ON TWO RESULTS ABOUT FIBRATIONS

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Abstract. Using equivariant methods, we provide straightforward proofs of a result of Chachólski and a result of Spivak about fibrations.

Introduction

Let $\tilde{h}_*$ denote a reduced homology theory on the category of based spaces,and let h_* denote the corresponding extension to an unreduced homology theory on the category of unbased spaces given by $h_*(X) = \tilde{h}_*(X_+)$, where X_+ denotes X with the addition of a disjoint basepoint.

The following was proven by W. Chachólski [2], using the elaborate machinery of 'cellular inequalities' developed by E. Dror Farjoun, et.al.:

Theorem A. *Suppose that*

$$F \to E \to B$$

is a fibration, with B a connected, based space. Assume that

$$h_*(F) \to h_*(E)$$

is an isomorphism for all degrees $*$. Then $\tilde{h}_*(B) = 0$.

Chacólski tells me that Dror Farjoun has asked if a 'classical proof' of Theorem A exists. The first part of this paper provides such a proof.

It is perhaps worth remarking here that if the fibration admits a section, the result is trivial, since the homology of B is a retract of the homology of the mapping cone of the inclusion $F \to E$ and the latter has trivial homology. This gives a clue as to how the result is to be proved in the general case: in effect, we will show that even when the fibration doesn't have a section a homological retraction still exists.

The second result which we give a new proof of was originally due to M. Spivak, reformulated by W. Browder ([3,4.3], [1,I.4.3]):

Theorem B. *Suppose that*

$$F \to E \xrightarrow{p} B$$

is a fibration, where B is 1-connected. Suppose also that the Thom isomorphism is satisfied, i.e., for some positive integer $n \geq 2$ there exists a class $u \in H^n(p)$ in singular homology for which

$$u\cap \colon H_*(p) \to H_{*-n}(B_+)$$

is an isomorphism for all $*$. Then F is a homology $(n-1)$-sphere. In particular, if F is 1-connected, then $F \simeq S^{n-1}$.

The Spivak-Browder proof of Theorem B was computational, making intricate use of the relative Serre spectral sequence.

In contrast, our methods of proving Theorems A and B will be to convert the statements into questions involving spectra equipped with the action of a topological group; this enables us to apply standard equivariant techniques.

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Outline of paper. §1 is preliminary. The main result of that section is 1.5. §2 contains the proof of Theorem A. §3 contains the proof of Theorem B. §4 is an addendum which contains variants of Theorems A and B which we state without proof.

§1 Preliminaries

The spaces in this paper are to be given the compactly generated topology.

1.1. Spectra with G-action. Let $G.$ be a simplicial group. Then its realization $G := |G.|$ (of its underlying simplicial set) is a topological group object in the category of compactly generated spaces.

A *pre-spectrum with G-action* $\mathbf{W}$ is a collection of based (left) G-spaces $\mathbf{W}_n$, $n \in \{0, 1, ...\}$, and equivariant maps $\Sigma\mathbf{W}_n \to \mathbf{W}_{n+1}$ where G acts trivially in the suspension coordinate in the source. We say that $\mathbf{W}$ is a *spectrum with G-action* when the adjoint map $\mathbf{W}_n \to \Omega\mathbf{W}_{n+1}$ is a weak homotopy equivalence (of underlying topological spaces).

Given a pre-spectrum with G-action $\mathbf{W}$, we can form its *spectrification* $\bar{\mathbf{W}}$ by taking $\bar{\mathbf{W}}_n$ to be

$$\operatorname*{hocolim}_k \Omega^k \mathbf{W}_{n+k} \, .$$

This is a spectrum with G-action.

Given a based G-space Y, we can form the spectrum with G-action $\mathbf{W} \wedge Y$: it is defined to be the spectrification of the pre-spectrum with G-action whose n-th space is $\mathbf{W}_n \wedge Y$, with G-acting diagonally.

If $\mathbf{W}$ is a pre-spectrum with G-action, we can form its *homotopy orbit* spectrum $\mathbf{W}_{hG}$; it is given by spectrifying the pre-spectrum with G-action whose n-th space is the reduced Borel construction on $\mathbf{W}_n$, i.e., $EG_+ \wedge_G \mathbf{W}_n$.

By a map of (pre-)spectra $f \colon \mathbf{X} \to \mathbf{Y}$ with G-action, we mean a collection of G-maps $f_n \colon \mathbf{X}_n \to \mathbf{Y}_n$ such that f_{n+1} composed with the structure map $\Sigma\mathbf{X}_n \to \mathbf{X}_{n+1}$ coincides with the structure map $\Sigma\mathbf{Y}_n \to \mathbf{Y}_{n+1}$ composed with Σf_n.

Given a (pre)-spectrum $\mathbf{X}$, recall that its *homotopy groups* $\pi_j(\mathbf{X})$ for $j \in \mathbb{Z}$ are given by $\operatorname{colim}_{j+n}(\mathbf{X}_n)$. We say $\mathbf{X}$ that *r-connected* if its homotopy groups vanish below degree r. We say that $\mathbf{X}$ is *bounded below* if it is r-connected for some integer r.

If $\mathbf{X}$ is a spectrum with G-action, its homotopy groups are a left $\mathbb{Z}[\pi_0(G)]$-module. The action of $[g] \in \pi_0(G)$ on $\pi_*(\mathbf{X})$ is defined by applying homotopy groups to the multiplication by g map.

Call a map $\mathbf{X} \to \mathbf{Y}$ of (pre-)spectra (with or without) G-action a *weak equivalence* if the induced map on homotopy groups is an isomorphism in every degree. Note that the natural map of a pre-spectrum $\mathbf{X}$ into its spectrification $\bar{\mathbf{X}}$ is a weak equivalence.

We use the notation $\mathbf{S}^0$ for the *sphere spectrum*, whose n-th space is $Q(S^n)$ (where $Q = \Omega^\infty \Sigma^\infty$, denotes the representing functor for stable homotopy). We also let $\mathbf{h}\mathbb{Z}$ denote the *Eilenberg-Mac Lane spectrum* with n-th space $K(\mathbb{Z}, n)$ (the Eilenberg-Mac Lane of type $(\mathbb{Z}, n)$).

Lemma 1.2. *Let $\mathbf{X}$ be $(j-1)$-connected for some $j \in \mathbb{Z}$. Then $\pi_j(\mathbf{X}_{hG})$ is naturally isomorphic to the coinvariants of $\pi_0(G)$ acting on $\pi_j(\mathbf{X}_{hG})$. I.e., there is an isomorphism*

$$\pi_j(\mathbf{X}_{hG}) \quad \cong \quad \mathbb{Z} \otimes_{\mathbb{Z}[\pi_0(G)]} \pi_j(\mathbf{X}),$$

where $\mathbb{Z}$ is considered as a right $\mathbb{Z}[\pi_0(G)]$ by letting $\pi_0(G)$ act trivially.

Proof. Consider the 0-connected map of pointed spaces

$$G_+ \to S^0$$

Since $\mathbf{X}$ is $(j-1)$-connected, it follows that smashing the map with $\mathbf{X}$ gives a j-connected map of spectra with G-action

$$\mathbf{X} \wedge G_+ \to \mathbf{X} \wedge S^0$$

Taking homotopy orbits, we see the map

$$\mathbf{X} \to \mathbf{X}_{hG}$$

is also j-connected and is also equivariant, where we give the target the trivial action. Taking homotopy groups in degree j, we get a surjective map of $\mathbb{Z}[\pi_0(G)]$-modules $\pi_j(\mathbf{X}) \twoheadrightarrow \pi_j(\mathbf{X}_{hG})$ or equivalently, a surjection

$$\mathbb{Z} \otimes_{\mathbb{Z}[\pi_0(G)]} \pi_j(\mathbf{X}) \twoheadrightarrow \pi_j(\mathbf{X}_{hG}) .$$

It therefore suffices to show that this map is an isomorphism.

By inspection, the map is an isomorphism if $\mathbf{X}$ is of the form $\mathbf{X} = \mathbf{S}^n \wedge G_+$ for $n \geq j$. On the other hand, $\mathbf{X}$ is obtainable up to weak equivalence as a filtered colimit of the form $\operatorname{colim}_k \mathbf{Y}_k$ where $\mathbf{Y}_k$ is obtained from $\mathbf{Y}_{k-1}$ by taking the mapping cone of a map $\mathbf{S}^n \wedge G_+ \to \mathbf{Y}_{k-1}$, where $\mathbf{Y}_0$ is trivial. The result for $\mathbf{X}$ now follows by induction using the fact the homotopy groups commute with filtered colimits. $\square$

Remark 1.3. The above result is also a straighforward consequence of homotopy orbit spectral sequence for G acting on $\mathbf{X}$.

Corollary 1.4. *Let $\mathbf{X}$ be bounded below and that G is connected, or more generally, that $\pi_0(G)$ acts trivially on $\pi_*(\mathbf{X})$. Furthermore, assume that X_{hG} has trivial homotopy groups. Then $\mathbf{X}$ has trivial homotopy groups.*

Proof. If $\mathbf{X}$ is bounded below, there exists an integer j such that $\mathbf{X}$ is $(j-1)$-connected. By induction, it will be sufficient to prove that $\mathbf{X}$ is also j-connected.

By 1.2, we have that

$$\pi_j(\mathbf{X}_{hG}) \quad = \quad \mathbb{Z} \otimes_{\mathbb{Z}[\pi_0(G)]} \pi_j(\mathbf{X}) .$$

By hypothesis the left-hand side is trivial, and the right-hand side is isomorphic to $\pi_j(\mathbf{X})$. Hence, $\mathbf{X}$ is j-connected. $\square$

Corollary 1.5. *If G is connected, then a map $\mathbf{X} \to \mathbf{Y}$ of bounded below spectra which induces a weak equivalence on homotopy orbits was also a weak equivalence to begin with.*

Proof. Let $\mathbf{C}$ denote the homotopy cofibre of the map $\mathbf{X} \to \mathbf{Y}$. Then $\mathbf{C}$ is bounded below, and $\mathbf{C}_{hG}$ has trivial homotopy groups. Applying the previous result, we find that $\mathbf{C}$ has trivial homotopy groups. Hence $\mathbf{X} \to \mathbf{Y}$ is a weak equivalence. □

1.6. Fibrations.

Let B be a connected, based space. It is well known that B is functorially weak homotopy equivalent to BG where $G = |G.|$ is the realization of a simplicial group. Namely, one defines $G.$ as the Kan loop group of the total singular complex of B. For this reason, we from now on assume that B is identified with BG.

Lemma 1.7. *Given a fibration $p\colon E \to B$ with fiber $F = p^{-1}(*)$, there exists an (unbased) G-space $\tilde{F}$ such that the Borel construction*

$$EG \times_G F' \to BG = B$$

is weak fiber homotopy equivalent to $p\colon E \to B$.

Proof. Form the pullback of p along the universal G-fibration $EG \to BG = B$. Let $E^* \to EG$ denote the resulting fibration, then the fiber of the later over the basepoint is F, and since EG is contractible, $F \to E^*$ is a weak homotopy equivalence. Morever E^* comes equipped with the structure of a G-space, and the orbit space E'/G is identified with E in a way such that $E'/G \to EG/G$ coincides with the projection $E \to B$. On the other hand, the action of G on E' is free, and consequently, the natural map $EG \times_G E' \to E'/G$ is a weak fiber homotopy equivalence. Hence we may take $\tilde{F}$ to be E'. □

Because of this lemma, we may assume from now on that our given fibration is a Borel construction.

§2 Proof of Theorem A

Let

$$\tilde{h}_*\colon \mathbf{Top}_* \to \mathbf{Ab}$$

be a reduced homology theory on the category of based spaces. And let h_* be the corresponding unreduced homology theory on the category of unbased spaces.

By the Brown representability theorem, there exists a spectrum $\mathbf{W}$ such that $\tilde{h}_*(Y) = \pi_*(\mathbf{W} \wedge Y)$ up to natural isomorphism. Using $\mathbf{W}$ we obtain a functor $h\colon \mathbf{Top} \to \mathbf{Sp}$ from unbased spaces to spectra by taking $Z \mapsto \mathbf{W} \wedge Z_+$. By taking homotopy groups, this recovers the unreduced theory h_*.

Given a fibration

$$F \xrightarrow{i} E \xrightarrow{p} B$$

satisfying the hypotheses of Theorem A, we may assume $B = BG$, F is a G-space and $E = EG \times_G F$ is a Borel construction.

Let $\mathbf{W}$ represent h_*. Define a left G-action on the spectrum $h(F) = \mathbf{W} \wedge F_+$ by letting G act trivially on $\mathbf{W}$ and taking the resulting diagonal action on the smash product.

Lemma 2.1. *Up to weak equivalence, $h(E)$ is given by the homotopy orbit spectrum*

$$h(F)_{hG}\,.$$

Moreover, with respect to this identification, the map $h(i)\colon h(F) \to h(E)$ coincides with the natural map $h(F) \to h(F)_{hG}$ from a spectrum with G-action to its homotopy orbit spectrum.

Proof.

$$\begin{aligned} h(E) &= h(EG \times_G F) \\ &= \mathbf{W} \wedge (EG \times_G F)_+ \\ &= \mathbf{W} \wedge (EG_+ \wedge_G F_+) \\ &= (\mathbf{W} \wedge F_+)_{hG} \\ &= h(F)_{hG} \end{aligned}$$

The establishes the first part. The second part is just a short diagram chase. □

Proof of Theorem A. By 2.1, the map

$$h(F) \to h(E)$$

is identified up to homotopy with the natural map

$$h(F) \to h(F)_{hG}\,.$$

By hypothesis, it is a weak equivalence. Moreover, it is G-equivariant, where the action on the target is defined to be trivial. Let us rename the target spectrum by $\mathbf{Z}$. Then the map $h(F) \to \mathbf{Z}$ is a weak equivalence of spectra with G-action, and therefore

$$h(F)_{hG} \quad \simeq \quad \mathbf{Z}_{hG} \quad = \quad BG_+ \wedge \mathbf{Z} \quad = \quad B_+ \wedge \mathbf{Z} \quad \simeq \quad B_+ \wedge h(F)\,,$$

where the equality in the middle uses the fact that G acts trivially on $\mathbf{Z}$. With respect to this chain of equivalences, the weak equivalence $h(F) \to h(F)_{hG}$ may be identified with the inclusion map

$$h(F) \to B_+ \wedge h(F)\,.$$

Now, using the fact that $h(F) = \mathbf{W} \wedge F_+$ we see that

$$B_+ \wedge h(F) = B_+ \wedge (\mathbf{W} \wedge F_+) = \mathbf{W} \wedge (B_+ \wedge F_+) = \mathbf{W} \wedge (B \times F)_+ = h(B \times F)\,.$$

We therefore have that $h(E) \simeq h(B \times F)$ and moreover, modulo this identification, the map $h(F) \to h(E)$ is given by applying h to the inclusion $F \to B \times F$.

Consequently, the hypotheses imply that

$$h(F) \to h(B \times F)$$

is a weak equivalence. Let $* \in F$ be a choice of basepoint. Then the map $* \to B \times *$ is a retract of the inclusion $F \to B \times F$. Hence $h(*) \to h(B)$ is also a weak equivalence. This means that the reduced homology of B, i.e., $\tilde{h}_*(B)$, is trivial. This completes the proof of Theorem A. □

§3 Proof of Theorem B

In order to prove Theorem B, it will be helpful to reformulate the Thom isomorphism for fibrations equipped with a section.

Let $F \to E \xrightarrow{p} B$ be a fibration equipped with a section $s\colon B \to E$ such that the section is a cofibration.

Define a diagonal map

$$\Delta\colon E/B \to (E/B) \wedge B_+$$

by taking induced quotients on the map of pairs $(E, B) \to (E \times B, B \times B)$ defined by $e \mapsto (e, p(e))$.

Then the cofibration sequence

$$E \cup_s C(B) \to E \cup_{\mathrm{id}} C(E) \to B \cup_p C(E)$$

shows that $\Sigma(E/B)$ is weak equivalent to $B \cup_p C(E)$. Consequently, there is an isomorphism $H^n(E \cup_s C(B)) \cong H^{n+1}(B \cup_p C(E))$.

Example 3.1. Let $F_0 \to E_0 \xrightarrow{p_0} B$ be a fibration. Taking the fiberwise join with S^0, we obtain a fibration $\Sigma F_0 \to E \to B$ with section $B \to E$. In this instance, we have a weak equivalence

$$E/B \simeq B \cup_{p_0} C(E_0)\,.$$

Note that the latter is often called the 'Thom complex' of the fibration $E_0 \to B$.

Definition 3.2. Let $\tilde{h}^*$ be reduced singular homology. We say that p has a *Thom class* (of degree n), if it is equipped with a class $u \in \tilde{h}^n(E/B)$ such that capping with u by means of the diagonal map Δ gives an isomorphism

$$u\cap\colon \tilde{h}_*(E/B) \quad \xrightarrow{\cong} \quad \tilde{h}_{*-n}(B_+)$$

for all degrees $*$.

Let $\tilde{h}\colon \mathbf{Top}_* \to \mathbf{Sp}$ represent reduced singular homology, i.e.,

$$\tilde{h}(Y) \quad = \quad \mathbf{h}(\mathbb{Z}) \wedge Y$$

where we recall that $\mathbf{h}(\mathbb{Z})$ denotes the Eilenberg-Mac Lane spectrum.

Lemma 3.3. *Suppose that p has a Thom class of degree n. Then there is a G-equivariant map*

$$\tilde{h}(F) \to \tilde{h}(S^n)$$

where the target has the trivial G-action. Moreover, this map yields a weak equivalence upon taking homotopy orbits.

Proof. Using the fact that $\tilde{h}^n(E/B) = [E/B, \mathbf{h}(\mathbb{Z})_n]$, we may represent the Thom class as a map of spectra

$$\mathbf{S}^0 \wedge E/B \to S^n \wedge \mathbf{h}(\mathbb{Z})\,.$$

Smash this on the right with $\mathbf{h}(Z)$ and the apply apply the multiplication map $\mathbf{h}(\mathbb{Z})^{\wedge 2} \to \mathbf{h}(\mathbb{Z})$ to give

$$E/B \wedge \mathbf{h}(\mathbb{Z}) \simeq \mathbf{S}^0 \wedge E/B \wedge \mathbf{h}(\mathbb{Z}) \to S^n \wedge \mathbf{h}(\mathbb{Z}) \wedge \mathbf{h}(\mathbb{Z}) \to S^n \wedge \mathbf{h}(\mathbb{Z})$$

Using the identification

$$F_{hG} \quad = \quad E/B$$

we therefore have a map

$$F_{hG} \wedge \mathbf{h}(\mathbb{Z}) \to S^n \wedge \mathbf{h}(\mathbb{Z}),$$

or equivalently, by giving $S^n \wedge \mathbf{h}(\mathbb{Z})$ the trivial G-action, an equivariant map

$$\tilde{h}(F) = F \wedge \mathbf{h}(\mathbb{Z}) \to S^n \wedge \mathbf{h}(\mathbb{Z}) = \tilde{h}(S^n)$$

Applying the homotopy orbit construction, yields a map

$$E/B \wedge \mathbf{h}(\mathbb{Z}) = (F \wedge \mathbf{h}(\mathbb{Z}))_{hG} \quad \to \quad (S^n \wedge \mathbf{h}(\mathbb{Z}))_{hG} = B_+ \wedge S^n \wedge \mathbf{h}(\mathbb{Z})$$

On homotopy groups, it is straightforward to check that the latter yields the cap product

$$u\cap \colon \tilde{h}_*(E/B) \to \tilde{h}_{*-n}(B_+)\,.$$

By assumption, the latter is an isomorphism for all $*$. Consequently, the map

$$\tilde{h}(F)_{hG} \to \tilde{h}(S^n)_{hG}$$

is a weak homotopy equivalence. $\square$

Proof of Theorem B. Let $F \to E \to B$ satisfy the Thom isomorphism, with B 1-connected. By taking fiberwise join with S^0, we see that the Thom isomorphism is also satisfied for the resulting fibration. We may therefore assume without loss in generality that the original fibration $E \to B$ is equipped with a section which is also a cofibration.

By the previous lemma, there exists a a G-equivariant map

$$\tilde{h}(F) \to \tilde{h}(S^n)$$

which induces an equivalence on homotopy orbits. Since $B = BG$ is 1-connected, G is connected. Therefore, applying 3.4 we conclude that

$$\tilde{h}(F) \to \tilde{h}(S^n)$$

is a weak equivalence. Hence F is a homology n-sphere. $\square$

§4 Addendum

Using the methods of this paper, it is possible without much trouble to vary Theorems A and B in several directions. We will state these results without proof.

Let

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a fibration, with B connected.

Suppose that ξ denotes a locally trivial twisted coefficient system of abelian groups on B. Let $H_*(B;\xi)$ ($\tilde{H}_*(B;\xi)$) denote the (resp. reduced) homology of B with coefficients in ξ. Let $p^*\xi$ denote the restriction of ξ to E; note that the further restriction to F is constant: let this module be denoted by ξ_0.

The following is a variant of Theorem A:

Assertion 4.1. *Suppose that the map*

$$H_q(F;\xi_0) \to H_q(E;p^*\xi)$$

is an isomorphism for all q. Then $\tilde{H}_q(B;\xi)$ is trivial for all q.

This is a special case of a more general result concerning homology with coefficients in a 'bundle of spectra' parametrized by points of B (however, we won't bother to define what this means).

We now give a generalization of Theorem B. Suppose that R denotes a connective ring spectrum. and suppose that the fibration is satisfies the Thom isomorphism with coefficients in the homology theory defined by R, i.e., there exists a class $u \in R^n(E)$ such that capping with u defines an isomorphism $R_*(p) \cong R_{n-*}(B_+)$.

Assertion 4.2. *Under these assumptions, it follows that F is a homology $(n-1)$-sphere with respect to the homology theory defined by R.*

References

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