Matrices whose inverses are generalized M–matrices

Reinhard Nabben
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
33 501 Bielefeld
Germany

Abstract

In 1972 T.L. Markham defined the so-called matrices of type D [M]. Markham proved that the inverse of a certain type D matrix is a tridiagonal M–matrix. Recently, several generalizations of this result were established, i.e. other classes of inverse M–matrices were introduced. Here we give another generalization. We consider block matrices whose blocks are hermitian positive semidefinite, and in some cases pairwise commute which leads to block type D matrices and block generalized ultrametric matrices. We will establish that the inverse of a block generalized ultrametric matrix is a generalized M–matrix. Moreover, we give an explicit formula for the inverse of a block type D matrix which shows that the inverse is a block tridiagonal generalized M–matrix.

1 Introduction and Preliminaries

Since Ostrowski introduced in 1937 the class of M–matrices, this class has been studied intensively and a lot of important properties of M–matrices have been established. Moreover, several generalizations of M–matrices have been introduced. Recently, Elsner and Mehrmann [EM] and Nabben [N1], [N2] defined the class of generalized M–matrices. These block matrices arise e.g. in the numerical solution of Euler- and Stokes equations (see [N2]).

Definition 1.1 *A block matrix* $A = [A_{ij}] \in \mathbb{C}^{\mathrm{nm,nm}}$ *with* $A_{ij} = A_{ij}^H \in \mathbb{C}^{\mathrm{m,m}}$, *for* $1 \leq i, j \leq n$, *and* A_{ij} *negative semidefinite for* $i \neq j$, *is called a generalized* Z*–matrix. If there exists a positive vector* $d = [d_i] \in \mathbb{R}^{\mathrm{n}}$ *such that*

$$\sum_{j=1}^{n} d_j A_{ij} \quad \textit{is positive definite for all } i$$

then A *is called a generalized* M*–matrix.*

If $m = 1$, then generalized Z–matrices are just Z–matrices and generalized M–matrices are M–matrices. Several equivalent definitions and properties of generalized M–matrices are given in [EM] and [N1] [N2].

An important problem related to M–matrices is the so called inverse M–matrix problem i.e. characterizing all nonnegative matrices whose inverses are M–matrices. In 1972 T.L. Markham introduced the first class of inverse M–matrices, the matrices of type D [M]. Recently this result is generalized in several directions. In [N3] matrices of type D with no restriction on the sign of the entries are discussed. It was shown that the inverses of such matrices are tridiagonal Z–matrices and one can deduce from the number of nonpositive diagonal entries the class of Z–matrix to which A^{-1} belongs. Moreover, McDonald, Neumann, Schneider, and Tsatsomeros [MNST] and Nabben and Varga [NV] introduced a new class of inverse M–matrices. This class of so called generalized ultrametric matrices contains the type D matrices and (strictly) ultrametric matrices.

Here we give another generalization of Markham's result [M] as well as generalizations of results of [MNST] and [NV]. We consider block matrices whose blocks are of the same size and hermitian positive semidefinite. If all blocks can be ordered by the positive definite ordering, we define block type D matrices and block generalized ultrametric matrices by replacing real nonnegative numbers with hermitian positive semidefinite matrices. We then show among other properties, that the inverses of these matrices are generalized M–matrices. For block type D matrices we will give an explicit formula for the inverse. Moreover we consider block matrices whose blocks commute. Matrices of this type arise e.g. in the numerical solution of the Laplace equation.

The paper is organized as follows. At the end of this section we mention some preliminaries and some notations. In Section 2 we consider first block matrices whose blocks commute. Then we introduce the class of block generalized ultrametric matrices. Section 3 deals with the class of block type D matrices.

If $A \in \mathbb{C}^{n,n}$ is hermitian positive (semi-) definite we write $A > 0$ ($A \geq 0$). Moreover for hermitian matrices $A, B \in \mathbb{C}^{n,n}$ the (partial) positive semidefinite ordering is given by $A > B$ ($A \geq B$) if $A - B > 0$ ($A - B \geq 0$).

We define $N := \{1, \ldots, n\}$.

Since we consider block matrices $A = [A_{i,j}] \in \mathbb{C}^{nm,nm}$, $A_{i,j} \in \mathbb{C}^{m,m}$, whose blocks are hermitian and positive (negative) semidefinite we use the following

Notation 1.2 *Let $A = [A_{i,j}] \in \mathbb{C}^{nm,nm}$, $A_{i,j} \in \mathbb{C}^{m,m}$. We define*

$$z_s := [I, I, \ldots, I]^T \in \mathbb{C}^{sm,s} \quad \text{where } I \in \mathbb{C}^{m,m}. \tag{1.1}$$

We then write

$$Az_n \succ 0, \quad Az_n \succeq 0$$

if all $m \times m$ blocks of Az_n are positive definite or positive semidefinite respectively.

For a matrix $A = [A_{i,j}] \in \mathbb{C}^{nm,nm}$, $A_{i,j} \in \mathbb{C}^{m,m}$, whose blocks can be ordered by the positive semidefinite ordering, we define

$$\left\{ \begin{array}{rcl} T(A) & := & min\{A_{i,j} : i, j \in N\}, \\ W(A) & := & min\{A_{i,j} : A_{j,i} = T(A)\}, \\ S(A) & := & W(A) - T(A), \quad \text{so that} \quad S(A) \geq 0. \end{array} \right. \tag{1.2}$$

Moreover, we make use of the Kronecker product of two matrices. If $C = [c_{ij}] \in \mathbb{C}^{n,n}$ and $D = [d_{ij}] \in \mathbb{C}^{m,m}$ then the right Kronecker product $C \otimes D$ is defined as

$$C \otimes D = \begin{bmatrix} c_{11}D & \ldots & c_{1n}D \\ \vdots & & \vdots \\ c_{n1}D & \ldots & c_{nn}D \end{bmatrix} \in \mathbb{C}^{nm,nm}.$$

We use two properties of the Kronecker product (see [LT]). If $A, C \in \mathbb{C}^{n,n}$ and $B, D \in \mathbb{C}^{m,m}$ then

$$\begin{array}{rcl} (A \otimes B)(C \otimes D) & = & AC \otimes BD \\ (A \otimes B)^{-1} & = & A^{-1} \otimes B^{-1}, \quad \text{provided that the inverses exist.} \end{array}$$

For a block-partitioned matrix $A \in \mathbb{R}^{n,n}$ with

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where $A_{1,1} \in \mathbb{R}^{r,r}$ is nonsingular and where $A_{2,2} \in \mathbb{R}^{n-r,n-r}$ with $1 \leq r < n$, then

$$A/A_{1,1} := A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2} \tag{1.3}$$

denotes the *Schur complement* of A with respect to $A_{1,1}$. Similarly, $A/A_{2,2}$ is defined.

2 Block generalized ultrametric matrices

We first consider a set of $m \times m$ matrices $\{A_i\}_{i=1}^n$ which pairwise commute. Obviously if $A_j \in \{A_i\}_{i=1}^n$ is nonsingular then A_j^{-1} commutes with all A_i. Moreover, any rational function of $\{A_i\}_{i=1}^n$ commutes with any A_i. For block matrices which consist of pairwise commuting blocks we have the following

Theorem 2.1 *Let $A = [A_{i,j}] \in \mathbb{C}^{nm,nm}$ be nonsingular with $A_{i,j} \in \mathbb{C}^{m,m}$ and $A_{i,j}$ pairwise commute. Let $A^{-1} =: C = [C_{i,j}]$, $C_{i,j} \in \mathbb{C}^{m,m}$. Then the blocks $C_{i,j}$ of A^{-1} pairwise commute and commute also with the blocks $A_{i,j}$ of A. Moreover, the $C_{i,k}$ are rational functions of the $A_{i,j}$.*

Proof. W.l.o.g. we may assume that the leading principal block submatrices of A are nonsingular. Otherwise we add εI to A and let $\varepsilon \to 0$.
If $n = 2$ we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and $A_{i,j} \in \mathbb{C}^{m,m}$ pairwise commute. We have

$$A^{-1} = \begin{bmatrix} (A/A_{22})^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix}$$

with

$$A/A_{22} = A_{11} - A_{12}A_{22}^{-1}A_{21} \quad \text{and} \quad A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

Obviously the statement of the theorem is true.

If $n > 2$ we partition A as follows

$$A = \begin{bmatrix} A_{11} & B_1 \\ B_2 & D \end{bmatrix}$$

with $A_{11} \in \mathbb{C}^{m,m}$. Since $A/A_{11} = D - B_2 A_{11}^{-1} B_1$, the blocks of A/A_{11} pairwise commute and commute also with A_{11} and A_{1j} and A_{j1}, $j > 1$. Thus, using the inductive hypothesis the blocks of $(A/A_{11})^{-1}$ pairwise commute and are combinations of the $A_{i,j}$ or matrices which commute with the $A_{i,j}$. The same holds for A/D. Hence using the formula for A^{-1} as for $n = 2$ we obtain Theorem 2.1. $\square$

Theorem 2.1 will be the main tool in considering block generalized ultrametric matrices.

Definition 2.2 *A matrix* $A = [A_{i,j}] \in \mathbb{R}^{nm,nm}$ *whose blocks can be ordered by the positive semidefinite ordering, is called a block generalized ultrametric matrix (block gum) if*

i) *all* $A_{i,j}$ *are symmetric positive semidefinite and pairwise commute;*

ii) $A_{i,j} \geq min\{A_{i,k}; A_{k,j}\}$ *for all* $i, k, j \in N$;

iii) $A_{i,i} \geq max\{A_{i,k} : k \in N \backslash \{i\}\}$ *for all* $i \in N$;

iv) $A_{i,i} \geq max\{A_{k,i} : k \in N \backslash \{i\}\}$ *for all* $i \in N$;

v) *each triple* $\{q, s, t\}$ *in* N^3 *can be reordered as a triple* $\{i, j, k\}$ *such that*

v.i) $A_{j,k} = A_{i,k}$ *and* $A_{k,j} = A_{k,i}$,

v.ii) $max\{A_{i,j}; A_{j,i}\} \geq max\{A_{i,k}; A_{k,i}\}$,

where, if $n = 1$, *iii) and iv) are interpreted as* $A_{1,1} \geq 0$. *A matrix* A *is called a strictly block generalized ultrametric matrix if the above conditions hold with strict inequality in iii) and iv).*

If A is symmetric then condition ii) and v) of Definition 2.2 are equivalent. Moreover, if a matrix A satisfies i), iii), iv) and v) of Definition 2.2, then condition ii) is equivalent to the statement:

(2.4) the same ordering used in v) satisfies $min\{A_{i,j}; A_{j,i}\} \geq min\{A_{i,k}; A_{k,i}\}$.

The definition of block generalized ultrametric matrices is similar to the definition of generalized ultrametric matrices. Here we consider block matrices whose blocks are hermitian positive semidefinite instead of nonnegative real numbers. The definition of generalized ultrametric matrices used above is due to Nabben and Varga [NV]. It can be also found in a slightly different form in [MNST].

An equivalent definition for a nonnegative matrix to be a generalized ultrametric matrix is given in [NV]. This definition characterize generalized ultrametric matrices using trees and distances between the leaves of the tree. In the same way we can characterize block generalized ultrametric matrices.

In [NV] and [MNST] the generalized ultrametric ordering is used to order nonnegative real numbers which leads to generalized ultrametric matrices. Here consider hermitian positive semidefinite matrices which pairwise commute. Therefore all theorems of [NV] and [MNST] which just use the generalized ultrametric ordering remain true. The next theorems are of this type. Therefore we omit the proofs.

The following theorem describes the nested structure of block generalized ultrametric matrices or generalized ultrametric matrices.

Theorem 2.3 *Let* $A = [A_{i,j}] \in \mathbb{R}^{\mathrm{nm,nm}}, \mathrm{A_{i,j}} \in \mathbb{R}^{\mathrm{m,m}}, \mathrm{n} > 1$, *be a block generalized ultrametric matrix. Then, there exists a positive integer* r, *with* $1 \leq r < n$, *and a permutation matrix* $P \in \mathbb{R}^{\mathrm{nm,nm}}$ *such that*

$$PAP^T = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} + uT(A)u^T + vS(A)w^T, \tag{2.5}$$

where $u = z_n$, $v = [0, \ldots, 0, z_{n-r}^T]^T \in \mathbb{R}^{\mathrm{nm,m}}$ *and* $w = (z_r^T, 0, \ldots,)^T \in \mathbb{R}^{\mathrm{nm,m}}$ *and* $C \in \mathbb{R}^{\mathrm{rm,rm}}$ *and* $D \in \mathbb{R}^{\mathrm{(n-r)m,(n-r)m}}$ *are block generalized ultrametric matrices. Moreover,*

$$W(A_{1,1}) \geq W(A), \text{ and } W(A_{2,2}) \geq W(A). \tag{2.6}$$

Conversely, if $C \in \mathbb{R}^{\mathrm{rm,rm}}$ *and if* $D \in \mathbb{R}^{\mathrm{(n-r)m,(n-r)m}}$, *with* $1 \leq r < n$, *are block generalized ultrametric matrices and if* $S \in \mathbb{C}^{\mathrm{m,m}}$ *is hermitian positive definite and commutes with all blocks of* C *and* D *and satisfies* $0 \leq S \leq W(C)$ *and* $0 \leq S \leq W(D)$, *then*

$$\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} + v_n T v_n^T + vSw^T$$

is a block generalized ultrametric matrix for all hermitian positive definite matrices $T \in \mathbb{C}^{\mathrm{m,m}}$ *which commute with* S.

Since the matrices C and D of (2.5) are again block generalized ultrametric matrices the reduction process of Theorem 2.3 can be applied to each of the matrices C and D . This reduction process stops with $m \times m$ matrices. An illustration of this process using binary trees is given in [NV]. If all permutation matrices during these reduction steps are the identities we say that A is in nested form.

One can see from Theorem 2.3 that we get in each reduction step for each submatrix C and D square matrices T, which are the minima of all blocks, square matrices S and rectangular matrices u, v, w. Thus Theorem 2.3 gives a representation of a block generalized ultrametric matrix $A \in \mathrm{I\!R}^{\mathrm{nm,nm}}$ as a sum of $2n-1$ matrices whose ranks are at most 2m, i.e.

$$PAP^T = \sum_{i=1}^{2n-1} (u_i T_i u_i^T + v_i S_i w_i^T). \tag{2.7}$$

In the following we consider conditions which characterize nonsingular block generalized ultrametric matrices. Again a theorem of [NV] which just uses the structure of generalized ultrametric matrices holds for the block case.

For the matrices T_i, S_i and $u_i = [(u_i)_j]_{j=1}^m$ with $(u_i)_j \in \mathrm{I\!R}^{\mathrm{nm}}$ of (2.7) we define

$$U_A = \{(u_i)_j : T_i + S_i > 0\},$$

i.e. U_A is the set of all columns of the u_i for which $T_i + S_i$ is positive definite. We then have

Theorem 2.4 *Let $A = [A_{i,j}]$ in $\mathrm{I\!R}^{nm,nm}$ be a block generalized ultrametric matrix. Then, the following are equivalent:*

$$\begin{aligned} I)&\quad A \text{ is nonsingular},\\ II)&\quad span U_A = \mathrm{I\!R}^{nm},\\ III)&\quad span U_{A+A^T} = \mathrm{I\!R}^{nm},\\ IV)&\quad A + A^T \text{ is nonsingular}. \end{aligned}$$

The proof of Theorem 2.4 is essentially the same as the proof of Theorem 3.4 in [NV].

As corollaries we get

Corollary 2.5 *Let A be a strictly block generalized ultrametric matrix. Then A is nonsingular.*

Corollary 2.6 *Let A be a nonsingular block generalized ultrametric matrix of the form* (2.5)

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} + uT(A)u^T + vS(A)w^T.$$

Then, $A_{1,1}$ and $A_{2,2}$ are nonsingular block generalized ultrametric matrices.

In [MNST] another necessary and sufficient condition for a generalized ultrametric matrix to be nonsingular is given. It is shown that a generalized ultrametric matrix is nonsingular if and only if it contains no zero row or zero column and no two rows or columns which are the same. The following example shows that this do not carry over to the block case.

Example 2.7 Let

$$A = \begin{bmatrix} A_1 & A_1 \\ A_1 & A_2 \end{bmatrix}$$

with

$$A_1 = \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 11 & 5 \\ 5 & 11 \end{bmatrix}.$$

We have $\det A = 0$ but no block rows or rows are the same.

In the following we give some more properties of block generalized ultrametric matrices before we state our main theorem.

Lemma 2.8 *Let $A = [A_{i,j}] \in \mathbb{R}^{\mathrm{nm,nm}}$ be nonsingular and assume that all $A_{i,j}$ are symmetric and pairwise commute. Moreover let $A^{-1}z \succeq 0$. Then for $T = [T_i] \in \mathbb{R}^{\mathrm{nm,m}}$ with*

$$T_i = \min_j A_{i,j}$$

we have

$$T_i z^T A^{-1} z \leq I.$$

Proof. Since all blocks of A commute we have

$$z = A(A^{-1}z) \succeq \begin{bmatrix} T_1 & \dots & T_1 \\ \vdots & \dots & \vdots \\ T_n & \dots & T_n \end{bmatrix} A^{-1}z = [T_1, \dots, T_n]^T z^T A^{-1} z.$$

$\square$

Lemma 2.9 *Let $A \in \mathbb{R}^{\mathrm{nm,nm}}$ be a nonsingular block generalized ultrametric matrix given by*

$$A = \begin{bmatrix} C & E_1 \\ E_2 & D \end{bmatrix}$$

$C \in \mathbb{R}^{\mathrm{rm,rm}}$ and $D \in \mathbb{R}^{\mathrm{(n-r)m,(n-r)m}}$ are block generalized ultrametric matrices and $E_1 = z_r T(A) z_{n-r}^T$ and $E_2 = z_{n-r} W(A) z_r^T$. Then A/C and A/D are also nonsingular block generalized ultrametric matrices.

Proof. Since A is nonsingular Corollary 2.6 gives that C and D are nonsingular. We have

$$\begin{aligned} A/C &= D - E_2 C^{-1} E_1 \\ &= D - z_{n-r} W(A) z_r^T C^{-1} z_r T(A) z_{n-r}^T \\ &= D - z_{n-r} T(A) W(A) (z_r^T C^{-1} z_r) z_{n-r}^T. \end{aligned}$$

Since with Lemma 2.8

$$W(A)(z_r^T C^{-1} z_r) \leq I$$

the blocks $(D - z_{n-r} T(A) z_{n-r})_{ij}$, $1 \leq i, j \leq n$ satisfy

$$(D - z_{n-r} T(A) z_{n-r})_{i,j} \geq (D - z_{n-r} T(A) W(A) (z_r^T C^{-1} z_r) z_{n-r}^T)_{i,j}.$$

Moreover since all blocks of $z_{n-r} T(A) W(A) (z_r^T C^{-1} z_r) z_{n-r}^T$ are the same and commute with the blocks of D. Thus A/C is a block generalized ultrametric matrix. Similarly we obtain that A/D is also a block generalized ultrametric matrix. □

Lemma 2.8 and Lemma 2.9 generalize results of [MNST]. In both Lemmata as well as in our main theorem the fact that the blocks of a block generalized ultrametric matrix commute is essential.

Theorem 2.10 *Let $A \in \mathbb{C}^{\mathrm{nm,nm}}$ be a nonsingular block generalized ultrametric matrix in nested form. Then A^{-1} is a generalized Z–matrix with*

$$A^{-1} z \succeq 0 \quad \textit{and} \quad z^T A^{-1} \succeq 0.$$

If A is a strictly block generalized ultrametric matrix then A^{-1} is a generalized M–matrix with

$$A^{-1} z \succ 0 \quad \textit{and} \quad z^T A^{-1} \succ 0.$$

Proof. Since A is in nested form we know with Theorem 2.3 that A is given by

$$A = \begin{bmatrix} C & E_1 \\ E_2 & D \end{bmatrix}$$

where $C \in \mathbb{R}^{\mathrm{rm,rm}}$ and $D \in \mathbb{R}^{\mathrm{(n-r)m,(n-r)m}}$ are block generalized ultrametric matrices of the same form and $E_1 = z_r T(A) z_{n-r}^T$, $E_2 = z_{n-r} W(A) z_r^T$. Moreover from Corollay 2.6 C and D are nonsingular. Then the inverse of A is given by

$$A^{-1} = \begin{bmatrix} (A/D)^{-1} & -C^{-1}E_1(A/C)^{-1} \\ -D^{-1}E_2(A/D)^{-1} & (A/C)^{-1} \end{bmatrix}$$

or

$$A^{-1} = \begin{bmatrix} (A/D)^{-1} & -(A/D)^{-1}E_1 D^{-1} \\ -(A/C)^{-1}E_2 C^{-1} & (A/C)^{-1} \end{bmatrix}.$$

With Lemma 2.9 A/D and A/C are block generalized ultrametric matrices. Hence, using an inductive argument, $(A/D)^{-1}$ and $(A/C)^{-1}$ are generalized M–matrices with $(A/D)^{-1}z_r \succeq 0$, $z_r^T(A/D^)-1 \succeq 0$ and $(A/C)^{-1}z_{n-r} \succeq 0$, $z_{n-r}^T(A/D)^{-1} \succeq 0$. Moreover C^{-1} and D^{-1} are block generalized M–matrices with $C^{-1}z_r \succeq 0$ and $D^{-1}z_{n-r} \succeq 0$.

Now consider $-C^{-1}E_1(A/C)^{-1}$ and $-D^{-1}E_2(A/D)^{-1}$. We have

$$\begin{aligned} -C^{-1}E_1(A/C)^{-1} &= -C^{-1}z_r T(A) z_{n-r}^T (A/C)^{-1}, \\ -D^{-1}E_2(A/D)^{-1} &= -D^{-1}z_{n-r} W(A) z_r^T (A/D)^{-1}. \end{aligned}$$

With Theorem 2.1 we know that all $m \times m$ blocks of C^{-1}, $(A/C)^{-1}$, D^{-1}, and $(A/D)^{-1}$ commute. Hence all $m \times m$ blocks of $-C^{-1}E_1(A/C)^{-1}$ and $-D^{-1}E_2(A/D)^{-1}$ are symmetric and negative semidefinite. Thus, A^{-1} is a generalized Z–matrix. Next we show that $A^{-1}z_n \succeq 0$. We have

$$\begin{aligned} & (A/D)^{-1}z_r - (A/D)^{-1}E_1 D^{-1} z_{n-r} \\ = \;& (A/D)^{-1}z_r - (A/D)^{-1} z_r T(A) z_{n-r}^T D^{-1} z_{n-r} \\ = \;& (A/D)^{-1}z_r (I - T(A) z_{n-r}^T D^{-1} z_{n-r}) \\ \succeq \;& 0. \end{aligned}$$

The last inequality follows from Lemma 2.8 and Theorem 2.1. Similarly we obtain $-(A/C)^{-1}E_2C^{-1}z_r + (A/C)^{-1}z_{n-r} \succeq 0$. Thus $A^{-1}z_n \succeq 0$.

Using the same techniques we obtain $z_n A^{-1} \succeq 0$ and strict inequalities for strictly block generalized ultrametric matrices. Thus, the inverse of a strictly block generalized ultrametric matrix is a generalized M–matrix. $\square$

Replacing the entries of a nonnegative matrix by positive semidefinite blocks of the same size, gives a generalization of the class of nonnegative matrices. In this meaning block generalized ultrametric matrices are generalized nonnegative matrices. However, one of the most famous property of an M–matrix , i.e. that the inverse is a nonnegative matrix, do not carry over to the block case. The inverse of a generalized M–matrix is not in general a matrix which consists of positive semidefinite blocks (see counterexamples in [EM]). Thus, the generalized nonnegativity of the inverse of a generalized Z–matrix do not implies that the matrix is an M–matrix. The following example gives a block generalized ultrametric matrix whose inverse is not a generalized M–matrix.

Example 2.11 Let

$$A = \begin{bmatrix} 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}.$$

Obviously, $A \in \mathbb{R}^{2\cdot 2,2\cdot 2}$ is a block generalized ultrametric matrix. The inverse of A is given by

$$A^{-1} = \begin{bmatrix} 0.5 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & -0.5 \\ -0.5 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

Since both block row sums are singular there is no vector such that the weighted block row sums are positive definite.

The next example shows that the assumption of commuting blocks in the definition of block generalized ultrametric matrices is necessary. Without this assumption the inverse is in general not a generalized M–matrix, i.e. it do not have negative semidefinite off-diagonal blocks in general.

Example 2.12 Let

$$A = \begin{bmatrix} A_3 & A_2 & A_1 & A_1 \\ A_2 & A_3 & A_1 & A_1 \\ A_1 & A_1 & A_3 & A_2 \\ A_1 & A_1 & A_2 & A_3 \end{bmatrix}$$

with

$$A_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -3 \\ 1 & -3 & 5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 7 & -6 \\ 2 & -6 & 11 \end{bmatrix} \quad A_3 = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 4 & -3 \\ 1 & -3 & 6 \end{bmatrix}$$

Obviously $A_3 > A_2 > A_1 > 0$. But the A_i do not commute. If we compute the inverse we see that $(A^{-1})_{1,4}$ is indefinite.

3 Block type D matrices

In 1972 T.L. Markham defined the so-called matrices of type D [M]. Markham defined a real matrix to be of type D if

$$a_{ij} = \begin{cases} a_i \ if \ i \leq j \\ a_j \ if \ i > j, \end{cases} \tag{3.1}$$

where $a_n > a_{n-1} > \ldots > a_1$.

Markham showed that the inverse of a type D matrix with $a_1 > 0$ is a tridiagonal M-matrix. Here we give generalization of Markhams result. Therefore we introduce the class of block type D matrices

Definition 3.1 *A block matrix $A = [A_{ij}]_{i,j=1}^n \in \mathbb{C}^{\mathrm{nm,nm}}$ is called of block type D if*

$$A_{ij} = \begin{cases} A_i \ if \ i \leq j \\ A_j \ if \ i > j, \end{cases} \tag{3.2}$$

where

$$A_n > A_{n-1} > \ldots > A_1$$

is a sequence of $m \times m$ symmetric positive semidefinite matrices.

Note that for $m = 1$ a block type D matrix is just a type D matrix.

Obviously, a block type D matrix whose blocks pairwise commute is a block generalized ultrametric matrix. Hence, for these matrices we already established that the inverse is a generalized M–matrix in the previous section. However, since the special structure of block type D matrices we do not need the assumption of commuting blocks.

We have seen seen in Section 2 that the inverse of a block strictly generalized ultrametric matrix is a strictly block diagonally dominant generalized M–matrix in the sense that the block row sums are positive definite matrices (i.e. the vector d of Definition 1.1 is the vector of all ones). This is not the case for the inverse of a block type D matrix. Therefore we need the next result for generalized M–matrices.

Theorem 3.2 [N1] *Let $A = [A_{ij}] \in \mathbb{C}^{\mathrm{nk,nk}}$ be a generalized Z–matrix. Then A is a generalized M–matrix if and only if A is of block generalized positive type.*

Definition 3.3 [N1] *Let $A = [A_{ij}] \in \mathbb{C}^{\mathrm{nk,nk}}$ be a generalized Z–matrix. A is of block generalized positive type if there exists a vector $d \in \mathbb{R}^{\mathrm{n}}_{+}$ such that:*

i) $R_i(A,d) := d_i A_{ii} + \sum\limits_{\substack{j=1\\ j\neq i}}^{n} d_j A_{ij} \geq 0$ *for all* $i \in\ < n >:= \{1,\ldots,n\}$,
and the set $J := \{i \in < n > | R_i(A,d) > 0\}$ *is nonempty.*

ii) *For each* $i_0 \in < n >$ *with* $i_0 \notin J$ *there exist indices* $i_1,\ldots,i_t \in < n >$ *with* $A_{i_l i_{l+1}} \neq 0, 0 \leq l \leq t-1$ *such that* $i_t \in J$.

In the following we use a representation of block type D matrices which make use of a sequence of vectors $\{u^{(i)}\}_{i=1}^{n} \in \mathbb{R}^{\mathrm{n,n}}$ defined as

$$u_j^{(i)} = \begin{cases} 0 & if\ j \leq i-1 \\ 1 & else \end{cases} \tag{3.3}$$

Then a block type D matrix can be written as

$$A = \sum_{i=1}^{n} u^{(i)}(u^{(i)})^T \otimes (A_i - A_{i-1}), \tag{3.4}$$

where $A_0 = 0$.

Before we state our main result of this section we will give some properties of block type D matrices which generalize results for type D matrices.

Proposition 3.4 *Let* $A \in \mathrm{IR}^{\mathrm{nm,nm}}$ *be of block type D partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathrm{IR}^{\mathrm{rm,rm}}$ *and* $A_{22} \in \mathrm{IR}^{\mathrm{(n-r)m,(n-r)m}}$ *for* $1 \leq r < n$. *If* A_{11} *is nonsingular the Schur complement* A/A_{11} *is again a matrix of block type D and*

$$A/A_{11} = A_{22} - ee^T \otimes A_r,$$

where $e \in \mathrm{IR}^{\mathrm{(n-r)m}}$ *is the vector of all ones.*

Proof. The proof is exactly the same as in the scalar case given in [N3]. Obviously, A_{11} is also of block type D. The block columns of A_{12} are all the same and equal to the last block row and block column of A_{11}. Since $A_{11}^{-1}A_{11} = I$, we obtain

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12} = A_{22} - ee^T \otimes A_r,$$

which is of block type D. □

Proposition 3.5 *Let* A *be a matrix of block type D. Then*

$$detA = \prod_{i=1}^{n} det(A_i - A_{i-1}), \tag{3.5}$$

where $A_0 = 0$. *If* A_1 *is positive definite then* A *is positive definite also.*

Proof. We use an induction on n, i.e. the number of diagonal blocks. For $n = 1, 2$, equation(3.5) is obviously true. Moreover,

$$detA = det(A_1) * det(A/A_1).$$

Since A/A_1 is again a matrix of block type D, with its first diagonal block $A_2 - A_1$, equation (3.5) follows immediately.

With (3.5) we obtain that A is nonsingular if A_1 is positive definite. Moreover, using the representation (3.4), A is the sum of positive semidefinite matrices. Hence, A is positive definite. □

In the next theorem we establish that the inverse of a matrix of type D is a block tridiagonal generalized M–matrix. Moreover, we give an explicit formula for the inverse. This formula make use of another sequence of vectors $\{v^{(i)}\}_{i=1}^{n}$, which are defined as

$$v_j^{(i)} = \begin{cases} -1 & if\ j = i-1 \\ 1 & if\ j = i \\ 0 & else \end{cases} \tag{3.6}$$

Theorem 3.6 *Let A be matrix of block type D with $A_1 > 0$. Then the inverse of A is a block tridiagonal generalized M–matrix. The sum of the blocks of the first row of A^{-1} is a positive definite matrix while the block row sum of the other rows is zero. Moreover,*

$$A^{-1} = \sum_{i=1}^{n} v^{(i)}(v^{(i)})^T \otimes (A_i - A_{i-1})^{-1}, \tag{3.7}$$

with $A_0 = 0$.

Proof. Let

$$\tilde{A} = \sum_{i=1}^{n} v^{(i)}(v^{(i)})^T \otimes (A_i - A_{i-1})^{-1},$$

We obtain

$$\begin{aligned} \tilde{A}A &= \sum_{i,j}(v^{(i)}(v^{(i)})^T \otimes (A_i - A_{i-1})^{-1})(u^{(j)}(u^{(j)})^T \otimes (A_j - A_{j-1})) \\ &= \sum_{i,j}(v^{(i)}(v^{(i)})^T u^{(j)}(u^{(j)})^T) \otimes ((A_i - A_{i-1})^{-1}(A_j - A_{j-1})). \end{aligned}$$

Since, $(v^{(i)})^T u^{(j)} = \delta_{i,j}$ where $\delta_{i,j}$ denotes the Kronecker symbol we obtain

$$\tilde{A}A = \sum_{i}(v^{(i)}(u^{(i)})^T) \otimes ((A_i - A_{i-1})^{-1}(A_i - A_{i-1})) = I.$$

Hence, $\tilde{A} = A^{-1}$. From the special structure of the rank one matrices $(v^{(i)})^T v^{(i)}$ we immediately obtain that A^{-1} is block tridiagonal. Moreover, it follows that the block row sum of all block rows except for the first one, is zero. Using the fact that $A^{-1}A = I$ and the special structure of A we obtain that the row sum of the first block row is just $A_{11}^{-1} > 0$. Thus, A^{-1} is of block generalized positive type. With Theorem 3.2 it follows that A^{-1} is a generalized M–matrix.

□

As a special case, i.e $m = 1$, Theorem 3.6 includes the result by Markham mentioned above. Moreover, the proof given here, is different from Markhams proof since we established the explicit formula for the inverse.

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