,i2^{-n})\right) \\
&\leq\; K\cdot d\left(\hat\omega(t,(i+1)2^{-n}),\hat\omega(t,i2^{-n})\right)\;,
\end{aligned}$$

where K is the constant from (1.1). Thus, by (3.5),

$$|Z^n_{t,s}(\hat\omega)-\hat\omega(t,s)|\leq K\cdot d(\hat\omega(t,(i+1)2^{-n}),\hat\omega(t,i2^{-n}))+|\hat\omega(t,(i+1)2^{-n})-\hat\omega(t,i2^{-n})|\,.$$

For $\varepsilon>0$ and $N\in\mathbb{N}$ we obtain

$$\begin{aligned}
&P_n[\{\hat\omega\in B_N; Z^n(\hat\omega)\neq\Delta \text{ and } \|Z^n(\hat\omega)-\hat\omega\|_\infty>\varepsilon\}] \\
&\leq P_n[\{\hat\omega\in B_N;\ \sup_{t\in[0,T]}\ \sup_{0\leq i<2^n}\ (K\cdot d(\hat\omega(t,(i+1)2^{-n}),\hat\omega(t,i2^{-n})) \\
&\qquad\qquad + |\hat\omega(t,(i+1)2^{-n})-\hat\omega(t,i2^{-n})|)>\varepsilon\}]\;,
\end{aligned}$$

which converges to 0 by the same arguments as above. Letting N tend to infinity, we see that (3.3) holds. This completes the proof of the corollary. □

4. An alternative tightness proof

We now prove Theorem 2. Apart from tightness, the argumentation is the same as in the proof of Theorem 1. Thus it only remains to show:

Proposition. *If (A 3) holds then $\{P_n; n \in \mathbb{N}\}$ is tight.*

Proof. We fix a countable dense subset $\{\omega_k; k \in \mathbb{N}\}$ of E, and define semi–metrics d_k, $k \in \mathbb{N}$, via

$$d_k(\omega, \omega') := |(\|\omega - \omega_k\|_\infty \wedge 1) - (\|\omega' - \omega_k\|_\infty \wedge 1)| \qquad (\omega, \omega' \in E) .$$

Observe that $\tilde{d}_\infty := \sum_{k=1}^{\infty} 2^{-k} d_k$ is a metric that generates the same topology as the supremum norm. It was noticed by K. Kuwae and T. Uemura [KuwU95] that for proving tightness it is useful to replace the given metric on a space by a metric similar to $\tilde{d}_\infty$. In fact, the tightness of a sequence of probability measures on $C([0,\infty), E)$ $(= \widehat{E})$ endowed with the supremum norm is independent of the choice of a metric on E that generates the usual topology, s. e.g. [Du89, § 11.5].

Fix $n, k \in \mathbb{N}$, and $p > 2$ such that $A_\Gamma \in L^{p/2}(\Omega, \mu)$, cf. (A 3). For $\hat{x} = (x_0, \ldots, x_{2^n}) \in M^{2^n+1}$ let $u(\hat{x}) := \|i_n(\hat{x}) - \omega_k\|_\infty \wedge 1$. We have

$$\begin{aligned} E_n\left[d_k\left(X_t, X_{t'}\right)^p\right] &= E^{(n)}\left[d_k(i_n(X_t^{(n)}), i_n(X_{t'}^{(n)}))^p\right] \\ &= E^{(n)}\left[|u(X_t^{(n)}) - u(X_{t'}^{(n)})|^p\right] . \end{aligned} \tag{4.1}$$

Moreover, for $\hat{x} \in M^{2^n+1}$,

$$u(\hat{x}) = \sup_{s \in \mathcal{D}} |i_n(\hat{x})(s) - \omega_k(s)| \wedge 1 = \max_{0 \le j \le 2^n - 1} \sup_{\alpha \in \mathcal{D}} f_{j,\alpha}(\hat{x}) , \tag{4.2}$$

where $\mathcal{D} := \bigcup_{\ell \in \mathbb{N}} \mathcal{D}_\ell$ denotes the set of dyadic rationals in $[0,1]$, and

$$f_{j,\alpha}(\hat{x}) := \left|(1-\alpha)x_j + \alpha x_{j+1} - \omega_k\left((j+\alpha)\cdot 2^{-n}\right)\right| \wedge 1 .$$

Below we will show that the first part of the following assertion is a more or less direct consequence of (A 3), and that the second part follows from the first:

Claim: 1) For any $0 \le j \le 2^n - 1$ and $\alpha \in \mathcal{D}$, $f_{j,\alpha}$ is in $D(\mathcal{E}^{(n)})$, and $\Gamma(f_{j,\alpha} \circ \Pi_n) \le mK^2 A_\Gamma$ μ–a.e., where K and A_Γ are as in (1.1) resp. (A 3).
2) u is in $D(\mathcal{E}^{(n)})$ and $\Gamma(u \circ \Pi_n) \le mK^2 A_\Gamma$ μ–a.e.

Take the claim for granted for the moment. Then we can apply the estimate from Lemma 1 to the right hand side of (4.1). For $t, t' \ge 0$ we obtain

$$E_n\left[d_k\left(X_t, X_{t'}\right)^p\right]^{1/p} \le B_p^{1/p} \cdot |t - t'|^{1/2} \cdot \left(\int \Gamma\left(u \circ \Pi_n\right)^{p/2} d\mu\right)^{1/p} \le c \cdot |t - t'|^{1/2} ,$$

where $c := m^{1/2} K \cdot \|A_\Gamma\|^2_{L^{p/2}(\Omega,\mu)} \cdot B_p^{1/p}$ is a finite constant independent of k and n. Hence

$$E_n\left[\tilde{d}_\infty\left(X_t, X_{t'}\right)^p\right]^{1/p} \leq \sum_{k=1}^\infty 2^{-k} E_n\left[d_k\left(X_t, X_{t'}\right)^p\right]^{1/p} \leq c \cdot |t-t'|^{1/2}$$

as well. The tightness of $\{P_n; n \in \mathbb{N}\}$ now follows again by standard arguments (cf. e.g. [EtKur86, Ch. 3, Theorems 7.2, 8.6, 8.8]) provided the one–dimensional marginals $P_n \circ X_t^{-1}$, $t \geq 0$, are tight. This, however, is true. In fact, $\mu_n = \mu \circ \Pi_n^{-1}$ is an invariant measure for the diffusion $\left(X_t^{(n)}, P^{(n)}\right)$. Hence

$$P \circ X_t^{-1} = P^{(n)} \circ \left(X_t^{(n)}\right)^{-1} \circ i_n^{-1} = \mu \circ \Pi_n^{-1} \circ i_n^{-1} = \mu \circ \left(i_n \circ \Pi_n\right)^{-1} \; .$$

The right hand side converges weakly to μ, since the linear interpolation $i_n \circ \Pi_n(\omega)$ converges to ω for any $\omega \in C([0,1], \mathbb{R}^m)$.

It remains to proof the claim:

Proof of the claim. 1) Fix j and α as in the claim. For $\ell \in \mathbb{N}$ let $\psi_\ell : \mathbb{R} \to [-\ell-1, \ell+1]$ and $\varphi_\ell : \mathbb{R}^m \to \left[\frac{1}{2\ell}, 1\right]$ be smooth functions with bounded derivatives such that $\psi_\ell(r) = r$ for all $r \in [-\ell, \ell]$, $|\psi_\ell'| \leq 1$, $\varphi_\ell(y) = |y|$ for all $y \in \mathbb{R}^m$ s.t. $\frac{1}{\ell} \leq |y| \leq 1 - \frac{1}{\ell}$, and $|\nabla \varphi_\ell| \leq 1$. For $y \in \mathbb{R}^m$ let $y^{(i)}$ $(1 \leq i \leq m)$ denote the i–th component. We define functions $h_\ell : M^{2^n+1} \to \mathbb{R}^m$ and $g_\ell : M^{2^n+1} \to \mathbb{R}$ via

$$\begin{aligned} h_\ell^{(i)}(\hat{x}) &:= (1-\alpha) \cdot \psi_\ell\left(x_j^{(i)}\right) + \alpha \cdot \psi_\ell\left(x_{j+1}^{(i)}\right) - \omega_k^{(i)}\left((j+\alpha) \cdot 2^{-n}\right) \quad (1 \leq i \leq m) \; , \\ g_\ell(\hat{x}) &:= \varphi_\ell\left(h_\ell(\hat{x})\right) \; . \end{aligned}$$

For each $\ell \in \mathbb{N}$ and $1 \leq i \leq m$, g_ℓ and $h_\ell^{(i)}$ are restrictions of functions in $C_b^\infty\left(\mathbb{R}^{m \cdot (2^n+1)}\right)$ to M^{2^n+1}, hence they are in $C_b^\infty\left(M^{2^n+1}\right)$ by (1.1). The sequence $(g_\ell)_{\ell \in \mathbb{N}}$ is uniformly bounded, and converges pointwise, thus in $L^2(\Omega_n, \mu_n)$, to $f_{j,\alpha}$. Let γ denote the matrix $(\gamma_{\lambda\nu})_{\lambda,\nu=1}^m$ where $\gamma_{\lambda\nu} := \Gamma\left(h_\ell^{(\lambda)} \circ \Pi_n, h_\ell^{(\nu)} \circ \Pi_n\right)$. By the chain rule for the carré du champ operator Γ (cf. [BH91, I.6]) we have

$$\Gamma\left(g_\ell \circ \Pi_n\right) = \left((\nabla\varphi_\ell) \circ h_\ell\right)^{tr} \gamma\left((\nabla\varphi_\ell) \circ h_\ell\right) \leq \left(\sum\nolimits_{\lambda,\nu=1}^m \gamma_{\lambda\nu}^2\right)^{1/2} \quad \mu\text{–a.e. .}$$

The Cauchy–Schwarz inequality for Γ implies $\gamma_{\lambda\nu}^2 \leq \gamma_{\lambda\lambda}\gamma_{\nu\nu}$, hence

$$\Gamma\left(g_\ell \circ \Pi_n\right) \leq \left(\sum\nolimits_{\lambda,\nu=1}^m \gamma_{\lambda\lambda}\gamma_{\nu\nu}\right)^{1/2} = \sum\nolimits_{\lambda=1}^m \gamma_{\lambda\lambda} \quad \mu\text{–a.e. .} \tag{4.3}$$

for all $\ell \in \mathbb{N}$. For $\hat{x} \in M^{2^n+1}$ and $1 \leq \lambda \leq m$ let $p_j^{(i)}(\hat{x}) := x_j^{(i)}$. Then $\psi_\ell \circ p_j^{(i)} \circ \Pi_n(\omega) = \psi_\ell\left(\omega^{(i)}\left(j \cdot 2^{-n}\right)\right)$ for all $\omega \in \Omega$. The differential of the smooth function mapping $y \in M$ to $\psi_\ell(y^{(i)})$ is bounded by 1 w.r.t. the Euclidean norm, thus (cf. (1.1)) by K w.r.t. the natural norm on T^*M. Hence, by (A 3), for all $1 \leq \lambda \leq m$,

$$\begin{aligned}
\gamma_{\lambda\lambda} &= \Gamma\left(h_\ell^{(\lambda)} \circ \Pi_n\right) \\
&= (1-\alpha)^2 \cdot \Gamma\left(\psi_\ell \circ p_j^{(i)} \circ \Pi_n\right) + 2\cdot(1-\alpha)\cdot\alpha\cdot\Gamma\left(\psi_\ell \circ p_j^{(i)} \circ \Pi_n, \psi_\ell \circ p_{j+1}^{(i)} \circ \Pi_n\right) \\
&\quad + \alpha^2 \cdot \Gamma\left(\psi_\ell \circ p_{j+1}^{(i)} \circ \Pi_n\right) \\
&\leq K^2 A_\Gamma \left((1-\alpha)^2 + 2(1-\alpha)\alpha + \alpha^2\right) = K^2 A_\Gamma \quad \mu\text{–a.e.}\ .
\end{aligned}$$

By (4.3) we obtain

$$\Gamma\left(g_\ell \circ \Pi_n\right) \leq mK^2 A_\Gamma \quad \mu\text{–a.e. for all } \ell \in \mathbb{N}\ . \tag{4.4}$$

In particular,

$$\sup_{\ell\in\mathbb{N}} \mathcal{E}^{(n)}(g_\ell) \ = \ \sup_{\ell\in\mathbb{N}} \int \Gamma\left(g_\ell \circ \Pi_n\right) d\mu \ \leq \ mK^2 \cdot \|A_\Gamma\|_{L^1(\Omega,\mu)} \ < \ \infty\ .$$

Thus by the theorems of Banach/Saks and Banach/Alaoglu (cf. e.g. [MR92, Appendix]), the $L^2(\Omega_n, \mu_n)$–limit $f_{j,\alpha}$ of $(g_\ell)_{\ell\in\mathbb{N}}$ is in $D(\mathcal{E}^{(n)})$ as well, and

$$\lim_{\ell\to\infty} \mathcal{E}^{(n)}\left(\omega_\ell - f_{j,\alpha}\right) \ = \ 0,$$

where ω_ℓ are the Césaro means of a suitable subsequence of $(g_\ell)_{\ell\in\mathbb{N}}$. Since Γ is a positive $L^1(\Omega,\mu)$–valued quadratic form we have

$$\begin{aligned}
&\|\Gamma\left(\omega_\ell \circ \Pi_n\right)^{1/2} - \Gamma\left(f_{j,\alpha} \circ \Pi_n\right)^{1/2}\|^2_{L^2(\Omega,\mu)} \\
&\quad \leq \|\Gamma\left(\omega_\ell \circ \Pi_n - f_{j,\alpha} \circ \Pi_n\right)^{1/2}\|^2_{L^2(\Omega,\mu)} = \mathcal{E}^{(n)}\left(\omega_\ell - f_{j,\alpha}\right)\ .
\end{aligned}$$

Thus $\Gamma\left(\omega_\ell \circ \Pi_n\right)^{1/2}$ converges to $\Gamma\left(f_{j,\alpha} \circ \Pi_n\right)^{1/2}$ in $L^2(\Omega,\mu)$ for $\ell \to \infty$. Because of (4.4), $\Gamma\left(\omega_\ell \circ \Pi_n\right) \leq mK^2 A_\Gamma$ μ–a.e. for all $\ell \in \mathbb{N}$. Hence the same estimate holds for $\Gamma\left(f_{j,\alpha} \circ \Pi_n\right)$.

2) For $\ell \in \mathbb{N}$ let $u_\ell := \max\limits_{0\leq j\leq 2^n-1} \max\limits_{\alpha\in\mathcal{D}_\ell} f_{j,\alpha}$. Part 1) of the claim implies that u_ℓ is in $D(\mathcal{E}^{(n)})$ and

$$\Gamma\left(u_\ell \circ \Pi_n\right) \ \leq \ \max_{0\leq j\leq 2^n-1} \max_{\alpha\in\mathcal{D}_\ell} \Gamma\left(f_{j,\alpha} \circ \Pi_n\right) \ \leq \ mK^2 A_\Gamma \tag{4.5}$$

μ–a.e., cf. [BH91, Ch. I, Ex. 7.2]. Obviously, $(u_\ell)_{\ell\in\mathbb{N}}$ is a uniformly bounded sequence of functions converging to u pointwise, thus in $L^2(\Omega_n,\mu_n)$. Now we can conclude in the same way as above that u is in $D(\mathcal{E}^{(n)})$ as well, and

$$\Gamma(u \circ \Pi_n) \leq mK^2 A_\Gamma \ \mu\text{–a.e.}$$

□

5. Examples

a) Diffusions on Wiener and pinned Wiener spaces of Riemannian manifolds.

Let (M,g) be a compact Riemannian manifold. We fix a point $o \in M$. Let $\Omega := \{\omega \in C([0,1],M);\ \omega(0) = o\}$ or $\Omega := \{\omega \in C([0,1],M);\ \omega(0) = \omega(1) = o\}$ be the based path resp. pinned loop space. Let μ be Wiener measure resp. pinned Wiener measure (cf. e.g. [D94]) on Ω, and let H denote the Cameron–Martin space consisting of all functions $h : [0,1] \to T_oM$ such that h is absolutely continuous, $\langle h,h\rangle_H := \int_0^1 g_o(h'(s),h'(s))\,ds < \infty$, and $h(0) = 0$ resp. $h(0) = h(1) = 0$. Let $F \in \mathcal{F}C_b^\infty(\Omega)$ such that $F(\omega) = f(\omega(s_1),\dots,\omega(s_n))$ with $n \in \mathbb{N}$, $s_1,\dots,s_n \in [0,1]$, and $f \in C_b^\infty(M^n)$. The gradient $DF : \Omega \to H$ is μ–a.e. defined by (cf. e.g. [DR92])

$$\langle DF(\omega),h\rangle_H = \sum_{i=1}^n g_o(h(s_i), \tau_{s_i}^{-1}(\omega)\,(\nabla_i f)\,(\omega(s_1),\dots,\omega(s_n)))\quad \forall h \in H. \tag{5.1}$$

Here $\tau_{s_i}(\omega)$ means stochastic parallel transport from T_oM to $T_{\omega(s_i)}M$ along the path ω, and ∇_i denotes the gradient w.r.t. the i–th component. Explicitly, DF is given by

$$DF(\omega) = \sum_{i=1}^n G(\cdot,s_i)\ \tau_{s_i}^{-1}(\omega)(\nabla_i f)(\omega(s_1),\dots,\omega(s_n))\ , \tag{5.2}$$

where $G : [0,1]^2 \to \mathbb{R}$, $G(s,s') := \min(s,s')$ in the path space case,
resp. $G(s,s') := \begin{cases} s(1-s') \text{ if } s \le s' \\ s'(1-s) \text{ if } s \ge s' \end{cases}$ in the loop space case, is the Green's function for the operator $-\frac{d^2}{ds^2}$ with Dirichlet boundary condition at $s = 0$, and Neumann resp. Dirichlet boundary condition at $s = 1$. $(D, \mathcal{F}C_b^\infty(\Omega))$ is a closable linear operator from $L^2(\Omega,\mu)$ to $L^2(\Omega,H,\mu)$, cf. e.g. [DR92].

Let $\mathcal{L}_\infty^{sym}(H)$ denote the space of all symmetric bounded linear operators on H with operator norm $\|\cdot\|_{\mathcal{L}(H)}$. Suppose $\omega \mapsto A(\omega)$, $\omega \in \Omega$, is a map from Ω to $\mathcal{L}_\infty^{sym}(H)$ such that $\omega \mapsto \langle A(\omega)h_1,h_2\rangle_H$ is measurable for all $h_1,h_2 \in H$. Furthermore, assume that there exists $\alpha \in (0,\infty)$ such that

$$\langle A(\omega)h,h\rangle_H \ge \alpha \cdot \|h\|_H^2 \qquad \text{for all } h \in H, \tag{5.3}$$

and that $\int \|A(\omega)\|_{\mathcal{L}(H)}\ \mu(d\omega) < \infty$. For $F, \widetilde{F} \in \mathcal{F}C_b^\infty(\Omega)$ let

$$\Gamma(F,\widetilde{F})(\omega) := \langle A(\omega)DF(\omega),\ D\widetilde{F}(\omega)\rangle_H \qquad (\omega \in \Omega)\ . \tag{5.4}$$

Clearly, $\Gamma(F,\widetilde{F})$ is in $L^1(\Omega,\mu)$, and Γ satisfies the product rule (1.2). Using the closability of D and (5.3), it is not difficult to show that the quadratic form $\mathcal{E}(F,\widetilde{F}) = \int \Gamma(F,\widetilde{F})d\mu$ $(F,\widetilde{F} \in \mathcal{F}C_b^\infty)$ is closable on $L^2(\Omega,\mu)$, cp. [MR92, II.3 e)]. If $A(\omega)$ is the identity map on H for all $\omega \in \Omega$, then the generator L of the closure $(\mathcal{E}, D(\mathcal{E}))$ is the **Ornstein–Uhlenbeck operator on the path resp. loop space** Ω. In general, L is an elliptic differential operator on Ω, which is formally given by $L = -D^*AD$.

Lemma. *1) If $\int \|A(\omega)\|_{\mathcal{L}(H)}^{1+\varepsilon}\ \mu(d\omega) < \infty$ for some $\varepsilon > 0$ then Assumption (A 3) holds (i.e. we obtain existence and finite dimensional approximation of the stationary diffusion generated by L).*

2) If $\int \|A(\omega)\|_{\mathcal{L}(H)}^{p/2}\ \mu(d\omega) < \infty$ for some $p \in (4,\infty)$ then (A 1) and (A 2) hold (i.e. the diffusion has Hölder–continuous paths of order α for all $\alpha \in (0, \frac{1}{2} - \frac{2}{p})$).

Proof. We have

$$\Gamma(F)^{1/2} \leq \|A\|_{\mathcal{L}(H)}^{1/2}\ \|DF\|_H \leq \|A\|_{\mathcal{L}(H)}^{1/2} \cdot \sup_{x\in M} |\nabla f|_{T_xM} \qquad \mu\text{–a.e.}\ ,$$

whenever $F(\omega) = f(\omega(s))$ for some $s \in [0,1]$ and $f \in C_b^\infty(M)$. Hence the integrability assumptions made imply (A 3) resp. (A 1). It is well-known that (A 2) holds for all $p \in (4,\infty)$, see e.g. [D94, Proof of Theorem 2.3]. □

In particular, the **Ornstein–Uhlenbeck process** on the based path or pinned loop space of a compact Riemannian manifold **exists** and has **Hölder–continuous paths** of order α for all $\alpha \in (0, \frac{1}{2})$.

Now fix $n \in \mathbb{N}$. We want to identify the approximating objects on M^{2^n+1} explicitly. The approximating state spaces are given by $\Omega_n = \{o\} \times M^{2^n}$ for the path space case resp. $\Omega_n = \{o\} \times M^{2^n-1} \times \{o\}$ for the loop space case. The measure μ_n is the corresponding finite dimensional distribution of Wiener resp. pinned Wiener measure, i.e.

$$\mu_n = \left(\prod_{i=0}^{2^n-1} p_{2^{-n}}(x_i, x_{i+1})\right) \cdot \delta_o(dx_0) dx_1 dx_2 \cdots dx_{2^n}, \quad \text{resp.}$$

$$\mu_n = \ (p_1(0,0))^{-1} \cdot \left(\prod_{i=0}^{2^n-1} p_{2^{-n}}(x_i, x_{i+1})\right) \cdot \delta_o(dx_0) dx_1 dx_2 \cdots dx_{2^n-1} \delta_o(dx_{2^n}),$$

where $p_t(\bullet,\bullet), t \in [0,\infty)$, denotes the heat kernel on M.

For $t \in (0,\infty)$ and $x, y \in M$ let $\nu_{x,y}^t$ denote the law of the Brownian bridge from x to y on $C_{x,y}^t := \{\omega \in C([0,t],M);\ \omega(0) = x,\ \omega(t) = y\}$, cf. [Bi84, Ch. II, e)]. For $n \in \mathbb{N}$ and $\hat{x} = (x_0, x_1, \ldots, x_{2^n}) \in M^{2^n+1}$ let $\nu_{\hat{x}}^{(n)} := \bigotimes_{i=0}^{2^n-1} \nu_{x_i,x_{i+1}}^{2^{-n}}$. Using the identification

$$\bigotimes_{i=0}^{2^n-1} C_{x_i,x_{i+1}}^{2^{-n}} \cong \left\{\omega \in C([0,1],M);\ \omega(i\cdot 2^{-n}) = x_i\ \ \forall\, 0 \leq i \leq 2^n\right\}\ ,$$

one verifies that $\hat{x} \mapsto \nu_{\hat{x}}^{(n)}$ is a regular version of the conditional expectation w.r.t. μ conditioned on $\Pi_n = \hat{x}$. (– note that both $\hat{x} \mapsto \nu_{\hat{x}}^{(n)}$ and $\hat{x} \mapsto \nu_{(o,x_1,x_2,\ldots,x_{2^n})}$ [resp. $\hat{x} \mapsto \nu_{(o,x_1,\ldots,x_{2^n-1},o)}$] are regular versions of the conditional expectation, since $\omega(0) = o$ [resp. $\omega(0) = \omega(1) = o$] for μ–a.e. ω).

We define the component maps $a_{ij}^{(n)}$ of A, $\ 0 \leq i,j \leq 2^n$, mapping Ω to the endomorphisms of T_oM, via

$$g_o\left(a_{ij}^{(n)}(\omega)v, \tilde{v}\right) = \langle A(\omega)\left(G\left(\cdot, i2^{-n}\right)v\right),\ G\left(\cdot, j2^{-n}\right)\tilde{v}\rangle_H \ \ \forall\, v,\tilde{v} \in T_oM,\ \omega \in \Omega\ .$$

Note that, for μ–a.e. $\omega \in \Omega$ and $0 \le i,j \le 2^n$, $\tau_{j\cdot 2^{-n}}(\omega) a_{ij}^{(n)}(\omega) \tau_{i\cdot 2^{-n}}^{-1}(\omega)$ is a linear map from $T_{\omega(i\cdot 2^{-n})}M$ to $T_{\omega(j\cdot 2^{-n})}M$. Hence, for $\hat{x} = (x_0, x_1, \ldots, x_{2^n}) \in M^{2^n+1}$, $\tau_{j\cdot 2^{-n}} \ a_{ij}^{(n)} \tau_{i\cdot 2^{-n}}^{-1}$ is $\nu_{\hat{x}}^{(n)}$–a.s. a linear map from $T_{x_i}M$ to $T_{x_j}M$, i.e.

$$\gamma_{ij}^{(n)}(\hat{x}) := \int \tau_{j\cdot 2^{-n}} \ a_{ij}^{(n)} \tau_{i\cdot 2^{-n}}^{-1} \ d\nu_{\hat{x}}^{(n)}$$

is a well–defined linear map from $T_{x_i}M$ to $T_{x_j}M$, too. The function $\gamma_{ij}^{(n)}$ can be viewed as a regular version of the conditional expectation w.r.t. μ of $\tau_{j\cdot 2^{-n}} \ a_{ij}^{(n)} \tau_{i\cdot 2^{-n}}^{-1}$ given Π_n.

Example. Suppose $A(\omega) = a(\omega) \cdot id_H \ \forall \omega \in \Omega$ for some measurable function $a : \Omega \to \mathbb{R}$. Then

$$\begin{aligned} a_{ij}^{(n)}(\omega) &= a(\omega) \ \mathrm{id}_{T_oM} \int_0^1 \frac{\partial}{\partial s} G(s, i2^{-n}) \frac{\partial}{\partial s} G(s, j2^{-n}) \ ds \\ &= a(\omega) \cdot G(i2^{-n}, j2^{-n}) \cdot id_{T_oM} \end{aligned}$$

for all $\omega \in \Omega$, $n \in \mathbb{N}$, and $0 \le i,j \le 2^n$. Here we used that G is a Green's function for $-\frac{d^2}{ds^2}$ on $[0,1]$. Hence

$$\tau_{j\cdot 2^{-n}} \ a_{ij}^{(n)} \tau_{i\cdot 2^{-n}}^{-1} = G(i2^{-n}, j2^{-n}) \cdot a \cdot \tau_{j\cdot 2^{-n}} \ \tau_{i\cdot 2^{-n}}^{-1} \ .$$

In particular, in the *Ornstein–Uhlenbeck case* (i.e. $a \equiv 1$), $\gamma_{ij}^{(n)}$ is $G(i \cdot 2^{-n}, j \cdot 2^{-n})$ times a regular version of the conditional expectation given Π_n of the stochastic parallel transport $\tau_{j\cdot 2^{-n}} \ \tau_{i\cdot 2^{-n}}^{-1}(\omega)$ from $T_{\omega(i\cdot 2^{-n})}M$ to $T_{\omega(j\cdot 2^{-n})}M$.

Lemma. *Let $n \in \mathbb{N}$. The Dirichlet form $\left(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)})\right)$ of the approximating diffusion is the closure in $L^2(\Omega_n, \mu_n)$ of*

$$\mathcal{E}^{(n)}(f, \tilde{f}) = \int \sum_{i,j=0}^{2^n} g_{x_j} \left(\gamma_{ij}^{(n)}(\hat{x}) (\nabla_i f)(\hat{x}), (\nabla_j \tilde{f})(\hat{x}) \right) \mu_n(d\hat{x}) \ , \tag{5.5}$$

$f, \tilde{f} \in C_b^\infty(M^{2^n+1})$.

Proof. Fix $f \in C_b^\infty(M^{2^n+1})$. By (5.2) we have

$$\begin{aligned} D(f \circ \Pi_n) &= \sum_{i=0}^{2^n} G(\cdot, i2^{-n}) \ \tau_{i\cdot 2^{-n}}^{-1} (\nabla_i f) \circ \Pi_n \qquad \mu\text{–a.e.,} \qquad \text{hence} \\ \Gamma(f \circ \Pi_n) &= \sum_{i,j=0}^{2^n} \langle A(\cdot)(G(\cdot, i2^{-n}) \tau_{i\cdot 2^{-n}}^{-1} (\nabla_i f) \circ \Pi_n), G(\cdot, j2^{-n}) \tau_{j\cdot 2^{-n}}^{-1} (\nabla_i f) \circ \Pi_n \rangle_H \\ &= \sum_{i,j=0}^{2^n} g_o \left(a_{ij}^{(n)} \tau_{i\cdot 2^{-n}}^{-1} (\nabla_i f) \circ \Pi_n \ , \ \tau_{j\cdot 2^{-n}}^{-1} (\nabla_j f) \circ \Pi_n \right) \qquad \mu\text{–a.e.} \end{aligned}$$

Since the stochastic parallel transport is an isometry we obtain

$$\begin{aligned}\mathcal{E}^{(n)}(f,f) &= \int \Gamma(f\circ\Pi_n)\ d\mu \\ &= \int \sum_{i,j=0}^{2^n} g_{\omega(j\cdot 2^{-n})}\left((\tau_{j\cdot 2^{-n}} a_{ij}^{(n)} \tau_{i\cdot 2^{-n}}^{-1})(\omega)(\nabla_i f)\circ\Pi_n(\omega), (\nabla_j f)\circ\Pi_n(\omega)\right)\ \mu(d\omega) \\ &= \int \sum_{i,j=0}^{2^n} g_{x_j}\left(\gamma_{ij}^{(n)}(\hat{x})(\nabla_i f)(\hat{x})\ ,\ (\nabla_j f)(\hat{x})\right)\ \mu_n(d\hat{x})\ .\end{aligned}$$

□

Remark. Integrating by parts in (5.5), one can now also calculate the generator $L^{(n)}$ of $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ provided the functions $\gamma_{ij}^{(n)}$ are sufficiently smooth. However, this smoothness and the explicit form of the derivatives of $\gamma_{ij}^{(n)}$ are highly non–trivial even in the case $A \equiv id_H$. In fact, sufficient smoothness of the $\gamma_{ij}^{(n)}$ turns out to be essentially equivalent to the existence of an integration by parts formula for the gradient D. It might be interesting to give an alternative proof of this formula by proving smoothness of the $\gamma_{ij}^{(n)}$.

b) Path spaces of $\mathbb{R}^m$ with non–Gaussian measures.

Let $M = \mathbb{R}^m$. Suppose $\varphi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^\ell$, $\ell \in \mathbb{N}$, is a linear map. Then the space $\Omega = \{\omega \in C([0,1],\mathbb{R}^m);\ \varphi(\omega(0),\omega(1)) = 0\}$ is a Banach space w.r.t. the supremum norm. Let H denote the Hilbert space consisting of all functions $h : [0,1] \to \mathbb{R}^m$ such that h is absolutely continuous, $\langle h,h\rangle_H := \int_0^1 |h'(s)|^2\ ds < \infty$, and $\varphi(h(0),h(1)) = 0$. H is densely and continuously embedded into E. Let

$$\mathcal{F}C_b^\infty := \{f(\ell_1,\ldots,\ell_n);\ n\in\mathbb{N}, f\in C_b^\infty(\mathbb{R}^n), \ell_1,\ldots,\ell_n\in\Omega'\}\ ,$$

where Ω' denotes the dual of Ω. The gradient $DF : \Omega \to H$ of a function $F \in \mathcal{F}C_b^\infty$ is defined by

$$\langle DF(\omega), h\rangle_H = \frac{\partial F}{\partial h}(\omega) := \frac{d}{ds}F(\omega+sh)|_{s=0}$$

for all $h \in H$ and $\omega \in \Omega$.

Suppose we are given a probability measure μ on Ω with supp $\mu = E$, and a map $A : \Omega \to \mathcal{L}_\infty^{sym}(H)$ such that (5.3) holds for some $\alpha > 0$, and $\int \|A(\omega)\|_{\mathcal{L}(H)}\ \mu(d\omega) < \infty$. Let $\Gamma(F,G) := \langle A(\cdot)DF, DG\rangle$. Again, by [MR92, II.3 e)], the quadratic form $\mathcal{E}(F,G) = \int \Gamma(F,G)\ d\mu$, $F,G \in \mathcal{F}C_b^\infty$, is closable on $L^2(\Omega,\mu)$ provided the corresponding form with $A \equiv id_H$ is closable. We assume that this is the case, cf. e.g. [MR92, Sect. II.3] for weak conditions on μ implying closability.

In the same way as above (cf. a)) we obtain that (A 3) holds provided $\int \|A(\omega)\|_{\mathcal{L}(H)}^{1+\varepsilon}\ \mu(d\omega) < \infty$ for some $\varepsilon > 0$. Under this condition Theorem 2 applies, giving an existence and finite dimensional approximation result. This result in particular extends Theorem 3.1 in [LRZ95], where tightness was proved under the condition $\|A(\cdot)\|_{\mathcal{L}(H)} \in L^\infty(\Omega,\mu)$. Similarly, Theorem 1 applies with $p \in (4,\infty)$ provided μ satisfies (A 2) (which e.g. is the case if μ itself is the law of a sufficiently nice diffusion process), and $\int \|A(\omega)\|_{\mathcal{L}(H)}^{p/2}\ \mu(d\omega) < \infty$.

Appendix

A. Hölder–continuity of path space valued functions.

Let M be a complete Riemannian manifold. Fix $T \in (0,\infty)$ and $\alpha \in (0,1)$. The space of all functions from $[0,T]\times[0,1]$ to M that are Hölder–continuous of order α w.r.t. the distance function on M is denoted by $C_\alpha([0,T]\times[0,1],M)$.

Lemma. *Let $\hat\omega \in C_\alpha([0,T]\times[0,1],M)$. Then the function $t\mapsto\hat\omega(t,\cdot)$ is $C_\beta([0,1],M)$–valued, and Hölder–continuous of order γ w.r.t. $\|\cdot\|_{C_\beta}$ for all $\beta,\gamma>0$ such that $\beta+\gamma<\alpha$. In particular, it is Hölder–continuous of order γ w.r.t. d_∞ for all $\gamma<\alpha$.*

Proof. Let K denote the range of $\hat\omega$. K is a compact connected subset of M. We plug M into $\mathbb{R}^m$ for some $m\in\mathbb{N}$ by Whitney's theorem. On K the Euclidean distance is equivalent to the Riemannian distance of M, i.e. there exists $\lambda\in[1,\infty)$ such that $\lambda^{-1}|x-y|\le d(x,y)\le\lambda|x-y|$ for all $x,y\in K$. Hence, on $\{\hat\omega(t,\cdot);\ t\in[0,T]\}$, the distance functions of the spaces $C_\beta([0,1],M), \beta\in(0,1)$, resp. $C([0,1],M)$, are equivalent to the distances of the corresponding spaces of $\mathbb{R}^m$–valued functions. Therefore it is enough to prove the assertion with M replaced by $\mathbb{R}^m$. We have

$$
\begin{aligned}
&|\hat\omega(t,s)-\hat\omega(t',s)-\hat\omega(t,s')+\hat\omega(t',s')| \qquad \text{(A.1)}\\
&\le \left(|\hat\omega(t,s)-\hat\omega(t,s')|+|\hat\omega(t',s)-\hat\omega(t',s')|\right)^{\frac{\alpha-\gamma}{\alpha}}\left(|\hat\omega(t,s)-\hat\omega(t',s)|+|\hat\omega(t,s')-\hat\omega(t',s')|\right)^{\frac{\gamma}{\alpha}}\\
&\le c\cdot|s-s'|^{\alpha-\gamma}\ |t-t'|^\gamma
\end{aligned}
$$

for all $s,s'\in[0,1]$, $t,t'\in[0,T]$, and $\gamma\in[0,\alpha]$, where c is some finite constant.

Fix $\gamma\in(0,\alpha)$. For $p\in(1,\infty)$, $\delta\in(0,1)$, and $\omega\in C([0,1],\mathbb{R}^m)$ define

$$\|\omega\|_{\delta,p} := \left[\int_0^1\int_0^1 \frac{|\omega(s)-\omega(s')|^p}{|s-s'|^{1+p\delta}}dsds'\right]^{1/p} .$$

By (A.1), we obtain for $t,t'\in[0,T]$, $\delta\in(0,\alpha-\gamma)$, and $p\in(1,\infty)$:

$$\|\hat\omega(0,\cdot)\|_{\delta,p}\le c\cdot\left(\int_0^1\int_0^1|s-s'|^{-1+(\alpha-\delta)p}dsds'\right)^{1/p}<\infty\ , \qquad \text{and}$$

$$
\begin{aligned}
&\|\hat\omega(t,\cdot)-\hat\omega(t',\cdot)\|_{\delta,p}\\
&=\left(\int_0^1\int_0^1\frac{|\hat\omega(t,s)-\hat\omega(t',s)-\hat\omega(t,s')+\hat\omega(t',s')|^p}{|s-s'|^{1+\delta p}}dsds'\right)^{1/p}\\
&\le c\cdot|t-t'|^\gamma\cdot\left(\int_0^1\int_0^1|s-s'|^{-1+(\alpha-\gamma-\delta)p}dsds'\right)^{1/p} .
\end{aligned}
$$

The last integral is finite since $\delta<\alpha-\gamma$. It is well–known that the norm in $C_\beta([0,1],\mathbb{R}^m)$ is dominated by a constant times $\|\cdot\|_{\delta,p}$ for all $\beta\in(0,\delta-\frac{1}{p})$, see e.g. [W84, Theorem 1.1]. By choosing δ and p sufficiently large we can conclude that for all $\beta\in(0,\alpha-\gamma)$, the function $t\mapsto\hat\omega(t,\cdot)$ is $C_\beta([0,1],\mathbb{R}^m)$–valued and Hölder–continuous of order γ w.r.t. $\|\cdot\|_{C_\beta}$. In particular, it is Hölder–continuous of order γ w.r.t. $\|\cdot\|_\infty$. □

B. The approximating forms and the associated diffusions.

Let $n \in \mathbb{N}$. We consider the situation from the introduction. We recall that the quadratic form $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ is called a **Dirichlet form** if $u^+ \wedge 1 \in D(\mathcal{E}^{(n)})$ and $\mathcal{E}^{(n)}(u^+ \wedge 1) \leq \mathcal{E}^{(n)}(u)$ whenever $u \in D(\mathcal{E}^{(n)})$. It is said to be **regular** if there is a core of continuous functions with compact support in $D(\mathcal{E}^{(n)})$ that is dense in $D(\mathcal{E}^{(n)})$ w.r.t. the inner product $(u,v) \mapsto \mathcal{E}^{(n)}(u,v) + \int uv \, d\mu_n$, and in $C_0(\Omega_n)$ w.r.t. uniform norm. We say that $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ has the **local property**, if $\mathcal{E}^{(n)}(u,v) = 0$ whenever $u, v \in D(\mathcal{E}^{(n)})$ such that $u \cdot v = 0$ μ_n–a.e. A regular Dirichlet form with local property has an associated symmetric diffusion process, cf. [FOT94], Theorem 7.2.2.

Lemma. *If (A 1) or (A 3) holds, then $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ is a regular Dirichlet form with local property. Moreover, the constant function 1 is in $D(\mathcal{E}^{(n)})$, and $\mathcal{E}^{(n)}(1,u) = 0$ for all $u \in D(\mathcal{E}^{(n)})$.*

Proof. The product rule (1.2) implies that $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ is a Dirichlet form with local property, cf. [Schm92]. Moreover, 1 is in $C_b^\infty(M^{2^n+1})$, thus in $D(\mathcal{E}^{(n)})$. By (1.2), $\Gamma(1 \circ \Pi_n, u \circ \Pi_n) = 0$ for all $u \in D(\mathcal{E}^{(n)})$, hence

$$\mathcal{E}^{(n)}(1,u) = \int \Gamma(1 \circ \Pi_n, u \circ \Pi_n) \, d\mu = 0.$$

To proof regularity we show that $C_0^\infty(M^{2^n+1})$ is dense in $D(\mathcal{E}^{(n)})$. Fix a point $y \in M$, and smooth decreasing functions $\psi_l : \mathbb{R} \to \mathbb{R}$, $l \in \mathbb{N}$, such that $\psi_l(x) = 1$ for $x \in (-\infty, l]$, $\psi_l(x) = 0$ for $x \in [l+2, \infty)$, and $|\psi_l'| \leq 1$. We define smooth mappings $f_l^{(i)} : M^{2^n+1} \to \mathbb{R}$, $l \in \mathbb{N}$, $0 \leq i \leq 2^n$, by $f_l^{(i)}(\hat{x}) := \psi_l(d(x_i, y))$, $\hat{x} = (x_0, x_1, \ldots, x_{2^n}) \in M^{2^n+1}$. For $l \in \mathbb{N}$ let $f_l := \prod_{i=0}^{2^n} f_l^{(i)}$. The product rule (1.2) implies

$$\begin{aligned}
\Gamma(f_l \circ \Pi_n) &= \sum_{i,j=0}^{2^n} \Gamma(f_l^{(i)} \circ \Pi_n, f_l^{(j)} \circ \Pi_n) \cdot \prod_{\alpha \neq i} (f_l^{(\alpha)} \circ \Pi_n) \prod_{\beta \neq j} (f_l^{(\beta)} \circ \Pi_n) \\
&\leq \sum_{i,j=0}^{2^n} \Gamma(f_l^{(i)} \circ \Pi_n)^{1/2} \, \Gamma(f_l^{(j)} \circ \Pi_n)^{1/2}
\end{aligned}$$

By Assumption (A 1) or (A 3), the $L^1(\Omega, \mu)$–norms of $\Gamma(f_l^{(i)} \circ \Pi_n)$, $l \in \mathbb{N}$, $0 \leq i \leq 2^n$, are bounded by a constant independent of l and i. Hence

$$\sup_{l \in \mathbb{N}} \int \Gamma(f_l \circ \Pi_n) \, d\mu < \infty.$$

Now let u be an arbitrary bounded function in $C_b^\infty(M^{2^n+1})$. Since M is complete and connected, $(u \cdot f_l)_{l \in \mathbb{N}}$ is a uniformly bounded sequence of functions in $C_0^\infty(M^{2^n+1})$ that converges to u pointwise, thus in $L^2(\Omega_n, \mu_n)$. Moreover, by (1.2), $\Gamma((u \cdot f_l) \circ \Pi_n) \leq 2(\|u\|_\infty^2 \cdot \Gamma(f_l \circ \Pi_n) + \|f_l\|_\infty^2 \cdot \Gamma(u \circ \Pi_n))$ μ–a.e. Hence

$$\sup_{l\in\mathbb{N}} \mathcal{E}^{(n)}(u\cdot f_l) \;=\; \sup_{l\in\mathbb{N}} \int \Gamma((u\cdot f_l)\circ\Pi_n)\,d\mu \;<\; \infty.$$

The theorems of Banach/Saks and Banach/Alaoglu (cf. e.g. [MR92], Appendix) imply, that the Césaro means w_l, $l \in \mathbb{N}$, of a suitable subsequence of $(u\cdot f_l)_{l\in\mathbb{N}}$ converge to u in $D(\mathcal{E}^{(n)})$. Obviously, the functions w_l are in $C_0^\infty(M^{2^n+1})$. Since $u \in C_b^\infty(M^{2^n+1})$ has been chosen arbitrarily, and $C_b^\infty(M^{2^n+1})$ is dense in $D(\mathcal{E}^{(n)})$, we obtain that $C_0^\infty(M^{2^n+1})$ is dense in $D(\mathcal{E}^{(n)})$ as well. Hence $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ is regular. □

As a consequence of the lemma we can apply Fukushima's method to construct a diffusion process generated by $L^{(n)}$:

Corollary. There exists a stationary Ω_n–valued diffusion $((X_t^{(n)}), P^{(n)})$ generated by $L^{(n)}$. The diffusion is conservative, i.e. $P^{(n)}$–a.e. path has infinite life–time.

Proof. Since $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ is a regular Dirichlet form with local property, there exists a μ_n–symmetric diffusion process generated by $L^{(n)}$, cf. [FOT94], Theorem 7.2.2. The conservativeness, and the stationarity of the diffusion started with μ_n follow because 1 is in $D(\mathcal{E}^{(n)})$, and $\mathcal{E}^{(n)}(1,u) = 0$ for all $u \in D(\mathcal{E}^{(n)})$. In fact, this implies that 1 is in the domain of $L^{(n)}$, and $L^{(n)}1 = 0$. Therefore $e^{tL^{(n)}}1 = 1$. □

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