ON PARABOLIC SUBGROUPS OF CLASSICAL GROUPS WITH A FINITE NUMBER OF ORBITS ON THE UNIPOTENT RADICAL

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Abstract. Let G be a classical algebraic group defined over an algebraically closed field of characteristic zero. In case the simple factors of G are of type A_r, B_r or C_r, all parabolic subgroups P of G with a finite number of orbits on the unipotent radical P_u are determined: this is the case precisely when the class of nilpotency of P_u is at most four. For groups of type D_r we obtain partial results, depending on the structure of P.

1. Introduction

Let G be a reductive algebraic group defined over an algebraically closed field k of characteristic 0. Let P be a parabolic subgroup of G with unipotent radical P_u. The general problem of determining each parabolic subgroup P of a reductive algebraic group G with a finite number of orbits on P_u was posed in [7]. The cases for Borel and semisimple rank one parabolic subgroups were classified in [6] and [7] respectively. In the case that the simple factors of G are of type A_r, B_r, or C_r, it was shown in [10, Theorem 3.1] that the number of orbits of P on P_u is infinite provided the class of nilpotency of P_u is at least 5. The converse of this statement which was conjectured in [10, 3.7] is proved in this paper.

Let $\ell(P_u)$ denote the nilpotency class of P_u. Our principal results are

Theorem 1.1. *Let G be a simple classical algebraic group and P a parabolic subgroup of G. The number of P-orbits on P_u is finite provided*

(i) *G is of type A_r, B_r or C_r, and $\ell(P_u) \leq 4$;*
(ii) *G is of type D_r and $\ell(P_u) \leq 3$;*
(iii) *G is of type D_r, $\ell(P_u) = 4$, and P is stable under the automorphism of G of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r.*

Combining Theorem 1.1 with Theorem 3.1 from [10] we immediately obtain

Theorem 1.2. *Let G be a simple classical algebraic group and P a parabolic subgroup of G. In the case that G is of type D_r assume that P is invariant under the graph automorphism of G of order two. Then the number of P-orbits on P_u is finite if and only if $\ell(P_u) \leq 4$.*

Remarks 1.3. Theorems 1.1 and 1.2 naturally extend to reductive classical groups by means of their simple components. A detailed discussion of parabolic subgroups with an Abelian unipotent radical, including the structure of orbit closures, appears in [8]. Parabolic subgroups of classical groups with $\ell(P_u) = 2$ are treated in [11]. Results for exceptional groups can be found in [5].

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In Section 2 we first recall some results from [7] and [10] and apply them to our situation in Section 3 in showing that for a large class of parabolic subgroups P of classical groups, the question of finiteness of the number of P-orbits on P_u can be reduced to the case of groups of type A_r. In the final part we prove the statement of Theorem 1.2 for $GL(V)$, where V is a finite-dimensional k-vector space, by translating the problem into a question of the representation-type of a particular category of Δ-filtered modules over a quasi-hereditary algebra.

Let G be a connected reductive algebraic group, T a maximal torus in G and Ψ the set of roots of G with respect to T. Fix a set of simple roots Π of Ψ and let B be the Borel subgroup of G corresponding to Π. A parabolic subgroup of G containing B is called standard. By $\Psi(P)$ we denote the set of roots of P. As every parabolic subgroup of G is conjugate to a standard one, we may if we wish assume that the parabolic subgroups under consideration are standard.

Because a reductive algebraic group is the direct product of simple ones over its center, and similarly for parabolic subgroups, we can reduce to the case of a simple algebraic group in our study of parabolic subgroups with the desired finiteness property. Henceforth we assume that G is a connected simple algebraic group.

Since $\operatorname{char} k = 0$, the exponential map is a P-equivariant isomorphism between the Lie algebra of P_u, denoted by $\mathfrak{p}_u$, and P_u. This allows us to work on $\mathfrak{p}_u$ instead of P_u, whenever convenient.

As a general reference for algebraic groups we cite Borel's book [1] and for information on root systems we refer the reader to Bourbaki [2]. Concerning the general theory of representations of finite-dimensional algebras we refer to the book by Gabriel and Roiter [4] and Ringel's monograph [9].

2. Some Recollections

We maintain the notation from the Introduction. The following result is part of Theorem 3.1 in [10].

Theorem 2.1. *Let G be a simple classical algebraic group. Let P be a parabolic subgroup of G. Then the number of P-orbits on P_u is infinite provided*

(i) *G is of type A_r, B_r or C_r, and $\ell(P_u) \geq 5$;*
(ii) *G is of type D_r and $\ell(P_u) \geq 6$;*
(iii) *G is of type D_r, $\ell(P_u) = 5$, and P is stable under the automorphism of G of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r.*

Let Θ be a semisimple automorphism of G and let P be a Θ-stable parabolic subgroup of G. As P_u is a characteristic subgroup of P, it is Θ-stable as well. Let G^Θ, P^Θ, and P_u^Θ be the Θ-fixed point subgroup of G, P, and P_u respectively. Our next result is Corollary 2.8 from [7].

Proposition 2.2. *Let Θ be a semisimple automorphism of G. Assume that P is a Θ-stable parabolic subgroup of G. If P has a finite number of orbits on P_u, then so does P^Θ on P_u^Θ.*

Remark 2.3. For P a standard parabolic subgroup of G, the class of nilpotency $\ell(P_u)$ is easily determined as follows, cf. [10, Lemma 2.5]. Let $\beta \in \Psi^+$. Write $\beta = \sum_{\alpha\in\Pi} c_\alpha(\beta)\alpha$, with $c_\alpha(\beta) \in \mathbb{Z}_0^+$ for each $\alpha \in \Pi$. Let ρ be the highest root of Ψ. Then $\ell(P_u) = \sum_{\alpha\in\Pi\setminus\Psi(P)} c_\alpha(\rho)$.

3. Reduction to Type A_r

Proposition 3.1. *Let $G = SL_n(k)$. Then each of the classical groups $SO_n(k)$ and $Sp_n(k)$ can be realized as a fixed point subgroup of G for a suitable semisimple automorphism Θ of G, i.e., G^Θ equals $SO_n(k)$ or $Sp_n(k)$. Moreover, each parabolic subgroup of G^Θ can be realized as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G. In case $G^\Theta = SO_n(k)$ with n even, this only works up to equivalence under the graph automorphism of G^Θ.*

For a proof of the assertion on G^Θ see Steinberg [12, §11 p. 169]. The remaining statements follow easily from the explicit description of Θ in *loc. cit.*

Theorem 3.2. *Let G be a simple algebraic group of type A_r and P a parabolic subgroup of G. Then the number of P-orbits on P_u is finite provided $\ell(P_u) \leq 4$.*

Theorem 3.2 follows from Theorem 4.1 below. Granting this for the moment, we are able to treat the remaining classical configurations of Theorem 1.1.

Corollary 3.3. *Let H be a simple classical algebraic group of type B_r, C_r, or D_r and let Q be a parabolic subgroup of H. Then the number of Q-orbits on Q_u is finite in each of the following cases:*

(i) *H is of type B_r or C_r and $\ell(Q_u) \leq 4$;*
(ii) *H is of type D_r and $\ell(Q_u) \leq 3$;*
(iii) *H is of type D_r, $\ell(Q_u) = 4$, and Q is stable under the automorphism of H of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r.*

Proof. Let $G = SL_n(k)$ and let Θ be a semisimple automorphism of G as in Proposition 3.1. Let Q be a parabolic subgroup of $H = G^\Theta$ as in Corollary 3.3. In each of the cases we can obtain Q as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G with $\ell(P_u) \leq 4$ by Proposition 3.1. The desired result for G^Θ then follows from Proposition 2.2 and Theorem 3.2.

Now Theorems 1.1 and 1.2 follow from Theorem 3.2 and Corollary 3.3.

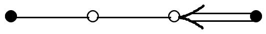
Remark 3.4. We may also obtain the result of Corollary 3.3 for groups of type D_r from the result for groups of type B_r. Let G be of type B_r and let H be the simple regular subgroup of G whose root system consists of the set of long roots in Ψ. Then H is of type D_r, e.g., see [12, §11]. Let Q be a parabolic subgroup of H as in Corollary 3.3. There exists a parabolic subgroup P of G with $\ell(P_u) \leq 4$ such that $P \cap H$ is either Q or the image of Q under the graph automorphism of H. The desired result for H then follows from the result for G and [7, Corollary 2.6].

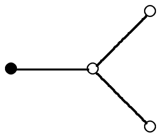
We illustrate the procedures in the proof of Corollary 3.3 and in Remark 3.4 by some examples.

Examples 3.5. Let P be a standard parabolic subgroup of G. The colored nodes indicate the simple roots in $\Psi(P)$ and likewise for $\Psi(G^\Theta)$. In our first example different semisimple automorphisms lead to different fixed point subgroups. For an explicit description of the automorphisms consult [12, §11].

1.

$G = SL_8(k) \supset P;\ \ell(P_u) = 4$

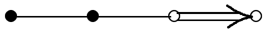




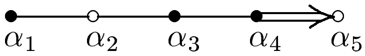
$G^{\Theta} = Sp_8(k) \supset P^{\Theta};\ \ell(P_u^{\Theta}) = 4$ $\qquad$ $G^{\Theta'} = SO_8(k) \supset P^{\Theta'};\ \ell(P_u^{\Theta'}) = 4$

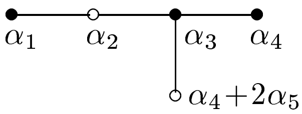
2.

$G = SL_9(k) \supset P;\ \ell(P_u) = 4$



$G^{\Theta} = SO_9(k) \supset P^{\Theta};\ \ell(P_u^{\Theta}) = 4$

3. 



$G = SO_{11}(k) \supset P;\ \ell(P_u) = 4$ $\qquad$ $H = SO_{10}(k) \supset Q,\ \ell(Q_u) = 3$

Remark 3.6. In view of Theorems 1.2 and 2.1, the only classical instances where the question of finiteness of orbits is still open are those when Q is a parabolic subgroup of $H = SO_{2n}(k)$ with $4 \leq \ell(Q_u) \leq 5$ and Q is not invariant under the graph automorphism of H of order 2. Such a Q may still be regarded as a fixed point subgroup of a certain parabolic subgroup P of $SL_{2n}(k)$ according to Proposition 3.1. However, in such a case $\ell(P_u) = 5$ and thus by Theorem 2.1 the number of P-orbits on P_u is known to be infinite; whence Proposition 2.2 does not yield the desired finiteness statement in this instance.

4. The case A_r

Note that we may consider parabolic subgroups in $GL_n(k)$ instead of $SL_n(k)$ for our finiteness considerations, since these two groups only differ by central elements. Let P be a parabolic subgroup of $GL_n(k)$. A Levi subgroup of P is (isomorphic to) a product of general linear groups, say $GL_{d_i}(k)$, for $1 \leq i \leq t$, with $n = \sum d_i$. Thus $d = (d_1, \ldots, d_t)$ determines the conjugate class of P. To indicate this, we often write $P = P(d)$. As $\ell(P_u) = t - 1$, Theorem 3.2 follows immediately from

Theorem 4.1. *Let V be a finite-dimensional k-vector space. Let $P = P(d)$ be a parabolic subgroup of $GL(V)$. Write $d = (d_1, \ldots, d_t)$. Then P acts with a finite number of orbits on $\mathfrak{p}_u$ if and only if $t \leq 5$.*

Proof. For a fixed $t \in \mathbb{N}$ we define a certain category $\mathcal{F}(t)$ as follows. The objects are pairs (F, f), where F is a flag $\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_t = V$ of length t of subspaces of some finite-dimensional k-vector space V, and f is an endomorphism of V with $f(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq t$. A morphism $\varphi : (F, f) \to (F', f')$ is a linear map $\varphi : V \to V'$ such that $\varphi(V_i) \subseteq V_i'$ for $1 \leq i \leq t$ and $\varphi f = f' \varphi$. For (F, f) in $\mathcal{F}(t)$ we set $d_i := \dim V_i - \dim V_{i-1}$ for $1 \leq i \leq t$ and call $d = (d_1, \ldots, d_t)$ the *dimension vector* of (F, f). Then $P = P(d)$ is the stabilizer of F in $GL(V)$ and f lies in $\mathfrak{p}_u$.

If $\varphi : (F, f) \xrightarrow{\sim} (F', f')$ is an isomorphism in $\mathcal{F}(t)$, then $d = d'$ and so $\dim V = \dim V'$. After identifying F with F' we see that $\varphi \in P = P(d)$ and $f' = \varphi f \varphi^{-1}$, i.e., f and f' are endomorphisms in $\mathfrak{p}_u$ conjugate under the action of P. Thus, for a fixed t the isomorphism classes of objects in $\mathcal{F}(t)$ of fixed dimension vector d correspond bijectively to the P-orbits on $\mathfrak{p}_u$ for $P = P(d)$.

Because of this correspondence and because $\mathcal{F}(t)$ is a Krull-Schmidt category, to show that the number of P-orbits on $\mathfrak{p}_u$ is finite, where $P = P(d)$ for any $d = (d_1, \ldots, d_t)$ with $t \leq 5$, it suffices to show that there are only finitely many indecomposable objects in $\mathcal{F}(t)$ for $t \leq 5$. Instead of working directly with $\mathcal{F}(t)$, we pass to an equivalent category where this is accomplished.

Let $\mathcal{M}(t)$ be the full subcategory of representations M of the quiver

$$\circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \quad \cdots \quad \circ \underset{\beta_{t-1}}{\overset{\alpha_{t-1}}{\rightleftarrows}} \circ$$

subject to the relations $\beta_1 \alpha_1 = 0$, $\beta_i \alpha_i = \alpha_{i-1} \beta_{i-1}$ for $1 < i < t$, and the condition that the linear maps $M(\alpha_i)$ are injective for $1 \leq i < t$. Observe that $\mathcal{F}(t)$ and $\mathcal{M}(t)$ are equivalent categories for any $t \in \mathbb{N}$. For, let (F, f) be in $\mathcal{F}(t)$ and define a representation $M(F, f)$ of this quiver by

$$V_1 \rightleftarrows V_2 \rightleftarrows V_3 \quad \cdots \quad V_{t-1} \rightleftarrows V_t \ ,$$

where the maps from left to right are simply the inclusions $V_{i-1} \hookrightarrow V_i$ and the maps in the opposite directions are simply the restrictions of f to V_i. One easily checks that $M(F, f)$ satisfies the conditions above, i.e., $M(F, f)$ is in $\mathcal{M}(t)$, and that the map $(F, f) \mapsto M(F, f)$ defines an equivalence of the two categories.

We observe that $\mathcal{M}(t)$ is the category of Δ-filtered modules over the Auslander algebra of the representation-finite algebra $k[x]/(x^t)$ and the Δ-filtration is given by the indecomposable objects of the form

$$\{0\} \rightleftarrows \{0\} \quad \cdots \quad \{0\} \rightleftarrows k \rightleftarrows k \quad \cdots \quad k \rightleftarrows k$$

where each of the maps is either zero or the identity. For results on categories of Δ-filtered modules over quasi-hereditary algebras we refer to [3]. It is known that $\mathcal{M}(t)$ is representation-finite if and only if $t \leq 5$; see for instance [3, Prop. 7.2]. Because of the equivalence above, the same holds for $\mathcal{F}(t)$. This completes the proof of Theorem 4.1. For $3 \leq t \leq 5$ the Auslander-Reiten quiver of $\mathcal{M}(t)$ is shown in [3]. In particular, there are 7, 16 and 45 indecomposable objects in $\mathcal{M}(t)$ for $t = 3, 4, 5$, respectively.

We are grateful to P. Gabriel who pointed out the equivalence of the categories $\mathcal{F}(t)$ and $\mathcal{M}(t)$ and the fact that $\mathcal{M}(t)$ is of finite representation type for $t \leq 5$. This provides a shorter and more appealing argument than our original proof of Theorem 4.1.

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