Smooth, Homogeneous and Isotropic Solutions of Quasilinear Stochastic Partial Differential Equations of McKean-Vlasov Type

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Abstract

Smooth, homogeneous and isotropic random fields are derived from the distribution of a system of diffusing particles with mean field interaction. The main tools are stochastic ordinary differential equations (SODE's) for the positions of the particles and stochastic partial differential equations (SPDE's) for the mass distribution of the particles. The methods and results used in this paper generalize methods and results of Kotelenez (1995b) from the case of conserved finite mass to the case of infinite mass and the derivation of homogeneous and isotropic fields as solutions of SPDE's.

Contents

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1 Introduction

Let N point particles be distributed over $\mathbf{R}^d$, where d is arbitrary and fixed. The position of the i-th particle at time t will be denoted by $r^i(t)$, which is by assumption in $\mathbf{R}^d$ for all i. We will assume that by some kind of adiabatic elemination we can neglect the momenta (cf. e.g., Kotelenez and Wang, (1994)). Thus we consider the position of the N-particle system at time t as a point in $\mathbf{R}^{dN}$, i.e.,

$$(r^1(t), \ldots, r^N(t)) \in \mathbf{R}^{dN}, \tag{1.1}$$

and a description of its time evolution will be a microscopic model for the particle distribution. Typically one derives from mechanical principles ordinary differential equations (ODE's) or stochastic ordinary differential equations (SODE's) for (1.1) (cf. e.g., Lebowitz and Rubin (1963), Il'in and Khasminskii (1964), Nelson (1972), Kotelenez and Wang (1994) etc.). We will call such equations (describing the position of each particle, usually in dependence on the position of the other particles) *microscopic equations.* Now let the mass of the i-th particle be a_i. Then the empirical mass distribution at time t of the N-particle system is given by

$$\mathcal{X}_N(t) := \sum_{i=1}^{N} a_i \delta_{r^i(t))}, \tag{1.2}$$

where δ_r is the point measure, concentrated in $r \in \mathbf{R}^d$. In other words, $\mathcal{X}_N(t)$ is a measure process and for $A \subset \mathbf{R}^d$, $\mathcal{X}_N(t,A) = \sum_{i=1}^N a_i 1_{\{r^i(t) \in A\}}$ describes the mass in A at t irrespective of which particular particles are in A at t, i.e., $\mathcal{X}_N(t)$ reduces information given by (1.1) to the relevant one as far as the mass distribution is concerned. On the other hand, if we agree to call a mass distribution *macroscopic* if the particle structure cannot be seen and stochastic fluctuations are absent then we may call a description of the time evolution of (1.2) a *mezoscopic model,* in particular, if stochastic fluctuations are present in (1.1), which we will always assume. In Kotelenez (1995b) a stochastic partial differential equation (SPDE) for (1.2) was derived, which was called a *mezoscopic equation.* Under physically and mathematically reasonable assumption on (1.1) this SPDE was extended from sums of weighted point measures to measures having densities with respect to the Lebesque measure dr on $\mathbf{R}^d$. Then as a certain parameter, the correlation length of the fluctuation forces in (1.1), tends to 0, the solutions of the SPDE were shown to tend to the solution of an (integro-) partial differential equation (PDE), which was called a *macroscopic equation* for the mass distribution.

This approach was motivated by the author's derivation of a stochastic Navier-Stokes equation for the vorticity of a two-dimensional fluid (Kotelenez (1995a)). The corresponding mezoscopic description $\mathcal{X}_N(t)$ in that case is a sum of point measures multiplied by intensities $a_i \in \mathbf{R}$, i.e., by point measures in the position of point vortices with positive and negative intensities. To include this case as well as other possible signed measure valued cases (cf. Fife (1992)) we will allow the a_i in (1.2) to be positive and negative and just call them positive and negative "masses". So let $a^+ > 0$ be the total "positive" mass of (1.2), $a^- \geq 0$ be the total negative mass and $\mathbf{a} := (a^+, a^-)$. Set

$$\mathbf{M_a} := \{\mu : \mu \text{ is a finite signed Borel measure on } \mathbf{R}^d, \mu^{\pm}(\mathbf{R}^d) = a^{\pm}\},$$

where μ^+ and μ^- is the Jordan decomposition of μ and for any Borel set $A \subset \mathbf{R}^d$ and nonnegative numbers $b^+, b^- \mu^{\pm}(A) = b^{\pm}$ if and only if $\mu^+(A) = b^+$ and $\mu^-(A) = b^-$. First we define a metric on $\mathbf{R}^d$ by

$$\rho(r,q) := (\bar{c}|r-q| \wedge 1), \tag{1.3}$$

where $r, q \in \mathbf{R}^d$, $|r-q|$ is the Euclidean distance on $\mathbf{R}^d$, $\bar{c}$ some positive constant and "$\wedge$" denotes "minimum". If $\mu, \tilde{\mu} \in \mathbf{M_a}$ we will call positive Borel measures $Q^{\pm}$ on $\mathbf{R}^{2d}$ joint representations of $(\mu^+, \tilde{\mu}^+)$, resp. $(\mu^-, \tilde{\mu}^-)$, if $Q^{\pm}(A \times \mathbf{R}^d) = \mu^{\pm}(A)a^{\pm}$ and $Q^{\pm}(\mathbf{R}^d \times B) = \tilde{\mu}^{\pm}(B)a^{\pm}$ for arbitrary Borel sets $A, B \subset \mathbf{R}^d$. The set of all joint representations of $(\mu^+, \tilde{\mu}^+)$, resp. $(\mu^-, \tilde{\mu}^-)$ will be denoted by $C(\mu^+, \tilde{\mu}^+)$, resp. $C(\mu^-, \tilde{\mu}^-)$. For $\mu, \tilde{\mu} \in \mathbf{M_a}$ and $m = 1, 2$ set

$$\gamma_m(\mu, \tilde{\mu}) := [\inf_{Q^+ \in C(\mu^+, \tilde{\mu}^+)} \int\int Q^+(dr, dq)\varrho^m(r, q) + \inf_{Q^- \in C(\mu^-, \tilde{\mu}^-)} \int\int Q^-(dr, dq)\varrho^m(r, q)]^{\frac{1}{m}}, \tag{1.4}$$

where the integration is taken over $\mathbf{R}^d \times \mathbf{R}^d$. We will not indicate the integration domain when integrating over $\mathbf{R}^d$.) It follows from Dudley (1989) and de Acosta (1982) that $(\mathbf{M_a}, \gamma_2)$ is a complete separable metric space.

Now let $\mathbf{M}$ be the set of finite signed Borel measures $(\mathbf{R}^d, \mathcal{B}^d)$, where $\mathcal{B}^d$ is the σ-algebra of Borel sets on $\mathbf{R}^d$. In other words,

$$\mathbf{M} = \bigcup_{\mathbf{a} \in \mathbf{R}^2} \mathbf{M_a}.$$

We extend the function γ_2 from $\mathbf{M_a} \times \mathbf{M_a}$ to $\mathbf{M} \times \mathbf{M}$ by using (1.4) for arbitrary $\mu \in \mathbf{M_a}$ and $\tilde{\mu} \in \mathbf{M_b}$. The following example, however, shows that γ_2 is not a metric on $\mathbf{M} \times \mathbf{M}$.

Example 1.1

Let $\mu, \tilde{\mu} \in \mathbf{M_a}$, $\mathbf{a} = (a^+, a^-) \neq (0, 0)$. For simplicity assume $a^- = 0$ and drop the "+"-sign in what follows. Choose $\alpha \in (0, 1)$ and set $\mu_\alpha := \alpha\mu$ and $\tilde{\mu}_{\frac{1}{\alpha}} := \frac{1}{\alpha}\tilde{\mu}$. Then $Q \in C(\mu, \tilde{\mu})$ iff $Q \in C(\mu_\alpha, \tilde{\mu}_{\frac{1}{\alpha}})$. Indeed, if $Q \in C(\mu, \tilde{\mu})$, then for any Borel set A and B, $Q(A \times \mathbf{R}^d) = \mu(A)\tilde{\mu}(\mathbf{R}^d) = \alpha\mu(A)\frac{1}{\alpha}\tilde{\mu}(\mathbf{R}^d)$ and $Q(\mathbf{R}^d \times B) = \mu(\mathbf{R}^d)\tilde{\mu}(B) = \alpha\mu(\mathbf{R}^d)\frac{1}{\alpha}\tilde{\mu}(B)$. Hence, $Q \in C(\mu_\alpha, \tilde{\mu}_{1/\alpha})$. The other direction is proved in the same way. These calculations imply, in particular, $\gamma_2(\mu, \beta\mu) = 0$ for all $\beta > 0$. ■

Set for $\mu \in \mathbf{M}$

$$\mu_1^{\pm} := \begin{cases} \frac{\mu^{\pm}}{a^{\pm}}, & \text{if } a^{\pm} > 0, \\ \delta_0, & \text{if } a^{\pm} = 0. \end{cases}$$

Then we define for $\mu \in \mathbf{M_a}$ and $\tilde{\mu} \in \mathbf{M_b}$

$$\tilde{\gamma}(\mu, \tilde{\mu}) := \gamma_2(\mu_1, \tilde{\mu}_1) \tag{1.5}$$

and

$$\gamma(\mu, \tilde{\mu}) := \tilde{\gamma}(\mu, \tilde{\mu}) + \rho(\mathbf{a}, \mathbf{b}), \tag{1.6}$$

where ρ is given by (1.3), assuming here $d = 2$. For $\varepsilon \in (0, 1)$ set

$$\mathbf{M}_{\varepsilon, \varepsilon^{-1}} := \{\mu \in \mathbf{M} : \mu^{\pm}(\mathbf{R}^d) \in [\varepsilon, \varepsilon^{-1}]\}. \tag{1.7}$$

Proposition 1.2

γ is a metric on $\mathbf{M}_{\varepsilon, \varepsilon^{-1}}$, and $(\mathbf{M}_{\varepsilon, \varepsilon^{-1}}, \gamma)$ is a complete and separable metric space.

Proof

We easily see that γ is a metric. The separability and completeness follows from the separability and completeness of $\mathbf{M}_{(1,0)}$ (resp. $\mathbf{M}_{(0,1)}$) (cf. de Acosta, loc. cit.) and of $[\varepsilon, \varepsilon^{-1}]$. ■

We also easily see that $(\mathbf{M},\gamma)$ itself defines a metric space, which is not complete, whereas the restriction of γ to $\bigcup_{\varepsilon\in(0,1)}\mathbf{M}_{\varepsilon,\varepsilon^{-1}}$ defines a complete metric space. Further, if $a^+\neq 0\neq a^-$ and $\mathbf{a}=(a^+,a^-)$, then γ restricted to $\mathbf{M_a}$ is equivalent to γ_2.

For the description of those $\mu\in\mathbf{M}$, (and more general measures), which have a density with respect to the Lebesgue measure (denoted "dr") we need the corresponding function spaces. First of all let $\mathcal{B}^n$ be the σ-algebra of Borel sets on $\mathbf{R}^n$. Secondly, we need the following weight function:

$$\lambda(r):=(1+|r|^2)^{-\alpha},$$

where $\alpha>\frac{d}{2}$ is fixed. 1 is the function which equals 1 for all $r\in\mathbf{R}^d$, and $\Phi\in\{1,\lambda\}$. Finally, to describe the order of differentiability of $f:\mathbf{R}^d\to\mathbf{R}$ we introduce the following notation on multiindices:

$$\mathbf{n}=(n_1,\dots n_d)\in(\mathbf{N}\cup\{0\})^d.$$

$|\mathbf{n}|:=n_1+\ldots+n_d$. If $\mathbf{n}=(n_1,\dots,n_d)$ and $\mathbf{m}=(m_1,\dots,m_d)$, then $\mathbf{n}\leq\mathbf{m}$ iff $n_i\leq m_i$ for $i=1,\dots,d$. $\mathbf{n}<\mathbf{m}$ iff $\mathbf{n}\leq\mathbf{m}$ and $|\mathbf{n}|<|\mathbf{m}|$. For arbitrary $\mathbf{n},\mathbf{m}$ we define

$$\mathbf{n}-\mathbf{m}:=(k_1,\dots,d_d),$$

where $k_i=(n_i-m_i)\vee 0$, $i=1,\dots,d$, with "$\vee$" denoting the maximum and $\mathbf{n}=(n_1,\dots,n_d)$, $\mathbf{m}=(m_1,\dots,m_d)$.

Set $\{f:f:\mathbf{R}^d\to\mathbf{R}$ is $\mathcal{B}^d-\mathcal{B}^1$ measurable and f has partial derivatives in the generalized sense up to order $m\in\mathbf{N}\cup\{0\}\}=:\mathbf{B}_{d,1,m}$. For $p>0$ and $f\in\mathcal{B}_{d,1,m}$ we set

$$\|f\|^p_{m,p,\Phi}:=\sum_{|\mathbf{j}|\leq m}\int|\partial^{\mathbf{j}}f|^p(r)\Phi(r)dr,\tag{1.8}$$

where $\partial^{\mathbf{j}}f=\frac{\partial^{\mathbf{j}}}{\partial r^{j_1}\ldots\partial r^{j_d}}f$ for $\mathbf{j}=(j_1,\dots,j_d)$, and set

$$\mathbf{W}_{m,p,\Phi}:=\{f\in\mathbf{B}_{d,1,m}:\|f\|_{m,p,\Phi}<\infty\}.\tag{1.9}$$

Further,

$$(\mathbf{H}_0,\|\cdot\|_0):=(\mathbf{W}_{0,2,1},\|\cdot\|_{0,2,1}),$$

$$(\mathbf{H}_{0,\lambda},\|\cdot\|_{0,\lambda}):=(\mathbf{W}_{0,2,\lambda},\|\cdot\|_{0,2,\lambda})$$

and

$$(\mathbf{H}_m,\|\cdot\|_m):=(\mathbf{W}_{m,2,1},\|\cdot\|_{m,2,1}).$$

Finally,

$$(\tilde{\mathbf{W}}_{m,p,\Phi}),\ \tilde{\|}\cdot\tilde{\|}_{m,p,\Phi}):=(\mathbf{W}_{m,p,\Phi}\times\mathbf{W}_{m,p,\Phi},\|\cdot\|_{m,p,\Phi}+\|\cdot\|_{m,p,\Phi})$$

and

$$(\tilde{\mathbf{H}}_{0,\lambda},\tilde{\|}\cdot\tilde{\|}_{0,\lambda}):=(\tilde{\mathbf{W}}_{0,2,\lambda},\tilde{\|}\cdot\tilde{\|}_{0,2,\lambda}).$$

Let $s\geq 0$. Then $\mathbf{H}_{0,\lambda,[s,\infty)}$ (resp. $\tilde{\mathbf{H}}_{0,\lambda,[s,\infty)}$, $\mathbf{M}_{[s,\infty)}$, $\mathbf{M}_{\mathbf{a},[s,\infty)}$) denotes the space of $\mathbf{H}_{0,\lambda}$–, (resp. $\tilde{\mathbf{H}}_{0,\lambda}$–, $\mathbf{M}$–, $\mathbf{M_a}$-) –valued continuous functions.

$$\mathbf{W}_{m,p,\Phi,[s,T]} := \{f : [s,T] \times \mathbf{R}^d \to \mathbf{R} : \int_s^T \|f(s)\|^p_{m,p,\Phi} ds < \infty\},$$

where we assume that for fixed s $f(s) \in \mathbf{W}_{m,p,\Phi}$ and that f is $\mathcal{B}_{s,t} \otimes \mathcal{B}^d - \mathcal{B}^1$-measurable with $\mathcal{B}_{s,t} := \mathcal{B}^1 \cap [s,T]$ (the restriction of $\mathcal{B}^1$ to $[s,T]$).

Next we introduce the stochastic set-up. $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a stochastic basis with right continuous filtration. All our stochastic processes are assumed to live on Ω and to be $\mathcal{F}_t$-adapted (including all initial conditions in stochastic ordinary differential equations (SODE's) and stochastic partial differential equations (SPDE's)). Moreover, the processes are assumed to be $dP \otimes dt$-measurable, where dt is the Lebesgue measure on $[0,\infty)$. Let $w_\ell(r,t)$ be i.i.d. real valued Brownian sheets on $\mathbf{R}^d \times \mathbf{R}_+$, $\ell = 1,\ldots,d$ (cf. Walsh (1986) and Kotelenez (1992)) with mean zero and variance $t|A|$, where A is a Borel set in $\mathbf{R}^d$ with finite Lebesgue measure $|A|$. Adaptedness for $w_\ell(r,t)$ means that $\int_A w_\ell(dp,t)$ is adapted for any Borel set $A \subset \mathbf{R}^d$ with $|A| < \infty$. $w(p,t) := (w_1(p,t),\ldots,w_d(p,t))^T$ where "T" denotes the transpose.

If $\mathbf{K}$ is some metric space, $\mathcal{K}_s$ is the space of $\mathbf{K}$-valued $\mathcal{F}_s$-measurable random variables. $\mathcal{W}_{m,p,\Phi,[s,T]}$ is the space of $\mathbf{W}_{m,p,\Phi}$-valued adapted $dt \otimes dP$-measurable processes $\eta(\cdot)$ such that

$$E\int_s^T \|\eta(u)\|^p_{m,p,\Phi} du < \infty.$$

Set $\mathcal{H}_{0,\lambda,[s,T]} := \mathcal{W}_{0,2,\lambda,[s,T]}$. $\tilde{\mathcal{W}}_{m,p,\Phi,[s,T]}$ and $\tilde{\mathcal{H}}_{0,\lambda,[s,T]}$ are defined by replacing $\mathbf{W}_{m,p,\Phi}$ by $\tilde{\mathbf{W}}_{m,p,\Phi}$ in the definition of $\mathcal{W}_{m,p,\Phi,[s,T]}$.

If $\mathbf{B}_i, i = 1,2$ are metric spaces, then $C(\mathbf{B}_1,\mathbf{B}_2)$ is the space of continuous functions from $\mathbf{B}_1$ into $\mathbf{B}_2$.

For $m \in \mathbf{N}$ let $C_b^m(\mathbf{R}^d,\mathbf{R})$ be the space of m times continuously differentiable bounded real valued functions on $\mathbf{R}^d$, where all derivatives up to order m are bounded, $C_0^m(\mathbf{R}^d,\mathbf{R})$ is the subspace of $C_b^m(\mathbf{R}^d,\mathbf{R})$ whose elements vanish at infinity and $C_c^m(\mathbf{R}^d,\mathbf{R})$ the subspace of $C_0^m(\mathbf{R}^d,\mathbf{R})$, whose elements have compact support. If $f \in C_b^m(\mathbf{R}^d,\mathbf{R})$ we set

$$|||f|||_m := \max_{\substack{\ell_1+\cdot+\ell_d=|\ell| \\ |\boldsymbol{\ell}|\le m}} \sup_{r\in\mathbf{R}^d} |\partial^{|\boldsymbol{\ell}|}_{\ell_1,\ldots\ell_d} f(r)|,$$

where $\partial^{|\boldsymbol{\ell}|}_{\ell_1\ldots\ell_d} f(r) = \frac{\partial^{\ell_1+\cdot+\ell_d}}{(\partial r_{\ell_1})^{\ell_1}\ldots(\partial r_{\ell_q})^{\ell_d}} f(r)$ and r_{ℓ_i}, is the ℓ_i-th coordinate of r. If we take only one partial derivative, say, with respect to r_ℓ, we will just write "∂_ℓ".

We now define the coefficients for the microscopic equations (SODE's):

$$F : \mathbf{R}^d \times \mathbf{M} \times \tilde{\mathbf{H}}_{0,\lambda} \to \mathbf{R}^d$$

and

$$\mathcal{J} : \mathbf{R}^d \times \mathbf{R}^d \times \mathbf{M} \times \tilde{\mathbf{H}}_{0,\lambda} \to \mathcal{M}_{d\times d},$$

where $\mathcal{M}_{d\times d}$ are the $d \times d$-matrices over $\mathbf{R}$. Endowing the domains and ranges of F and $\mathcal{J}$ with the Borel σ-algebras, we assume F and $\mathcal{J}$ to be measurable. We will consider two types of microscopic equations:

$$\left.\begin{array}{rl} dr(t) = & F(r(t),\tilde{\mathcal{Y}}(t),\tilde{Z}(t))dt + \int \mathcal{J}(r(t),p,\tilde{\mathcal{Y}}(t),\tilde{Z}(t))w(dp,dt) \\ r(s) = & r_s \in \mathcal{R}_{d,s}, \tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[0,\infty)}, \tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}, \end{array}\right\} \tag{1.10}$$

where $\mathcal{R}_{d,s}$ is the space of $\mathbf{R}^d$-valued, $\mathcal{F}_s$-measurable random variables r_s such that $E|r_s|^2 < \infty$, and $\tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}$ is the inductive limit of $\tilde{\mathcal{W}}_{0,\lambda,[0,T]}, T \uparrow \infty$ (and the same for $\mathcal{M}_{\mathbf{b},[0,\infty)}$). (1.10) describes the motion of a diffusing particle in a random environment (represented by $(\tilde{\mathcal{Y}}, \tilde{Z})$). The integration in the stochastic integral is over $\mathbf{R}^d$, and in what follows we will never indicate the integration domain, if it is $\mathbf{R}^d$.

$$\left.\begin{array}{rl} dr^i(t) = & F(r^i(t), \mathcal{X}_N(t), \tilde{Z}(t))dt + \int \mathcal{J}(r^i(t), p, \mathcal{X}_N(t), \tilde{Z}(t))w(dp, dt) \\ r^i(s) = & r^i_s \in \mathcal{R}_{d,s},\ i = 1, \ldots N,\ \mathcal{X}_N(t) := \sum_{i=1}^N b_i \delta_{r^i(t)},\ \tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}, \end{array}\right\} \tag{1.11}$$

where $b_i \in \mathbf{R}$ (implying $\mathcal{X}_N(t, w) \in \mathbf{M_b}$ for all (t, ω), if $\mathbf{b} = (b^+, b^-)$ and $b^+ := \sum_{b_i \geq 0} b_i$, $b^- = -\sum_{b_i < 0} b_i \cdot$). (1.11) represents the motion of N interacting and diffusing particles in a random environment (represented by $\tilde{Z}$). δ_{r^i} is the point measure concentrated in r^i.

Let us assume that the following functions are given:

$B \in C_b^3(\mathbf{R}; \mathbf{R})$ such that for all $x \in \mathbf{R}$, $|B(x)| \leq |x|$, $B(x) = B(-x)$, $B'(x) > 0$ and $|B^{(3)}(x)x^3| + |B''(x)x^2| \leq c_1|B'(x)| \cdot |x| \leq c_2|B(x)|$, where $B^{(3)}$, B'' and B' are the third, second and first derivatives of B, respectively, and c_1 and c_2 are finite constants.
$\Lambda_L \in C_b(\mathbf{R}^{2d}; \mathbf{R}_+)$ such that $\Lambda_L^2(r, \cdot)\lambda^{-1}(\cdot)$ as a function of r belongs to $C(\mathbf{R}^d; \mathbf{W}_{0,1,1})$, $L \in \{F, \mathcal{J}\}$;
$K \in C_b^1(\mathbf{R}^{3d}; \mathbf{R}_+)$;
$\Gamma \in C_b^2(\mathbf{R}^{4d}; \mathbf{R}_+)$, and $\sup_{q,\overline{p}} \int (\Gamma(q, \tilde{p}, p, \overline{p}) - \Gamma(q, \hat{p}, p, \overline{p}))^2 dp \leq c_{\mathcal{J}} \varrho^2(\tilde{p}, \hat{p})$. (1.12)

All constants will denoted by $c_F, c_{\mathcal{J}}, c_{\mathcal{J},T}$ etc. and will be assumed to be nonnegative and finite.

Hypothesis 1.3

Suppose $(r_\ell, \mu_\ell, \tilde{\eta}_\ell) \in \mathbf{R}^d \times \mathbf{M_b} \times \tilde{\mathbf{H}}_{0,\lambda}, \ell = 1, 2$, where $\mathbf{b} = (b^+, b^-)$ and $b^+ \neq 0 \neq b^-$. Then with $\tilde{\eta}_\ell = (\eta_\ell, \hat{\eta}_\ell)$

$$\left.\begin{array}{rl} & |F(r, \mu_1, \tilde{\eta}_1) - F(r_2, \mu_2, \tilde{\eta}_2)| \\ \leq & c_F\{\rho(r_1, r_2) + \|B(\eta_1) - B(\eta_2)\|_{0,\lambda} \\ & \quad + \sup_{\overline{p}} \sum_{+,-} |\int K(r_2, \tilde{p}, \overline{p})(\mu_1^{\pm}(d\tilde{p}) - \mu_2^{\pm}(d\tilde{p}))| \\ & \quad + \int \Lambda_F(r_2, p)|B(\hat{\eta}_1(p)) - B(\hat{\eta}_2(p))|dp\}; \end{array}\right\} \tag{1.13}$$

$$\left.\begin{array}{l} \sum_{k,\ell=1}^d \int (\mathcal{J}_{k\ell}(r_1, p, \mu_1, \tilde{\eta}_1) - \mathcal{J}_{k\ell}(r_2, p, \mu_2, \tilde{\eta}_2))^2 dp \\ \leq c_{\mathcal{J}}\{\rho^2(r_1, r_2) + \|B(\eta_1) - B(\eta_2)\|^2_{0,\lambda} \\ \quad + \sup_{\overline{p}} \sum_{+,-} \int (\int \Gamma(r_2, p, \tilde{p}, \overline{p})(\mu_1^{\pm}(d\tilde{p}) - \mu_2^{\pm}(d\tilde{p})))^2 dp \\ \quad + (\int \Lambda_{\mathcal{J}}(r_2, p)|B(\hat{\eta}_1(p)) - B(\hat{\eta}_2(p))|dp)^2\}. \end{array}\right\} \tag{1.14}$$

Example 1.4

$$F(r, \mu, \tilde{\eta}) := F^1(r, \mu) + F^3(r, \hat{\eta}) \tag{1.15}$$

We assume:

$$F^1(r, \mu) \text{ is the functional from Kotelenez (1995b, (1.11)) and} \tag{1.16}$$

$$F^3(r, \hat{\eta}) := \int K(r - p)B(\hat{\eta}(p))dp \tag{1.17}$$

where $K_\ell(r) := c\exp(-|r|^2/\gamma^2)$, $\quad c > 0, \gamma^2 > 0, \ell = 1, \ldots, d$.

Since by Kotelenez (loc. cit.) $F^1(r, \mu)$ implies (1.13) with $B \equiv 0$ we have only to verify (1.13) for $F^3(r, \hat{\eta})$. We easily see that $\Lambda_F(r, p) := K_1(r - p)$ satisfies (1.12), whence

$$|F^3(r,\hat\eta_1) - F^3(r,\hat\eta_2)| \le c_F \int \Lambda_F(r,p)|B(\hat\eta_1(p)) - B(\hat\eta_2(p))|dp. \tag{$*$}$$

Next,

$$\begin{aligned}
& |\exp(-\frac{|r-p|^2}{\gamma^2}) - \exp(-\frac{|q-p|^2}{\gamma^2})| \\
= & \left| \int_{\frac{|q-p|}{\gamma}}^{\frac{|r-p|}{\gamma}} \alpha e^{-\frac{\alpha^2}{2}} d\alpha \right| \wedge 1 \\
= & \left| \int_0^{\frac{|r-p|-|q-p|}{\gamma}} (\beta + \frac{|q-p|}{\gamma}) \exp(-(\beta + \frac{|q-p|}{\gamma})^2 \frac{1}{2}) d\beta \right| \wedge 1 \\
& (\text{if } |r-p| \ge |q-p|) \\
\le & \left(2 \int_0^{\frac{|r-q|}{\gamma}} \exp(-(\beta + \frac{|q-p|}{\gamma})^2 \frac{1}{4}) d\beta \right) \wedge 1 \\
\le & (2 \frac{|r-q|}{\gamma} \exp(-\frac{|q-p|^2}{\gamma^2}\frac{1}{4})) \wedge 1 \ .
\end{aligned}$$

So by symmetry in r and q we obtain

$$\left. \begin{aligned}
& |\exp(-\tfrac{|r-p|^2}{\gamma^2}) - \exp(-\tfrac{|q-p|^2}{\gamma^2})| \\
& \le (2\tfrac{|r-q|}{\gamma}(\exp(-\tfrac{|q-p|^2}{\gamma}\tfrac{1}{4}) + \exp(-\tfrac{|r-p|^2}{\gamma}\tfrac{1}{4}))) \wedge 1 \ .
\end{aligned} \right\} \tag{$**$}$$

Hence,

$$|\int (K_1(r-p) - K_1(q-p))B(\hat\eta(p))dp| \le c_{B,\gamma}\rho(r,q) \ . \tag{$***$}$$

Now $(*)$ and $(***)$ in addition to the choice of F^1 imply (1.13). ∎

Example 1.5

$$\mathcal{J}(r,p,\mu,\tilde\eta) := \mathcal{J}^1(r,p,\mu) + \mathcal{J}^2(r,p,\eta) \tag{1.18}$$

Again we assume

$$\mathcal{J}^1(r,p,\mu) \text{ is the functional from Kotelenez, loc. cit. ((1.11)).} \tag{1.19}$$

$$\mathcal{J}^2(r,p,\eta) = \int \Gamma(r-p-\tilde p)\lambda(\tilde p)B(\eta(\tilde p))d\tilde p, \tag{1.20}$$

where $\Gamma = (\Gamma_{k\ell}) \in \mathcal{M}_{d\times d}$ and $\Gamma_{k\ell}(r) := c\exp(-\frac{|r|^2}{\gamma^2})$ as in Example 1.4.

$$\int (\mathcal{J}^2_{k\ell}(r,p,\eta) - \mathcal{J}^2_{k\ell}(q,p,\eta))^2 dp \le c_{\mathcal{J},B}\rho^2(r,q) \tag{$*$}$$

by $(**)$ of Example 1.4 and the integrability of λ. Further,

$$\left.\begin{array}{l}\int(\mathcal{J}^2_{k\ell}(r,p,\eta_1)-\mathcal{J}^2_{k\ell}(r,p,\eta_2)^2)dp\\ \le c_{\mathcal{J},\lambda}\|B(\eta_1)-B(\eta_2)\|^2_{0,\lambda}\end{array}\right\}\tag{$**$}$$

by the Cauchy-Schwarz inequality and the integrability of λ and $\Gamma^2_{k\ell}$.

Again $(*)$ and $(**)$ in addition to the choice of $\mathcal{J}^1$ imply (1.14).

Example 1.6

$$F(r,\mu,\tilde{\eta}) := F^1(r,\mu)+F^2(r,\eta)\tag{1.21}$$

where

$$F^2(r,\eta) := \int K(r-p)\lambda(p)B(\eta(p))dp\tag{1.22}$$

with K and $F^1(r,\mu)$ as in Example 1.4.

Example 1.7

$$\mathcal{J}(r,p,\mu,\tilde{\eta}) := \mathcal{J}^1(r,p,\mu)+\mathcal{J}^3(r,p,\hat{\eta}),\tag{1.23}$$

where

$$\mathcal{J}^3(r,p,\hat{\eta}) := \Gamma(r-p)\int\Gamma(p-\tilde{p})B(\hat{\eta}(\tilde{p}))d\tilde{p}.\tag{1.24}$$

Again we take Γ and $\mathcal{J}^1(r,p,\mu)$ as in Example 1.5. Set $f(\tilde{p}) := |B(\hat{\eta}_1(\tilde{p}))-B(\hat{\eta}_2(\tilde{p}))|$ and note that

$$\lambda(r)\lambda^{-1}(q)\le c\cdot\lambda^{-1}(r-q).\tag{1.25}$$

Then

$$\left.\begin{array}{l}\Gamma_{kj}(r-p)\int\Gamma_{j\ell}(p-\tilde{p})f(\tilde{p})d\tilde{p}\\ =\Gamma_{kj}(r-p)\int\Gamma_{j\ell}(p-\tilde{p})\lambda^{-1}(r-\tilde{p})\lambda(\tilde{p}-p)\lambda^{-1}(\tilde{p}-p)\lambda(r-\tilde{p})f(\tilde{p})d\tilde{p}\\ \le c\Gamma_{kj}(r-p)\lambda^{-1}(r-p)\int\Gamma_{j\ell}(p-\tilde{p})\lambda^{-1}(p-\tilde{p})\lambda(r-\tilde{p})f(\tilde{p})d\tilde{p}.\end{array}\right\}\tag{$*$}$$

Since $|||\Gamma_{j\ell}\lambda^{-1}|||+\|\Gamma_{kj}\lambda^{-1}\|^2_0<\infty$, $(*)$ implies

$$\left.\begin{array}{l}\int(\Gamma_{kj}(r-p)\int\Gamma_{j\ell}(p-\tilde{p})f(\tilde{p})d\tilde{p})^2dp\\ \le c_{\Gamma,\lambda}(\int\lambda(r-\tilde{p})f(\tilde{p})d\tilde{p})^2.\end{array}\right\}\tag{1.26}$$

Setting $\Lambda_{\mathcal{J}}(r,p) := \lambda(r-p)$, we verify (by (1.25))

$$\int\lambda^2(r-p)\lambda^{-1}(p)dp\le c\lambda^{-1}(r)\int\lambda(r-p)dp=\overline{c}\lambda^{-1}(r).\tag{$**$}$$

Hence, $\mathcal{J}$ satisfies (1.14).

Remark 1.8

(i) Let $\{\tilde{\phi}_n\}_{n\in N}$ be a complete orthonormal system (CONS) in $\mathbf{H}_0$ and define an $\mathcal{M}_{d\times d}$-valued function ϕ_n whose entries on the main diagonal are all $\tilde{\phi}_n$ and whose other entries are all 0. Then for any $\tilde{\mathcal{Y}}\in\mathcal{M}_{b,[0,\infty)}$, $\tilde{Z}\in\tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}$ and $r(\cdot)\in\mathcal{R}_{d,[0,\infty)}$ (the spaces of $\mathbf{R}^d$-valued adapted continuous processes)

$$\int \mathcal{J}(r(t),p,\tilde{\mathcal{Y}}(t),\tilde{Z}(t))w(dp,dt) = \sum_{n=1}^{\infty}\int \mathcal{J}(r(t),p,\tilde{\mathcal{Y}}(t),\tilde{Z}(t))\phi_n(p)dpd\beta^n(t), \tag{1.27}$$

where $\beta^n(t)$ are $\mathbf{R}^d$-valued i.i.d. standard Wiener processes. The right hand side of (1.27) defines the increment of an $\mathbf{R}^d$-valued square integrable continuous martingale M, which can be verified as the corresponding statement in Kotelenez (1995b), Remark 1.2. In particular, (1.14) implies for the mutual quadratic variation of the one-dimensional components of $M(t) := M(r(t),\tilde{\mathcal{Y}}(t),\tilde{Z}(t))$

$$[M_k(t), M_\ell(t)] \leq c_{\mathcal{J}} t. \tag{1.28}$$

(ii) For the analysis of equivalent distributions it is convenient to have a nice state space for $w(dp,dt)$. The representation (1.27) suggests for fixed t a state space on which we can define spatial Gaussian standard white noise as a countably additive Gauss measure. One way is to set for some fixed $\alpha > d$

$$\mathbf{H}_w := Dom((-\triangle + |r|^2)^{-\alpha}),$$

where $\triangle$ is the Laplacian on $\mathbf{H}_0$, $|r|^2$ is a multiplication operator, and "Dom" is the domain of the fractional power $(-\alpha)$ of $(-\triangle + |r|^2)$, which itself determines a "natural" Hilbert norm $\|\cdot\|_w$ on $\mathbf{H}_w$. Then with $\mathbf{1} = (1,\ldots,1)$

$$\partial^{\mathbf{1}} w(\cdot,t) \in C([0,\infty);\mathbf{H}_w) =: \mathbf{H}_{w,[0,\infty)} \text{ a.s.} \tag{1.29}$$

(cf., e.g., Kotelenez (1985)).

Following Kotelenez (1995b) we now introduce the mezoscopic equations (or stochastic partial differential equations (SPDE's)) associated with (1.10) and (1.11). The empirical process associated with (1.10) is given by

$$\mathcal{Y}_N(t) := \sum_{i=1}^{N} a_i \delta_{r(t,r_s^i)}, \tag{1.30}$$

where $r(t,r_s^i)$ is the solution of (1.10) starting at r_s^i and $a_i \in \mathbf{R}$ (assuming that a solution exists). The empirical process associated with (1.11) is $\mathcal{X}_N(t)$, as given in (1.11).

We easily see that the mutual quadratic variation $[M_k(t), M_\ell(t)]$ is differentiable in t. Set for $(\mu,\tilde{\eta},r) \in \mathbf{M} \times \tilde{\mathbf{H}}_{0,\lambda} \times \mathbf{R}^d$

$$D_{k\ell}(\mu,\tilde{\eta},r) := \frac{d}{dt}[M_k(r,\mu,\tilde{\eta}), M_\ell(r,\mu,\tilde{\eta})], \tag{1.31}$$

and abbreviate

$$\partial^2_{k\ell} := \frac{\partial^2}{\partial r_k \partial r_\ell}.$$

Then the empirical process associated with (1.10), $\mathcal{Y}_N(t)$, is a solution of the bilinear SPDE

$$\left.\begin{aligned} d\mathcal{Y} = \quad & (\tfrac{1}{2}\textstyle\sum_{k,\ell=1}^{d} \partial^2_{k\ell}(D_{k\ell}(\tilde{\mathcal{Y}},\tilde{Z})\mathcal{Y}) - \nabla\cdot(\mathcal{Y}F(\tilde{\mathcal{Y}},\tilde{Z})))dt \\ & - \nabla\cdot(\mathcal{Y}\textstyle\int \mathcal{J}(\cdot,p,\tilde{\mathcal{Y}},\tilde{Z})w(dp,dt)) \end{aligned}\right\} \tag{1.32}$$

with initial condition $\mathcal{Y}(s) = \sum_{i=1}^{N} a_i \delta_{r_s^i}$.

The empirical process associated with (1.11), $\mathcal{X}_N(t)$, is a solution of the quasilinear SPDE

$$\left.\begin{array}{ll} d\mathcal{X} & = (\frac{1}{2}\sum_{k,\ell=1}^d \partial^2_{k\ell,r}(D_{k\ell}(\mathcal{X},\tilde{Z})\mathcal{X}) - \bigtriangledown\cdot(\mathcal{X}F(\mathcal{X},\tilde{Z})))dt \\ & -\bigtriangledown\cdot(\mathcal{X}\int\mathcal{J}(\cdot,p,\mathcal{X},\tilde{Z})w(dp,dt)), \end{array}\right\} \tag{1.33}$$

with initial condition $\mathcal{X}(s) = \sum_{i=1}^N a_i\delta_{r_s^i}$.

A solution of (1.32) and (1.33) will always be a weak solution (in the sense of partial differential equations, cf. Kotelenez (1995b)), where the test functions are from $C_b^3(\mathbf{R}^d;\mathbf{R})$.

Remark 1.9

(i) If the coefficients F and $\mathcal{J}$ are independent of $\tilde{\eta} = (\eta,\hat{\eta})$, then (1.32) and (1.33) are a straightforward generalization of the SPDE's considered by Kotelenez (1995b). The only remarkable difference then is our more abstract formulation which allows us also to consider, e.g., interaction of three and more particles by using the scheme (for the one-dimensional components of the forces)

$$\int\ldots\int K_1(r-q_1)K_2(q_1-q_2)\ldots K_n(q_{n-1}-q_n)\mu(dq_n)\ldots\mu(dq_1), \tag{1.34}$$

which was not explicitly considered in Kotelenez (loc. cit.). Moreover, we allow here spatially inhomogeneous "interaction" for the microscopic and mezoscopic models.

(ii) The real novelty, however, is the dependence on $\tilde{\eta} = (\eta,\hat{\eta})$, which can be considered as densities of possibly non-finite measures. Note that the derivation of the mezoscopic SPDE's from the particle distribution in Kotelenez (loc. cit.) crucially depended on the Wasserstein metric, which is a metric on probability measures, but can be easily extended to finite measures with the same mass. Since we do not know a (convenient) extension of the Wasserstein metric to non-finite measures (which we need, if we want to derive spatially homogeneous random fields as solutions!), we have introduced $\tilde{\eta}$ as "external" fields at the microscopic level. This means that (1.33) would be bilinear, if the coefficients were independent of μ (resp. $\tilde{\mathcal{Y}}$). Once we derive "smooth" solutions for (1.32) and (1.33), generalizing the procedure of Kotelenez (1995b), we may afterwards try to "identify", say, η with an "extended" version of the smooth solution. This strategy is successfully applied in Section 6 (Theorem 6.13) under the assumption that F and $\mathcal{J}$ are independent of μ. So, basically μ and η represent the same physical phenomenon, namely the mass distribution of the "heavy" particles, whose positions are described by the microscopic SODE's.

(iii) The rôle of $\hat{\eta}$ in $\tilde{\eta} = (\eta,\hat{\eta})$ is at this stage confined to act as a shift absorbing medium for (1.32) or (1.33). Indeed, let us define the shift operator U_h (acting on suitable functions $f:\mathbf{R}^d\to\mathbf{R}$) by

$$(U_hf)(r) := f(r+h), \qquad h\in\mathbf{R}^d. \tag{$*$}$$

Then, with F^3 given by (1.17) and $\hat{Z}\in\mathcal{H}_{0,\lambda,[0,\infty)}$,

$$\left.\begin{array}{ll} U_hF^3(r,\hat{Z}(t)) & = \int K(r+h-p)B(\hat{Z}(t,p))dp \\ & = \int K(r-p)B(\hat{Z}(t,p+h))dp. \end{array}\right\} \tag{1.35}$$

So, if the distribution of $\hat{Z}(t,p)$ is the same as that of $\hat{Z}(t,p+h)$ for all t and p, i.e., if $\hat{Z}(t)$ is spatially shift invariant or homogeneous for all t, then we might hope to obtain homogeneous solutions for (1.32), provided similar calculations hold for the diffusion part and $F_k^1\equiv\mathcal{J}_{k\ell}^1\equiv 0$ for $k,\ell=1,\ldots d$. We also see that the preceding arguments do not apply to F^2 given by (1.22) because of the factor $\lambda(p)$.

Now take $\mathcal{J}^2$, defined by (1.20), $Z\in\mathcal{H}_{0,\lambda,[0,\infty)}$, integrate $\mathcal{J}^2(r,p,Z(t))$ against the Brownian sheet and apply U_h;

$$\left.\begin{array}{l} U_h\int\mathcal{J}^2(r,p,Z(t))w(dp,dt) \\ = \int\int\Gamma(r+h-p-\tilde{p})\lambda(\tilde{p})B(Z(t,\tilde{p}))d\tilde{p}\,w(dp,dt) \\ = \int\int\Gamma(r-p-\tilde{p})\lambda(\tilde{p})B(Z(t,\tilde{p}))d\tilde{p}w_h(dp,dt), \end{array}\right\} \tag{1.36}$$

with $w_h(r,t) := w(r+h,t)$ (using (1.27) and Fubini's theorem). In other words, we could absorb the shift into the Brownian sheet and leaving $\lambda(\tilde{p})B(Z(t,\tilde{p}))$ unchanged. Since we easily see that $w_h \sim w$ (are equivalent in distribution—cf. Kotelenez (1995c)) one may expect to derive spatially homogeneous solutions for (1.33) provided $F = F^3$ and $\mathcal{J} = \mathcal{J}^2$ both in the "bilinear" case (Z is an external field) and in the quasilinear case (Z is "identified" with the solution). Under some additional assumptions we actually derive homogeneous solutions both for the bilinear and the quasilinear cases in Section 7 (Theorems 7.2 and 7.3). ■

Remark 1.10

(I) A particle approach to derive mezoscopic equations was, to our knowledge, first used by Dawson (1975) for a system of branching and diffusing particles. In a recent paper Dawson and Vaillaneourt (1995) derived mezoscopic equations for diffusing particles with mean field interaction by methods different from ours. The total mass in that model equals 1. Our approach was first developed for a model of interacting and diffusing vortices in a $2D$-fluid (Kotelenez (1995a)).

(II) Since in this paper we derive homogeneous and isotropic solutions of quasilinear SPDE's describing the mass distribution of diffusing particles with mean field interaction there are several natural problems arising:

(i) Shift invariance in t in addition to the spatial homogeneity (s. Dorogovtsev and Kotelenez (in preparation))

(ii) Analysis of the spectral properties of the fields and comparing to "inverse" approaches, where one starts with the spectrum etc. (cf. Dobrushin (1979)).

(iii) Introduction of more singular interaction (cf., for the microscopic equations, Skorohod (in preparation))

(iv) Inclusion of a reaction scheme (creation and annihilation). For a simple model of diffusing and branching particles this was done by Goncharuk and Kotelenez (1996). In that paper a fractional step method was used to include creation and annihilation, since the Wasserstein metric approach seems to work only for fixed (finite) mass. It would be nice to use the fractional step method to include nonlinear reaction using, e.g., the recent diffusion approximation results by Blount (1995).

Let us we now briefly describe the content. In Section 2 we analyze the microscopic equations. Apart from existence and uniqueness we obtain versions of the solutions which are measurable with respect to $(\mu(\cdot), \tilde{\eta}(\cdot), r, s) \in \mathbf{M}_{\mathbf{b},[0,\infty)} \times \tilde{\mathbf{H}}_{0,\lambda,[0,\infty)} \times \mathbf{R}^d \times [0,t)$, where the σ-algebras are the cylinder set σ-algebras on the first two spaces and the Borel σ-algebras on $\mathbf{R}^d \times [0,t)$. This generalizes the well known result for Markovian SODE's to the non-Markovian case. We also derive the backward SODE in Section 2. Complicated, resp. technical proofs are given in Appendix II (Section 10).

In Section 3 we derive general solutions for the mezoscopic equations. In Section 4 we derive smoothness for the mezoscopic equations under smoothness assumptions on the coefficients and assuming $\mathcal{J}(r,p,\ldots) = \mathcal{J}(r-p,\ldots)$, i.e., assuming homogeneity in r and p. The latter assumption can be easily discarded, and we just assumed it to reduce the calculations to a minimum. The proofs for Section 4 are given in Appendix I (Section 9). In Section 5 we derive the Itô-formula for $\|Y(t)\|^p_{m,p,\Phi}$, where Y is the function valued solution of (1.32) (we identify densities Y with the measure $\mathcal{Y}$—cf. Kotelenez (1995b)). In Section 6 we extend the mezoscopic equations (1.32) and (1.33) to initial conditions from $\mathcal{W}_{m,p,\Phi,s}$, where for (1.33) we have to assume the independence of F and $\mathcal{J}$ of μ. In Section 7 we derive homogeneous and isotropic solutions for (1.32) and (1.33).

Section 8 provides a simple comparison to the variational approach (cf. Pardoux (1975) as well as Krylov and Rozovski (1979)). Moreover, we also show that (1.32) is equivalent to a first order SPDE, if we use Stratonovich integration instead of Itô integration. That this should be the case was first pointed out to the author by J.P. Fouque (1994). Sections 9 and 10 are the appendices with technical proofs.

Some more notation:

Differential operators with respect to the space variable are usually denoted by $\partial^{\mathbf{j}}$, $\mathbf{j} \in (\mathbf{N} \cup \{0\})^d$, but if there are several space variables, like (r, p, q) etc.), we will write $\partial_r^{\mathbf{j}}, \partial_q^{\mathbf{j}}$ etc. to indicate the variable on which $\partial^{\mathbf{j}}$ is acting. Partial derivatives of first and second order may also be denoted by $\partial_k, \partial_{k\ell}^2$, resp. $\partial_{k,r}, \partial_{k\ell,r}^2$ (indicating again in the latter cases the relevant space variables). $\mathcal{G}_{s,t}$ (resp. $\mathcal{G}_t$) is the σ-algebra generated by the Brownian sheet between s and t (resp. 0 and t) for $t \geq s$. $\mathcal{G}_{w,s,t}$ and $\mathcal{G}_{w,t}$ are the corresponding cylinder set σ-algebras on $\mathbf{H}_{w,[0,\infty)}$. We will use a bar on top of a σ-algebra to indicate that this σ-algebra is complete. The corresponding cylinder set filtrations on $\mathbf{M}_{\mathbf{a},[0,\infty)}$, $\mathbf{H}_{0,\lambda,[0,\infty)}$, $\tilde{\mathbf{H}}_{0,\lambda,[0,\infty)}$ are denoted $\mathcal{F}_{\mathbf{a},s,t}$, $\mathcal{F}_{0,\lambda,s,t}$, $\tilde{\mathcal{F}}_{0,\lambda,s,t}$ etc. $[\cdot]$ will denote the (scalar) quadratic variation of $\mathbf{R}^m$-valued martingales, where $m \geq 1$. Moreover, for $m \in \mathbf{N}$, $\mathbf{R}_{m,[s,\infty)} := C([s,\infty); \mathbf{R}^m)$. Finally, if $\mathbf{B}_i$, $i = 1, 2$ are Banach spaces, $\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2)$ are the bounded linear operators from $\mathbf{B}_1$ into $\mathbf{B}_2$. If $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}$, then $\mathcal{L}(\mathbf{B}) = \mathcal{L}(\mathbf{B}, \mathbf{B})$. Finally, we set

$$W(t) := \partial^{\mathbf{1}} w(\cdot,\ t)\ , \tag{1.37}$$

where $\mathbf{1} = (1, \ldots, 1)$ (cf. (1.29)). Note that W is an $\mathbf{H}_0$-valued standard cylindrical Brownian motion.

2 The Microscopic Equations

Existence and uniqueness are derived for a class of SODE's of type (1.11) as well as the measurability of its solution with respect to $(\omega, \mu, \tilde{\eta}, r)$. This generalizes the well known result for SODE's of Markovian type (measurability in (ω, r)) to our setting. As a consequence, equivalence in distribution of the solutions of SODE's driven by different "inputs" $(w_i, \tilde{Y}_i, \tilde{Z}_i, r_{s,i})$, $i = 1, 2$ is derived, provided that the "inputs" are equivalent in distribution (Theorem 2.2). In Theorem 2.3 the SODE for interacting particles is solved, and in Theorem 2.7 the "backward" SODE is derived. The proofs of Theorem 2.7 and of a part of Theorem 2.1 are given in Appendix II.

Let $\tilde{F}$ and $\tilde{\mathcal{J}}$ satisfy the assumptions (1.13) and (1.14), $\tilde{w}$ be an $\mathcal{F}_t$-adapted Brownian sheet (as in (1.27)) and consider the following $\mathbf{R}^d$-valued SODE on $[s, \infty)$:

$$\left.\begin{aligned} dr(t) = &\ \tilde{F}(r(t), \tilde{\mathcal{Y}}(t), \tilde{Z}(t))dt + \int \tilde{\mathcal{J}}(r(t), p, \tilde{\mathcal{Y}}(t), \tilde{Z}(t))\tilde{w}(dp, dt) \\ &\ r(s) = r_s, \tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[0,\infty]},\ \tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[0,\infty)}. \end{aligned}\right\} \tag{2.1}$$

Denote a solution of (2.1), if it exists, by $r(t, \tilde{\mathcal{Y}}, \tilde{Z}, r_s, s)$. Further, if f is a stochastic process on $[s, \infty)$ with values in some metric space, we set for $t \geq s$

$$(\pi_{s,t} f)(u) := f(u \wedge t), (u \geq s),$$

where "$\wedge$" denotes "minimum".

Theorem 2.1

1) To each $s \geq 0$, $r_s \in \mathcal{R}_{d,s}$, $\tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[s,\infty)}$, $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ (2.1) has a unique solution $r(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, r_s, s) \in \mathcal{R}_{d,[s,\infty)}$.

2) Let $\tilde{\mathcal{Y}}_i \in \mathcal{M}_{\mathbf{b},[s,\infty)}$, $\tilde{Z}_i \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ and $r_{s,i} \in \mathcal{R}_{d,s}$, $i = 1, 2$. Then for any $T \geq s$ and any stopping time $\tau \geq s$

$$\left.\begin{aligned} &E \sup_{s \leq t \leq T \wedge \tau} \rho^2(r(t, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{s,1}, s), r(t, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{s,2}, s)) \\ \leq\ &c_{T,F,\mathcal{J}}\{E\rho^2(r_{s,1}, r_{s,2}) \\ +\ &E\int_s^{T\wedge\tau} (\gamma_2^2(\tilde{\mathcal{Y}}_1(u), \tilde{\mathcal{Y}}_2(u)) + \|Z_1(u) - Z_2(u)\|_{0,\lambda}^2)du \\ +\ &\sum_{L \in \{\tilde{F}\tilde{\mathcal{J}}\}} E\int_s^{T\wedge\tau} (\int \Lambda_L(r(u, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{s,2}, s), p)|B(\hat{Z}_1(u,p)) - B(\hat{Z}_2(u,p))|dp)^2 du\}. \end{aligned}\right\} \tag{2.2}$$

Further, with probability 1 uniformly in $t \in [s, \infty)$

$$r(t, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{s,1}, s) \equiv r(t, \pi_{s,t}\tilde{\mathcal{Y}}_1, \pi_{s,t}\tilde{Z}_1, r_{s,1}, s). \tag{2.3}$$

3) For any $N \in \mathbf{N}$ there is a $\mathbf{R}^{dN}$-valued map in $(t, \omega, \mu(\cdot), \tilde{\eta}(\cdot), r_N, s)$, $0 \le s \le t < \infty$ such that for any fixed $s \ge 0$

$$\overline{r}_N(\cdot, \dots, \cdot, s) : \Omega \times \mathbf{M}_{\mathbf{b},[s,\infty)} \times \tilde{\mathbf{H}}_{0,\lambda,[s,\infty)} \times \mathbf{R}^{dN} \to C([s,\infty); \mathbf{R}^{dN}),$$

and the following holds:

(i) For any $t \ge s$

$\overline{r}_N(t, \cdot, \dots, \cdot, s)$ is $\overline{\mathcal{G}}_{s,t} \otimes \mathcal{F}_{\mathbf{b},s,t} \otimes \tilde{\mathcal{F}}_{0,\lambda,s,t} \otimes \mathcal{B}^{dN} \otimes \mathcal{B}_{[0,t]} - \mathcal{B}^{dN}$-measurable.

(ii) The i-th d-vector of $\overline{r}_N = (\overline{r}^1, \dots, \overline{r}^i, \dots \overline{r}^N)$ depends only on the i-th d-vector initial condition $r_s^i \in \mathcal{R}_{d,s}$, and with probability 1 (uniformly in $t \in [s, \infty)$)

$$\overline{r}^i(t, \cdot, \tilde{\mathcal{Y}}, \tilde{Z}, r_s^i, s) \equiv r(t, \tilde{\mathcal{Y}}, \tilde{Z}, r_s^i, s), \tag{2.4}$$

where the right hand side of (2.4) is the solution of (2.1).

(iii) If $u \ge s$ is fixed, then with probability 1 (uniformly in $t \in [u, \infty)$)

$$\left.\begin{array}{l} \overline{r}_N(t, \cdot, \pi_{u,t}\tilde{\mathcal{Y}}, \pi_{u,t}\tilde{Z}, \overline{r}_N(u, \cdot, \pi_{s,u}\tilde{\mathcal{Y}}, \pi_{s,u}\tilde{Z}, r_{N,s}, s), u) \\ \equiv \overline{r}_N(t, \cdot, \pi_{s,t}\tilde{\mathcal{Y}}, \pi_{s,t}\tilde{Z}, r_{N,s}, s). \end{array}\right\} \tag{2.5}$$

Proof

We will prove statements 1) and 2) without loss of generality on $[0, T]$ for fixed $T > 0$ and $s = 0$ and statement 3) on $[s, T]$, and whenever possible we will suppress the dependence on $s = 0$ in our notation.

(i) Let $q_\ell \in \mathcal{R}_{d,[0,\infty)}$ and set

$$\tilde{q}_\ell(t) := q_\ell(0) + \int_0^t \tilde{F}(q_\ell(s), \tilde{\mathcal{Y}}_\ell(s), \tilde{Z}_\ell(s))ds + \int_0^t \int \tilde{\mathcal{J}}(q_\ell(s), p, \tilde{\mathcal{Y}}_\ell(s), \tilde{Z}_\ell(s))\tilde{w}(dp, ds).$$

(ii) By (1.13) we obtain

$$\left.\begin{array}{rl} & \rho(\int_0^t \tilde{F}(q_1(s), \tilde{\mathcal{Y}}_1(s), \tilde{Z}_1(s))ds, \int_0^t \tilde{F}(q_2(s), \tilde{\mathcal{Y}}_2(s), \tilde{Z}_2(s))ds) \\ \le & c_F\{\int_0^t \rho(q_1(s), q_2(s))ds + \int_0^t \|B(Z_1(s)) - B(Z_2(s))\|_{0,\lambda}ds \\ + & \int_0^t \sup_{\overline{p}} \sum_{+,-} |\int K(q_2(s), \tilde{p}, \overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s, d\tilde{p}) - \tilde{\mathcal{Y}}_2^{\pm}(s, d\tilde{p}))|ds \\ + & \int_0^t \int \Lambda_F(q_2(s), p)|B(\hat{Z}_1(s,p)) - B(\hat{Z}_2(s,p))|dpds\}. \end{array}\right\} \tag{2.6}$$

(iii) By (1.14)

$$\left.\begin{array}{l} [\int_0^t \int \{\tilde{\mathcal{J}}(q_1(s), p, \tilde{\mathcal{Y}}_1(s), \tilde{Z}_1(s)) - \tilde{\mathcal{J}}(q_2(s), p, \tilde{\mathcal{Y}}_2(s), \tilde{Z}_2(s))\}\tilde{w}(dp, ds)] \\ \le c_{\tilde{\mathcal{J}}}\{\int_0^t \rho^2(q_1(s), q_2(s))ds + \int_0^t \|B(Z_1(s)) - B(Z_2(s))\|_{0,\lambda}^2 ds \\ \quad + \int_0^t \sup_{\overline{p}} \sum_{+,-} \int (\int \Gamma(q_2(s), p, \tilde{p}, \overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s, d\tilde{p}) - \tilde{\mathcal{Y}}_2^{\pm}(s, d\tilde{p})))^2 dpds \\ \quad + \int_0^t (\int \Lambda_{\tilde{\mathcal{J}}}(q_2(s), p)|B(\hat{Z}_1(s,p)) - B(\hat{Z}_2(s,p))|dp)^2 ds\}. \end{array}\right\} \tag{2.7}$$

(2.6), (2.7), the Cauchy-Schwarz and Doob inequalities imply

$$\left.\begin{array}{l}E\sup_{0\le t\le T\wedge\tau}\rho^2(\tilde q_1(t),\tilde q_2(t))\\ \le \tilde c_{F,\mathcal{J},T}\{E\rho^2(q_1(0),q_2(0))+E\int_0^{T\wedge\tau}(\rho^2(q_1(s),q_2(s))+\|B(Z_1(s))-B(Z_2(s))\|^2_{0,\lambda})ds\\ \quad E\int_0^{T\wedge\tau}\sup_{\overline{p}}\sum_{+,-}[(\int K(q_2(s),\tilde p,\overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s,d\tilde p)-\tilde{\mathcal{Y}}_2^{\pm}(s,d\tilde p)))^2\\ \qquad +\int(\int\Gamma(q_2(s),\tilde p,p,\overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s,d\tilde p)-\tilde{\mathcal{Y}}_2^{\pm}(s,d\tilde p)))^2dp]ds\\ \quad +\sum_{L\in\{\tilde F,\tilde{\mathcal{J}}\}}E\int_0^{T\wedge\tau}(\int\Lambda_L(q_2(s),p)|B(\hat Z_1(s,p))-B(\hat Z_2(s,p))|dp)^2ds\}.\end{array}\right\}\quad(2.8)$$

(iv) Next we note that

$$\begin{array}{rl} & (\displaystyle\int\Gamma(q_2(s),\tilde p,p,\overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s,d\tilde p)-\tilde{\mathcal{Y}}_2^{\pm}(s,d\tilde p)))^2\\ = & \displaystyle\frac{1}{(b^{\pm})^2}(\int\int Q^{\pm}(s,d\tilde p,d\hat p)[\Gamma(q_2(s),\tilde p,p,\overline{p})-\Gamma(q_2(s),\hat p,p,\overline{p})])^2\\ & \text{with } Q^{\pm}(s)\in C(\tilde{\mathcal{Y}}_1^{\pm}(s),\tilde{\mathcal{Y}}_2^{\pm}(s)) \text{ arbitrary})\\ \le & \displaystyle\int\int Q^{\pm}(s,d\tilde p,d\hat p)[\Gamma(q_2(s),\tilde p,p,\overline{p})-\Gamma(q_2(s),\hat p,p,\overline{p})]^2\\ & \text{(by the Cauchy-Schwarz inequality).}\end{array}$$

Hence, by (1.12)

$$\begin{array}{rl} & \displaystyle\sum_{+,-}\int(\int\Gamma(q_2(s),\tilde p,p,\overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s,d\tilde p)-\tilde{\mathcal{Y}}_2^{\pm}(s,d\tilde p)))^2dp\\ \le & \displaystyle\sum_{+,-}\int\int Q^{\pm}(s,d\tilde p,d\hat p)\int(\Gamma(q_2(s),\tilde p,p,\overline{p})-\Gamma(q_2(s),\hat p,p,\overline{p}))^2dp\\ \le & \displaystyle c_{\mathcal{J}}\sum_{+,-}\int\int Q^{\pm}(s,d\tilde p,d\hat p)\rho^2(\tilde p,\hat p).\end{array}$$

Estimating similarly the term with K in (2.8) we obtain by (1.12)

$$\left.\begin{array}{rl} & \sup_{\overline{p}}\sum_{+,-}[(\int K(q_2(s),\tilde p,\overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s,d\tilde p)-\tilde{\mathcal{Y}}_2^{\pm}(s,d\tilde p)))^2\\ & \quad +\int(\int\Gamma(q_2(s),\tilde p,p,\overline{p})(\tilde{\mathcal{Y}}_1^{\pm}(s,d\tilde p)-\tilde{\mathcal{Y}}_2^{\pm}(s,d\tilde p)))^2dp]\\ \le & c_{F,\mathcal{J}}\sum_{+,-}\int\int Q^{\pm}(s,d\tilde p,d\hat p)\rho^2(\tilde p,\hat p)\\ = & c_{F,\mathcal{J}}\gamma_2^2(\tilde{\mathcal{Y}}_1(s),\tilde{\mathcal{Y}}_2(s)).\end{array}\right\}\quad(2.9)$$

since $Q^{\pm}(s)\in C(\tilde{\mathcal{Y}}_1^{\pm}(s),\tilde{\mathcal{Y}}_2^{\pm}(s))$ was arbitrary.

(v) (2.8) and (2.9) imply

$$\left.\begin{array}{l}E\sup_{0\le t\le T\wedge\tau}\rho^2(\tilde q_1(t),\tilde q_2(t))\\ \le\tilde c_{F,\mathcal{J}}\{E\rho^2(q_1(0),q_2(0))+E\int_0^{T\wedge\tau}\rho^2(q_1(s),q_2(s))ds\\ \quad +E\int_0^{T\wedge\tau}\gamma_2^2(\tilde{\mathcal{Y}}_1(s),\tilde{\mathcal{Y}}_2(s))ds+E\int_0^{T\wedge\tau}\|B(Z_1(s))-B(Z_2(s))\|^2_{0,\lambda}ds\\ \quad +E\int_0^{T\wedge\tau}\sum_{L\in\{\tilde F,\tilde{\mathcal{J}}\}}(\int\Lambda_L(q_2(s),p)|B(\hat Z_1(s,p))-B(\hat Z_2(s,p))|dp)^2ds\}\ .\end{array}\right\}\quad(2.10)$$

(vi) Choosing first $\tilde{\mathcal{Y}}_1\equiv\tilde{\mathcal{Y}}_2$, $Z_1\equiv Z_2$, $\hat Z_1\equiv\hat Z_2$ and $\tau=T$ the existence of a unique continuous solution follows from (2.10) and the contraction mapping principle. Having thus established the existence of unique (continuous) solutions $r(\cdot,\tilde{\mathcal{Y}}_\ell,\tilde Z_\ell,r_{0,\ell})$, $\ell=1,2$, (2.2) follows from (2.10) and Gronwall's lemma.

(vii) The relation (2.3) follows immediately from the construction.
(viii) Statement 3 is proved in Addendum 10.1 of Appendix II. ■

If f and g are random variables with values in some measurable space, we will write

$$f \sim g,$$

if f and g have the same distribution. Recall that by (1.37) W is an element of $\mathcal{H}_{w,[0,\infty)}$. Now suppose for $s \geq 0$ there are two sets of random variables $(w_i, \tilde{\mathcal{Y}}_i, Z_i, r_{s,i})$, $i = 1, 2$, on $(\Omega, \mathcal{F}, P)$ such that

(i) w_i are Brownian sheets with W_i in $\mathcal{H}_{w,[0,\infty)}$ such that in the (weak) representation (1.27) $W_i(\cdot, t) = \sum_{n=1}^{\infty} \phi_n(\cdot)\beta_{n,i}(t)$, where $\{\beta_{n,i}(\cdot)\}_{n\in\mathbf{N}}$ are two families of independent $\mathbf{R}^d$-valued standard Brownian motions;

(ii) $\tilde{\mathcal{Y}}_i \in \mathcal{M}_{\mathbf{b},[s,\infty)}$ and for any $t \geq u \geq s$

$$\pi_{u,t}(\tilde{\mathcal{Y}}_1) \sim \pi_{u,t}(\tilde{\mathcal{Y}}_2);$$

(iii) $\tilde{Z}_i \in \tilde{\mathcal{H}}_{0,\lambda,[s,\infty)}$ and for any $t \geq u \geq s$

$$\pi_{u,t}(\tilde{Z}_1) \sim \pi_{u,t}(\tilde{Z}_2);$$

(iv) $r_{i,s} \in \mathcal{R}_{d,s}$ and

$$r_{1,s} \sim r_{2,s}.$$

In what follows we will denote by $r(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{s,i}, s)$ the unique continuous solutions of (2.1) with input variables $(w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{i,s})$, $i = 1, 2$. Moreover, $r_N(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{N,i,s}, s) := (r(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r^1_{s,i}, s), \ldots, r(\cdot, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r^N_{s,i}, s))$ is a collection of N solutions, considered as an $\mathbf{R}^{dN}$-valued process (cf. (2.4)).

Theorem 2.2
Suppose

$$(W_1, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{N,1,s}) \sim (W_2, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{N,2,s}) \text{ on } C([s,\infty); \mathbf{H}_w \times \mathbf{M}_{\mathbf{b}} \times \tilde{\mathbf{H}}_{0,\lambda}) \times \mathbf{R}^{dN}. \tag{2.11}$$

Then

$$r_N(\cdot, w_1, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{N,1,s}, s) \sim r_N(\cdot, w_2, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{N,2,s}, s) \text{ on } C[s,\infty); \mathbf{R}^{dN}). \tag{2.12}$$

In particular, for any $t \geq s$

$$r_N(t, w_1, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, r_{N,1,s}, s) \sim r_N(t, w_2, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r_{N,2,s}, s). \tag{2.13}$$

Proof
Assume without loss of generality $s = 0$ and $N = 1$. Set $\mathcal{G}_{t,i} := \mathcal{G}(w_i(s), s \leq t)$. Further, let $\overline{\mathcal{G}}_{w,t}$ be the completion of $\mathcal{G}_{w,t}$ with respect to $\pi_t W_1 P$ (the probability measure induced by $\pi_t W_1$ on $(\mathcal{H}_{w,[0,\infty)}, \mathcal{G}_{w,\infty})$, which by equivalence equals $\pi_t W_2 P$). $\overline{r}(\cdot, w_i(\cdot), \cdot, \cdot, \cdot)$ are the general solution maps from Theorem 2.1, $i = 1, 2$.

It is now more convenient to "lift" the stochastic analysis to the "canonical" probability space $(\mathbf{H}_{w,[0,\infty)}), \overline{\mathcal{G}}_{w,\infty}, \overline{\mathcal{G}}_{w,t}, w_1 P)$. Then, by the results of Theorem 2.1, there is a map

$$\bar{\bar{r}}\colon \mathbf{H}_{w,[0,\infty)} \times \mathbf{M}_{\mathbf{b},[0,\infty)} \times \tilde{\mathbf{H}}_{0,\lambda,[0,\infty)} \times \mathbf{R}^d \to C([0,\infty); \mathbf{R}^d)$$

such that for all $t \geq 0$

$$(\pi_t \, \bar{\bar{r}})(\cdot,\cdot,\cdot,\cdot) \text{ is } \overline{\mathcal{G}}_{w,t} \otimes \mathcal{F}_{\mathbf{b},t} \otimes \tilde{\mathcal{F}}_{0,\lambda,t} \otimes \mathcal{B}^d - \mathcal{F}_{\mathbf{R}^d,t}\text{-measurable,} \tag{2.14}$$

where $\mathcal{F}_{\mathbf{R}^d,t}$ is the σ-algebra of cylinder sets on $C([0,t];\mathbf{R}^d)$. Moreover, with probability 1 uniformly in $t \geq 0$

$$\bar{\bar{r}} \, (t, \pi_t W_i, \pi_t \tilde{\mathcal{Y}}, \pi_t \tilde{Z}_i, r_{i,0}) \equiv r(t, w_i, \tilde{\mathcal{Y}}_i, \tilde{Z}_i, r_{i,0}), \quad i \equiv 1,2, \tag{2.15}$$

(cf. Ikeda and Watanabe (1981), Ch. IV). (2.11), (2.14) and (2.15) imply (2.12) and (2.13). ■

Next, we assume

$$\mathbf{M_a} = \mathbf{M_b} \text{ (i.e., } a^{\pm} = b^{\pm})$$

and take $\tilde{\mathcal{Y}}(t) := \mathcal{X}_N(t) := \sum_{i=1}^N a_i \delta_{r^i(t)}$, i.e., we consider the $\mathbf{R}^{dN}$-valued system of coupled SODE's (1.11). By a direct generalization of the proof of Theorem 2.2 in Kotelenez (1995b) we obtain:

Theorem 2.3

To each $\mathcal{F}_s$-adapted initial condition $r_N(s) \in \mathbf{R}^{dN}$ and $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ (1.11) has a unique solution $r_N(\cdot, \tilde{Z}, r_N(s)) \in \mathcal{R}_{d,[s,\infty)}$.

Let us now choose $\tilde{F} := F$, $\tilde{\mathcal{J}} := \mathcal{J}$ and $\tilde{w} := w$. Then (2.1) becomes the SODE (1.10) for the random "forward" flow. Next we will follow the procedure in Ikeda and Watanabe (1981), Ch. V, to construct the random "backward" flow. Fix $T > s$ and consider

$$\left.\begin{aligned} dr(t) = \quad & -F(r(t), \tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t))dt \\ & + \int \mathcal{J}(r(t), p, \tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t)) \check{w}(dp,dt), \\ & r(s) = r \text{ (deterministic)}, \; t \in [s,T], \end{aligned}\right\} \tag{2.16}$$

where

$$\check{w}(dp,t) := w(dp, T-t) - w(dp,T). \tag{2.17}$$

Setting now $\tilde{F}(r, \tilde{\mathcal{Y}}(t), \tilde{Z}(t)) := -F(r, \tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t)), \tilde{\mathcal{J}}(r,p,\tilde{\mathcal{Y}}(t),\tilde{Z}(t)) := \mathcal{J}(r,p,\tilde{\mathcal{Y}}(T+s-t), \tilde{Z}(T+s-t))$ and $\tilde{w} := \check{w}$ Theorem 2.1 implies the existence of a unique Itô-solution of (2.16). Moreover, setting $\overline{\mathcal{G}}_t := \check{\mathcal{G}}_t := \overline{\sigma}(\check{W}(u), u \leq t)$ (where $\overline{\sigma}(\cdot)$ is the completed σ-algebra) this solution can be represented through the $\check{\mathcal{G}}_t \otimes \mathcal{F}_{\mathbf{b},t} \otimes \tilde{\mathcal{F}}_{0,\lambda,t} \otimes \mathcal{B}^d$– $\mathcal{B}^d$-measurable map $\check{r}(t,\cdot,\cdot,\cdot,s,\check{w})$ of part 3 of Theorem 2.1 with $N=1$, provided $\tilde{Z} \in \tilde{\mathcal{H}}_{0,\lambda,[s,\infty)}$. Then the measurability properties of $\check{r}$ allow us to define

$$\xi \mapsto \check{r}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \xi, s, \check{w}) \tag{2.18}$$

for any $\mathbf{R}^d$-valued random variable ξ such that for deterministic $\xi = r$ $\check{r}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, r, s, \check{w})$ is the unique Itô-solution of (2.16) with $\check{r}(s, \tilde{\mathcal{Y}}, \tilde{Z}, r, s, \check{w}) = r$.

Remark 2.4

(i) In general, $\check{r}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \xi, s, \check{w})$ cannot be interpreted as an Itô-solution of (2.16) with initial condition ξ at time s, since ξ can be anticipating with respect to $\check{w}$.

(ii) Since in (2.18) we do not need the measurability in μ and $\tilde{\eta}$, we may always assume $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[0,\infty)}$ etc. and assume that $\check{r}(t, \tilde{\mathcal{Y}}, \tilde{Z}, \cdot, s, \check{w})$ is measurable with respect to (ω, r). ■

We will now show that (2.16) is indeed the SODE for the backward flow under an additional smoothness assumption on the coefficients of (2.1).

Abbreviate for $u \in [0,T]$

$$\left.\begin{array}{ll}\tilde{F}_k(r,u) := \tilde{F}_k(r,\tilde{\mathcal{Y}}(u),\tilde{Z}(u)), & k=1,\ldots,d; \\ \tilde{\mathcal{J}}_{k\ell}(r,p,u) := \tilde{\mathcal{J}}_{k\ell}(r,p,\tilde{\mathcal{Y}}(u),\tilde{Z}(u)), & k,\ell=1,\ldots,d.\end{array}\right\} \tag{2.19}$$

Hypothesis 2.5

For some $m \geq 1$

$$\max_{1\leq k,\ell\leq d} ess \sup_{\omega\in\Omega,\ 0\leq u\leq T} \{|||\tilde{F}_k(\cdot,u,\omega)|||_m + |||\int \tilde{\mathcal{J}}^2_{k\ell}(\cdot,p,u,\omega)dp|||_{m+1}\} < \infty. \tag{2.20}$$

Remark 2.6

For homogeneous kernels $\mathcal{J}_{k\ell}(r-p,u)$ condition (2.20) reduces to (4.2) on $\hat{D}$ (cf. Section 4). Indeed, using the homogeneity in the first step of the calculations below we obtain for any $\tilde{q}(u) \in \mathbf{R}_{d,[0,\infty)}$ (assuming $|\mathbf{j}| \leq m+1$)

$$\begin{array}{rl} & \int (\partial_r^{\mathbf{j}}\mathcal{J}_{k\ell})^2(\tilde{q}(u)-p,u)dp \\ = & \int (\partial_r^{\mathbf{j}}\mathcal{J}_{k\ell})^2(q,u)dq \\ = & (-1)^{|\mathbf{j}|}\int (\partial_r^{2\mathbf{j}}\mathcal{J}_{k\ell}(q,u))\mathcal{J}_{k\ell}(q,u)dq \\ \leq & ess \sup\limits_{\omega\in\Omega,\ u\in[0,T]} |||\hat{D}_{k\ell}(u,\omega)|||_{2(m+1)} \leq d_{1,T} < \infty. \end{array}$$

■

Theorem 2.7

Fix $T > s$ and assume (2.20) with $m \geq 1$. Then with probability 1

$$\bar{r}(T-t,\tilde{\mathcal{Y}},\tilde{Z},r,s,w) = \check{r}(t,\tilde{\mathcal{Y}},\tilde{Z},\bar{r}(T-s,\tilde{\mathcal{Y}},\tilde{Z},r,s,w),\check{w}), \tag{2.21}$$

uniformly in $r \in \mathbf{R}^d$ and $t \in [s,T]$, where the left hand side in (2.21) is the (ω,r)-measurable version of the solution of the "forward" SODE (1.10), and the right hand side is the measurable version (in (ω,r)) of the "backward" SODE (2.16).

Proof

(10.35), (10.40) and (10.41) in Appendix II imply (2.21). ■

3 The Mezoscopic Equation–Existence

Existence for the mezoscopic equation (1.32) is obtained through extension by continuity on the space of finite signed measures. The "mass" is conserved but it can depend on ω. Finally, we derive existence for the quasilinear SPDE (1.33) on $\mathbf{M}_{[s,\infty)}$.

Let $\tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[0,\infty)}, \tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[0,\infty)}, r_s^i \in \mathcal{R}_{d,s}, i=1,\ldots N$. Consider on $[s,\infty)$ the N-system of $\mathbf{R}^d$-valued SODE's:

$$\left.\begin{array}{rl} dr^i(t) = & F(r^i(t),\tilde{\mathcal{Y}}(t),\tilde{Z}(t))dt + \int \mathcal{J}(r^i(t),p,\tilde{\mathcal{Y}}(t),\tilde{Z}(t))w(dp,dt) \\ & r^i(s) = r^i_s,\ i=1,\ldots,N; \end{array}\right\} \tag{3.1}$$

where F and $\mathcal{J}$ satisfy assumptions (1.13) and (1.14). By Theorem 2.1 (3.1) has unique continuous solutions $r^i(t)$, $i=1,\ldots,N$. Set

$$\mathcal{Y}_N(t) := \sum_{i=1}^{N} a_i \delta_{r^i(t)}, \tag{3.2}$$

where $a_i \in \mathbf{R}$. Further, set $a^+ := \sum_{a_i \geq 0} a_i$, $a^- := \sum_{a_i<0} a_i$ and $\mathbf{a} := (a^+, a^-)$. Then

$$\mathcal{Y}_N \in \mathcal{M}_{\mathbf{a},[s,\infty)}, \tag{3.3}$$

and (by Itô's formula, cf. Kotelenez (1995b)) $\mathcal{Y}_N$ is a solution of (1.32) with initial condition

$$\mathcal{Y}_N(s) = \sum_{i=1}^{N} a_i \delta_{r^i(s)} =: \mathcal{X}_{\mathbf{a},s}. \tag{3.4}$$

Now let $\mathcal{X}_{\mathbf{a},s}$ be any initial condition which can be represented by (3.4) and $\mathcal{Y}(t,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_{\mathbf{a},s})$ the empirical process $\mathcal{Y}_N(t,\mathcal{X}_{\mathbf{a},s})$ given by (3.2). Let $B \in \mathcal{F}_s$. Then both $\mathcal{X}^{\pm}_{\mathbf{a},s}1_B$ and $\mathcal{X}^{\pm}_{\mathbf{a},s}1_{B^c}$ are $\mathcal{F}_s$-measurable (where $B^c := \Omega \backslash B$). Set

$$\mathcal{Y}^{\pm}(t,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}^{\pm}_{\mathbf{a},s}1_B) := 1_B \mathcal{Y}^{\pm}(t,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}^{\pm}_{\mathbf{a},s}).$$

We immediately verify that $\mathcal{Y}^{\pm}(\cdot,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}^{\pm}_{\mathbf{a},s}1_B)$ solves (1.32) with initial condition $\mathcal{X}^{\pm}_{\mathbf{a},s}1_B$, where now for all $t \geq s$

$$\mathcal{Y}^{\pm}(t,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}^{\pm}_{\mathbf{a},s}1_B) \in \begin{cases} \mathbf{M}_{\mathbf{a}^{\pm}}, & \text{if } \omega \in B, \\ \{0_\mu\}, & \text{if } \omega \notin B. \end{cases} \tag{3.5}$$

Here, $\mathbf{M}_{\mathbf{a}^{\pm}}$ are the non-negative Borel measures of mass $\mathbf{a}^{\pm}$, and $0_\mu(A) = 0$ for all Borel sets A. This observation together with the fact that the solution of a bilinear equation (on some vector space) defines a linear operator on the space of initial conditions allows us to "extend" our solutions of (1.32) with deterministic mass $\mathbf{a}$ to solutions with $\mathcal{F}_s$-measurable random mass $\mathbf{a}$ as follows: Let $\mathbf{a}_i \in \mathbf{R}^2$, $B_i \in \mathcal{F}_s$ and $\mathcal{X}_{\mathbf{a}_i,s}$ given by (3.4), $i=1,\ldots n$. Then

$$\mathcal{Y}(t,\tilde{\mathcal{Y}},\tilde{Z},\sum_{i=1}^{n} 1_{B_i}\mathcal{X}_{\mathbf{a}_i,s}) := \sum_{i=1}^{n} 1_{B_i}\mathcal{Y}(t,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_{\mathbf{a}_i,s}) \tag{3.6}$$

is a solution of (1.32) with initial condition

$$\mathcal{X}_s := \sum_{i=1}^{n} 1_{B_i}\mathcal{X}_{\mathbf{a}_i,s}, \quad \mathbf{a}(\omega) := \sum_{i=1}^{n} \mathbf{a}_i 1_{B_i}(\omega). \tag{3.7}$$

Set

$$\mathcal{M}_{s,d} := \{\mathcal{X}_s \in \mathcal{M}_s : \mathcal{X}_s =: \mathcal{X}_{s,n,N} \text{ can be represented by (3.7), (3.4); } n,N \in \mathbf{N}\}. \tag{3.8}$$

We restrict the metric $(E\gamma^2(\cdot,\cdot))^{\frac{1}{2}}$ on $\mathcal{M}_s$ to $\mathcal{M}_{s,d}$.

Theorem 3.1

(I) For any fixed $\tilde{\mathcal{Y}} \in \mathcal{M}_{\mathbf{b},[s,\infty)}, \tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ the map

$$\mathcal{X}_{s,n,N} \mapsto \mathcal{Y}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_{s,n,N}) \text{ from } \mathcal{M}_{s,d} \text{ into } \mathcal{M}_{[s,\infty)}$$

extends uniquely to a map

$$\mathcal{X}_s \mapsto \mathcal{Y}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_s) \text{ from } \mathcal{M}_s \text{ into } \mathcal{M}_{[s,\infty)}.$$

Moreover, if $a^{\pm}(\omega) := \mathcal{X}_s^{\pm}(\omega, \mathbf{R}^d)$ is the random mass at time s, then with probability 1 uniformly in $t \in [s, \infty)$

$$\mathcal{Y}(t, \omega, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_s(\omega)) \in \mathbf{M}_{\mathbf{a}(\omega)}, \tag{3.9}$$

where $\mathbf{a}(\omega) := (a^+(\omega), a^-(\omega))$,

(II) Let $\mathcal{X}_{s,\ell} \in \mathcal{M}_s, \tilde{\mathcal{Y}}_\ell \in \mathcal{M}_{\mathbf{b},[s,\infty)}, \tilde{Z}_\ell \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$ a.s., $\ell = 1, 2$, where $\mathcal{X}_{s,1}^{\pm}(\omega, \mathbf{R}^d) = \mathcal{X}_{s,2}^{\pm}(\omega, \mathbf{R}^d) =: a^{\pm}(\omega)$. Then for any bounded stopping time $\tau \geq s$

$$\left.\begin{array}{l}
E \sup_{s \leq t \leq \tau} \gamma^2(\mathcal{Y}(t, \tilde{\mathcal{Y}}_1, \tilde{Z}_1, \mathcal{X}_{s,1}), \mathcal{Y}(t, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, \mathcal{X}_{s,2})) \\
\leq c_{\tau,F,\mathcal{J}} \{ E\gamma^2(\mathcal{X}_{s,1}, \mathcal{X}_{s,2}) \\
\quad E|\mathbf{a}|^2 \int_s^\tau (\gamma^2(\tilde{\mathcal{Y}}_1(u), \tilde{\mathcal{Y}}_2(u)) + \|B(Z_1(u)) - B(Z_2(u))\|_{0,\lambda}^2) du \\
\quad + \sum_{L \in \{F,\mathcal{J}\}} \sum_{+,-} E|\mathbf{a}| \int_s^\tau \int \mathcal{Y}_2^{\pm}(u, dr) (\int \Lambda_L(r(u, \tilde{\mathcal{Y}}_2, \tilde{Z}_2, r, s) - p) |B(\hat{Z}_1(u,p)) - B(\hat{Z}_2(u,p))| dp)^2 du.
\end{array}\right\} \tag{3.10}$$

Proof

(i) Suppose the mass parameters of $\mathcal{X}_s$, $\mathcal{X}_{s,1}$ and $\mathcal{X}_{s,2}$ are discrete random variables $\mathbf{a}_n$, $\mathbf{a}_{n,1}$ and $\mathbf{a}_{n,2}$ respectively (i.e., they take only finitely many values). Then we can restrict the analysis to the case, where $\mathbf{a}_n$, $\mathbf{a}_{n,1}$ and $\mathbf{a}_{n,2}$ are all constant in ω. However, for constant mass parameters statements (I) and (II) are straight forward generalizations of Lemma 3.1 and Corollary 3.2 of Kotelenez (1995b), where the main difference comes from the last term on the right hand side of (2.2).

(ii) First we bound the mass away from 0 and ∞. Let $\varepsilon \in (0,1)$ and recall

$$(\mathbf{M}_{\varepsilon,\varepsilon^{-1}}, \gamma) \text{ is complete and separable.} \tag{3.11}$$

Clearly,

$$\bigcup_{\varepsilon \in (0,1)} \mathbf{M}_{\varepsilon,\varepsilon^{-1}} = \{\mu \in \mathbf{M} : \mu^{\pm}(\mathbf{R}^d) > 0\}\ . \tag{3.12}$$

(iii) Assume for notational convenience $s = 0$. Let $\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$ and $\mathcal{M}_{\varepsilon,\varepsilon^{-1},[0,\infty)}$ the restrictions of $\mathcal{M}_0$ and $\mathcal{M}_{[0,\infty)}$, respectively, to random variables, resp. random processes, with values in $\mathbf{M}_{\varepsilon,\varepsilon^{-1}}$. Let $\mathcal{X}_0 \in \mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$ with mass $\mathbf{a}$. Take a sequence of discrete random variables $\mathbf{a}_n$ with $a_n^+, a_n^- \in (\varepsilon, \varepsilon^{-1})$ a.s. and $E\rho^2(\mathbf{a}_n, \mathbf{a}) \to 0$, as $u \to \infty$. Set

$$\mathcal{X}_{0,n}^{\pm} := \frac{a_n^{\pm}}{a^{\pm}} \mathcal{X}_0^{\pm}.$$

Then, $\tilde{\gamma}^2(\mathcal{X}_0, \mathcal{X}_{0,n}) = 0$, whence

$$E\gamma^2(\mathcal{X}_{0,n}, \mathcal{X}_0) = E\rho^2(\mathbf{a}_n, \mathbf{a}) \to 0, \text{ as } n \to \infty. \tag{3.13}$$

Hence,

$$\mathcal{M}_{0,d} \text{ dense in } \mathcal{M}_0$$

with respect to the metric $(E\gamma^2(\cdot,\cdot))^{\frac{1}{2}}$.

The completeness of $(\mathbf{M}_{\varepsilon,\varepsilon^{-1}},\gamma)$ implies the completeness of $\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$ and $\mathcal{M}_{\varepsilon,\varepsilon^{-1},[0,\infty)}$. Step (i) implies

$$\left.\begin{array}{l} E\sup_{0\le t\le\tau}\gamma^2(\mathcal{Y}(t,\tilde{\mathcal{Y}}_1,\tilde{Z}_1,\mathcal{X}_{0,n}),\mathcal{Y}(t,\tilde{\mathcal{Y}}_2,\tilde{Z}_2,\mathcal{X}_{0,m})) \\ \le c_{\tau,F,\mathcal{J},\varepsilon}\{E\gamma^2(\mathcal{X}_{0,n},\mathcal{X}_{0,m}) \\ \quad +E\int_0^\tau |\mathbf{a}|^2(\gamma^2(\tilde{\mathcal{Y}}_1(s),\tilde{\mathcal{Y}}_2(s))+\|B(Z_1(s))-B(Z_2(s))\|^2_{0,\lambda})ds \\ \quad \sum_{L\in\{F,\mathcal{J}\}}\sum_{+,-}E\int_0^\tau\int|\mathbf{a}|\mathcal{Y}_2^{\pm}(s,dr)(\int\Lambda_L(r(u,\tilde{\mathcal{Y}}_2,\tilde{Z}_2,r)-p)|B(\hat{Z}_1(s,p))-B(\hat{Z}_2(s,p)))dp)^2ds\} \end{array}\right\} \tag{3.14}$$

for any $\mathcal{X}_{0,n},\mathcal{X}_{0,m}\in\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}\cap\mathcal{M}_{0,d}$. Hence for any $\mathcal{X}_0\in\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$, $\tilde{\mathcal{Y}}\in\mathcal{M}_{\mathbf{b},[s,\infty)}$ and $\tilde{Z}\in\tilde{W}_{0,2,\lambda,[s,\infty)}$ there is a unique extension $\mathcal{Y}(\cdot,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_0)\in\mathcal{M}_{\varepsilon,\varepsilon^{-1},[0,\infty)}$. Moreover, the inequality extends by continuity to arbitrary initial conditions $\mathcal{X}_{0,1}$ and $\mathcal{X}_{0,2}$ from $\mathcal{M}_{\varepsilon,\varepsilon^{-1},0}$.

(iv) Since $\varepsilon\in(0,1)$ was arbitrary and $\mathcal{Y}(t,\tilde{\mathcal{Y}},\tilde{Z},\mathcal{X}_s)$ is linear in the initial condition (cf. (3.6)) (3.10) extends to all $\mathcal{X}_s\in\mathcal{M}_s$ (where the right hand side in (3.10) may be infinite and where we set $\mathcal{Y}^{\pm}(t,\tilde{\mathcal{Y}},\tilde{Z},0_\mu)\equiv 0_\mu$). (3.9) follows from the construction of the extension. ∎

Next, we consider the mezoscopic equation (1.33) associated with the coupled system of SODE's (1.11). Let for $b>0$

$$\mathbf{M}_{0,b} := \{\mu\in\mathbf{M} : \mu^+(\mathbf{R}^d)+\mu^-(\mathbf{R}^d)\le b\}$$

and let $\mathcal{M}_{0,b,s}$ denote the restriction of $\mathcal{M}_s$ to $\mathbf{M}_{0,b}$. By a trivial generalization of Theorem 2.1 we obtain that (1.11) has a unique solution even if $\tilde{\mathcal{Y}}\in\mathcal{M}_{0,b,[s,\infty)}$ (the restriction of $\mathcal{M}_{[s,\infty)}$ to $\mathbf{M}_{0,b}$-valued processes) such that (2.2) holds. We denote the empirical process of the N-particle system by $\mathcal{X}(t,\tilde{Z},\mathcal{X}_{s,N})$, with $\mathcal{X}_{s,N}$ being its initial state. A combination of such states as given by (3.7) will be denoted $\mathcal{X}_{s,N,n}$. Note that $\mathcal{X}(t,\tilde{Z},\mathcal{X}_{s,N,n})=\mathcal{Y}(t,\mathcal{Y},\tilde{Z},\mathcal{X}_{s,N,n})$, if we set $\mathcal{Y}:=\mathcal{X}(\cdot,\tilde{Z},\mathcal{X}_{s,N,n})$.

Theorem 3.2

(I) Let $\tilde{Z}\in\tilde{W}_{0,2,\lambda,[s,\infty)}$. Then for any $b>0$ the map $\mathcal{X}_{s,N,n}\mapsto\mathcal{X}(\cdot,\tilde{Z},\mathcal{X}_{s,N,n})$ from $\mathcal{M}_{0,b,s}\cap\mathcal{M}_{s,d}$ into $\mathcal{M}_{[s,\infty)}$ extends uniquely to a map $\mathcal{X}_s\mapsto\mathcal{X}(\cdot,\tilde{Z},\mathcal{X}_s)$ from $\mathcal{M}_{0,b,s}$ into $\mathcal{M}_{[s,\infty)}$. Moreover, if $a^{\pm}(\omega):=\mathcal{X}_s^{\pm}(\omega,\mathbf{R}^d)$ is the random mass at time s, $\mathbf{a}(\omega):=(a^+(\omega),a^-(\omega))$, then with probability 1 for all $t\ge s$

$$\mathcal{X}(t,\omega,\tilde{Z},\mathcal{X}_s(\omega))\in\mathbf{M}_{\mathbf{a}(\omega)}. \tag{3.15}$$

(II) Suppose for $\ell=1,2$ $\tilde{Z}_\ell\in\tilde{W}_{0,2,\lambda,[s,\infty)}$ and $\mathcal{X}_{s,\ell}\in\mathcal{M}_{0,b,s}$ such that $\mathcal{X}^{\pm}_{s,1}(\omega,\mathbf{R}^d)=\mathcal{X}^{\pm}_{s,2}(\omega,\mathbf{R}^d)=a^{\pm}(\omega)$ a.s. Then, for any bounded stopping time $\tau\ge s$

$$\left.\begin{array}{l} E\sup_{s\le t\le\tau}\gamma^2(\mathcal{X}(t,\tilde{Z}_1,\mathcal{X}_{s,1}),\mathcal{X}(t,\tilde{Z}_2,\mathcal{X}_{s,2})) \\ \le c_{\tau,F,\mathcal{J},b}\{E\gamma^2(\mathcal{X}_{s,1},\mathcal{X}_{s,2}) \\ \quad +E|\mathbf{a}|^2\int_s^\tau\|B(Z_1(u))-B(Z_2(u))\|^2_{0,\lambda}du \\ \quad +\sum_{L\in\{F,\mathcal{J}\}}\sum_{+,-}E|\mathbf{a}|\int_s^\tau\int\mathcal{X}_2^{\pm}(u,dq)(\int\Lambda_L(r(u,\mathcal{X}_2,\tilde{Z}_2,q,s)-p)|B(\hat{Z}_1(u,p))-B(\hat{Z}_2(u,p))|dp)^2du\} \end{array}\right\} \tag{3.16}$$

with $\mathcal{X}_2:=\mathcal{X}(\cdot,\tilde{Z}_2,\mathcal{X}_2)$.

Proof

Theorem 3.1 and Kotelenez (1995b), Theorem 3.4. ∎

Theorem 3.3

Let $\tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[s,\infty)}$. Then the following statements hold:

(I) Suppose $\tilde{\mathcal{Y}} \in \mathcal{M}_{0,b,[s,\infty)}$, $\mathcal{X}_s \in \mathcal{M}_s$. Then $\mathcal{Y}(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, \mathcal{X}_s)$, the process obtained in Theorem 3.1., is a solution of (1.32) (cf. Kotelenez (1995b), (3.9)).

(II) Suppose $\mathcal{X}_s \in \mathcal{M}_{0,b,s}$ for some $b > 0$. Then $\mathcal{X}(\cdot, \tilde{Z}, \mathcal{X}_s)$, the process obtained in Theorem 3.2., is a solution of (1.33) (cf. Kotelenez (1995b), (1.27)).

The **proof** is a simple generalization of the proofs of Theorems 3.5 and 3.6 in Kotelenez (1995b). ∎

4 The Mezoscopic Equation—Smoothness

Under the smoothness assumptions (4.2) on the coefficients of (1.10) the solution of (1.32) (resp. of (1.33)) is shown to live on corresponding smooth Sobolev-Hilbert spaces $\mathbf{H}_m$ provided the initial condition lives on $\mathbf{H}_m$. The proof is given in Appendix I.

Note that smoothness for the bilinear SPDE (1.32) implies smoothness for the quasilinear SPDE (1.33) (by taking $\mathcal{Y} \equiv \mathcal{X} \in \mathcal{M}_{0,b,[s,\infty)}$, provided $\mathcal{X}_s \in \mathcal{M}_{0,b,s}$ for some $b > 0$). In Kotelenez (1995b) $\mathbf{H}_0$-valued solutions for both SPDE's were derived under the absence of the "input" $(\hat{Z}, Z)$ and for deterministic mass parameter $\mathbf{a}$. Moreover, the kernel $\mathcal{J}$ was assumed to be homogeneous. Since the arguments become more transparent in the homogeneous case, we will make the same assumption here. It will, however, become obvious that with minor changes in the proof one can also derive smoothness for inhomogeneous $\mathcal{J}$.

Hypothesis 4.1

$$\mathcal{J}(r,p,\cdot) = \mathcal{J}(r-p,\cdot) \text{ for all } r,p \in \mathbf{R}^d. \tag{4.1}$$

Then we abbreviate:

$$\left.\begin{aligned} D_{k\ell}(s) &:= D_{k\ell}(\tilde{\mathcal{Y}}(s), \tilde{Z}(s)), \\ F_k(r,s) &:= F_k(r, \tilde{\mathcal{Y}}(s), \tilde{Z}(s)), \\ \mathcal{J}(r,s) &:= \mathcal{J}(r, \tilde{\mathcal{Y}}(s), \tilde{Z}(s)), \\ dM(s) &:= \int \mathcal{J}(\cdot - p, s) w(dp, ds), \\ \tilde{D}_{k\ell}(s, r-q) &:= \tfrac{d}{ds}[M_k(s,r), M_\ell(s,q)], \\ \hat{D}_{k\ell}(s, r-q) &:= D_{k\ell}(s) - \tilde{D}_{k\ell}(s, r-q). \end{aligned}\right\} \tag{$*$}$$

Hypothesis 4.2

Fix $m \in \mathbf{N} \cup \{0\}, T > 0$ and suppose there are finite constants $d_{i,T}, i = 1,2,3$ such that

$$\left.\begin{aligned} &\max_{1\le k,\ell\le d} ess \sup_{(t,\omega)\in[0,T]\times\Omega} |||\hat{D}_{k\ell}(t,\omega)|||_{(m+1)2} \le d_{1,T}; \\ &\max_{1\le k,\ell\le d} ess \sup_{(t,\omega)\in[0,T]\times\Omega} |D_{k\ell}(t,\omega)| \le d_{2,T}; \\ &\max_{1\le k\le d} ess \sup_{(t,\omega)\in[0,T]\times\Omega} |||F_k(t,\omega)|||_{m+1} \le d_{3,T}\ . \end{aligned}\right\} \tag{4.2}$$

We will call m the smoothness degree of the coefficients F and $\mathcal{J}$.

Theorem 4.3

Let $m \in \mathbf{N} \cup \{0\}$ and suppose (4.2) in addition to (4.1) and

$$Y_s \in \mathcal{W}_{m,2,1,s} \cap \mathcal{M}_{0,b,s} \text{ for some } b > 0. \tag{4.3}$$

Then, the unique (weak) solution of (1.32) has a density with respect to the Lebesgue measure $Y(\cdot, Y_s) := Y(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s)$ such that for any $T > s$

$$Y(\cdot, s) \in \mathcal{W}_{m,2,1,[s,T]} \cap \mathcal{M}_{0,b,[s,T]}. \tag{4.4}$$

Moreover, there is a finite constant $c_{T,m} := c_{T,m}(d_1, d_2, d_3)$ such that uniformly in $\tilde{\mathcal{Y}} \in \mathcal{M}_{0,b,[s,T]}$ and $\tilde{Z} \in \tilde{\mathcal{W}}_{0,\lambda,[s,\infty)}$.

$$\sup_{s \leq t \leq T} E\|Y(t, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s)\|_m^2 \leq c_{T,m} E\|Y_s\|_m^2, \tag{4.5}$$

where d_1, d_2 and d_3 are the bounds in (4.2) evaluated at T.

The **proof** is given in the Appendix I (Section 9). ∎

5 The Itô formula for $\|\cdot\|_{m,p,\Phi}^p$

The Itô formula for $\|Y(t)\|_{m,p,\Phi}^p$ is derived under smoothness assumption on the coefficients and the initial conditions.

Theorem 5.1 (Itô formula)

Let $\Phi \in \{1, \lambda\}$, $p > 0$, if $\Phi = \lambda$ and $p \geq 2$, if $\Phi \equiv 1$, $m \in \mathbf{N} \cup \{0\}$, $\overline{m}$ an even integer such that $\overline{m} > \frac{d}{2} + m + 2$ and assume (4.2) with smoothness degree $\overline{m}$ (instead of m). in addition to (4.1). Further, assume

$$Y_s \in \mathcal{W}_{\overline{m},2,1,s} \cap \mathcal{M}_{0,b,s} \text{ for some } b > 0. \tag{5.1}$$

Let $Y(\cdot) := Y(\cdot, Y_s) := Y(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s)$ be the (smooth) solution of (1.32) (derived in Section 4). Then for any multiindex $\mathbf{m}$ with $|\mathbf{m}| \leq m$

$$\left.\begin{array}{rl}
 & \int (\partial^{\mathbf{m}} Y(t,r))^p \Phi(r) dr \\
= & \int (\partial^{\mathbf{m}} Y(s,r))^p \Phi(r) dr \\
+ & \frac{1}{2} \sum_{k,\ell=1}^d \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^p \partial_{k\ell}^2 \Phi(r) dr D_{k\ell}(s) ds \\
- & \frac{p}{2} \sum_{k,\ell=1}^d \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-1} \{ \sum_{\mathbf{n}<\mathbf{m}_k} (\partial^{\mathbf{n}} Y(u,r)) \mathcal{L}_{k\ell,\mathbf{m}_k-\mathbf{n}}(u) \partial_\ell \Phi(r) \\
 & \qquad + \sum_{\mathbf{n}<\mathbf{m}_\ell} \partial^n Y(u,r) (-1)^{|\mathbf{m}_\ell - \mathbf{n}|+1} \mathcal{L}_{k\ell,\mathbf{m}_\ell-\mathbf{n}}(u) \partial_k \Phi(r) \} dr du \\
+ & \frac{p(p-1)}{2} \sum_{k,\ell=1}^d \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-2} \sum_{\substack{\mathbf{n}<\mathbf{m}_k \\ \tilde{\mathbf{n}}<\mathbf{m}_\ell}} (\partial^{\mathbf{n}} Y(u,r)) (\partial^{\tilde{\mathbf{n}}} Y(u,r)) (-1)^{|\mathbf{m}_\ell - \tilde{\mathbf{n}}|+1} \cdot \\
 & \qquad \cdot \mathcal{L}_{k,\ell,\mathbf{m}_k+\mathbf{m}_\ell-\mathbf{n}-\tilde{\mathbf{n}}}(u) \Phi(r) dr du \\
+ & \sum_{\ell=1}^d \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^p ((\partial_\ell F_\ell)(r,u) \Phi(r) + F_\ell(r,u) \partial_\ell \Phi(r)) dr du \\
- & p \sum_{\ell=1}^d \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-1} \sum_{\mathbf{n}<\mathbf{m}_\ell} (\partial^{\mathbf{n}} Y(u,r)) (\partial^{\mathbf{m}_\ell-\mathbf{n}} F_\ell(r,u)) \Phi(r) dr du \\
+ & \sum_{\ell=1}^d \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^p ((d\partial_\ell M_\ell(u,r)) \Phi(r) + dM_\ell(u,r) \partial_\ell \Phi(r)) dr \\
- & p \sum_{\ell=1}^d \int_s^t \int (\partial^{\mathbf{m}} Y(u,r))^{p-1} \sum_{\mathbf{n}<\mathbf{m}_\ell} (\partial^{\mathbf{n}} Y(u,r)) d(\partial^{\mathbf{m}_\ell-\mathbf{n}} M_\ell(u,r)) \Phi(r) dr, \\
=: & \sum_{i=1}^8 C_{\mathbf{m},i}(t),
\end{array}\right\} \tag{5.2}$$

where $\mathbf{m}_k := \mathbf{m} + 1_k, \mathbf{m}_\ell := \mathbf{m} + 1_\ell$ and

$$\mathcal{L}_{k,\ell,\mathbf{n}}(s) := \partial^{\mathbf{n}} \hat{D}_{k\ell}(s,r)|_{r=0}.$$

Proof

(i) If $p \geq 2$, Theorem 4.3 and the definition of $\overline{m}$ imply $Y(t)$, $\partial_k Y(t)$, $\partial^2_{k\ell} Y(t)$ are in $\mathbf{W}_{m,p,1} \subset \mathbf{W}_{m,p,\Phi}$ for $k,\ell = 1,\ldots,d$. For $\Phi = \lambda$ Hölder's inequality implies $\mathbf{W}_{m,p,\lambda} \supset \mathbf{W}_{m,2,\lambda}$ if $0 < p \leq 2$. Again we assume $s = 0$. Further, since we can always choose a decomposition $\{B_n\}_{n\in\mathbf{n}}$ of Ω with $B_n \in \mathcal{F}_0$ such that $ess\sup_\omega \|Y_0\|_{\overline{m}} 1_{B_n} < \infty$ for $n \geq 2$ and $P(B_1) = 0$ we may without loss of generality assume

$$ess\sup_{\omega} \|Y_0(\omega)\|_{\overline{m}} \leq c < \infty. \tag{$*$}$$

By (5.1) we may stop $\|Y(t)\|_{\overline{m}-2}$ at the first exit time $\tau := \tau_N$ for the ball with center 0 and radius $N > c$ (where $\tau_N(\omega) = \infty$, if $\sup_{0\leq t<\infty} \|Y(t)\|_{\overline{m}-2} < \infty$). Hence, $\|Y(t \wedge \tau_N)\|_{\overline{m}-2} \leq N$ a.s. for all $t \geq 0$. By the smoothness result we obtain the analogue of (9.11) with $\mu = \infty$, i.e., $R_\mu^{\frac{m}{2}}$ replaced by I and $t \wedge \tau$ instead of t. We now proceed as in the proof of Theorem 4.3. Let us also use the same abbreviations as in that proof (with $\mu = \infty$, $t \mapsto t \wedge \tau$ and setting here $Y_{\mathbf{m}}(t) := \partial^{\mathbf{m}} Y(t)$).

(ii) Integration by parts yields

$$\left.\begin{array}{rl} & A_{2,\mathbf{m}}(t) \\ = & -\frac{p(p-1)}{2} \sum_{k,\ell=1}^d \int_0^{t\wedge\tau} \int (Y_{\mathbf{m}}(s,r))^{p-2} (\partial_k Y_{\mathbf{m}}(s,r))(\partial_\ell Y_{\mathbf{m}}(s,r)) \Phi(r) dr D_{k\ell}(s) ds \\ + & \frac{1}{2} \sum_{k,\ell=1}^d \int_0^{t\wedge\tau} \int (Y_{\mathbf{m}}(s,r))^p \partial^2_{k\ell} \Phi(r) dr ds \end{array}\right\} \tag{5.3}$$

$$=: A_{2,\mathbf{m},1}(t) + A_{2,\mathbf{m},2}(t),$$

where the second term comes from

$$p \int Y^{p-1}(s,r)(\partial_\ell Y_{\mathbf{m}}(s,r))\partial_k \Phi(r) dr$$

$$= \int \partial_\ell (Y^p_{\mathbf{m}}(s,r))\partial_k \Phi(r) dr$$

and integration by parts. Next, note that the $\mathbf{n} \leq \mathbf{m}_k$, $\tilde{\mathbf{n}} \leq \mathbf{m}_\ell$

$$\left.\begin{array}{rl} & d[\partial^{\mathbf{m}_k - \mathbf{n}} M_k(s,r), \partial^{\mathbf{m}_\ell - \tilde{\mathbf{n}}} M_\ell(s,r)] \\ = & \mathcal{L}_{k\ell,\mathbf{m}_k + \mathbf{m}_\ell - \mathbf{n} - \tilde{\mathbf{n}}}(s)(-1)^{|\mathbf{m}_\ell - \tilde{\mathbf{n}}|+1} ds, \end{array}\right\} \tag{5.4}$$

and

$$-\mathcal{L}_{k\ell,0}(s) = D_{k\ell}(s).$$

Therefore,

$$\left.\begin{array}{rl} & [\partial_k \partial^{\mathbf{m}}(Y(s,r)dM_k(s,r)), \partial_\ell \partial^{\mathbf{m}}(Y(s,r)dM_\ell(s,r))] \\ = & (\partial_k Y_{\mathbf{m}}(s,r))(\partial_\ell Y_{\mathbf{m}}(s,r)) D_{k\ell}(s) ds \\ + & \sum_{\substack{\mathbf{n}<\mathbf{m}_k \\ \tilde{\mathbf{n}}<\mathbf{m}_\ell}} (\partial^{\mathbf{n}} Y(s,r))(\partial^{\tilde{\mathbf{n}}} Y(s,r))(-1)^{|\mathbf{m}_\ell - \mathbf{n}|+1} \mathcal{L}_{k\ell,\mathbf{m}_k + \mathbf{m}_\ell - \mathbf{n} - \tilde{\mathbf{n}}}(s) ds \\ + & \sum_{\tilde{\mathbf{n}}<\mathbf{m}_\ell} (\partial_k Y_{\mathbf{m}}(s,r))(\partial^{\tilde{\mathbf{n}}} Y(s,r))(-1)^{|\mathbf{m}_\ell - \tilde{\mathbf{n}}|+1} \mathcal{L}_{k\ell,\mathbf{m}_\ell - \tilde{\mathbf{n}}}(s) ds \\ + & \sum_{\mathbf{n}<\mathbf{m}_k} (\partial_\ell Y_{\mathbf{m}}(s,r))(\partial^{\mathbf{n}} Y(s,r)) \mathcal{L}_{k\ell,\mathbf{m}_k - \mathbf{n}}(s)\, ds \\ =: & \sum_{i=1}^4 B_{k\ell,i}(s,r) ds. \end{array}\right\} \tag{5.5}$$

The decomposition of the left hand side of (5.5) into $\sum_{i=1}^4 B_{k\ell,i}(s,r)ds$ induces a decomposition

$$A_{5,\mathbf{m}}(t) = \sum_{i=1}^{4} A_{5,\mathbf{m},i}(t).$$

Clearly,

$$A_{2,\mathbf{m},1}(t) + A_{5,\mathbf{m},1}(t) \equiv 0. \tag{5.6}$$

By the same trick, which lead to a simplification of $A_{2,\mathbf{m},2}(t)$ we easily see that

$$\sum_{i=3}^{4} A_{5,\mathbf{m},i}(t) = C_{\mathbf{m},3}(t \wedge \tau).$$

Moreover,

$$A_{5,\mathbf{m},2}(t) = C_{\mathbf{m},4}(t \wedge \tau).$$

(iii) Similarly to (ii) (in particular (5.3)) we obtain

$$A_{3,\mathbf{m}}(t) = C_{\mathbf{m},5}(t \wedge \tau) + C_{\mathbf{m},6}(t \wedge \tau) \tag{5.7}$$

and

$$A_{4,\mathbf{m}}(t) = C_{\mathbf{m},7}(t \wedge \tau) + C_{\mathbf{m},8}(t \wedge \tau). \tag{5.8}$$

So we obtain (5.2) with t replaced by $t \wedge \tau$.

(iv) Since $\tau_N \uparrow \infty$ a.s., as $N \to \infty$, (5.2) holds for any $t \geq s$. ∎

6 Extension of the Mezoscopic Equations to $\mathbf{W}_{m,p,\Phi}$ – Existence and Uniqueness

First, under certain smoothness assumptions the solution of the bilinear SPDE (1.32) is extended to initial conditions in $\mathcal{W}_{m,p,\Phi,s}$ and shown to live on $\mathcal{W}_{m,p,\Phi,[s,\infty)}$. (Theorem 6.2). Then, F and $\mathcal{J}$ are assumed to depend only on r and $\tilde{Z} = (\hat{Z}, Z)$. A representation of the solution of (1.32) in terms of the backward SODE (2.16) is derived (Lemma 6.4). Hence, microscopic estimates can be used to derive a unique solution X of the quasilinear SPDE (1.33), where $\hat{Z}$ is treated as a fixed random input, the initial condition X_0 does not depend on r and is bounded and Z in the bilinear SPDE is replaced by X (Theorem 6.13).

Theorem 5.1 implies:

Lemma 6.1

Under the assumptions of Theorem 5.1 there is a finite $c_{m,F,\mathcal{J},\Phi,p}$ such that for any $T > s$

$$\sup_{s \leq t \leq T} E\|Y(t, Y_s)\|^p_{m,p,\Phi} \leq \exp((T - s)c_{m,F,\mathcal{J},\Phi,p}) E\|Y_s\|^p_{m,p,\Phi}. \tag{6.1}$$

Proof

Apply (5.2), Hölder's inequality as well as Gronwall's and Fatou's lemmas. ∎

Theorem 6.2

Let $\Phi \in \{1, \lambda\}$, p an even number ≥ 2, $m \in \mathbf{N} \cup \{0\}$, $\overline{m}$ an even integer with $\overline{m} > \frac{d}{2} + m + 2$ and suppose (4.2) with smoothness degree $\overline{m}$ in addition to (4.1). Further, let $\tilde{\mathcal{Y}} \in \mathcal{M}_{0,b,[s,\infty)}$ for some $b > 0$ and $\tilde{Z} \in \tilde{\mathcal{W}}_{0,2,\lambda,[s,\infty)}$. Assume

$$Y_s \in \mathcal{W}_{m,p,\Phi,s}. \tag{6.2}$$

Then, there is a unique weak solution $Y(\cdot) := Y(\cdot, Y_s) := Y(\cdot, \tilde{\mathcal{Y}}, \tilde{Z}, Y_s) \in \mathcal{W}_{m,p,\Phi,[s,\infty)}$ of (1.32) with $Y(s) = Y_s$, and for any $T > s$ (6.1) holds with $c_{m,F,\mathcal{J},\Phi,p}$ independent of $\tilde{\mathcal{Y}}$ and $\tilde{Z}$.

Proof

(i) Again we may without loss of generality assume $s = 0$. Since $Y(\cdot, Y_0)$ is linear in the initial condition, (6.1) implies that $Y(\cdot, Y_0)$ is a uniformly continuous map from $\mathcal{D} \subset \mathcal{W}_{m,p,\Phi,0}$ into $\mathcal{W}_{m,p,\Phi,[0,T]}$ for any $T > 0$. Here $\mathcal{D}$ is the set of linear combinations of initial conditions satisfying (5.1)with smoothness degree $\overline{m}$ and $s = 0$, endowed with the norm of $\mathcal{W}_{m,p,\Phi,0}$. It is shown in Lemma 9.6 that $\mathcal{D}$ is dense in $\mathcal{W}_{m,p,\Phi,0}$, whence there is a unique extension of the map $Y(\cdot, Y_0)$ to $\mathcal{W}_{m,p,\Phi,0}$, which also satisfies (6.1).

(ii) In this step we show that the extension $Y(\cdot, Y_0)$ also satisfies (1.32). Let $Y_{0,N}$ be the approximation of Y_0 as constructed in Lemma 9.6 and set $h_N(t) := Y(t, Y_{0,N}) - Y(t, Y_0)$. Further, abbreviate $\mathcal{J}(s, r) := \mathcal{J}(r, \tilde{Y}(s), \tilde{Z}(s))$, let $\varphi \in C_c^2(\mathbf{R}^d, \mathbf{R})$ and set $c_{\Phi,\varphi} := \sup_{r \in supp\,\varphi} \Phi^{-2}(r)$, where "*supp* φ" is the support of φ. Then, in case $\Phi = \lambda$, we obtain

$$
\begin{aligned}
& [\int_0^t \langle h_N(s), \int \mathcal{J}(s, \cdot - p) w(dp, ds) \cdot \nabla\varphi\rangle] \\
= \;& \sum_{j,k,\ell=1}^{d} \int_0^t \int \int h_N(s,r) h_N(s,q) \int \mathcal{J}_{kj}(s, r-p) \mathcal{J}_{\ell j}(s, q-p) dp \partial_k \varphi(r) \partial_\ell \varphi(q) dq dr ds \\
\leq \;& d^2 c_{\Phi,\varphi} |||\varphi|||_1^2 c \int_0^t (\int |h_N(s,r)| \Phi(r) dr)^2 ds \\
& (\text{using } |\sum_{j=1}^{d} \int \mathcal{J}_{kj}(s, r-p) \mathcal{J}_{\ell j}(s, q-p) dp| \leq |\tilde{D}_{k\ell}(s, r-q)| \leq c) \\
\leq \;& d^3 c_{\Phi,\varphi} |||\varphi|||_1^2 \tilde{c} \int_0^t \|1\|_{0,\frac{p-1}{p},\Phi} \cdot \|h_N(s)\|_{0,p,\Phi}^2 ds.
\end{aligned}
$$

But $\sup_{0 \leq s \leq T} E\|h_N(s)\|_{0,p,\Phi}^2 \leq \sup_{0 \leq s \leq T} (E\|h_N(s)\|_{0,p,\Phi}^p)^{\frac{2}{p}} \to 0$ by step (i).

The convergence of the deterministic integrals can be proved by the same method. Since $Y(\cdot, Y_{0,N})$ satisfies (1.32) for all N the same holds for $Y(\cdot, Y_0)$. The case $\Phi \equiv 1$ can be proved by similar arguments.

(iii) The uniqueness is a trivial generalization of Theorem 5.1 in Kotelenez (1995b) and Theorem 5.2 in Kotelenez (1995c). ∎

To solve the quasilinear SPDE (1.32) on $\mathbf{H}_{0,\lambda}$ we assume

Hypothesis 6.3

$$\text{The functions } F \text{ and } \mathcal{J} \text{ from (1.11) depend only on } r \text{ and } \tilde{Z}. \tag{6.3}$$

In what follows next we will treat $\tilde{Z}$ as a fixed parameter and suppress in the notation the explicit dependence of $F, \mathcal{J}$ etc. on $\tilde{Z}$. $\check{r}(u,r)$ will be the measurable version of the backward SODE (2.16) on $[0,t]$ (i.e., with t instead of T) in the sense of Remark 2.4 (ii). We abbreviate

$$\psi(u,r) := |det \frac{\partial}{\partial r}\check{r}(u,r)|, \tag{6.4}$$

i.e., the left hand side is the absolute value of the Jacobian determinant of $\check{r}(u,r)$, which is well-defined by Lemma 10.4 of Appendix II, if we suppose (2.20), resp. (4.2), with $m \geq 1$. Suppose

$$X_0 \in \mathcal{W}_{0,2,\Phi,0} \cap L_0(\Omega; C(\mathbf{R}^d;\mathbf{R})). \tag{6.5}$$

Lemma 6.4

Suppose (4.2) with $m > \frac{d}{2}+2$, (4.1), (6.3) and (6.5). Let $Y(t) := Y(t,\tilde{Z},X_0)$ be the weak solution of (1.32) with $Y(0) = X_0$. Then, for fixed t $dr \otimes dp$ a.e.

$$Y(t,r) = X_0(\check{r}(t,r))\psi(t,r) \tag{6.6}$$

with $\psi(t,r)$ defined by (6.4).

Proof

Let A be a compact subset of $\mathbf{R}^d$. Further, set $r(t,A) := \cup_{q\in A} r(t,q)$. Since $\check{r}(t,r(t,A)) = A$, $r(t,A)$ is also compact. Moreover, by continuity, if $diam\ (A) \to 0$, then $diam\ (r(t,A)) \to 0$, for a sequence of compact sets A, where $diam\ (A) := \sup_{r,q\in A}|r-q|$. Then we have a.s.

$$\int_{r(t,A)} Y(t,q)dq = \int_A X_0(q)dq = \int_{r(t,A)} X_0(\check{r}(t,q))\psi(t,q)dq. \tag{$*$}$$

Here, the first equality results from the definition of our measure process $\mathcal{Y}(t,A)$ and the backward flow. The second equality is just the change of variable formula. Since our assumptions imply by Corollary 10.10 that also $\check{r}_\ell(t,\cdot)$, $\ell = 1,\dots,d$, and $\psi(t,\cdot)$ are in $L_0(\Omega;C(\mathbf{R}^d;\mathbf{R}))$, differentiation in (*) yields (6.6) for all $r \in \mathbf{R}$ and almost all ω. ■

Let δ be the Fréchet derivative of differentiable functions defined on $\mathbf{R}^d$. Set for $u \in [0,t]$

$$\left.\begin{aligned} F_i^-(r,u) :=&\ -F_i(r,t-u)\\ A(r,u) :=&\ (\delta F^-)(\check{r}(u,r),u)\\ B_\ell(r,p,u) :=&\ (\delta\mathcal{J}_{\cdot\ell})(\check{r}(u,r)-p,t-u), \quad \ell=1,\dots,d, \end{aligned}\right\} \tag{6.7}$$

where $\mathcal{J}_{\cdot\ell}$ is the ℓ-th column of the matrix process $\mathcal{J}$, $F_i(r,t-u) := F_i(r,\tilde{Z}(t-u))$, $\mathcal{J}_{ij}(r,t-u) := \mathcal{J}_{ij}(r,\tilde{Z}(t-u))$ etc. (cf. Section 4). Let "Tr" denote the trace of a $d\times d$-matrix.

Lemma 6.5

Suppose (4.2) with $m \geq 2$ and (4.1). Then, with probability 1 uniformly in $u \in [0,t]$ and $r \in \mathbf{R}^d$

$$\left.\begin{aligned} &\psi(u,r)\\ &= \exp(\textstyle\int_0^u Tr\{(A(r,v) - \frac{1}{2}\sum_{\ell=1}^d \int B_\ell^2(r,p,v)dp)dv + \sum_{\ell=1}^d \int B_\ell(r,p,v)\check{w}_\ell(dp,dv)\}), \end{aligned}\right\} \tag{6.8}$$

where $\check{w}$ is given by (2.17) (with t instead of T).

Proof

(i) Assumption (4.2), $m \geq 2$, implies that $A(r,u)$, $\int B_\ell^2(r,p,u)dp$ etc. are bounded uniformly in $u \in [0,t], \omega \in \Omega, r \in \mathbf{R}^d$. Set $\varphi(u,r) := \frac{\partial}{\partial r}\check{r}(u,r)$. Then, by (10.23), $\varphi(\cdot,r)$ is the solution of the bilinear $\mathcal{M}_{d\times d}$-valued SODE (on $[0,t]$)

$$\left.\begin{array}{rl} d\varphi(u,r) = & A(u,r)\varphi(u,r)du + \sum_{\ell=1}^d \int B_\ell(r,p,u)\varphi(u,r)\check{w}_\ell(dp,du) \\ & \varphi(0,r) = I. \end{array}\right\} \tag{6.9}$$

We consider (6.9) as an SODE on $\mathbf{R}^{d^2}$ and approximate it by random ODE's as defined in Appendix II (Section 10) (and Ikeda and Watanabe (1981)–cf. our Remark 10.6). As in Appendix II we expand the martingale term into an infinite expansion, cut this off at level M (obtaining thus M i.i.d. $\mathbf{R}^d$-valued Brownian motions β_n) and replace these β_n by the (smooth) piecewise linear approximations $\beta_{n,L}$ from (10.42). The resulting random ODE is given by

$$\left.\begin{array}{l} \dot{\varphi}_{M,L}(u,r) \\ = \tilde{A}_M(r,u)\varphi_{M,L}(u,r) + \sum_{n=1}^M \sum_{\ell=1}^d (\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u)\frac{d\check{\beta}_{n,L,\ell}}{du}(u)\varphi_{M,L}(u,r) \\ \varphi_{M,L}(0,r) = I, \end{array}\right\} \tag{6.10}$$

with

$$\tilde{A}_M(r,u) := A(r,u) - \frac{1}{2}\sum_{n=1}^M \sum_{\ell=1}^d ((\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u))^2. \tag{6.11}$$

(ii) Next, we easily see that

$$\sum_{n=1}^M ((\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u))^2 \to \int B_\ell^2(r,p,u)dp, \tag{6.12}$$

as $M \to \infty$ (on $\mathcal{M}_{d\times d}$ with the Euclidean norm), which implies (as $M \to \infty$)

$$Tr\sum_{n=1}^M ((\delta\sigma_{n,\cdot\ell})(\check{r}(u,r),t-u))^2 \to \int Tr(B_\ell^2(r,p,u)dp. \tag{6.13}$$

(iii) From Appendix II by (4.2), $m \geq 2$ (cf. Remark 2.6), (10.56) and (10.50)) for any $K > 0$

$$E \sup_{|r|\leq K} \sup_{0\leq u\leq t} |\varphi_{M,L}(u,r) - \varphi(u,r)| \to 0, \text{ as } L \to \infty, M \to \infty, \tag{6.14}$$

which implies convergence of the sequence ($\sup_{|r|\leq K}\ldots$) in probability.

(iv) Now (6.8) is correct, if we replace in the left hand side $\det\varphi$ by $\det\varphi_{M,L}$ and in $\exp\{\ \}$ the right hand side of (6.13) by the approximating trace in the left hand side of (6.13), which is just a classical result of ODE's (cf. Coddington and Levinson (1955), Ch. I, Section 7, (7.18)). Application of step (iii) and (6.12) finishes the proof. ∎

Corollary 6.6

Suppose (4.2) with $m \geq 2$ and (4.1). Then $\psi(u,r)$ defined by (6.4) is the unique solution of the $\mathbf{R}$-valued bilinear Itô SODE (on $[0,t]$)

$$\left.\begin{array}{rl} d\psi(u,r) = & Tr(A(r,u))\psi(u,r)du + \sum_{\ell=1}^{d} \int Tr(B_\ell(r,p,u))\psi(u,r)\breve{w}_\ell(dp,du) \\ + & \frac{1}{2}\sum_{\ell=1}^{d} \int ((Tr(B_\ell(r,p,u)))^2 - TrB_\ell^2(r,p,u))dp\psi(u,r)du \\ & \psi(0,r) = 1. \end{array}\right\} \qquad (6.15)$$

Proof

The definition of the independent Brownian sheets $\breve{w}_\ell, \ell = 1,\ldots,d$, and Itô's formula imply (6.15). ■

Let us now check, whether the coefficients in (6.15) as functions of $\check{r}(u,r)$ and $\tilde{Z}(t-u)$ are Lipschitz. Let $q_i \in \mathbf{R}^d, \eta_i \in \tilde{\mathbf{H}}_{0,\lambda}, i = 1,2$. Then, a typical element of $(TrB_\ell(r,p,u))^2$ or $TrB_\ell^2(r,p,u)$ with $q_i = \check{r}(u,\tilde{Z}_i,r), \eta_i := \tilde{Z}_i(t),\ i = 1,2)$ can be written as follows:

$$\int \partial_m \mathcal{J}_{k\ell}(q_i - p,\eta_i)\partial_n \mathcal{J}_{\tilde{k}\ell}(q_i - p,\eta_i)dp. \qquad (**)$$

An elementary calculation—using integration by part and the Cauchy-Schwarz inequality—yields

$$\left.\begin{array}{rl} & |\int\{(\partial_m \mathcal{J}_{k\ell})(q_1 - p,\eta_1)(\partial_n \mathcal{J}_{\tilde{k}\ell})(q_1 - p,\eta_1) - (\partial_m \mathcal{J}_{k\ell})(q_2 - p,\eta_2)\partial_n \mathcal{J}_{\tilde{k}\ell}(q_2 - p,\eta_2)\}dp| \\ \le & \max_{k=1,\ldots,d}(\int\{(\partial^2_{m,n}\mathcal{J}_{k\ell})^2(q_1 - p,\eta_1) + (\partial^2_{m,n}\mathcal{J}_{k\ell})^2(q_2 - p,\eta_2)\}dp)^{\frac{1}{2}}\cdot \\ & \cdot\max_{\tilde{k}=1,\ldots,d}(\int(\mathcal{J}_{\tilde{k}\ell}(q_1 - p,\eta_1) - \mathcal{J}_{\tilde{k}\ell}(q_2 - p,\eta_2))^2dp)^{\frac{1}{2}} \\ \le & \max_{\tilde{k}=1,\ldots,d}(2d_{1,T})^{\frac{1}{2}}(\int(\mathcal{J}_{\tilde{k}\ell}(q_1 - p,\eta_1) - \mathcal{J}_{\tilde{k}\ell}(q_2 - p,\eta_2))^2dp)^{\frac{1}{2}}, \end{array}\right\} \qquad (6.16)$$

assuming (4.2) with $m \ge 2$. So the Lipschitz assumption (1.14) on $\mathcal{J}_{\tilde{k}\ell}$ in addition to (4.1) and (4.2) with $m \ge 2$, implies the (global) Lipschitz property for $(TrB_\ell)^2$ and TrB_ℓ^2.

The Lipschitz property of $\partial_k F_k(q,\eta), k = 1,\ldots,d$, cannot be directly derived from (4.1), (4.2) and the Lipschitz property of F_k. So we will assume it and verify that this condition holds in our Examples 1.4, 1.6, if we take smooth kernels.

Hypothesis 6.7

Suppose that for any $k = 1,\ldots,d, (q_i,\tilde{\eta}_i) \in \mathbf{R}^d \times \tilde{\mathbf{H}}_{0,\lambda}, i = 1,2$

$$\left.\begin{array}{rl} & \rho^2(\partial_k F_k(q_1,\tilde{\eta}_1),\partial_k F_k(q_2,\tilde{\eta}_2)) \\ \le & c_F\{\rho^2(q_1,q_2) + \|B(\eta_1) - B(\eta_2)\|^2_{0,\lambda} + (\int \Lambda_F(q_2 - p)|B(\hat{\eta}_1(p)) - B(\hat{\eta}_2(p))|dp)^2\}. \end{array}\right\} \qquad (6.17)$$

Lemma 6.8

Suppose (4.2), $m \ge 2$, in addition to (4.1) and (6.17). Let $\tilde{Z}_i \in \tilde{\mathcal{W}}_{0,2,\lambda,[0,\infty)}$ and set $\psi_i(u,r) := \frac{\partial}{\partial r}\check{r}(u,\tilde{Z}_i,r), i = 1,2$. Then, for $u \le t\,(\le T)$

$$\left.\begin{array}{rl} & E\sup_{0\le v\le u}(B(\psi_1(v,r)) - B(\psi_2(v,r)))^2 \\ \le & c_{F,\mathcal{J},B,T}E\int_0^u\{\|B(Z_1(t-v)) - B(Z_2(t-v))\|^2_{0,\lambda} \\ & +\sum_{L\in\{F,\mathcal{J}\}}(\int \Lambda_L(\check{r}(v,\hat{Z}_2,r) - p)|B(\hat{Z}_1(t-v,p)) - B(\hat{Z}_2(t-v,p))|dp)^2\}dv. \end{array}\right\} \qquad (6.18)$$

Proof

(i) The coefficients in (6.15) now explicitly depend on $\tilde{Z}_i$, $i = 1,2$. Let $a_i(u)$ be the sum of all 3 drift coefficients from (6.15), where we suppress the dependence on r. Further, we abbreviate $b_{\ell,i}(u,p) := Tr(B_{\ell,i}(r,p,u))$. Itô's formula yields:

$$\left.\begin{array}{rl} & (B(\psi_1(u)) - B(\psi_2(u)))^2 \\ = & \int_0^u 2(B(\psi_1(v)) - B(\psi_2(v)))d(B(\psi_1(v)) - B(\psi_2(v))) \\ + & \int_0^u d\langle B(\psi_1(v)) - B(\psi_2(v))\rangle, \end{array}\right\} \tag{6.19}$$

where, again by Itô's formula,

$$\left.\begin{array}{rl} & B(\psi_i(u)) = \int_0^u B'(\psi_i(v))a_i(v)\psi_i(v)dv + \sum_{\ell=1}^d \int_0^u B'(\psi_i(v)) \int b_{\ell,i}(v,p)\psi_i(v)\check{w}_\ell(dp,dv) \\ + & \frac{1}{2}\sum_{\ell=1}^d \int_0^u B''(\psi_i(v)) \int b_{\ell,i}^2(v,p)dp\psi_i^2(v)dv. \end{array}\right\} \tag{6.20}$$

(ii) Hence,

$$\left.\begin{array}{rl} & \langle B(\psi_1(u)) - B(\psi_2(u))\rangle \\ = & \sum_{\ell=1}^d \int_0^u \int (B'(\psi_1(v))\psi_1(v)b_{\ell,1}(v,p) - B'(\psi_2(v))\psi_2(v)b_{\ell,2}(v,p))^2 dpdv \\ \le & 2\sum_{\ell=1}^d \int_0^u (B'(\psi_1(v))\psi_1(v) - B'(\psi_2(v))\psi_2(v))^2 \int b_{\ell,1}^2(v,p)dpdv \\ + & 2\sum_{\ell=1}^d \int_0^u B'(\psi_2(v))\psi_2(v) \int (b_{\ell,1}(v,p) - b_{\ell,2}(v,p))^2 dpdv. \end{array}\right\} \tag{6.21}$$

Now the properties of B (cf. (1.12)), the boundedness of $\int b_{\ell,1}^2(v,p)dp$ and (6.16) in addition to (1.14) imply for $u \le t(\le T)$

$$\left.\begin{array}{rl} & \langle B(\psi_1(u)) - B(\psi_2(u))\rangle \\ \le & c_{\mathcal{J},B,T} \int_0^u (B(\psi_1(v)) - B(\psi_2(v)))^2 dv \\ + & c_{\mathcal{J},B,T} \int_0^u \{\rho^2(\check{r}(v,\tilde{Z}_1,r), \check{r}(v,\tilde{Z}_2,r)) \\ & \quad + \|B(Z_1(t-v)) - B(Z_2(t-v))\|_{0,\lambda}^2 \\ & \quad + \sum_{L\in\{F,\mathcal{J}\}} (\int \Lambda_L(\check{r}(v,\hat{Z}_2,r) - p)|B(\hat{Z}_1(t-v,p)) - B(\hat{Z}_2(t-v,p))|dp)^2\} \\ & \quad \cdot B'(\psi_2(v))\psi_2(v)dv. \end{array}\right\} \tag{6.22}$$

(iii) Let us abbreviate the second integral of the right-hand side of (6.22) by $H_{1,2}(u)$. Note that our assumptions on B imply (by the mean value theorem)

$$|B'(y)y - B'(x)x| + |B''(y)y^2 - B''(x)x^2| \le c_B|B(y) - B(x)|$$

for all $x, y \in \mathbf{R}$. In particular,

$$\sup_{x\in\mathbf{R}}\{|B'(x)x + B''(x)x^2|\} < \infty \ .$$

Hence, by an elementary calculation, using the Cauchy-Schwarz inequality and $|ab| \le \frac{a^2}{2} + \frac{b^2}{2}$, $a, b \in \mathbf{R}$, (1.4), (6.16) and (6.17) imply

$$\begin{array}{rl} & 2\int_0^u |B(\psi_1(v)) - B(\psi_2(v))| \cdot \\ & \quad \cdot\{|B'(\psi_1(v))\psi_1(v)a_1(v) - B'(\psi_2(v))\psi_2(v)a_2(v)| \\ & \quad + \frac{1}{2}\sum_{l=1}^d |B''(\psi_1(v))\psi_1(v)\int b_{l,1}^2(v,p)dp - B''(\psi_2(v))\psi_2^2(v)\int b_{l,2}^2(v,p)dp|\}dv \\ \le & c_{F,\mathcal{J},B,T}\left\{\int_0^u (B(\psi_1(v)) - B(\psi_2(v)))^2 + H_{1,2}(u)\right\} \ . \end{array} \tag{6.23}$$

(iv) Let $M_i(v)$ be the martingale part of $B(\psi_i(v))$, $i=1,2$. Since $\langle M_1(v)-M_2(v)\rangle = \langle B(\psi_1(v)) - B(\psi_2))\rangle$ we obtain (using, e.g., Ikeda and Watanabe, loc. cit., Ch. II, §2, (2.11) and an elementary calculation)

$$\left.\begin{array}{rl} & c[\int_0^u 2(B(\psi_1(v))-B(\psi_2(v)))d(M_1(v)-M_2(v))]^{\frac{1}{2}} \\ \le & \frac{1}{2}\sup_{v\le u}(B(\psi_1(v))-B(\psi_2(v)))^2 + 8c^4\langle B(\psi_1(u))-B(\psi_2(u))\rangle, \end{array}\right\} \tag{6.24}$$

where $c>0$ is an arbitrary positive constant.

(v) Since $|B'(x)x|$ is bounded we obtain from (6.19)-(6.24), the Burkholder-Davis-Gundy inequality and Gronwall's lemma

$$\left.\begin{array}{rl} & E\sup_{v\le u}(B(\psi_1(v))-B(\psi_2(v)))^2 \\ \le & c_{F,\mathcal{J},B,T}E\int_0^u\{\rho^2(\check{r}(v,\tilde{Z}_1,r),\check{r}(v,\tilde{Z}_2,r)) + \|B(Z_1(t-v))-B(Z_2(t-v))\|^2_{0,\lambda} \\ & \quad +\sum_{L\in\{F,\mathcal{J}\}}(\int\Lambda_L(\check{r}(v,\hat{Z}_2,r)-p)|B(\hat{Z}_1(t-v,p))-B(\hat{Z}_2(t-v,p))|dp)^2\}dv. \end{array}\right\} \tag{6.25}$$

Now, (2.2) in addition to (6.25) implies (6.18). ■

In order to derive a solution of the quasilinear SPDE (1.33) we will now assume:

Hypothesis 6.9

$$\hat{Z}\in\mathcal{W}_{0,2,\lambda,[0,\infty)} \text{ is a fixed random input.} \tag{6.26}$$

Remark 6.10

Hypothesis 6.9 excludes F^3 and $\mathcal{J}^3$ from Examples 1.4 and 1.7 as coefficients for the quasilinear SPDE (1.33). A derivation of quasilinear SPDE's with those coefficients is planned for another paper. ■

Since we want to be conceptual rather than computational we make an assumption on the initial condition which could be relaxed with some additional effort.

Hypothesis 6.11

X_0 is $\mathcal{F}_0$-measurable, and it does not depend on $r\in\mathbf{R}^d$. Moreover, there are constants $0<\underline{c}<\overline{c}<\infty$ such that a.s.

$$\underline{c}\le|X_0(\omega)|\le\overline{c}. \tag{6.27}$$

Lemma 6.12

Suppose (4.2) with $m>\frac{d}{2}+2$, in addition to (4.1), (6.3), (6.26) and (6.27). Let $Z_i\in\mathcal{W}_{0,2,\lambda,[0,\infty)}$ and set $Y_i(t,r):=Y(t,Z_i,X_0,r):=Y(t,\hat{Z},Z_i,X_0,r)$ $i=1,2$. Then, for any $t>0$

$$\sup_{0\le t\le T}E\|B(Y_1(t))-B(Y_2(t))\|^2_{0,\lambda}\le c_{F,\mathcal{J},\lambda,T,X_0}\int_0^T E\|B(Z_1(s))-B(Z_2(s))\|^2_{0,\lambda}ds. \tag{6.28}$$

Proof

Fix $t\in[0,T]$. Then, by (6.6) and the properties of B

$$(B(Y_1(t,r))-B(Y_2(t,r)))^2\le c(B(\psi_1(t,r))-B(\psi_2(t,r)))^2.$$

Application of (6.18) with change of variables $t-v\to s$ and integration against $\lambda(r)dr$ implies (6.28). ■

Now we can solve the quasilinear SPDE (1.33) on $\mathbf{H}_{0,\lambda}$ under the previous assumptions.

Theorem 6.13

Suppose (4.1), (4.2), $m > \frac{d}{2}+2$, (6.3), (6.26) and (6.27). Then (1.33) has a unique weak (Itô) solution $X(\cdot, X_0) \in \mathcal{H}_{0,\lambda,[0,\infty)}$.

Proof

(i) By Theorem 6.2 the solution of the bilinear SPDE (1.32) satisfies the relation $Y(\cdot, Z, X_0) := Y(\cdot, \hat{Z}, Z, X_0) \in \mathcal{H}_{0,\lambda,[0,\infty)}$ for any $\hat{Z}, Z \in \mathcal{W}_{0,2,\lambda,[0,\infty)}$. Hence, suppressing the dependence on $\hat{Z}$ in our notation, we iteratively define

$$Y_{n+1}(\cdot, Y_n, X_0), \quad n \geq 1, \quad Y_1 \equiv X_0.$$

(6.28) implies that this sequence has a unique fixed point $\tilde{X} := \lim_{n\to\infty} B(Y_n)$ on $\mathcal{W}_{0,2,\lambda,[0,\infty)}$. Set

$$X(t,r) := B^{-1}(\tilde{X}(t,r)). \tag{6.29}$$

(ii) Let $f : \mathbf{R} \to \mathbf{R}$ be bounded and continuous. Then we easily verify that

$$E\|f(Y_n(t)) - f(X(t))\|_{0,\lambda}^2 \to 0, \text{ as } n \to \infty. \tag{6.30}$$

In particular, choosing f such that $|f|(x) \leq |x|$ for all $x \in \mathbf{R}$, we obtain by (6.1)

$$E\|f(X(t))\|_{0,\lambda}^2 \leq \lim\inf_{n\to\infty} E\|Y_n(t)\|_{0,\lambda}^2 \leq c_{F,\mathcal{J},\lambda,T} E|X_0|^2. \tag{6.31}$$

Choosing $f_m \geq 0$ such that $f_m(x) \leq |x|$ and $f_m(x) \uparrow |x|$ for all $x \in \mathbf{R}$, monotone convergence implies

$$\sup_{0\leq t\leq T} E\|X(t)\|_{0,\lambda}^2 \leq c_{F,\mathcal{J},\lambda,T} E|X_0|^2. \tag{6.32}$$

In particular,

$$X \in \mathcal{W}_{0,2,\lambda,[0,\infty)}. \tag{6.33}$$

Next, set $\mu(dr) := \lambda(r)dr$ and let d_T be the Lebesgue measure on $[0,T]$. Then μ and d_T are finite Borel measures and by Chebyshev's inequality and (6.1)

$$d_T \otimes \mu \otimes P\{(t,r,\omega) : |Y_n(t,r,\omega)| \geq N\} \leq c_{F,\mathcal{J},\lambda,T} \frac{E|X_0|^2}{N^2}. \tag{6.34}$$

Hence, $|Y_n|^2$ is uniformly integrable on $L_1([0,T] \times \mathbf{R}^d \times \Omega, d_T \otimes \mu \otimes p)$ By (6.30) and (6.32) this implies

$$\int_0^T E\|Y_n(t) - X(t)\|_{0,\lambda}^2 dt \to 0, \text{ as } n \to \infty. \tag{6.35}$$

(iii) Repeating the arguments of the proof of Theorem 6.2, it follows that $X(\cdot, X, X_0)$ is a solution of the bilinear SPDE (1.32) in $\mathbf{H}_{0,\lambda}$ with $Z \equiv X$ and that it is an element of $\mathcal{W}_{m,2,\lambda,[0,\infty)}$. Hence, $X(\cdot, X, X_0)$ is a strong solution of (1.33) on $\mathbf{H}_{0,\lambda}$ (for the definition, cf. Da Prato and Zabczyk (1992)) and, therefore, $X(\cdot, X, X_0) \in \mathcal{H}_{0,\lambda,[0,\infty)}$. The uniqueness follows directly from Theorem 6.2. ∎

7 Homogeneous and Isotropic Solutions

First, using Theorem 2.2, homogeneous and isotropic solutions of the bilinear SPDE (1.32) are derived (Theorem 7.2). This result is used to show that under the assumptions of Theorem 6.13 the solution of the quasilinear SPDE (1.33) is homogeneous and isotropic, provided the distribution of the ensemble of "input" processes is shift and rotational invariant (Theorem 7.3).

Let $SO(d) := \{\sigma \in \mathcal{M}_{d\times d} : \det\sigma = \pm 1\}$. For $\sigma \in SO(d)$ we define the rotation operator S_σ on $\mathbf{H}_{0,\lambda}$ by

$$(S_\sigma f)(r) := f_\sigma(r) := f(\sigma r).$$

Similarly, we define the shift operator U_h on $\mathbf{H}_{0,\lambda}$, where $h \in \mathbf{R}^d$, by

$$(U_h f)(r) := f_h(r) := f(r+h).$$

Moreover, we will use $f_{\sigma h}(r)$ to denote the image of the composition $S_\sigma \circ U_h$:

$$(S_\sigma \circ U_h f)(r) := f_{\sigma h}(r) := f(\sigma(r+h)).$$

If the coefficients F and $\mathcal{J}$ are generated by homogeneous kernels, then they may give rise to homogeneous solutions. If, in addition, $F(r,\cdot)$ and $\mathcal{J}(r,\cdot)$ do depend only on $|r|$ (as, e.g., $G(u,r)$, the fundamental solution of the heat equation), we also expect to obtain isotropic solutions of (1.32).

In this section we will distinguish between $\hat{Z}$ and Z.

For $h \in \mathbf{R}^d$ and $\sigma \in SO(d)$ $\tilde{h}$ and $\tilde{\sigma}$ will denote arbitrary elements from $\{h, 0\}$ and $\{\sigma, I\}$, respectively, where I is the identity matrix. We define for $f \in \mathbf{H}_{0,\lambda}$

$$\left.\begin{array}{rl} U_{\tilde{h}} f := & f_{\tilde{h}}, \\ S_{\tilde{\sigma}} f := & f_{\tilde{\sigma}}. \end{array}\right\} \tag{7.1}$$

Set

$$\left.\begin{array}{l} Q(r, \hat{Z}(t), Z(t), w(d\cdot, dt)) \\ \quad := (F(r, \hat{Z}(t), Z(t))dt, \int \mathcal{J}(r-p, \hat{Z}(t), Z(t))w(dp, dt)). \end{array}\right\} \tag{7.2}$$

$\hat{Z}$, Z and w will be called the coordinates of Q or of the pair $(F, \mathcal{J})$.

Definition 7.1

Let $\sigma \in SO(d)$ and $h \in \mathbf{R}^d$. The pair $(F, \mathcal{J})$ is called *σ-h-admissible,* if $\mathcal{J}$ satisfies (4.1) and

$$(S_\sigma U_h Q)(r, \hat{Z}(t), Z(t), w(d\cdot, dt)) = Q(r, \hat{Z}_{\tilde{\sigma}\tilde{h}}(t), Z_{\tilde{\sigma}\tilde{h}}(t), w_{\tilde{\sigma}\tilde{h}}(d\cdot, dt)), \tag{7.3}$$

where $\tilde{\sigma}$ and $\tilde{h}$ may vary with coordinates. $(F, \mathcal{J})$ is called *admissible,* if it is σ-h-admissible for all $(\sigma, h) \in SO(d) \times \mathbf{R}^d$. $(F^2, \mathcal{J}^i)$, F^2 from (1.22) and $\mathcal{J}^i$, $i = 2, 3$, from (1.20), resp. (1.24) are not admissible pairs (unless Z is independent of r). On the other hand $(F^3, \mathcal{J}^i)$, $i = 2, 3$, F^3 from (1.17), are admissible. The coordinates in (7.3), where $\tilde{h} = h$, combined with the coordinates, where $\tilde{\sigma} = \sigma$, determine a *type* of the admissible pair $(F, \mathcal{J})$.

We easily see that a given pair may have several types. Since we can obviously restrict the notion of a type to any of the two components in the right hand side of (7.2), admissibility implies that, neglecting the additional coordinate $w(d\cdot, dt)$ of the second component of Q, F and $\int \mathcal{J}(\cdot - p, \cdot, \cdot)w(dp, dt)$ must have at least one common type. Next note that U_h and S_σ are homomorphisms (on the group of functions with pointwise multiplication). Therefore, the types of the mutual quadratic variations $d[M_k(t), M_\ell(t)]$ coincides with the types of $dM(t) := \int \mathcal{J}(\cdot - p, \cdot, \cdot) w(dp, dt)$.

Theorem 7.2

Suppose (4.2), $m \geq 0$, (6.3) and $\hat{Z}, Z \in \mathcal{H}_{0,\lambda,[0,\infty)}$. Further, let $h \in \mathbf{R}^d$ and $\sigma \in SO(d)$ and assume $F, \mathcal{J}$ is σ-h-admissible. Let $Y(\cdot, \hat{Z}, Z, Y_0, w)$ be the solution of (1.32) with $Y_0 \in \mathcal{W}_{0,2,\lambda,0}$ and w the Brownian sheet from (1.10). Let $(\tilde{\sigma}, \tilde{h})$ be a collection of (coordinate-dependent) values so that (7.3) holds. In addition to the preceding assumptions suppose that on $\mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda} \times \mathbf{H}_{w,[0,\infty)}$

$$(S_\sigma^\sim U_h^\sim \hat{Z}, S_\sigma^\sim U_h^\sim Z, S_\sigma U_h Y_0, S_\sigma^\sim U_h^\sim W) \sim (\hat{Z}, Z, Y_0, W). \tag{7.4}$$

Then, for any $t \geq 0$ on $\mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{W}_{0,2,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda} \times \mathbf{H}_{w,[0,\infty)}$

$$\left.\begin{array}{l} (S_\sigma^\sim U_h^\sim \hat{Z}, (\pi_t S_\sigma U_h Y)(\cdot, \hat{Z}, Z, Y_0, W), S_\sigma^\sim U_h^\sim Z, S_\sigma U_h Y_0, S_\sigma U_h W) \\ \sim (\hat{Z}, (\pi_t Y)(\cdot, \hat{Z}, Z, Y_0, W), Z, Y_0, W). \end{array}\right\} \tag{7.5}$$

Proof

(i) We will only show the shift invariance, assuming $\sigma = I$, since the proof for $\sigma \neq I$ follows the same pattern. Again we assume $s = 0$.

(ii) Further, let $\varphi \in C_c^2(\mathbf{R}^d; \mathbf{R})$. Since differentiation and shift commute on $C_c^2(\mathbf{R}^d; \mathbf{R})$ we obtain from the preceding remarks

$$\begin{aligned}
& \langle (U_h Y)(t, \hat{Z}, Z, Y_0, W), \varphi \rangle_0 \\
= \; & \langle Y(t, \hat{Z}, Z, Y_0, W), U_{-h}\varphi \rangle_0 \\
= \; & \langle Y_0, U_{-h}\varphi \rangle_0 \\
+ \; & \int_0^t \frac{1}{2} \sum_{k,\ell=1}^d \langle Y(s, \hat{Z}, Z, Y_0, W), D_{k\ell}(\hat{Z}(s), Z(s)) U_{-h} \partial_{k\ell}^2 \varphi \rangle_0 ds \\
+ \; & \int_0^t \langle Y(s, \hat{Z}, Z, Y_0, W), F(\cdot, \hat{Z}(s), Z(s)) \cdot (U_{-h} \bigtriangledown \varphi) \rangle_0 ds \\
+ \; & \int_0^t \langle Y(s, \hat{Z}, Z, Y_0, W), \int \mathcal{J}(\cdot - p, \hat{Z}(s), Z(s)) w(dp, ds) \cdot (U_{-h} \bigtriangledown \varphi) \rangle_0 ds \\
= \; & \langle Y_{0,h}, \varphi \rangle_0 \\
+ \; & \int_0^t \frac{1}{2} \sum_{k,\ell=1}^d \langle Y_h(s, \hat{Z}, Z, Y_0, W), D_{k\ell}(\hat{Z}_h^\sim(s), Z_h^\sim(s)) \partial_{k\ell}^2 \varphi \rangle_0 ds \\
+ \; & \int_0^t \langle Y_h(s, \hat{Z}, Z, Y_0, W), F(\cdot, \hat{Z}_h^\sim(s), Z_h^\sim(s)) \cdot \bigtriangledown \varphi \rangle_0 ds \\
+ \; & \int_0^t \langle Y_h(s, \hat{Z}, Z, Y_0, W), \int \mathcal{J}(\cdot - p, \hat{Z}_h^\sim(s), Z_h^\sim(s)) w_h(dp, ds) \cdot \bigtriangledown \varphi \rangle_0.
\end{aligned}$$

Here, $w_h^\sim(r, t) = w(r + \tilde{h}, t)$ (the shifted Brownian sheet). So, a.s.,

$$(U_h Y)(\cdot, \hat{Z}, Z, Y_0, W) = Y(\cdot, \hat{Z}_h^\sim, Z_h^\sim, Y_{0,h}, W_h^\sim). \tag{7.6}$$

(iii) We want to apply Theorem 2.2. Therefore, we discretize $\mathbf{H}_{0,\lambda}$ as follows: Set $C_n := \{r \in \mathbf{R}^d : -n \leq r_k < n \text{ for } k = 1, \ldots d\}$, where $n \in \mathbf{N}$. Then we partition C_n into N boxes of equal volume $\frac{(2n)^d}{N}$ (cf. Kotelenez (1988)), where we assume that the left end points of those boxes (or d-dimensional intervals) belong to the box and the right end points are excluded. Let us denote the left end points by $\{r^1, \ldots, r^N\}$ and identify the i-th box with r^i, $i = 1, \ldots, N$. If $f \in \mathbf{H}_{0,\lambda}$ we set

$$(\pi_{N,n} f)(r) := \left\{ \begin{array}{ll} \frac{N}{(2n)^d} \int_{[r^i)} f(q) dq, & r \in [r^i), i = 1, \ldots N, \\ 0, & r \notin C_n, \end{array} \right\} \tag{7.7}$$

where $[r^i)$ is the i-th box. Lebesgue's differentiation theorem and Lebesgue's dominated convergence theorem imply for any $f \in \mathbf{H}_{0,\lambda}$

$$\|f - \pi_{N,n} f\|_{0,\lambda} \to 0, \text{ as } N \to \infty, n \to \infty, \tag{7.8}$$

and also for any $X_0 \in \mathcal{W}_{0,2,\lambda,0}$

$$E\|X_0 - \pi_{N,n} X_0\|^2_{0,\lambda} \to 0, \text{ as } N \to \infty, n \to \infty. \tag{7.9}$$

Moreover, for fixed n and N we define a map

$$\left. \begin{array}{ll} g_N : \pi_{N,n} \mathbf{H}_{0,\lambda} & \to \mathbf{R}^N \\ \pi_{N,n} f & \mapsto a(N, f) \end{array} \right\} \tag{7.10}$$

where the i-th component of $a(N, f)$ is the integral in (7.7) (the "weight"). We easily check that g_N is continuous, where $\pi_{N,n} \mathbf{H}_{0,\lambda}$ is endowed with $\| \cdot \|_{0,\lambda}$. Since $\pi_{N,n}$ is a projection into a closed subspace of $\mathbf{H}_{0,\lambda}$, (7.4) implies

$$(U_h^{\sim} \hat{Z}, U_h^{\sim} Z, g_N \pi_{N,n} U_h Y_0, U_h^{\sim} W) \sim (\hat{Z}, Z, g_N \pi_{N,n} Y_0, W) \tag{7.11}$$

on $\mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{R}^N \times \mathbf{H}_{w,[0,\infty)}$.

(iv) By the proof of Theorem 2.2 there is a solution map $\bar{\bar{r}}_N$ for (2.1) (on the "canonical" probability space) such that for any $t \geq 0$ the following map is measurable as indicated:

$$\left. \begin{array}{rl} (\hat{\eta}, a(N), r_N, \eta, \overline{W}) & \mapsto (\hat{\eta}, a(N), (\pi_t\, \bar{\bar{r}}_N)(\cdot, \overline{W}, \hat{\eta}, \eta, r_N), \eta, \overline{W}) \\ \mathcal{F}_{0,\lambda,\infty} \otimes \mathcal{B}^N \otimes \mathcal{B}^{dN} \otimes \mathcal{F}_{0,\lambda,\infty} \otimes \overline{\mathcal{G}}_{w,\infty} & - \mathcal{F}_{0,\lambda,\infty} \otimes \mathcal{B}^N \otimes \mathcal{F}_{\mathbf{R}^{dN},t} \otimes \mathcal{F}_{0,\lambda,\infty} \otimes \overline{\mathcal{G}}_{w,\infty}, \end{array} \right\} \tag{7.12}$$

where $\overline{W}$ denotes a typical element of $\mathbf{H}_{w,[0,\infty)}$. Then, by (7.11), for any $t \geq 0$

$$\left. \begin{array}{l} (U_h^{\sim} \hat{Z}, g_N \pi_{N,n} U_h Y_0, (\pi_t\, \bar{\bar{r}}_N)(\cdot, U_h^{\sim} W, U_h^{\sim} \hat{Z}, U_h^{\sim} Z, r_N), U_h^{\sim} Z, U_h^{\sim} W) \\ \sim (\hat{Z}, g_N \pi_{N,n} Y_0, (\pi_t\, \bar{\bar{r}}_N)(\cdot, W, \hat{Z}, Z, r_N), Z, W) \end{array} \right\} \tag{7.13}$$

on $\mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{R}^N \times \mathbf{R}_{dN,[0,\infty)} \times \mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{w,[0,\infty)}$.

(v) Let $a(N) := (a_1, \ldots, a_N) \in \mathbf{R}^N$ (the "weights") and $q_N := (q^1, \ldots, q^N) \in \mathbf{R}^{dN}$ (the "positions"). Then,

$$h_N(a(N), q_N) := \sum_{i=1}^N a_i \delta_{q^i} \tag{7.14}$$

is continuous from $\mathbf{R}^N \times \mathbf{R}^{dN}$ into $(\mathbf{M}, \gamma)$. Let $\mathcal{Y}(\cdot)$ denote the empirical process associated with $\bar{\bar{r}}_N(\cdot)$. Then (7.13) and (7.14) imply

$$\left.\begin{array}{l}(U_h^{\sim}\hat{Z}, \pi_{N,n}U_hY_0, (\pi_t\mathcal{Y})(\cdot, U_h^{\sim}W, U_h^{\sim}\hat{Z}, U_h^{\sim}Z, \tilde{\pi}_{N,n}U_hY_0), U_h^{\sim}Z, U_h^{\sim}W) \\ \sim (\hat{Z}, \pi_{N,n}Y_0, (\pi_t\mathcal{Y})(\cdot, W, \hat{Z}, Z, \tilde{\pi}_{N,n}Y_0), Z, W)\end{array}\right\} \tag{7.15}$$

on $\mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda} \times \mathbf{M}_{[0,\infty)} \times \mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{w,[0,\infty)}$, where we used the fact that g_N from (7.10) has a continuous inverse. Moreover, $\tilde{\pi}_{N,n}Y_0$ and $\tilde{\pi}_{N,n}U_hY_0$ are the point measures associated with the discretization.

(vi) The Cauchy-Schwarz inequality implies that Y_0 (and U_hY_0), restricted to C_n is in $\mathcal{W}_{0,1,1,0}(\subset \mathcal{M}_0!)$. Therefore,

$$E\gamma^2(\tilde{\pi}_{N,n}Y_0, 1_{C_n}Y_0) \to 0, \text{ as } N \to \infty \tag{7.16}$$

and, trivially,

$$E\|1_{C_n}Y_0 - Y_0\|_{0,\lambda}^2 \to 0 \text{ as } n \to \infty. \tag{7.17}$$

The same holds for U_hY_0.

Since $\mathcal{Y}$ is the solution of the bilinear SPDE (1.32), it follows from Theorem 4.2 that $\mathcal{Y}(\cdot, \ldots, 1_{C_n}Y_0) = Y(\cdot, \ldots, 1_{C_n}Y_0) \in \mathcal{M}_{[0,\infty)} \cap \mathcal{W}_{0,2,\lambda,[0,\infty)}$, and similarly for $1_{C_n}U_hY_0$. Hence, (7.16) implies the equivalence, as stated in (7.15), but with $\pi_{N,n}$ and $\tilde{\pi}_{N,n}$ replaced by 1_{C_n} and $\mathbf{M}_{[0,\infty)}$ replaced first by $\mathbf{M}_{[0,\infty)} \cap \mathbf{W}_{0,2,\lambda,[0,\infty)}$ and then by $\mathbf{W}_{0,2,\lambda,[0,\infty)}$. Then, (7.17) in addition to (7.6) yields (7.5). ∎

We now extend the results of Theorem 7.2 to the solution X of the quasilinear SPDE (1.33) under the assumptions of Theorem 6.13.

Theorem 7.3

Suppose (4.2), $m > \frac{d}{2} + 2$, (6.3), (6.26), (6.27). Let $h \in \mathbf{R}^d$ and $\sigma \in SO(d)$ and assume that $(F, \mathcal{J})$ is σ-h-admissible. Further, assume $\hat{Z} \in \mathcal{H}_{0,\lambda,[0,\infty)}$ and let $(\tilde{\sigma}, \tilde{h})$ be a collection of (coordinate-dependent) values so that (7.3) holds with $Z := X$. Further, suppose we have that on $\mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda} \times \mathbf{H}_{w,[0,\infty)}$

$$(S_\sigma U_h\hat{Z}, X_0, S_\sigma U_hW) \sim (\hat{Z}, X_0, W). \tag{7.18}$$

Then, for any $t \geq 0$ on $\mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda,[0,\infty)} \times \mathbf{H}_{0,\lambda} \times \mathbf{H}_{w,[0,\infty)}$

$$\left.\begin{array}{l}(S_\sigma^{\sim}U_h^{\sim}\hat{Z}, (\pi_tS_\sigma U_hX)(\cdot, \hat{Z}, X, X_0, W), X_0, S_\sigma^{\sim}U_h^{\sim}W) \\ \sim (\hat{Z}, (\pi_tX)(\cdot, \hat{Z}, X, X_0, W), X_0, W).\end{array}\right\} \tag{7.19}$$

In particular, (7.19) implies (on $\mathbf{H}_{0,\lambda,[0,\infty)}$)

$$(S_\sigma U_hX)(\cdot, \hat{Z}, X, X_0, W) \sim X(\cdot, \hat{Z}, X, X_0, W). \tag{7.20}$$

Proof

We approximate $X(\cdot, \hat{Z}, X, X_0)$ by $Y_{n+1}(\cdot, \hat{Z}, Y_n, X_0)$ with $Y_1 \equiv 0$, as done in the proof of Theorem 6.13. Although this procedure uses the topology of $\mathcal{W}_{0,2,\lambda,[0,\infty)}$, the smoothness of X_0 in addition to (4.2) implies that for each n $Y_{n+1}(\cdot, \hat{Z}, Y_n, X_0) \in \mathcal{H}_{0,\lambda,[0,\infty)}$. So we may apply (7.5), with $\mathbf{W}_{0,2,\lambda,[0,\infty)}$ replaced by $\mathbf{H}_{0,\lambda,[0,\infty)}$. Then (7.5), (7.18) and the independence of X_0 of r imply for any $t \geq 0$

$$\left.\begin{array}{l}(S_\sigma^{\sim}U_h^{\sim}\hat{Z}, (\pi_tS_\sigma U_hY_{n+1})(\cdot, \hat{Z}, Y_n, Y_0, W), S_\sigma^{\sim}U_h^{\sim}Y_n, X_0, S_\sigma^{\sim}U_h^{\sim}W) \\ \sim (\hat{Z}, (\pi_tY_{n+1})(\cdot, \hat{Z}, Y_n, Y_0, W), Y_n, X_0, W).\end{array}\right\} \tag{7.21}$$

Since by (6.35) $Y_n \to X$, $S_\sigma U_hY_n \to S_\sigma U_hX$, $S_\sigma^{\sim}U_h^{\sim}Y_n \to S_\sigma^{\sim}U_h^{\sim}X$ on $\mathcal{W}_{0,2,\lambda,[0,\infty)}$ and since $X \in \mathcal{H}_{0,\lambda,[0,\infty)}$, (7.21) implies (7.19) and (7.20). ∎

Remark 7.4

The presence of the shift invariant $\hat{Z}$ allows to absorb the shift $F(r+h,\hat{Z},\cdot)$ into $F(r,\hat{Z}_h,\cdot)$. This is not necessary for $\mathcal{J}$, since the shift can be absorbed by w, which is shift invariant. So we could simplify the SPDE (1.33) by assuming that $\mathcal{J}$ is independent of $\hat{Z}$. Then $\hat{Z}$ would play the same role for the "slowly varying" forces, modeled by F, as w does for the "rapidly varying" forces, modeled by $\mathcal{J}$, namely the role of a shift absorbing medium (cf. (1.35), (1.36)).

8 Relations to other SPDE's

Under the assumption of nondegeneracy of the diffusion coefficients of the bilinear SPDE (1.32) we show that this SPDE can be considered as the limit of a sequence of SPDE's, which satisfy the variational assumptions. Further, we show that the solution of the bilinear SPDE (1.32) is the solution of a bilinear first order SPDE if we use Stratonovich differentials.

First we want to show the relation to the variational approach of Pardoux (1975) and Krylov and Rozovski (1979). We will confine ourselves to the bilinear SPDE (1.32) with solution $Y(\cdot) := Y(\cdot, Y_0)$ and use the abbreviations from (*) in Section 4.

Proposition 8.1

Suppose (4.1), (4.2), $m \geq 1$, and $Y_0 \in \mathcal{W}_{0,2,1,0}$. Further, assume

$$ess \inf_{w\in\Omega;\ t\in[0,\infty)} \min_{k,\ell=1,\dots d} D_{k\ell}(t,\omega) > 0. \tag{8.1}$$

Then Y is the solution of the following "variational" integral equation:

$$\left.\begin{array}{ll} Y_\alpha(t) & = Y_0 + \frac{1}{2}\int_0^t \sum_{k,\ell=1}^d D_{k\ell}(s)\partial^2_{k\ell}Y_\alpha(s)ds - \int_0^t \triangledown(Y_\alpha(s)F(s))ds \\ & -(1-\alpha)\int_0^t \triangledown(Y_\alpha(s)\int \mathcal{J}(\cdot - p,s))w(dp,ds) \\ & +\varphi_\alpha(t) \end{array}\right\} \tag{8.2}$$

where

$$\varphi_\alpha(t) := -\alpha\int_0^t \triangledown\cdot(Y(s)\int \mathcal{J}(\cdot - p,s))w(dp,ds)$$

and $\alpha \in [0,1)$.

Proof

The Itô-formula (5.2) (or its simpler version for $n=1$ in Kotelenez (1995b), Theorem 4.13) implies that the right hand side of (8.2) without φ_α satisfies the monotonicity and coercivity conditions of Krylov and Rozovski (loc. cit.), Section 2, whence (8.2) has a unique solution Y_α in the variational sense by Theorem 2.1 of Krylov and Rozovski (loc. cit.). But $Y(t)$ is (trivially) a solution of (8.2). So, by uniqueness,

$$Y(t) = Y_\alpha(t) \text{ for all } \alpha\in[0,1). \tag{8.3}$$

■

Next, let "$\circ$" denote Stratonovich integration.

Proposition 8.2

Suppose $\mathcal{Y}_0 \in \mathcal{M}_0$ and let $\mathcal{Y}(\cdot) := \mathcal{Y}(\cdot, \mathcal{Y}_0)$ be the Itô-solution of the bilinear SPDE (1.32). Then $\mathcal{Y}(\cdot)$ is a solution of the bilinear first order SPDE in Stratonovich form:

$$\left.\begin{array}{rl} d\mathcal{Y} = & -\bigtriangledown \cdot (\mathcal{Y}F(t)) - \bigtriangledown \cdot (\mathcal{Y} \int \mathcal{J}(\cdot, p, t)) \circ w(dp, dt) \\ & \mathcal{Y}(0) = \mathcal{Y}_0. \end{array}\right\} \tag{8.4}$$

Proof

Let $r_N(t, r_N(0))$ be the solution of (3.1) at $s = 0$, $r_N = (r^1, \ldots, r^N)$. For sufficiently smooth φ we obtain

$$\left.\begin{array}{rl} d\varphi(r^i(t)) = & \sum_{k=1}^d \partial_k \varphi(r^i(t)) \circ dr^i_k(t) =: \bigtriangledown\varphi(r^i(t)) \circ dr^i(t) \\ & i = 1, \ldots, N \end{array}\right\} \tag{8.5}$$

(cf. Ikeda and Watanabe, loc. cit., Ch. III, 2) (8.5) implies—as in Section 3—that the empirical process $\mathcal{Y}_N(t)$ associated with (3.1) is a solution of (8.4) (and, of course, of (1.32)). The same holds for the extensions.

∎

9 Appendix I

For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbf{Z}^d$ we set

$$\pi_{\mathbf{n}}(r) := \begin{cases} \Pi_{i=1}^d r_i^{n_i}, & \text{if for all } i = 1, \ldots, d \quad n_i \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha \in \mathbf{R}$, then we set $[\alpha] := \max\{k \in \mathbf{Z} : k \leq \alpha\}$. Further, we abbreviate

$$G(u, r) := (2\pi u)^{-\frac{d}{2}} \exp\left(\frac{-|r|^2}{2u}\right), u > 0 \tag{9.1}$$

with the other usual assignments for the cases $u = 0$ and ($|r| = 0$ or $|r| > 0$).

Lemma 9.1

For any $\mathbf{n} \geq \mathbf{0}$ there are finite constants $\gamma_{\mathbf{i}}$, with $\gamma_{\mathbf{0}} = (-1)^{|\mathbf{n}|}$,

$$\partial^{\mathbf{n}} G(u, r) = \sum_{\mathbf{0} \leq 2\mathbf{i} \leq \mathbf{n}} \gamma_{\mathbf{i}} \frac{\pi_{\mathbf{n}-2\mathbf{i}}(r)}{u^{|\mathbf{n}|-|\mathbf{i}|}} G(u, r). \tag{9.2}$$

Proof

We use induction in $|\mathbf{n}|$. (9.2) is obviously correct for $|\mathbf{n}| = 0$. Assume (9.2) is correct for all $\mathbf{n}$ such that $|\mathbf{n}| \leq m - 1$ and $m - 1 \geq 0$. Take $\mathbf{n}$ with $|\mathbf{n}| = m$. Then,

$$\partial^{\mathbf{n}} G(u, r) = \partial_k \partial^{\mathbf{n}-1_k} G(u, r)$$

for some $k \in \{1, \ldots, d\}$

$$= \partial_k \left\{ \sum_{|\mathbf{i}|=0}^{[\frac{|\mathbf{n}|-1}{2}]} \gamma_{\mathbf{i}} \frac{\pi_{\mathbf{n}-1_k-2\mathbf{i}}(r)}{u^{|\mathbf{n}|-1-|\mathbf{i}|}} G(u, r) \right\}$$

(by induction hypothesis)

$$= \sum_{|\mathbf{i}|=0}^{\frac{|\mathbf{n}|-1}{2}} \gamma_{\mathbf{i}} \frac{1}{u^{|\mathbf{n}|-1-|\mathbf{i}|}} \left(c_k \pi_{\mathbf{n}-2\cdot 1_k - 2\mathbf{i}}(r) - \frac{\pi_{\mathbf{n}-1_k-2\mathbf{i}}(r) r_k}{u} \right) G(u,r)$$

(where $c_k := (n_k - 1) \vee 0$)

$$= \sum_{|\mathbf{i}|=0}^{[\frac{\mathbf{n}}{2}]} \tilde{\gamma}_{\mathbf{i}} \frac{\pi_{\mathbf{n}-2\mathbf{i}}(r)}{u^{|\mathbf{n}|-|\mathbf{i}|}} G(u,r)$$

for some finite constants $\tilde{\gamma}_{\mathbf{i}}$ (depending on $\mathbf{n}$), where $\tilde{\gamma}_{\mathbf{0}} = (-1)\gamma_{\mathbf{0}}$. ∎

Lemma 9.2

For any $\mathbf{n} \geq \mathbf{0}$ there are finite constants $\gamma_{\mathbf{i}}$ such that

$$G(u,r)\pi_{\mathbf{n}}(r) = \sum_{\mathbf{0} \leq 2\mathbf{i} \leq \mathbf{n}} u^{|\mathbf{n}|-|\mathbf{i}|} \gamma_{\mathbf{i}} \partial^{\mathbf{n}-2\mathbf{i}} G(u,r). \tag{9.3}$$

Proof

(9.3) is obvious for $|\mathbf{n}| = 0$ and $|\mathbf{n}| = 1$. Assume (9.3) is correct for all $\mathbf{n} \geq \mathbf{0}$ with $|\mathbf{n}| \leq m-1$, $m - 1 \geq 1$. Choose $\mathbf{n}$ with $|\mathbf{n}| = m$. Then, by Lemma 9.1

$$\begin{aligned} & G(u,r)\frac{\pi_{\mathbf{n}}(r)}{u^{|\mathbf{n}|}} - (-1)^{|\mathbf{n}|}\partial^{\mathbf{n}} G(u,r) \\ = & \sum_{\mathbf{0} < 2\mathbf{i} \leq \mathbf{n}} \frac{\gamma_{\mathbf{i}}}{u^{|\mathbf{n}|-|\mathbf{i}|}} \pi_{\mathbf{n}-2\mathbf{i}}(r) G(u,r) \\ = & \sum_{\mathbf{0} < 2\mathbf{i} \leq \mathbf{n}} \frac{\gamma_i}{u^{|\mathbf{n}|-|\mathbf{i}|}} \left(\sum_{0 \leq 2\mathbf{j} \leq \mathbf{n}-2\mathbf{i}} u^{|\mathbf{n}|-2|\mathbf{i}|-|\mathbf{j}|} \tilde{\gamma}_{\mathbf{j}} \partial^{\mathbf{n}-2\mathbf{i}-2\mathbf{j}} G(u,r) \right) \end{aligned}$$

(by induction hypothesis with finite constants $\tilde{\gamma}_{\mathbf{j}}$ and by $|\mathbf{n}-2\mathbf{i}| = |\mathbf{n}| - 2|\mathbf{i}|$ for $\mathbf{0} \leq 2\mathbf{i} \leq \mathbf{n}$)

$$= \sum_{\mathbf{0} < 2\mathbf{k} \leq \mathbf{n}} \beta_{\mathbf{k}} u^{-|\mathbf{k}|} \partial^{\mathbf{n}-2\mathbf{k}} G(u,r)$$

(with $\mathbf{k} = \mathbf{i} + \mathbf{j}$ and $\beta_{\mathbf{k}}$ finite constants). This implies (9.3). ∎

Lemma 9.3

Let $\mathbf{n}, \mathbf{m} \geq \mathbf{0}$. Then there are finite constants $\beta_{\mathbf{k}}$ such that

$$\left(\partial^{\mathbf{m}} G(u,r)\right) \pi_{\mathbf{n}}(r) = \sum_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{m}} \beta_k \partial^{\mathbf{m}-\mathbf{k}} \left(G(u,r)\pi_{\mathbf{n}}(r)\right). \tag{9.4}$$

Proof

(9.4) is correct for $|\mathbf{m}| = 0$ and arbitrary $\mathbf{n} \geq \mathbf{0}$. Suppose (9.4) holds for all $\mathbf{k} \geq 0$ such that $|\mathbf{k}| \leq |\mathbf{m}| - 1$, $|\mathbf{m}| - 1 \geq 0$, and arbitrary $\mathbf{n} \geq 0$. Then,

$$(\partial^{\mathbf{m}} G(u,r))\pi_{\mathbf{n}}(r) = \partial^{\mathbf{m}}(G(u,r)\pi_{\mathbf{n}}(r)) - \sum_{\mathbf{0} < \mathbf{k} \leq \mathbf{m}} \gamma_{\mathbf{k}} (\partial^{\mathbf{m}-\mathbf{k}} G(u,r)) \pi_{\mathbf{n}-\mathbf{k}}(r),$$

where $\gamma_{\mathbf{k}}$ are finite constants. Using the induction hypothesis finishes the proof. ∎

Corollary 9.4

Let $\mathbf{n}, \mathbf{m} \geq \mathbf{0}$. Then there are finite constants $\gamma_{\mathbf{i}}$ such that

$$(\partial^{\mathbf{m}} G(u,r))\pi_{\mathbf{n}}(r) = \sum_{\mathbf{0} \leq 2\mathbf{i} \leq \mathbf{n}} u^{|\mathbf{n}|-|\mathbf{i}|} \sum_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{m}} \gamma_{\mathbf{i}} \beta_{\mathbf{k}} \partial^{\mathbf{m}+\mathbf{n}-\mathbf{k}-2\mathbf{i}} G(u,r). \tag{9.5}$$

Proof

(9.3) and (9.4). ∎

Addendum 9.5—Proof of Theorem 4.3

(i) Assume for notational simplicity $s = 0$. Note that $Y_0 \in \mathcal{W}_{0,2,1,0}$ implies by a simple generalization of Kotelenez (1995b), Theorem 4.11, $Y(\cdot, Y_0) \in \mathcal{W}_{0,2,1,[0,T]}$. Throughout the proof let $\mu \geq 1$, and let $(\mu - \frac{1}{2}\triangle)^{-1}$ be the resolvent of $\frac{1}{2}\triangle$ at μ. Set $R_\mu := \mu(\mu - \frac{1}{2}\triangle)^{-1}$ and $G(t,r) := (2\pi t)^{d/2} e^{-\frac{|r|^2}{2t}}$. The Hausdorff-Young inequality implies that the heat semigroup $T(t)$, defined for "suitable" f by

$$(T(t)f)(r) := \int G(t, r-q) f(q) dq \ ,$$

is a strongly continuous semigroup on $\mathbf{W}_{n,p,\Phi}$, $n \in \mathbf{N} \cup \{0\}$, $p \in [1,\infty)$, $\Phi \in \{1, \lambda\}$. The n-th power of the resolvent of $\frac{1}{2}\triangle$ has the representation

$$(\mu - \frac{1}{2}\triangle)^{-n} = \frac{1}{(n-1)!} \int_0^\infty u^{n-1} e^{-\mu u} T(u) du \tag{9.6}$$

(cf. Pazy (1983)), and we set

$$R_\mu^n := \mu^n (\mu - \frac{1}{2}\triangle)^{-n}.$$

The definition of R_μ^n and the Hausdorff-Young inequality imply for $f \in \mathbf{W}_{0,p,\Phi}$, $p \geq 2$

$$\|R_\mu^n f\|_{0,p,\Phi} \leq c_\Phi \|f\|_{0,p,\Phi}, \tag{9.7}$$

with $c_\Phi = 1$, if $\Phi \equiv 1$. By Triebel (1978) (cf. also Kotelenez (1995b)) for any $n > \frac{d}{2} + 1$

$$\mathbf{M}_a \subset \mathbf{H}_{-n} \tag{9.8}$$

with continuous inclusion, where $\mathbf{H}_{-n}$ is the strong dual of $\mathbf{H}_n$. For $\frac{m}{2} \geq n$ the extension of $R_\mu^{\frac{m}{2}}$ to $\mathbf{H}_{-n}$ is a bounded linear operator from $\mathbf{H}_{-n}$ into $\mathbf{H}_{m-n}$. (cf. Kotelenez (1985)). Abbreviate

$$\overline{m} := \min\{n \in \mathbf{N} : n \text{ even and } n > m + 3 + d\}. \tag{9.9}$$

Let $f \in C_c^{\overline{m}}(\mathbf{R}^d; \mathbf{R})$. The definition of $\| \cdot \|_{i,p,\Phi}$, the Sobolev imbedding theorem and an elementary calculation imply for $i = 0,1,2$, $p \geq 2$ and $\tilde{m} := \overline{m} - \frac{d}{2} - 1$

$$\begin{aligned} \|f\|_{i,p,\Phi}^p \leq \|f\|_{i,p}^p &\leq |||f|||_i^{p-2} \cdot \|f\|_i^2 \\ &\leq c_2 \|f\|_{\tilde{m}-2+i-m}^{p-2} \cdot \|f\|_i^2 \\ &\leq c_2 \|f\|_{\tilde{m}-2+i-m}^p. \end{aligned}$$

Hence, for $i = 0, 1, 2$ and $p \geq 2$

$$\mathbf{H}_{\tilde{m}-m} \subset \mathbf{H}_{\tilde{m}-2+i-m} \subset \mathbf{W}_{i,p,\Phi} \tag{9.10}$$

with continuous (and dense) imbeddings. Further, let $\mathbf{m} \geq \mathbf{0}$ such that $|\mathbf{m}| = m$. For $0 \leq \mathbf{n} \leq \mathbf{m}$ we set

$$Q_\mu^{\mathbf{n}} := \partial^{\mathbf{n}} R_\mu^{\frac{\overline{m}}{2}}, Y_{\mu,\mathbf{n}}(t) := Q_\mu^{\mathbf{n}} Y(t).$$

We obtain (on $\mathbf{H}_0$–cf. Kotelenez (1995b))

$$\begin{aligned} Y_{\mu,\mathbf{n}}(t) &= Y_{\mu,\mathbf{n}}(0) + \int_0^t \frac{1}{2} \sum_{k,\ell=1}^d D_{k\ell}(s) \partial_{k\ell}^2 Y_{\mu,\mathbf{n}}(s) ds \\ &- \int_0^t \bigtriangledown \cdot Q_\mu^{\mathbf{n}}(Y(s)F(s)) ds - \int_0^t \bigtriangledown \cdot Q_\mu^{\mathbf{n}}(Y(s) dM(s)). \end{aligned} \tag{9.11}$$

We easily verify that $Y_{\mu,\mathbf{n}}(\cdot) \in C([0,\infty); \mathbf{H}_{\overline{m}-m-2})$ a.s. (since $Q_\mu^{\mathbf{n}} \in \mathcal{L}(\mathbf{H}_0, \mathbf{H}_{\overline{m}-m})$ and $Y(\cdot) \in \mathcal{W}_{0,2,1,[0,T]}$), whence $\|Y_{\mu,\mathbf{n}}(\cdot)\|_{\overline{m}-m-2} \in C([0,\infty); \mathbf{R})$ a.s. Moreover, by the choice of $\overline{m}$ and $\tilde{m}$ there is a $c_\mu < \infty$ such that uniformly in (t,ω), $\mathbf{0} \leq \mathbf{n} \leq \mathbf{m}$

$$\|Y_{\mu,\mathbf{n}}(t)\|_{\tilde{m}-m-2} \leq c_\mu \|Y(t)\|_{-\alpha}, \quad \text{for some} \quad \alpha > \frac{d}{2} + 1.$$

Since $Y(t,\omega) \in \mathbf{M}_{0,b}$ for some $b > 0$ (9.8) implies

$$\max_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{m}} ess \sup_\omega \sup_{t \geq s} \|Y_{\mu,\mathbf{n}}(t,\omega)\|_{\tilde{m}-m-2} \leq c_{\mu,b,\rho} < \infty. \tag{9.12}$$

Then by (9.10) $\|Y_{\mu,\mathbf{n}}(t,\omega)\|_{0,p,\Phi}^p$ is integrable with respect to $dt \otimes dP$. Hence, we may apply Itô's formula:

$$\left.\begin{array}{rl} \|Y_{\mu,\mathbf{m}}(t)\|_{0,p,\Phi}^p = & \|Y_{\mu,\mathbf{m}}(0)\|_{0,p,\Phi}^p \\ + & \frac{p}{2} \sum_{k,\ell=1}^d \int_0^t \langle Y_{\mu,\mathbf{m}}^{p-1}(s), \partial_{k\ell}^2 Y_{\mu,\mathbf{m}}(s) \rangle_{0,\Phi} D_{k\ell}(s) ds \\ - & p \int_0^t \langle Y_{\mu,\mathbf{m}}^{p-1}(s), \bigtriangledown \cdot Q_\mu^{\mathbf{m}}(Y(s)F(s)) \rangle_{0,\Phi} ds \\ - & p \int_0^t \langle Y_{\mu,\mathbf{m}}^{p-1}(s), \bigtriangledown \cdot Q_\mu^{\mathbf{m}}(Y(s) dM(s)) \rangle_{0,\Phi} \\ + & \frac{p(p-1)}{2} \int_0^t \langle Y_{\mu,\mathbf{m}}^{p-2}(s), [\bigtriangledown \cdot Q_\mu^{\mathbf{m}}(Y(s) dM(s))] \rangle_{0,\Phi} \\ =: & \sum_{i=1}^5 A_{i,\mathbf{m}}(t), \end{array}\right\} \tag{9.13}$$

where we now assumed p to be an even integer ≥ 2. We now specialize to $p = 2$ and $\Phi \equiv 1$.

(ii) Following Kotelenez (1995b) we first bound

$$\sum_{\mathbf{0} \leq \mathbf{n} \leq \mathbf{m}} (A_{2,\mathbf{n}}(t) + A_{5,\mathbf{n}}(t)) \ .$$

Integration by parts implies

$$\left.\begin{array}{rl} & [\int_0^t \bigtriangledown Q_\mu^{\mathbf{n}}(Y(s) dM(s))] \\ = & \sum_{k,\ell=1}^d \frac{\mu^{\overline{m}}}{(\overline{m}-1)!} \int_0^t \int_0^\infty du u^{\overline{m}-1} e^{-\mu u} (-1)^{|\mathbf{n}|} \cdot \\ & \cdot \int \int (\partial_{k\ell,r}^2 \partial_r^{2\mathbf{n}} G(u, r-p)) Y(s,p) Y(s,r) \{\hat{D}_{k\ell}(s, r-p) - D_{k\ell}(s)\} dp dr ds \\ =: & \sum_{k,\ell=1}^d \frac{\mu^{\overline{m}}}{(\overline{m}-1)!} \int_0^t \int_0^\infty du u^{\overline{m}-1} e^{-\mu u} \{B_{5,1}(s,u) - B_{5,2}(s,u)\} ds. \end{array}\right\} \tag{9.14}$$

We easily see that

$$A_{2,\mathbf{n}}(t) + \sum_{k,\ell=1}^{d} \frac{\mu^{\overline{m}}}{(\overline{m}-1)!} \int_0^t \int_0^\infty du u^{\overline{m}-1} e^{-\mu u} B_{5,2}(s,u) ds = 0. \tag{9.15}$$

Next we define for $\mathbf{n} \geq \mathbf{0}$ with $|\mathbf{n}| \leq 2(m+1)$

$$\mathcal{L}_{k\ell,\mathbf{n}}(s) := \partial^{\mathbf{n}} \hat{D}_{k\ell}(s,r)|_{r=0}. \tag{9.16}$$

Since $\hat{D}_{k\ell}(s,r) = \hat{D}_{k\ell}(s,-r)$, $\mathcal{L}_{k\ell,\mathbf{n}}(s) = 0$, if $|\mathbf{n}|$ is odd. Taylor's expansion yields

$$\hat{D}_{k\ell}(s,r) = \sum_{i=1}^{m} \sum_{|\mathbf{j}|=2i} \frac{\mathcal{L}_{k\ell,\mathbf{j}}(s)\pi_{\mathbf{j}}(r)}{|\mathbf{j}|!} + R_{2(\mathbf{m}+1)}(s,r). \tag{9.17}$$

We first estimate the terms coming from the substitution of $\mathcal{L}_{k\ell,\mathbf{j}}(s)\pi_{\mathbf{j}}(r)$ for $\hat{D}_{k\ell}(s,k)$ in (9.14).

Note that by assumption

$$\max_{0\leq|\mathbf{n}|\leq 2(m+1)} ess\sup_{\omega} \sup_{0\leq s\leq T} |\mathcal{L}_{k\ell,\mathbf{n}}(s,\omega)| < \infty \text{ for any } T > 0. \tag{9.18}$$

Further, by (9.5)

$$(\partial^2_{k\ell}\partial^{2\mathbf{n}} G(u,r))\pi_{\mathbf{j}} = \sum_{\mathbf{0}\leq 2\mathbf{i}\leq\mathbf{j}} u^{|\mathbf{j}|-|\mathbf{i}|} \sum_{\mathbf{0}\leq\tilde{\mathbf{n}}\leq 2\mathbf{n}+1_k+1_\ell} \gamma_{i,\tilde{\mathbf{n}}} \partial^{2\mathbf{n}+1_k+1_\ell+\mathbf{j}-2\mathbf{i}-\tilde{\mathbf{n}}} G(u,r). \tag{9.19}$$

Suppose there are $\hat{\mathbf{n}}, \hat{\mathbf{m}} \geq \mathbf{0}$ such that $|\hat{\mathbf{n}}| \leq |\mathbf{j}|$, $|\hat{\mathbf{m}}| \leq |\mathbf{m}|$ and $2\mathbf{i} \leq \mathbf{j}$. Then, $|\hat{\mathbf{n}}+\mathbf{j}-2\mathbf{i}| \leq |\hat{\mathbf{n}}|+|\mathbf{j}|-2|\mathbf{i}| \leq 2|\mathbf{j}|-2|\mathbf{i}|$. Hence, (by extension)

$$\sup_{\mu\geq 1} \|\partial^{\hat{\mathbf{n}}+\mathbf{j}-2\mathbf{i}}(\mu - \frac{1}{2}\triangle)^{-(|\mathbf{j}|-|\mathbf{i}|)}\|_{\mathcal{L}(H_0)} \leq c_{\hat{\mathbf{n}},\mathbf{j},\mathbf{i}} < \infty. \tag{9.20}$$

Let us abbreviate the multiindex in the right hand side of (9.19)

$$\tilde{\mathbf{m}} := 2\mathbf{n} + 1_k + 1_\ell + \mathbf{j} - 2\mathbf{i} - \tilde{\mathbf{n}} \tag{$*$}$$

Then,

$$|\tilde{\mathbf{m}}| - 2|\mathbf{n}| \leq 2 + |\mathbf{j}| - 2|\mathbf{i}|,$$

whence there is an $\hat{\mathbf{n}} \geq 0$, with $|\hat{\mathbf{n}}| \leq 2(\leq |\mathbf{j}|)$, and $\hat{\mathbf{m}} \geq 0$ with $|\hat{\mathbf{m}}| \leq |\mathbf{m}|$ such that

$$\begin{aligned} & \frac{\mu^{\overline{m}}}{(\overline{m}-1)!} \int_0^\infty u^{|\mathbf{j}|-|\mathbf{i}|+\overline{m}-1} e^{-\mu u} \int \partial_r^{\tilde{\mathbf{m}}} G(u, r-q) Y(s,q) dq \\ = \; & c_{\overline{m},\hat{\mathbf{m}}} \partial^{\hat{\mathbf{n}}+\mathbf{j}-2\mathbf{i}} (\mu - \frac{1}{2}\triangle)^{-(|\mathbf{j}|-|\mathbf{i}|)} \partial^{\mathbf{n}} R_\mu^{\overline{m}/2} Y_{\mu,\hat{\mathbf{m}}}(s). \end{aligned}$$

So substituting $\mathcal{L}_{k\ell,\mathbf{j}}(s)\pi_{\mathbf{j}}(r)$ for $\hat{D}_{k\ell}(s,r)$ in (9.14) yields

$$\begin{aligned}
&\frac{\mu^{\overline{m}}}{(\overline{m}-1)!}\int_0^t\int_0^\infty duu^{\overline{m}-1}e^{-\mu u}\mathcal{L}_{k\ell,\mathbf{j}}(s)\int\int(\partial^2_{k\ell,q}\partial_q^{2\mathbf{n}}G(u,q-p))Y(s,p)Y(s,q)\pi_{\mathbf{j}}(q-p)dpdqds\\
=\ & c_{\overline{m},\hat{\mathbf{m}}}\sum_{\mathbf{0}\le 2\mathbf{i}\le\mathbf{j}}(-1)^{|\mathbf{n}|}\int_0^t\mathcal{L}_{k\ell,\mathbf{j}}\langle\partial^{\hat{\mathbf{n}}+\mathbf{j}-2\mathbf{i}}(\mu-\frac{1}{2}\triangle)^{-(|\mathbf{j}|-|\mathbf{i}|)}Y_{\mu,\hat{\mathbf{m}}}(s),Y_{\mu,\mathbf{n}}(s)\rangle_0 ds\\
\le\ & \overline{c}_{\overline{m},\hat{\mathbf{m}}}\int_0^t\|Y_{\mu,\hat{\mathbf{m}}}(s)\|_0\cdot\|Y_{\mu,\mathbf{n}}(s)\|_0 ds
\end{aligned}$$

by (9.18) and (9.20) with $|\hat{\mathbf{m}}|\le|\mathbf{m}|$. This implies

$$\left.\begin{aligned}
&|\tfrac{\mu^{\overline{m}}}{(\overline{m}-1)!}\textstyle\int_0^t\int_0^\infty duu^{\overline{m}-1}e^{-\mu u}\int\int(\partial^2_{k\ell,q}\partial_q^{2\mathbf{n}}G(u,q-p))Y(s,p)Y(s,q)\cdot\\
&\quad\cdot(\hat{D}_{k\ell}(s,q-p)-R_{2(\mathbf{m}+1)}(s,q-p))dpdqds\,|\\
\le\ &\tilde{c}_m\textstyle\int_0^t\sum_{|\mathbf{i}|\le|\mathbf{m}|}\|Y_{\mu,i}(s)\|_0^2 ds\ .
\end{aligned}\right\}\qquad(9.21)$$

(iv) Set

$$\hat{\mathbf{n}}:=2\mathbf{n}+1_k+1_\ell.\qquad(**)$$

Then, $|\hat{\mathbf{n}}|\le 2|\mathbf{m}|+2$. We have the following decomposition:

$$R_{2(\mathbf{m}+1)}(s,r)=\alpha(s,r)\sum_{|\tilde{\mathbf{m}}|=|\mathbf{m}|}\pi_{2(\tilde{\mathbf{m}}+1)}(r),\qquad(9.22)$$

where by assumption

$$ess\sup_{\omega,r;s\in[0,T]}|\alpha(s,r,\omega)|<\infty\text{ for any }T>0.\qquad(9.23)$$

We can factor $\pi_{2(\tilde{\mathbf{m}}+1)}(r)$ by $\pi_{\tilde{\mathbf{n}}}(r)\cdot\pi_{\mathbf{j}}(r)$ with $|\tilde{\mathbf{n}}|=|\hat{\mathbf{n}}|$ and $|\mathbf{j}|\ge 0$. Then, by (9.2) and

$$\sup_{r;u\ge 0}|\frac{\pi_{\mathbf{k}}(r)}{u^{|\mathbf{k}|/2}}\exp(-\frac{|r|^2}{u})|\le c_{\mathbf{k}}<\infty\qquad(9.24)$$

for any $\mathbf{k}$, we obtain

$$|(\partial^{\hat{\mathbf{n}}}G(u,r))\pi_{2(\tilde{\mathbf{m}}+1)}(r)|\le c_m u^{\frac{|\mathbf{j}|}{2}}G(2u,r).\qquad(9.25)$$

So (with some finite constants $\gamma_{\mathbf{j}}$)

$$\left.\begin{aligned}
&|\textstyle\sum_{k,\ell=1}^d\frac{\mu^{\overline{m}}}{(\overline{m}-1)!}\int_0^t\int_0^\infty duu^{\overline{m}-1}e^{-\mu u}\int\int(\partial^2_{k\ell,q}\partial_q^{2\mathbf{n}}G(u,q-p)Y(s,p))Y(s,q)\cdot\\
&\quad\cdot R_{2(\mathbf{m}+1)}(s,q-p)dpdqds|\\
\le\ &\textstyle\sum_{|\mathbf{j}|\le 2|\mathbf{m}|+1}\gamma_{\mathbf{j}}\int_0^t\frac{\mu^{\overline{m}}}{(\overline{m}-1)!}\int_0^\infty duu^{\frac{|\mathbf{j}|}{2}+\overline{m}-1}e^{-\mu u}\int\int G(2u,q-p)|Y(s,p)||Y(s,q)|dpdqds\\
\le\ &\overline{c}_{\mathbf{m}}\textstyle\int_0^t\|R_\mu^{\overline{m}/2}|Y(s)|\,\|_0^2 ds.
\end{aligned}\right\}\qquad(9.26)$$

Altogether, we obtain

$$|\sum_{\mathbf{0}\le\mathbf{n}\le\mathbf{m}}A_{2,\mathbf{n}}(t)+A_{5,\mathbf{n}}(t)|\le c_{\mathbf{m}}\int_0^t(\sum_{|\mathbf{i}|\le|\mathbf{m}|}\|Y_{\mu,\mathbf{i}}(s)\|_0^2+\|R_\mu^{\overline{m}/2}|Y(s)|\,\|_0^2)ds.\qquad(9.27)$$

(v) Now we estimate $\sum_{\mathbf{n}\le\mathbf{m}} A_{3,\mathbf{n}}$.
Taylor's expansion yields

$$F_\ell(s,q) = F_\ell(s,r) + \sum_{|\mathbf{j}|=1}^{|\mathbf{m}|} \gamma_{\mathbf{j}}(\partial^{\mathbf{j}} F_\ell(s,r))\pi_{\mathbf{j}}(q-r) + R_{|\mathbf{m}|+1}(s,q-r). \tag{9.28}$$

Then, for $\mathbf{n}\le\mathbf{m}$ (cf. Kotelenez (1995b), (4.11))

$$\langle Y_{\mu,\mathbf{n}}(s), (\partial_\ell Y_{\mu,\mathbf{n}}(s))F_\ell(s)\rangle_0 = -\frac{1}{2}\langle Y^2_{\mu,\mathbf{n}}(s), \partial_\ell F_\ell(s)\rangle_0. \tag{9.29}$$

Next, we use (9.5) to derive a convenient representation of $(\partial_\ell \partial^{\mathbf{n}} G(u,r))\pi_{\mathbf{j}}$ as in (9.19). We abbreviate (cf. (9.19) and (*))

$$\tilde{\mathbf{m}} := \mathbf{n} + 1_\ell + \mathbf{j} - 2\mathbf{i} - \tilde{\mathbf{n}}. \tag{$***$}$$

Then we have

$$|\tilde{\mathbf{m}}| - |\mathbf{n}| \le 1 + |\mathbf{j}| - 2|\mathbf{i}|\ ,$$

whence there is an $\hat{\mathbf{n}}\ge\mathbf{0}$ with $|\hat{\mathbf{n}}|\le 1$ and $\hat{\mathbf{m}}\ge\mathbf{0}$, with $|\hat{\mathbf{m}}|\le|\mathbf{m}|$, such that (cf. (9.20))

$$\partial^{\tilde{\mathbf{m}}} R_\mu^{\frac{\overline{m}}{2}} Y(s) = c_{\overline{m},\hat{\mathbf{m}}}\partial^{\hat{\mathbf{n}}+\mathbf{j}-2\mathbf{i}}(\mu - \frac{1}{2}\triangle)^{-(|\mathbf{j}|-|\mathbf{i}|)} Y_{\mu,\hat{\mathbf{m}}}(s). \tag{9.30}$$

Since $|\mathbf{j}|\ge|\hat{\mathbf{n}}|$ and $\mathbf{0}\le 2\mathbf{i}\le\mathbf{j}$, we have $|\hat{\mathbf{n}}+\mathbf{j}-2\mathbf{i}|\le 2|\mathbf{j}|-2|\mathbf{i}|$. So substituting $(\partial^{\mathbf{j}}F_\ell(s,r))\pi_{\mathbf{j}}(q-r)$ for $F_\ell(s,q)$ in $A_{3,\mathbf{n}}(t)$ and using (9.21) yields

$$\left.\begin{array}{rl}
& \frac{\mu^{\overline{m}}}{((\overline{m}/2-1)!)^2}\int_0^t\int_0^\infty\int_0^\infty dudvu^{\frac{\overline{m}}{2}-1}v^{\frac{\overline{m}}{2}-1}e^{-\mu(u+v)}\cdot \\
& \quad \int\int\int(\partial_r^{\mathbf{n}}G(u,r-p)Y(s,p))(\partial_{\ell,r}\partial_r^{\mathbf{n}}G(v,r-q))Y(s,q)(\partial^{\mathbf{j}}F_\ell(s,r))\pi_{\mathbf{j}}(q-r)dqdpdrds \\
= & c_{\overline{m},\hat{\mathbf{m}}}\langle(\partial^{\mathbf{j}}F_\ell(s))Y_{\mu,\mathbf{n}}(s), \partial^{\hat{\mathbf{n}}+\mathbf{j}-2\mathbf{i}}(\mu-\frac{1}{2}\triangle)^{-(|\mathbf{j}|-|\mathbf{i}|)}Y_{\mu,\hat{\mathbf{m}}}(s)\rangle_0 \\
\le & \bar{c}_{\overline{m},\hat{\mathbf{m}}}|||F_\ell(s)|||_{|\mathbf{j}|}\cdot\|Y_{\mu,\mathbf{n}}(s)\|_0\cdot\|Y_{\mu,\hat{\mathbf{m}}}(s)\|_0.
\end{array}\right\} \tag{9.31}$$

Finally, we write

$$R_{|\mathbf{m}|+1}(s,r) = \beta(s,r)\sum_{|\hat{\mathbf{m}}|=|\mathbf{m}|}\pi_{\hat{\mathbf{m}}+1}(r), \tag{9.32}$$

where by assumption

$$ess \sup_{\omega,r;s\in[0,T]} |\beta(s,r,\omega)| < \infty \text{ for any } T>0. \tag{9.33}$$

Now, as in (9.26),

$$\left.\begin{array}{rl}
& |\sum_{\ell=1}^d \frac{\mu^{\overline{m}}}{((\overline{m}/2-1)!)^2}\int_0^t\int_0^\infty\int_0^\infty dudvu^{\frac{\overline{m}}{2}-1}v^{\frac{\overline{m}}{2}-1}e^{\mu(u+v)}\cdot \\
& \quad \cdot\int\int\int(\partial_r^{\mathbf{n}}G(u,r-p))Y(s,p)(\partial_{\ell,r}\partial_r^{\mathbf{n}}G(v,r-q))Y(s,q)R_{|\mathbf{m}|+1}(s,q-r)dqdpdrds| \\
\le & \sum_{|\mathbf{j}|\le|\mathbf{m}|+1}\gamma_{\mathbf{j}}\int_0^t\frac{\mu^{\overline{m}/2}}{((\overline{m}/2-1)!)^2}\int_0^\infty dv\ v^{\frac{|\mathbf{j}|}{2}+\frac{\overline{m}}{2}-1}e^{-\mu v}\int\int G(2v,r-p)|Y(s,p)||Y_{\mu,\mathbf{n}}(s,r)|dpdrds \\
\le & c_{\mathbf{m}}\int_0^t\|R_\mu^{\frac{\overline{m}}{2}}|Y(s)|\ \|_0\cdot\|Y_{\mu,\mathbf{n}}(s)\|_0 ds.
\end{array}\right\} \tag{9.34}$$

Altogether, (the substitution of $F_\ell(s,r)$ for $F_\ell(s,q)$ implying an estimate similar to (9.31) – cf. Kotelenez (1995b, (4.1)))

$$|\sum_{\mathbf{n}\le\mathbf{m}} A_{3,\mathbf{n}}(t)| \le \hat{c}_{\mathbf{m}} \int_0^t |\sum_{|\mathbf{i}|\le|\mathbf{m}|} \|Y_{\mu,\mathbf{i}}(t)\|_0^2 + \|R_\mu^{\frac{\overline{m}}{2}}|Y(s)|\ \|_0^2)ds. \tag{9.35}$$

(vi) The previous steps imply

$$\left.\begin{aligned} E\sum_{\mathbf{n}\le\mathbf{m}}\|Y_{\mu,\mathbf{n}}(t)\|_0^2 \le &\quad c_{\mathbf{m}} E\sum_{\mathbf{n}\le\mathbf{m}}\|Y_{\mu,\mathbf{n}}(0)\|_0^2 \\ &+ \quad c_{\mathbf{m}}\int_0^t (E\sum_{\mathbf{n}\le\mathbf{m}}\|Y_{\mu,\mathbf{n}}(s)\|_0^2 + E\|R_\mu^{\frac{\overline{m}}{2}}|Y(s)|\ \|_0^2)ds, \end{aligned}\right\} \tag{9.36}$$

with $c_{\mathbf{m}} = c_{\mathbf{m}}(F_\ell, \hat{D}_{k\ell}, k, \ell = 1, \ldots, d) < \infty$. Since by Kotelenez (1995b) and (9.7)

$$\sum_{0\le s\le T} E\|R_\mu^{\frac{\overline{m}}{2}}|Y(s)|\ \|_0^2 \le c_T E\|Y(0)\|_0^2, \tag{9.37}$$

the definition of $\|\cdot\|_m$ and the Gronwall lemma imply

$$\sup_{0\le t\le T} E\|R_\mu^{\frac{\overline{m}}{2}} Y(t)\|_m^2 \le e^{c_{\mathbf{m}}T}\{E\|R_\mu^{\frac{\overline{m}}{2}} Y(0)\|_m^2 + c_T E\|Y(0)\|_0^2\}. \tag{9.38}$$

Fatou's lemma implies (as $\mu\to\infty$) (4.5). The statement (4.4) is a simple consequence of the fact that $\sup_{0\le t\le T} E\|f(t)\|_m^2$ is stronger than $\int_0^T E\|f(t)\|_m^2 dt$ for suitable $f(\cdot)$ and the $dP\otimes dt$-measurability of $R_\mu^{\frac{\overline{m}}{2}} Y(\cdot)$ as an $\mathbf{H}_m$-valued process. ∎

Let $\overline{m}$ be given by (9.9), where $m\in\mathbf{N}\cup\{0\}$.

Lemma 9.6
For any $p\ge 2$ and $\Phi\in\{1,\lambda\}$

$$\mathcal{M}_s \cap \mathcal{W}_{\overline{m},2,1,s} \subset \mathcal{W}_{m,p,\Phi,s} \tag{9.39}$$

with dense and continuous imbedding.

Proof
(i) For notational simplicity let $s=0$ and $Y_0\in\mathcal{W}_{m,p,\Phi,0}$. Choose $\eta_N\in C^{\overline{m}+1}(\mathbf{R};\mathbf{R})$ such that

$$\eta_N(x) := \begin{cases} x, & |x|\le N, \\ N+1, & x > N+1, \\ -(N+1), & -x > N+1 \end{cases}$$

and such that

$$\max_{1\le n\le\overline{m}+1} \sup_N \sup_{x\in\mathbf{R}} |\frac{d^n}{dx^n}\eta_N(x)| < \infty.$$

Set

$$\tilde{\eta}_N(r) := \frac{d}{dx}\eta_N(x)_{|x=|r|}.$$

Next, define for $M\in\mathbf{N}$ mollifiers $\varphi_M\in C^\infty(\mathbf{R}^d;\mathbf{R})$ through

$$\varphi_M(r) := \begin{cases} cM^d \exp\{\frac{-1}{1-|r|^2M^2}\}, |r| < \frac{1}{M}, \\ 0, \quad |r| > \frac{1}{M}, \end{cases}$$

where $c := \|\varphi_1\|_{0,1}^{-1}$ (with $\varphi_1 := \varphi_{M|M=1}$). Set

$$Y_{0,N} := Y_0 \cdot \tilde{\eta}_N$$
$$Y_{0,N,M}(r,\omega) := \varphi_M * Y_{0,N}(\omega)(r),$$

where "$*$" denotes the convolution. By the boundedness of $\tilde{\eta}_N$ and the fact that $\tilde{\eta}_N$ has compact support as well as by the Hausdorff-Young and Hölder inequalities

$$E(\int |Y_{0,N,M}(r)|dr)^2 \le c_N(E\int |Y_0|^p(r)dr)^{\frac{2}{p}}. \tag{9.40}$$

Since by assumption the right hand side of (9.40) is finite

$$Y_{0,N,M} \in \mathcal{M}_0. \tag{9.41}$$

Define finite constants by

$$\begin{aligned} c_{\Phi,N} &:= \sup_{|r|\le N+1} \Phi^{-1/p}(r); \\ c_{\overline{m},q,N} &:= \begin{cases} \|\tilde{\eta}_N\|_{\overline{m},\frac{p}{p-q}\cdot q}, & \text{if } q \in [1,p) \\ |||\tilde{\eta}_N|||_{\overline{m}}, & \text{if } q = p; \end{cases} \\ c_{\overline{m},N} &:= \sup_{1\le q\le p} c_{\overline{m},q,N}. \end{aligned}$$

Let $\mathbf{i}$ be a multiindex with $|\mathbf{i}| \le \overline{m}$.

Then, $1 \le c_{\Phi,N}^q \Phi^{\frac{q}{p}}(r)$ on $\{|r| \le N+1\}$, the support of $\partial^{\mathbf{i}}\tilde{\eta}_N$. Hölder's inequality for $q < p$ and the definition of $c_{\overline{m},N}$ imply for $q \in [1,p]$ and $f \in \mathbf{W}_{0,p,\Phi}$

$$\|f \cdot \partial^{\mathbf{i}}\tilde{\eta}_N\|_{0,q} \le c_{\Phi,N} c_{\overline{m}} \|f\|_{0,p,\Phi}. \tag{9.42}$$

Now let $\mathbf{m}$ be a multiindex with $|\mathbf{m}| \le m$ and $\mathbf{j}$ be an arbitrary multiindex. Integration by parts implies

$$\partial^{\mathbf{j}+\mathbf{m}}\varphi_M * Y_{0,N} = \partial^{\mathbf{j}}\varphi_M * \partial^{\mathbf{m}} Y_{0,N} \tag{$*$}$$

and

$$\partial^{\mathbf{m}} Y_{0,N} = \sum_{\mathbf{i}\le\mathbf{m}} \partial^{\mathbf{m}-\mathbf{i}} Y_0 \cdot \partial^{\mathbf{i}}\tilde{\eta}_N. \tag{$**$}$$

Choosing $f := f_{\mathbf{i}} := \partial^{\mathbf{m}-\mathbf{i}} Y_0(\omega)$, (9.42) and the Hausdorff-Young inequality imply for $q \in [1,p]$

$$\|\partial^{\mathbf{j}+\mathbf{m}} Y_{0,N,M}(\omega)\|_{0,q}^q \le (c_{\Phi,N} c_{\overline{m},N})^q \|\partial^{\mathbf{j}}\varphi_M\|_{0,1}^q \cdot \sum_{\mathbf{i}\le\mathbf{m}} \|\partial^{\mathbf{i}} Y_0\|_{0,p,\Phi}^q. \tag{9.43}$$

Hence for $q \in [1,p]$

$$E\|Y_{0,N,M}\|_{\overline{m},q}^q \le (c_{\Phi,N,\overline{m},p})^q \cdot \|\varphi_M\|_{\overline{m}-m,1}^q \cdot (E\|Y_0\|_{m,p,\Phi}^p)^{\frac{q}{p}}. \tag{9.44}$$

Again, by assumption the right hand side of (9.44) is finite, whence

$$Y_{0,N,M} \in \mathcal{W}_{\overline{m},p,\Phi}. \tag{9.45}$$

Let $\varepsilon > 0$ be given. By (9.42) and $(**)$ $Y_{0,N} \in \mathcal{W}_{m,p,\Phi}$ and by Lebesgue's dominated convergence theorem there is an $N(\varepsilon)$ such that for $N \geq N(\varepsilon)$

$$(E\|Y_{0,N} - Y_0\|_{m,p,\Phi}^p)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}. \tag{9.46}$$

Let $\mathbf{m}$ be a multiindex with $|\mathbf{m}| \leq m$. By $(*)$ and $(**)$

$$\partial^{\mathbf{m}} Y_{0,N,M} - \partial^{\mathbf{m}} Y_{0,N} = \sum_{\mathbf{n}+\mathbf{i}=\mathbf{m}} (\varphi_M * \partial^{\mathbf{n}} Y_0 \partial^{\mathbf{i}} \tilde{\eta}_N - \partial^{\mathbf{n}} Y_0 \partial^{\mathbf{i}} \tilde{\eta}_N).$$

Set $f_{\mathbf{n}} := \partial^{\mathbf{n}} Y_0$, $\tilde{\eta}_{N,\mathbf{i}} := \partial^{\mathbf{i}} \tilde{\eta}_N$. Then

$$\left.\begin{array}{rl} \varphi_M * f_{\mathbf{n}}\tilde{\eta}_{N,\mathbf{i}} - f_{\mathbf{n}}\tilde{\eta}_{N,\mathbf{i}} = & \varphi_M * (f_{\mathbf{n}}\tilde{\eta}_{N,\mathbf{i}} - \eta_N(f_{\mathbf{n}})\tilde{\eta}_{N,\mathbf{i}}) \\ + & \varphi_M * (\eta_N(f_{\mathbf{n}})\tilde{\eta}_{N,\mathbf{i}}) - \eta_N(f_{\mathbf{n}})\tilde{\eta}_{N,\mathbf{i}} \\ + & \eta_N(f_{\mathbf{n}})\tilde{\eta}_{N,\mathbf{i}} - f_{\mathbf{n}}\tilde{\eta}_{N,\mathbf{i}} \end{array}\right\} \tag{9.47}$$

$$=: \sum_{\ell=1}^{3} A_{M,N,\ell}(r).$$

The following simple observation will be used to estimate the first term. Let $\alpha \geq 0$. Then, by change of variables $r_i := q_i M^{-1}$, $i = 1, \ldots, d$,

$$\left.\begin{array}{rl} \int \varphi_M(r)|r|^\alpha dr \leq & M^{-\alpha} c \int_{|q|<1} \exp(\frac{-1}{1-|q|^2})|q|^\alpha dq \\ \leq & M^{-\alpha} \leq 1. \end{array}\right\} \tag{9.48}$$

Moreover, we easily see that there is an $\alpha > 0$ and a $\overline{c} \geq 1$ such that

$$\lambda^{-\frac{1}{p}}(r) \leq \overline{c}(1 + |r|^\alpha) \tag{$***$}$$

Then,

$$\begin{aligned} & \int |A_{M,N,1}(r)|^p \Phi(r) dr \\ \leq\ & c_\Phi \int | \int \varphi_M(r-q)\Phi^{-1/p}(r-q)(f_{\mathbf{n}}(q)\tilde{\eta}_{N,\mathbf{i}}(q) - \eta_N(f_{\mathbf{n}}(q)) \cdot \tilde{\eta}_{N,\mathbf{i}}(q))\Phi^{\frac{1}{p}}(q) dq|^p dr \\ \leq\ & c_\Phi \cdot \overline{c} \|f_{\mathbf{n}}\tilde{\eta}_{N,\mathbf{i}} - \eta_N(f_{\mathbf{n}}) \cdot \tilde{\eta}_{N,\mathbf{i}}\|_{0,p,\Phi}^p \\ & \text{(by the Hausdorff-Young inequality, (***) and (9.48))} \\ \leq\ & c_\Phi c_{m,p} \cdot \|f_{\mathbf{n}} - \eta_N(f_{\mathbf{n}})\|_{0,p,\Phi}^p \end{aligned}$$

($c_{m,p} := \overline{c}|||\tilde{\eta}_{N,\mathbf{i}}|||^p$). Since $f_{\mathbf{n}} \in \mathcal{W}_{0,p,\Phi}$

$$f_{\mathbf{n}}(r,\omega) - \eta_N(f_{\mathbf{n}}(r,\omega)) \to 0 \qquad dr \otimes dP \text{ a.e.}$$

Since $|f_{\mathbf{n}} - \eta_N(f_{\mathbf{n}})|(r) \leq (1+c)|f_{\mathbf{n}}|(r)$, Lebesgue's dominated convergence theorem in addition to the previous estimates implies

$$\left.\begin{array}{rl}\sup_M E\int |A_{M,N,1}(r)|^p\Phi(r)dr \leq & c_\Phi c_{m,p}E\|\partial^{\mathbf{n}}Y_0-\eta_N(\partial^{\mathbf{n}}Y_0)\|^p_{0,p,\Phi} \\ \to & 0,\ \text{as } N\to\infty.\end{array}\right\} \tag{9.49}$$

$|||\Phi|||=1$ and the fact that $\|\varphi_M\|_{0,1}=1$ yield

$$\begin{array}{rl} & \int |A_{M,N,2}(r)|^p\Phi(r)dr \\ \leq & \int |\int \varphi_M(r-q)(\eta_N(f_{\mathbf{n}}(q))\cdot\tilde{\eta}_{N,\mathbf{i}}(q)-\eta_N(f_{\mathbf{n}}(r))\tilde{\eta}_{N,\mathbf{i}}(r))dq|^p dr\end{array}$$

$\to 0$, as $M\to\infty$ for any $N\in\mathbf{N}$, $\mathbf{n}+\mathbf{i}=\mathbf{m}$, a.s. (cf. Adams (1975), Lemma 2.18). For $\delta>0$ set

$$B_{M,N,2,\delta}:=\{\omega:\int |A_{M,N,2}(r,\omega)|^p dr\geq\delta\}.$$

By the boundedness of η_N and $\tilde{\eta}_{N,\mathbf{i}}$ and the fact that $\tilde{\eta}_{N,\mathbf{i}}$ has compact support, $\eta_N(f_{\mathbf{n}})\tilde{\eta}_{N,\mathbf{i}}\in\mathcal{W}_{0,2p,1}$. Moreover, for all ω the support of $A_{M,N,2}(r,\omega)$ in $\mathbf{R}^d$ is contained in $B(0,N+2)$, the ball with center 0 and radius $N+2$. So the Cauchy-Schwarz and Hausdorff-Young inequalities imply

$$\left.\begin{array}{l}(E\int |A_{M,N,2}(r)|^p dr)^{\frac{1}{p}} \\ \leq (E1_{B_{M,N,2,\delta}})^{\frac{1}{2p}}(c_{N,p}c_{\mathbf{m}})^{\frac{1}{p}}(E\|\eta_N(f_{\mathbf{n}})\|^{2p}_{0,2p})^{\frac{1}{2p}}+\delta^{\frac{1}{p}}.\end{array}\right\} \tag{9.50}$$

$A_{M,N,3}(r)$ does not depend on M and we obtain the same bound as for $A_{M,N,1}(r)$ (cf. (9.49)). Now we choose $\overline{N}(\varepsilon)$ such that for all $N\geq\overline{N}(\varepsilon)$

$$2\cdot 4^{(m+1)^d}\cdot c_\Phi\cdot c_{m,p}\max_{|\mathbf{n}|\leq m}E\|\partial^n Y_0-\eta_N(\partial^n Y_0)\|^p_{0,p,\Phi}\leq\left(\frac{\varepsilon}{4}\right)^p \tag{9.51}$$

(where $4^{(m+1)^d}=(2^{(m+1)^d})^2$ comes from all possible multiindices $\mathbf{m}$ with $|\mathbf{m}|\leq m$ and decompositions $\mathbf{n}+\mathbf{i}=\mathbf{m}$, and the factor 2 comes from using the right hand side of (9.49) both for $A_{N,M,1}$ and $A_{N,M,3}$). Then choose $\delta:=(\frac{\varepsilon}{8})^p$ in (9.50) and $M=M(\hat{N},\delta)=M(\hat{N},\delta(\varepsilon))=M(\hat{N},\varepsilon)$ large enough ($\hat{N}:=\overline{N}(\varepsilon)\vee N(\varepsilon)$) so that

$$(E1_{B_{M,N,2,\delta}})^{\frac{1}{2p}}(4^{(m+1)^d}c_{N,p}c_{\mathbf{m}})^{\frac{1}{p}}\max_{|\mathbf{n}|\leq m}(E\|\eta_N(f_{\mathbf{n}})\|^{2p}_{0,2p})^{\frac{1}{2p}}\leq\frac{\varepsilon}{8}. \tag{9.52}$$

Altogether, we obtain $\hat{N}:=\hat{N}(\varepsilon)$ and $M(\hat{N},\varepsilon)$ such that

$$E\|Y_{0,M(\hat{N},\varepsilon),\hat{N}(\varepsilon)}-Y_0\|^p_{m,p,\Phi}\leq\varepsilon. \tag{9.53}$$

∎

10 Appendix II

Addendum 10.1—Proof of Statement 3 of Theorem 2.1

(i) For this proof we may without loss of generality assume $N=1$ and F and $\mathcal{J}$ independent of μ, since the following proof can be easily generalized from $\tilde{\eta}$ to $(\mu,\tilde{\eta})$ taking values in the separable metric space $\mathbf{M}_{\mathbf{b},[0,T]}\times\hat{\mathbf{H}}_{0,\lambda,[0,T]}$. We adjust the classical proof of the Markov property for certain SODE's to our setting (cf., e.g., Dynkin (1965), Ch. VI, §2). Let "diameter" be defined as usually for metric spaces (cf.

"*diam*" in the proof of Lemma 6.4). $C^1([0,T];\tilde{\mathbf{H}}_{0,\lambda})$ is the space of continuously differentiable functions from $[0,T]$ into $\tilde{\mathbf{H}}_{0,\lambda}$, and note that $C^1([0,T];\tilde{\mathbf{H}}_{0,\lambda})$ is dense in $\tilde{\mathbf{H}}_{0,\lambda,[0,T]}$, which itself is separable. So for each $n\in\mathbf{N}$ we may define a countable decomposition of $\tilde{\mathbf{H}}_{0,\lambda,[0,T]}$ into disjoint Borel sets D^n_ℓ of diameter $\le 3^{-n}$, $\ell,n\in\mathbf{N}$, such that each D^n_ℓ contains an $\tilde{\eta}^n_\ell\in C^1([0,T];\tilde{\mathbf{H}}_{0,\lambda})$. By the separability of $\mathbf{R}^d$ there is a sequence of countable decompositions $\{E^m_k\}_{k\in\mathbf{N}}$, where E^m_k are non-empty Borel sets of diameter $\le 3^{-m}$ for all $k\in\mathbf{N}$ and $m\in\mathbf{N}$. In each E^m_k we choose an arbitrary but fixed element $r^m_k, k,m\in\mathbf{N}$. Finally, we decompose $[0,T)$ into intervals $[s^N_j,s^N_{j+1}), j=0,1,\ldots$, where $s^N_j:=(3^{-N}j)\wedge T$. Now we define maps:

$$\left.\begin{array}{l} f_m:\mathbf{R}^d\to\mathbf{R}^d, f_m(r)=r^m_k,\ \text{if } r\in E^m_k;\\ h_n:\tilde{\mathbf{H}}_{0,\lambda,[0,T]}\to\tilde{\mathbf{H}}_{0,\lambda,[0,T]}, h_n(\tilde{\eta})=\tilde{\eta}^n_\ell\in C^1([0,T];\tilde{\mathbf{H}}_{0,\lambda})\cap D^n_\ell,\ \text{if } \tilde{\eta}\in D^n_\ell;\\ g_N:[0,T)\to[0,T), g_N(s)=s^N_{j-1},\ \text{if } s\in[s^N_{j-1},s^N_j);\\ k,\ell,j=1,2,\ldots \end{array}\right\}\tag{10.1}$$

Let $\tilde{\mathbf{H}}_{0,\lambda,[0,T],N}$ be the space of $\tilde{\mathbf{H}}_{0,\lambda}$-valued functions on $[0,T]$, which are constant on $[s^N_{j-1},s^N_j), j=1,2,\ldots$ and set

$$\hat{\mathbf{H}}_{0,\lambda,[0,T]}:=\tilde{\mathbf{H}}_{0,\lambda,[0,T]}\cup\bigcup_{N=1}^\infty\tilde{\mathbf{H}}_{0,\lambda,[0,T],N}.$$

We endow $\hat{\mathbf{H}}_{0,\lambda,[0,T]}$ with the metric $|||\tilde{\eta}_1-\tilde{\eta}_2|||_T:=\sup_{0\le t\le T}\|\tilde{\eta}_1(t)-\tilde{\eta}_2(t)\|_{\tilde{\mathbf{H}}_{0,\lambda}}$. Then we define

$$\left.\begin{array}{l} \pi_N:\tilde{\mathbf{H}}_{0,\lambda,[0,T]}\to\tilde{\mathbf{H}}_{0,\lambda,[0,T],N}\subset\hat{\mathbf{H}}_{0,\lambda,[0,T]}\\ \tilde{\eta}(\cdot)\mapsto\pi_N\tilde{\eta}(\cdot):=\tilde{\eta}(g_N(\cdot))\ . \end{array}\right\}\tag{10.2}$$

Let $\tilde{\mathcal{B}}_{0,\lambda,[0,t]}$ and $\tilde{\mathcal{F}}_{0,\lambda,t}$ be the σ-algebras of the Borel sets and the cylinder sets on $\tilde{\mathbf{H}}_{0,\lambda,[0,t]}$, respectively, $t\le T$. The corresponding σ-algebras on $\hat{\mathbf{H}}_{0,\lambda,[0,t]}$ will be denoted $\hat{\mathcal{B}}_{0,\lambda,[0,t]}$ and $\hat{\mathcal{F}}_{0,\lambda,t}$, respectively. By the separability of $\tilde{\mathbf{H}}_{0,\lambda}$ and the definition of $|||\cdot|||_T$ we have for all $t\le T$

$$\left.\begin{array}{rl} \tilde{\mathcal{B}}_{0,\lambda,[0,t]}= & \tilde{\mathcal{F}}_{0,\lambda,t},\\ \hat{\mathcal{B}}_{0,\lambda,[0,t]}= & \hat{\mathcal{F}}_{0,\lambda,t} \end{array}\right\}\tag{10.3}$$

(cf., e.g., Kuo (1975), Ch. 1, §3, whose proof can be easily generalized to our setting). The continuity of the imbedding

$$\tilde{\mathbf{H}}_{0,\lambda,[0,t]}\hookrightarrow\hat{\mathbf{H}}_{0,\lambda,[0,t]}$$

implies

$$\hat{\mathcal{B}}_{0,\lambda,[0,t]}\cap\tilde{\mathbf{H}}_{0,\lambda,[0,t]}=\tilde{\mathcal{B}}_{0,\lambda,[0,t]},\tag{10.4}$$

where the left hand side of (10.4) is the trace of $\hat{\mathcal{B}}_{0,\lambda,[0,t]}$ in $\tilde{\mathbf{H}}_{0,\lambda,[0,t]}$. Since for $\pi_t:=\pi_{0,t}$

$$\pi_t\tilde{\mathcal{F}}_{0,\lambda,T}=\tilde{\mathcal{F}}_{0,\lambda,t},\qquad\pi_t\hat{\mathcal{F}}_{0,\lambda,T}=\hat{\mathcal{F}}_{0,\lambda,t}$$

we obtain

$$D^n_{k,t}:=\pi_tD^n_k\in\tilde{\mathcal{F}}_{0,\lambda,t},\qquad D^{n,N}_{k,t}:=\pi_t\pi_ND^n_k\in\hat{\mathcal{F}}_{0,\lambda,t}.\tag{10.5}$$

(ii) Denote by $\tilde{r}(\cdot,\omega,\pi_Nh_n(\tilde{\eta}),f_m(r),g_N(s))$ the unique continuous (Itô-)solution of (2.1) with fixed deterministic input $(\pi_Nh_n(\tilde{\eta}),f_m(r),g_N(s))$. By (2.3) with probability 1 (uniformly in $t\in[g_N(s),T]$)

$$\tilde{r}(t,\omega,\pi_N h_n(\tilde{\eta}), f_m(r), g_N(s)) \equiv \tilde{r}(t,\omega,\pi_t\pi_N h_n(\tilde{\eta}), f_m(r), g_N(s)). \tag{10.6}$$

Let $B \in \mathcal{B}^d$. By adaptedness

$$\left.\begin{array}{l} A_{t,\pi_N h_n(\tilde{\eta}),f_m(r),g_N(s)} \\ \quad := \{\omega : \tilde{r}(t,\omega,\pi_N h_n(\tilde{\eta}), f_m(r), g_N(s)) \in B\} \in \mathcal{G}_t. \end{array}\right\} \tag{10.7}$$

Set

$$\tilde{r}^{-1}(t,\cdot,\cdot,\cdot,\cdot)(B) := \bigcup_{k,\ell,j} A_{t,\pi_N\eta_\ell^n, r_k^m, s_j^N} \times D_{\ell,t}^{n,N} \times E_k^m \times [s_{j-1}^N, s_j^N)$$

(the subindex t at $D_\ell^{n,N}$ is justified by (10.6) for the inverse image). Then, by (10.5)

$$\tilde{r}^{-1}(t,\cdot,\cdot,\cdot,\cdot)(B) \in \mathcal{G}_t \otimes \hat{\mathcal{F}}_{0,\lambda,t} \otimes \mathcal{B}^d \otimes \mathcal{B}_{[0,t]}. \tag{10.8}$$

(iii) We need a "nice" continuous dependence of the solution of (2.1) on all "parameters" $\tilde{\mathcal{Y}}, \tilde{Z}, r_0$. This would follow from (2.2) provided we can "nicely" estimate the convolution integral with kernel $\Lambda_L, L \in \{F, \mathcal{J}\}$. Set

$$\Psi_L(r) := \int \Lambda_L^2(r-p)\lambda^{-1}(p)dp. \tag{10.9}$$

Let $\partial^{\mathbf{j}}$ be an arbitrary differential operator of order $|\mathbf{j}|$. Since $|\partial^{\mathbf{j}}\lambda^{-1}(p)| \le c_{\mathbf{j}}\lambda^{-1}(p)$ with $c_{\mathbf{j}} \in (0,\infty)$ we obtain for $m = 1,2$ $|\partial^{\mathbf{j}}\psi_L^m(r)| \le c_{\mathbf{j},m}\psi_L^m(r)$ with finite $c_{\mathbf{j},m}$. Suppose temporarily

$$ess\sup_\omega |r_0(\omega)| =: M < \infty. \tag{$***$}$$

Abbreviate $r(\cdot) := r(\cdot,\tilde{Z},r_0)$ (which is the solution of (2.1) with "input" $(\tilde{Z},r_0)$). By our assumptions of F and $\mathcal{J}$ ((1.13) and (1.14)), Itô's formula and $(***)$ we obtain for $m = 1,2$

$$\sup_{0\le s\le T} E\Psi_L^m(r(s)) \le c_{T,m}E\Psi_L^m(r_0) \le c_{T,m,M} < \infty \tag{10.10}$$

with finite $c_{T,m} := c_{T,m}(\lambda, F, \mathcal{J})$ (independent of $\tilde{\mathcal{Y}}, \tilde{Z}, r_0$), $c_{M,m} := \max_{L\in\{F,\mathcal{J}\}}(\sup_{|r|\le M}\Psi_L(r))^m < \infty$ and $c_{T,m,M} := c_{T,m}\cdot c_{M,m}$.

(iv) We will now use (2.2) and the Borel-Cantelli lemma to remove step by step the discretization of our input variables, extending the proof given in Dynkin (loc. cit., Ch. XI, §2, (11.44), (11.45)) to our setting. The first step is the limit $m \to \infty$, and we may directly copy the proof in Dynkin (loc. cit.). This step yields the existence of a function

$$\tilde{\tilde{r}} : 0 \le s \le t \le T \times \Omega \times \hat{\mathbf{H}}_{0,\lambda,[0,T]} \times \mathbf{R}^d \to \mathbf{R}^d$$

and sets $\tilde{\Omega}_{\tilde{\eta},r,s}$ with $P(\tilde{\Omega}_{\tilde{\eta},r,s}) = 1$ such that uniformly in $t \in [g_N(s), T]$

$$\tilde{\tilde{r}}\,(t,\omega,\pi_N h_n(\tilde{\eta}), f_m(r), g_N(s)) :\equiv \left\{\begin{array}{l} \tilde{r}(t,\omega,\pi_N h_n(\tilde{\eta}), f_m(r), g_N(s)),\ \text{if } \omega \in \tilde{\Omega}_{\tilde{\eta},r,s} \\ 0,\ \text{if } \omega \notin \tilde{\Omega}_{\tilde{\eta},r,s}) \end{array}\right\} \tag{10.11}$$

and for any $\omega, \tilde{\eta}, r, s$ uniformly on $[g_N(s), T]$

$$\lim_{m\to\infty} \widetilde{\widetilde{r}}\,(\cdot,\omega,\pi_N h_n(\tilde{\eta}),f_m(r),g_N(s)) = \widetilde{\widetilde{r}}\,(\cdot,\omega,\pi_N h_n(\tilde{\eta}),r,g_N(s)). \tag{10.12}$$

(10.11) and (10.12) imply that both $\widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\cdot),f_m(\cdot),g_N(\cdot))$ and $\widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\cdot),\cdot,g_N(\cdot))$ are $\overline{\mathcal{G}}_t \otimes \hat{\mathcal{F}}_{0,\lambda,t} \otimes \mathcal{B}^d \otimes \mathcal{B}_{[0,t]} - \mathcal{B}^d$ measurable. Further,

$$\left.\begin{array}{l} \widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{\eta}),r_{g_N(s)},g_N(s)) \text{ is a solution of (2.1) for any} \\ r_{g_N(s)} \in \mathcal{R}_{d,g_N(s)},\, \tilde{\eta} \in \tilde{\mathbf{H}}_{0,\lambda,[0,T]}. \end{array}\right\} \tag{10.13}$$

(v) In this step we will show that $\widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_{g_N(s)},g_N(s))$ is a solution of (2.1) for $r_{g_N(s)} \in \mathcal{R}_{d,g_N(s)}, \tilde{Z} \in \tilde{\mathcal{H}}_{0,\lambda,[0,T]}$. Without loss of generality let $g_N(s) = 0$. Since on $[0,s_1^N)$ $\pi_N h_n(\tilde{Z})(s) = h_n(\tilde{Z})(0)$ is $\mathcal{F}_0$-measurable we may again copy the proof in Dynkin (loc. cit.) to show that $\widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_0,0)$ is a solution of (2.1) on $[0,s_1^N)$ and, by a continuity argument, also on $[0,s_1^N]$.

Set $r_{s_1^N} := \lim_{t\uparrow s_1^N} \widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_0,0)$. As before we conclude that

$$\widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(Z),r_{s_1^N}) \text{ solves (2.1) on } [s_1^N,s_2^N].$$

Set

$$\Omega_{k_0} := \{\omega : \pi_N h_n(\tilde{Z})(0) = \eta^n_{k_0}(0)\},$$

$$\Omega_{k_1} := \{\omega : \pi_N h_n(\tilde{Z})(s_1^N) = \eta^n_{k_1}(s_1^N)\}$$

Then, since $\Omega_{k_0} \in \mathcal{F}_0$ and $\Omega_{k_1} \in \mathcal{F}_{s_1^N}$, a.s. for $t \in [s_1^N,s_2^N]$

$$\begin{aligned} & \widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_{s_1^N})1_{\Omega_{k_0}\cap\Omega_{k_1}} \\ = \; & \widetilde{\widetilde{r}}\,(t,\cdot,\pi_N \eta^n_{k_1}\,\widetilde{\widetilde{r}}\,(s_1^N,\cdot,\pi_N\eta^n_{k_0},r_0,0),s_1^N)1_{\Omega_{k_0}\cap\Omega_{k_1}} \\ = \; & \widetilde{\widetilde{r}}\,(t,\cdot,\pi_N\eta^n_{k_0,k_1},r_0,0)1_{\Omega_{k_0}\cap\Omega_{k_1}} \end{aligned}$$

with $\eta^n_{k_0,k_1} \in \hat{\mathbf{H}}_{0,\lambda,[0,T]}$ such that $\pi_N\eta^n_{k_0,k_1}(0) = \eta^n_{k_0}(0)$ and $\pi_N\eta^n_{k_0,k_1}(s_1^N) = \eta^n_{k_1}(s_1^N)$. The second identity comes from the unique continuation property of solutions of (2.1) and (10.13). So we obtain a.s. for $t \in [s_1^N,s_2^N]$

$$\widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_{s_1^N},s_1^N)1_{\Omega_{k_0}\cap\Omega_{k_1}} = \widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_0,0)1_{\Omega_{k_0}\cap\Omega_{k_1}}.$$

Since $\Omega = \bigcup_{k_0,k_1} \Omega_{k_0}\cap\Omega_{k_1}$, this implies that a.s. for any $t \in [s_1^N,s_2^N]$

$$\widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_{s_1^N},s_1^N) \equiv \widetilde{\widetilde{r}}\,(t,\cdot,\pi_N h_n(\tilde{Z}),r_0,0),$$

whence $\widetilde{\widetilde{r}}\,(\cdot,\cdot,\pi_N h_n(\tilde{Z}),r_0,0)$ is a solution of (2.1) on $[0,s_2^N]$. Repeating this procedure a finite number of times finishes step (v).

(vi) We easily verify that for any n and k

$$\left.\begin{array}{l} (\alpha)\int_0^T \|\pi_N\eta^n_k(s) - \eta^n_k(s)\|_{\tilde{\mathbf{H}}_{0,\lambda}}\,ds \le c_T|||\frac{\partial}{\partial t}\eta^n_k|||^2_T e^{-N} \\ \text{and for any } s,\tilde{\eta}, \text{ and } r \\ (\beta)E|r(s,\tilde{\eta},r,g_N(s)) - r|^2 \le c_T 3^{-N}. \end{array}\right\} \tag{10.14}$$

Hence, as in step (iv) using (2.2) and the Borel-Cantelli lemma we obtain a function

$$\bar{\bar{r}}: 0 \le s \le t \le T \times \Omega \times \hat{\mathbf{H}}_{0,\lambda,[0,T]} \times \mathbf{R}^d \to \mathbf{R}^d$$

and sets $\tilde{\tilde{\Omega}}_{\tilde{\eta},r,s}$ with $P(\tilde{\tilde{\Omega}}_{\tilde{\eta},r,s}) = 1$ such that uniformly in $t \in [g_N(s), T]$

$$\bar{\bar{r}}\,(t, \omega, \pi_N h_n(\tilde{\eta}), f_N(r), g_N(s)) :\equiv \left\{ \begin{array}{l} \tilde{\tilde{r}}\,(t, \omega, \pi_N h_n(\tilde{\eta}), f_N(r), g_N(r)), \text{ if } \omega \in \tilde{\tilde{\Omega}}_{\tilde{\eta},r,s} \\ 0, \text{ if } \omega \notin \tilde{\tilde{\Omega}}_{\tilde{\eta},r,s}; \end{array} \right\} \tag{10.15}$$

and for any $\omega, \tilde{\eta}, r, s$ uniformly in $t \in [g_N(s), T]$

$$\lim_{N\to\infty} \bar{\bar{r}}\,(\cdot, \omega, \pi_N h_n(\tilde{\eta}), f_N(r), g_N(s)) = \bar{\bar{r}}\,(\cdot, \omega, h_n(\tilde{\eta}), r, s). \tag{10.16}$$

Clearly, $\bar{\bar{r}}$ has the same measurability properties as $\tilde{\tilde{r}}$. Moreover, setting $\tilde{\Omega} := \bigcap_{\ell,k,j} \tilde{\tilde{\Omega}}_{\tilde{\eta}^n_\ell, r^N_k, s^N_j}$ we have $P(\tilde{\Omega}) = 1$, whence $\bar{\bar{r}}\,(\cdot, \cdot, \pi_N h_n(\tilde{Z}), f_N(r_{g_N(s)}), g_N(s))$ is a solution of (2.1) (being a.s. equal to $\tilde{\tilde{r}}\,(\cdot, \cdot, \pi_N h_n(\tilde{Z}), f_N(r_{g_N}(s)), g_N(s)))$. Next, we may suppose $E|r_{g_N(s)}|^2 \to E|r_s|^2$, as $N \to \infty$. Using the usual technique of cutting off we may without loss of generality assume $ess\sup_\omega \sup_{0\le t\le T} \|\tilde{Z}(t,\omega)\|_{\tilde{H}_{0,\lambda}} < \infty$. Then we choose a subsequence $\tilde{N}(N) \to \infty$ such that

$$E\int_0^T \|\pi_{\tilde{N}} h_n(\tilde{Z}(t)) - h_n(\tilde{Z}(t))\|^2 dt \le c_T 3^{-N}. \tag{10.17}$$

The choice of $\tilde{N}(N)$ implies that a.s. uniformly on $[s, T]$

$$\bar{\bar{r}}\,(\cdot, \cdot, \pi_{\tilde{N}} h_n(\tilde{Z}), f_{\tilde{N}}(r_{g_{\tilde{N}}(s)}), g_N(s)) \to r(\cdot, h_n(\tilde{Z}), r_s, s), \tag{10.18}$$

the solution of (2.1).

(10.15) and (10.18) imply that a.s.

$$\bar{\bar{r}}\,(\cdot, \cdot, h_n(\tilde{Z}), r_s, s) \equiv r(\cdot, h_n(\tilde{Z}), r_s, s),$$

i.e., $\bar{\bar{r}}\,(\cdot, \cdot, h_n(\tilde{Z}), r_s, s)$ solves (2.1).

(vii) Now we proceed as before. By (2.2) and the Borel-Cantelli lemma we obtain

$$\bar{r} : 0 \le s \le t \le T \times \hat{\mathbf{H}}_{0,\lambda,[0,T]} \times \mathbf{R}^d \times [0,t] \to \mathbf{R}^d$$

and sets $\overline{\Omega}_{\tilde{\eta},r,s}$ with $P(\overline{\Omega}_{\tilde{\eta},r,s}) = 1$ such that uniformly in $t \in [g_N(s), T]$

$$\bar{r}(t, \omega, h_N(\tilde{\eta}), f_N(r), g_N(s)) := \left\{ \begin{array}{l} \bar{\bar{r}}\,(t, \omega, h_N(\tilde{\eta}), f_N(r), g_N(s)), \text{ if } \omega \in \overline{\Omega}_{\tilde{\eta},r,s}, \\ 0, \text{ if } \omega \notin \overline{\Omega}_{\tilde{\eta},r,s}; \end{array} \right\} \tag{10.19}$$

and for any $\omega, \tilde{\eta}, r, s$ uniformly in $t \in [g_N(s), T]$

$$\lim_{N\to\infty} \bar{r}(\cdot, \omega, h_N(\tilde{\eta}), f_N(r), g_N(s)) \equiv \bar{r}(\cdot, \omega, \tilde{\eta}, r, s). \tag{10.20}$$

This implies the $\overline{\mathcal{G}}_t \otimes \hat{\mathcal{F}}_{0,\lambda,t} \otimes \mathcal{B}^d \otimes \mathcal{B}_{[0,t]} - \mathcal{B}^d$-measurability of $\bar{r}(t, \cdot, \cdot, \cdot, \cdot)$, and, as before, $\bar{r}(t, \cdot, h_N(\tilde{Z}), f_N(r_{g_N(s)}), g_N(s))$ is a solution of (2.1).

Again, by (2.2) and the Borel-Cantelli lemma we obtain now that $\overline{r}(t,\cdot,\tilde{Z},r_s,s)$ is a solution of (2.1). This finishes the proof of properties (i) and (ii) under the assumption (***).

(viii) Note that for $r_{s,N} := r_s \cdot 1_{\{|r_s|\le N\|}$ satisfying (***) (at s)

$$\overline{r}(\cdot,\cdot,\tilde{Z},r_{s,N},s) \equiv r(\,\cdot\,,\tilde{Z},r_{s,N},s) \quad \text{a.s.}$$

and that for some subsequence $\tilde{N}$, $r(\cdot,\tilde{Z},r_{s,\tilde{N}},s) \to r(\cdot,\tilde{Z},r_s,s)$ a.s. Hence, $\overline{r}(t,\cdot,\tilde{Z},r_s,s)$ solves (2.1). (2.5) follows from uniqueness. ∎

Let $(\mathbf{B},\|\cdot\|_B)$ be some Banach space with norm $\|\cdot\|_B$. A $\mathbf{B}$-valued random field is by definition a $\mathcal{B}^d\otimes\mathcal{F}-\mathcal{B}_B$-measurable map from $\mathbf{R}^d\times\Omega$ into $\mathbf{B}$, where $\mathcal{B}_{\mathbf{B}}$ is the Borel σ-algebra on $\mathbf{B}$.

Definition 10.2

Let $p\ge 1$. A $\mathbf{B}$-valued random field is said to be an element of $L_{p,loc}(\mathbf{R}^d\times\Omega;\mathbf{B})$, if for any $L<\infty$

$$E\int_{|q|\le L}\|f(q)\|_B^p\,dq<\infty.$$

Let $\overline{r}(\cdot,q) := \overline{r}(\cdot,\tilde{\mathcal{Y}},\tilde{Z},s,q,\tilde{w})$ be the measurable version of the solution of (2.1) with $\overline{r}(s,q)=q$, as derived by Theorem 2.1.

Lemma 10.3

(I) For any $p\ge 2$ and $T>s$, $\overline{r}(\cdot,\cdot)\in L_{p,loc}(\mathbf{R}^d\times\Omega;C([s,T];\mathbf{R}^d)$.

(II) Let $q,\tilde{q}\in\mathbf{R}^d$. Then, for any $p\ge 2$ and $T>s$

$$E\sup_{s\le t\le T}|\overline{r}(t,q)-\overline{r}(t,\tilde{q})|^p\le c_{T,F,\mathcal{J},p}|q-\tilde{q}|^p. \tag{10.21}$$

Proof

(i) Theorem 2.1 implies that $\overline{r}(\cdot,\cdot)$ is $C([s,T];\mathbf{R}^d)$-valued random field for any $T>s$. The integrability with respect to (q,ω) follows from the boundedness and integrability assumptions on the coefficients $\tilde{F}$ and $\tilde{\mathcal{J}}$.

(ii) (10.21) follows from our assumptions and the Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities. ∎

Endow $\mathcal{M}_{d\times d}$, the space of $d\times d$-matrices over $\mathbf{R}$, with the Euclidean metric of $\mathbf{R}^{d^2}$ and let δ denotes the Fréchet derivative.

Lemma 10.4

Suppose the coefficients of (2.1) satisfy condition (2.20) with $m\ge 1$. Then, $\overline{r}(t,\cdot)$ is at least once differentiable with respect to q_k in the generalized sense, $k=1,\dots,d$, and for any $p\ge 2$, $T>s$

$$\varphi(\cdot,\cdot) := \frac{\partial}{\partial r}\overline{r}(\cdot,\cdot)\in L_{p,loc}(\mathbf{R}^d\times\Omega;C([s,T];\mathcal{M}_{d\times d})). \tag{10.22}$$

Moreover, for any $q\in\mathbf{R}^d$, $\varphi(\cdot,q)$ is the Itó-solution of the following bilinear $\mathcal{M}_{d\times d}$-valued SODE:

$$d\varphi(t,q)=\left.\begin{array}{l}(\delta\tilde{F})(\overline{r}(t,q),t)\varphi(t,q)dt+\sum_{\ell=1}^d\int(\delta\tilde{\mathcal{J}}_\ell)(\overline{r}(t,q),p,t)\varphi(t,q)\tilde{w}_\ell(dp,dt)\\ \varphi(s,q)=I \text{ for all } q\in\mathbf{R}^d,\end{array}\right\} \tag{10.23}$$

where I is the identity matrix in $\mathcal{M}_{d\times d}$.

Proof

We adjust the proof of Theorem 12, Ch. 4, of Gihman and Skorohod (1982) to our setting.

(i) Assumption (2.20) ($m \geq 1$) implies that for each $q \in \mathbf{R}^d$ (10.23) has a unique solution $\varphi(\cdot, q) \in C([s,T]; \mathcal{M}_{d\times d})$ a.s. Moreover, for each $p \geq 2$ there is a finite $c_{T,F,\mathcal{J},p,d}$ such that

$$\sup_{q\in\mathbf{R}^d} E \sup_{s\leq t\leq T} |\varphi(t,q)|^p \leq c_{T,F,\mathcal{J},p,d}. \tag{10.24}$$

(ii) First we need some notation. For $q \in \mathbf{R}^d$ and $\triangle_j \in \mathbf{R}$ set $q_{+\triangle_j} := (q_1, \ldots, q_{j-1}, q_j + \triangle_j, q_{j+1}, \ldots, q_d)^T$. Now let $f : \mathbf{R}^d \to \mathbf{R}$. Set $\partial_{\triangle,j} f(q) = \frac{1}{\triangle_j}(f(q_{+\triangle_j}) - f(q))$. Next, assume f to be differentiable and define the "quotient" map for f, $Q_\triangle f : \mathbf{R}^{2d} \to \mathbf{R}^d$, as follows:

$$(r, \hat{r}) \mapsto (Q_\triangle f)_k(r, \tilde{r}) = \left\{ \begin{array}{l} \frac{f(\tilde{r}_1,\ldots,\tilde{r}_{k-1},r_k,\ldots r_d) - f(\tilde{r}_1,\ldots,\tilde{r}_k,r_{k+1},\ldots r_d)}{r_k - \tilde{r}_k}, \text{ if } r_k \neq \tilde{r}_k \\ \partial_k f(\tilde{r}_1, \ldots, \tilde{r}_k, r_{k+1}, \ldots, r_d), \text{ if } r_k = \tilde{r}_k, \end{array} \right\} \tag{10.25}$$

$k = 1, \ldots, d$, where $f(\tilde{r}_1, \ldots, \tilde{r}_j, r_{j+1}, \ldots, r_d)$ means: the first j coordinates are those of $\tilde{r}$ and the last $d-j$ coordinates are those of r. Moreover, if $j = 0$, then we have $f(r_1, \ldots r_d)$ and for $j = d$ we obtain $f(\tilde{r}_1, \ldots, \tilde{r}_d)$. By assumption

$$(Q_\triangle \tilde{F}_i)_k(\overline{r}(u, q_{+\triangle_j}), \overline{r}(u,q), u),$$

$$(Q_\triangle \tilde{\mathcal{J}}_{i\ell})_k(\overline{r}(u, q_{+\triangle_j}), \tilde{p}, \overline{r}(u,q), \tilde{p}, u)$$

are well defined for $i, j, \ell, k = 1, \ldots d$. They are also adapted processes and (for fixed $\triangle_j$) measurable in (u, q, ω) (as a consequence of the measurability and adaptedness of $\overline{r}(\cdot,\cdot)$ with respect to (u, q, ω) and the continuous differentiability of $\tilde{F}$ and $\tilde{\mathcal{J}}$). We have to verify that these "quotients" are also "nicely" integrable.

By Taylor's formula

$$\left. \begin{array}{rl} & (Q_\triangle \tilde{\mathcal{J}}_{i\ell})_k(\overline{r}(u, q_{+\triangle_j}), \tilde{p}, \overline{r}(u,q), \tilde{p}, u) \\ = & \partial_{k,r} \tilde{\mathcal{J}}_{i\ell}(\tilde{r}, \tilde{p}, u) + \frac{1}{2}\partial^2_{kk,r} \tilde{\mathcal{J}}_{i\ell}(\hat{r}, \tilde{p}, u)(\overline{r}_k(u, q_{+\triangle_j}) - \overline{r}_k(u,q)), \end{array} \right\} \tag{10.26}$$

where

$$\tilde{r} = (\overline{r}_1(u,q), \ldots, \overline{r}_k(u,q), \overline{r}_{k+1}(u, q_{+\triangle_j}), \ldots, \overline{r}_d(u, q_{+\triangle_j})) =: \tilde{r}(u,q,j) \tag{$*$}$$

and

$$\hat{r} = (\overline{r}_1(u,q), \ldots, \overline{r}_{k-1}(u,q), \theta(\overline{r}_k(u, q_{+\triangle_j}) - \overline{r}_k(u,q)) + \overline{r}_k(u,q), \overline{r}_{k+1}(u, q_{+\triangle_j}), \ldots, \overline{r}_d(u, q_{+\triangle_j})) \tag{$**$}$$

$$=: \hat{r}(u,q,j)$$

with $\theta \in (0,1)$.

By (2.20)

$$\int (\partial^2_{kk,r} \tilde{\mathcal{J}}_{i\ell})^2(r, \tilde{p}, u) d\tilde{p}_{|_{r=\hat{r}(u,q,j)}} \leq d_{1,T} \tag{10.27}$$

uniformly in i, k, ℓ, j, ω, r and $u \leq T$.

(10.27) allows us to reduce the integrability properties of $(Q_\triangle \tilde{\mathcal{J}}_{i\ell})_k(\overline{r}(u, q_{+\triangle_j}), \tilde{p}, \overline{r}(u,q), \tilde{p}, u)$ to those of $\partial_k \tilde{\mathcal{J}}_{i\ell}(\tilde{r}(u,q,j), \tilde{p})$. In particular, (10.26) and (10.27) imply

$$\left.\begin{array}{ll} & \int\{(Q_\triangle\tilde{\mathcal{J}}_{i\ell})_k(\overline{r}(u,q_{+\triangle_j}),\tilde{p},\overline{r}(u,q),\tilde{p},u)-\partial_k\tilde{\mathcal{J}}_{i\ell}(\tilde{r}(u,q,j),\tilde{p},u)\}^2d\tilde{p} \\ \le & d_{1,T}|\overline{r}_k(u,q_{+\triangle_j})-\overline{r}_k(u,q)|^2 \end{array}\right\} \tag{10.28}$$

for $i,j,k,\ell\in\{1,\ldots,d\}$, where we used the abbreviations from (*) and (**). With the same abbreviations we obtain immediately

$$\left.\begin{array}{l} |(Q_\triangle\tilde{F}_i)_k(\overline{r}(u,q_{+\triangle_j}),\overline{r}(u,q),u))-\partial_k\tilde{F}_i(\tilde{r}(u,q,j),u)| \\ \le d_{3,T}|\overline{r}(u,q_{+\triangle_j})-\overline{r}(u,q)|. \end{array}\right\} \tag{10.29}$$

(iii) Now we define the $\mathcal{M}_{d\times d}$-valued process $\eta_\triangle(t,q)$ by

$$\eta_{\triangle,i,j}(t,q)=\partial_{\triangle,j}\overline{r}_i(t,q), \tag{10.30}$$

where $\partial_{\triangle,j}$ acts on the j-th component of q.

Then, with $\tilde{\mathcal{J}}_{\cdot\ell}$ being the ℓ-th column of $\tilde{\mathcal{J}}$,

$$\left.\begin{array}{rl} \eta_\triangle(t,q)= & I+\int_s^t(Q_\triangle\tilde{F})(\overline{r}(u,q_{+\triangle_j}),\overline{r}(u,q),u)\eta_\triangle(u,q)du \\ + & \sum_{\ell=1}^d\int_s^t\int(Q_\triangle\tilde{\mathcal{J}}_{\cdot\ell})(\overline{r}(u,q_{+\triangle_j}),\tilde{p},\overline{r}(u,q),\tilde{p},u)\eta_\triangle(u,q)\tilde{w}_\ell(d\tilde{p},du), \end{array}\right\} \tag{10.31}$$

which can be rewritten as an integral equation associated with (10.23) plus two error terms. Setting

$$\psi_\triangle(t,q):=\varphi(t,q)-\eta_\triangle(t,q),$$

we have

$$\left.\begin{array}{rl} \psi_\triangle(t,q)= & \int_s^t(\delta\tilde{F})(\overline{r}(u,q),u)\psi_\triangle(u,q)dq \\ + & \sum_{\ell=1}^d\int_s^t\int(\delta\tilde{\mathcal{J}}_{\cdot\ell})(\overline{r}(u,q),\tilde{p},u)\psi_\triangle(u,q)\tilde{w}_\ell(d\tilde{p},du) \\ + & \int_s^t D_{\tilde{F}}(\overline{r}(u,q),\triangle,u)du \\ + & \sum_{\ell=1}^d\int_s^t\int D_{\tilde{\mathcal{J}}}(\overline{r}(u,q),\tilde{p},\triangle,u)\tilde{w}_\ell(d\tilde{p},du). \end{array}\right\} \tag{10.32}$$

(iv) Now, by the Burkholder-Davis-Gundy inequality in addition to (10.21), (10.28) and (10.29) we obtain

$$E\sup_{s\le t\le T}|\psi_\triangle(t,q)|^p\le c_{T,F,\mathcal{J},p}|(\triangle_1,\ldots,\triangle_d)^T|^p. \tag{10.33}$$

Thus, $\varphi(\cdot,\cdot)=\frac{\partial}{\partial r}\overline{r}(\cdot,\cdot)$. The measurability of φ with respect to q follows from the measurability of $\eta_\triangle(\cdot,\cdot)$ in q, since by (10.33) $\eta_\triangle(t,q)$ approximates $\varphi(t,q)$ uniformly in q, and so, a fortiori, also in $L_{p,loc}$ ( ), where the local integrability is trivial. ■

Next, we set

$$\tilde{\mathcal{J}}_M(r,p,u):=\sum_{n=1}^M\int\tilde{\mathcal{J}}(r,q,u)\phi_n(q)dq\phi_n(p), \tag{10.34}$$

with ϕ_n from (1.27). This implies

$$\int\tilde{\mathcal{J}}_M(r,p,u)\tilde{w}(dp,du)=\sum_{n=1}^M\tilde{\sigma}_n(r,u)d\beta_n(u), \tag{10.35}$$

where $\tilde{\sigma}_n(r,u) := \int \tilde{\mathcal{J}}(r,q,u)\phi_n(q)dq$. Hence (2.1) with $\tilde{\mathcal{J}}_M$ instead of $\tilde{\mathcal{J}}$, becomes an SODE driven by finitely many Brownian motions. We easily see that the validity of the conditions (1.14) and (2.20) on $\tilde{\mathcal{J}}$ entail the validity of the same conditions for $\tilde{\mathcal{J}}_M$. Hence, Theorem 2.1 and Lemmas 10.3 and 10.4 hold for (2.1), if we replace $\tilde{\mathcal{J}}$ by $\tilde{\mathcal{J}}_M$. Moreover, all these results also carry over to the SODE for the backward flow (2.16) associated with $(\check{\mathcal{J}}_M, \check{r})$ and $\check{w}(dp,t)$ defined by (2.17) for fixed $T > 0$, and $t \in [0,T]$. We will denote the measurable versions of the solutions of the forward and backward flows associated with $\tilde{\mathcal{J}}_M$ and derived by Theorem 2.1 by

$$\overline{r}_M(\cdot,q) := \overline{r}_M(\cdot,\tilde{\mathcal{Y}},\tilde{Z},s,q,\tilde{w})$$

and

$$\check{r}_M(\cdot,q) = \check{r}(\cdot,\tilde{\mathcal{Y}},\tilde{Z},s,q,\check{w}),$$

respectively. The solutions of (10.23) associated with $\overline{r}_M$ and $\check{r}_M$ will be denoted by $\overline{\varphi}_M$ and $\check{\varphi}_M$, respectively.

If $\mathbf{B}$ is some metric space, then $L_0(\Omega;\mathbf{B})$ is by definition the space of $\mathbf{B}$-valued random variables. Fix $T > s$.

Lemma 10.5

Let $m \ge 1$ and suppose (2.20). Let $\mathbf{j}$ and $\boldsymbol{\ell}$ be multiindices with $|\mathbf{j}| \le m$ and $|\boldsymbol{\ell}| \le m-1$. Then for any $\tilde{r}_M \in \{\overline{r}_M, \check{r}_M\}$ and $p \ge 2$

$$\partial^{\mathbf{j}}\tilde{r}_M \in L_{p,loc}(\mathbf{R}^d \times \Omega; C([s,T];\mathbf{R}^d)) \tag{10.36}$$

and

$$\partial^{\boldsymbol{\ell}}\tilde{r}_M \in L_0(\Omega; C(\mathbf{R}^d \times [s,T];\mathbf{R}^d). \tag{10.37}$$

Proof

(10.36) for $|\mathbf{j}| \le 1$ is a consequence of Lemmas 11.2 and 11.3. For $|\mathbf{j}| > 1$ we can (repeatedly) apply the procedure of the proof of Lemma 10.4 to SODE's in increasing dimensions. So to prove (10.36) for $m = 2$ we consider the $\mathbf{R}^{d^2+d}$-valued SODE for $(\overline{r},\varphi)$ and obtain a bilinear SODE for $\bigtriangledown((\overline{r},\varphi)^T)$ on $\mathcal{M}_{(d^2+d)\times(d^2+d)}$ ($\bigtriangledown$ now being the (d^2+d)-dimensional gradient). (10.37) follows from a straightforward generalization of Lemma 2.1 and Propositions 2.1 and 2.2 in Ch. V of Ikeda and Watanabe (1981). ■

Next, we are going to show that $\check{r}_M$ indeed defines the flow inverse to $\overline{r}_M$. We will follow the procedure of Ikeda and Watanabe (loc. cit) and approximate $\overline{r}_M$ and $\check{r}_M$ by solutions of suitably chosen random ODE's (Wong-Zakai approximations), for which the invertibility will follow from uniqueness.

Remark 10.6

Note that for a time-variant diffusion coefficient there are inconsistent definitions of Stratonovich SODE's in the literature (cf. Arnold (1973), Ch. 10.3 and Ikeda and Watanabe (loc. cit.), Ch. III, 1, Theorem 1.4). For time-independent coefficients Ikeda and Watanabe (loc. cit.) provide nice approximations of Itô SODE's by random ODE's. An analysis of the proof of the approximation theorems in Ikeda and Watanabe (Ch. V, 2, Ch. VI, 7) shows the following: (i) No explicit use of Stratonovich SODE's is made. Instead, random ODE's with different coefficients (containing the "correction" term) approximate the Itô-SODE's (ii). The independence of time and chance of the coefficients is not necessary—the "correction" term has the same form for time (and chance)-dependent coefficients and for those which do not depend on t or ω (cf. the following (10.38)). (iii) Arnold's definition of a Stratonovich SODE gives this correction term in both the time-invariant and the time-variant cases. ■

Now we define the coefficients for the drift of the forward random ODE, resp. of the backward random ODE:

$$\left.\begin{array}{rl} \tilde{F}_i^+(r,t) = & F_i(r,t) - \frac{1}{2}\sum_{n=1}^M \sum_{j,k=1}^d (\partial_k \tilde{\sigma}_{n,ij}(r,t))\tilde{\sigma}_{n,kj}(r,t)) \text{ (forward)}, \\ \tilde{F}_i^-(r,t) = & F_i(r,T-t) + \frac{1}{2}\sum_{n=1}^M \sum_{j,k=1}^d (\partial_k \tilde{\sigma}_{n,ij}(r,T-t))\tilde{\sigma}_{n,kj}(r,T-t) \text{ (backward)}. \end{array}\right\} \tag{10.38}$$

Lemma 10.7

Suppose (2.20) with $m \geq 1$. Then $\tilde{F}_i^+$ and $\tilde{F}_i^-$ are globally Lipschitz with Lipschitz constants independent of t and ω.

Proof

(i)

$$\begin{array}{rl} & |(\partial_k \tilde{\sigma}_{n,ij}(r,t))\sigma_{n,kj}(r,t) - (\partial_k \tilde{\sigma}_{n,ij}(q,t))\tilde{\sigma}_{n,kj}(q,t)| \\ \leq & |\partial_k \tilde{\sigma}_{n,ij}(r,t) - \partial_k \tilde{\sigma}_{n,ij}(q,t)| \cdot |\tilde{\sigma}_{n,kj}(r,t)| \\ + & |\tilde{\sigma}_{n,kj}(r,t) - \tilde{\sigma}_{n,kj}(q,t)| \cdot |\partial_k \tilde{\sigma}_{n,ij}(q,t)| \\ =: & A_{1,n}(t,r,q) + A_{2,n}(t,r,q). \end{array}$$

(ii)

$$\sum_{n=1}^M A_{1,n}(t,r,q) \leq (\sum_{n=1}^\infty \tilde{\sigma}_{n,kj}^2(r,t))^{\frac{1}{2}} (\sum_{n=1}^\infty (\partial_k \tilde{\sigma}_{n,ij}(r,t) - \partial_k \tilde{\sigma}_{n,ij}(q,t))^2)^{\frac{1}{2}}.$$

Further,

$$\sum_{n=1}^\infty \tilde{\sigma}_{n,kj}^2(r,t) = \int \tilde{\mathcal{J}}_{kj}^2(r,p,t)d\hat{p} \leq d_{1,T}$$

uniformly in $t \in [s,T]$ and ω by (2.36). Next,

$$(\partial_k \tilde{\sigma}_{n,ij}(r,t) - \partial_k \tilde{\sigma}_{n,ij}(q,t))^2 = (\int (\partial_k \tilde{\mathcal{J}}_{ij}(r,\tilde{p},t) - \partial_k \tilde{\mathcal{J}}_{ij}(q,\tilde{p},t))\phi_n(\tilde{p})d\tilde{p})^2$$

$$= \left(\sum_{\ell=1}^d \int_{q_\ell}^{r_\ell} \int \partial_{k\ell}^2 \tilde{\mathcal{J}}_{ij}((q,r,\alpha)_\ell, \tilde{p}, t)\phi_n(\tilde{p})d\tilde{p}d\alpha\right)^2$$

(where $(q,r,\alpha)_\ell = (q_1, \ldots, q_{\ell-1}, \alpha, r_{\ell+1}, \ldots r_d)$ with the usual interpretation for $\ell = 1$ and $\ell = d$ - cf. (10.25))

$$\leq d|r-q| \cdot \sum_{\ell=1}^d \int_{\overline{q}_\ell}^{\overline{r}_\ell} (\int \partial_{k\ell}^2 \tilde{\mathcal{J}}_{ij}((q,r,\alpha)_\ell, \tilde{p}, t)\phi_n(\tilde{p})d\tilde{p})^2 d\alpha$$

by the Cauchy-Schwarz inequality, where $\overline{r}_\ell = r_\ell$, $\overline{q}_\ell = q_\ell$ if $r_\ell \geq q_\ell$ and $\overline{r}_\ell = q_\ell$, $\overline{q}_\ell = r_\ell$ if $q_\ell > r_\ell$). Summing up from $n = 1, \ldots, \infty$ implies:

$$
\begin{aligned}
&\sum_{n=1}^{\infty}(\partial_k\tilde{\sigma}_{n,ij}(r,t)-\partial_k\tilde{\sigma}_{n,ij}(q,t))^2\\
\le\ & d|r-q|\cdot\sum_{\ell=1}^{d}\int_{\overline{q}_\ell}^{\overline{r}_\ell}\int(\partial^2_{k\ell}\tilde{\mathcal{J}}_{ij}((q,r,\alpha)_\ell,\tilde{p},t))^2d\tilde{p}d\alpha\\
\le\ & d_{1,T}d^2|r-q|^2\\
& \text{by (2.20).}
\end{aligned}
$$

Hence,

$$
\sum_{n=1}^{M}(A_{1,n}(t,r,q))\le\tilde{d}_{1,T}d|r-q|. \tag{10.39}
$$

uniformly in t and ω (and M).

(iii)

$$
\sum_{n=1}^{\infty}(\partial_k\tilde{\sigma}_{n,ij})^2(r,t)=\int(\partial_k\tilde{\mathcal{J}}_{ij}(r,\tilde{p},t))^2d\tilde{p}\le d_{1,T}.
$$

So, repeating the derivation of (10.39) (with the obvious changes)

$$
\sum_{n=1}^{M}|A_{2,n}(t,r,q)|\le\tilde{d}_{1,T}d|r-q| \tag{10.40}
$$

(uniformly in t,ω and M).

(iv) Since F_i, $i=1\dots d$, are globally Lipschitz by assumption, the lemma is proven. ∎

Lemma 10.8

Under the assumption (2.20), $m\ge1$, for any $T>s$ with probability 1

$$
\overline{r}_M(T-t,r)=\check{r}_M(t,\overline{r}_M(T-s,r)) \tag{10.41}
$$

uniformly in $r\in\mathbf{R}^d$ and $t\in[s,T]$.

Proof

(i) We take the Wong-Zakai approximations to the SODE's for $\overline{r}_M$ and $\check{r}_M$ with piecewise linear approximations $\beta_{n,L}(t)$ of β_n, $n=1,\dots M$, $L\in\mathbf{N}$, where

$$
\left.\begin{aligned}
\beta_{n,L}(t)=\ & L\{(\tfrac{j+1}{L}-t)\beta_n(\tfrac{j}{L})+(t-\tfrac{j}{L})\beta_n(\tfrac{j+1}{L})\}\\
& \text{if } \tfrac{j}{L}\le t\le\tfrac{j+1}{L},\quad j=0,1,\dots\dots\dots
\end{aligned}\right\} \tag{10.42}
$$

By Lemma 10.7 the Wong-Zakai approximations of the forward and backward flows have unique solutions $\overline{r}_{M,L}(t,r)$ and $\check{r}_{M,L}(t,r)$, respectively. Moreover, $\overline{r}_{M,L}(T-t,r)$, the solution of the approximate forward random ODE, evaluated at $T-t$, satisfies the random ODE

$$
\frac{dr(t)}{dt}=-\tilde{F}^{+}(r(t),T-t)-\sum_{n=1}^{M}\tilde{\sigma}_n(r(t),T-t)\frac{d\beta_{n,L}}{dt}(t), \tag{10.43}
$$

which is the backward random ODE for $\check{r}_{M,L}(t,\cdot)$. By uniqueness (10.41) follows if we replace $\overline{r}_M$ and $\check{r}_M$ by $\overline{r}_{M,L}$ and $\check{r}_{M,L}$, respectively, uniformly in $r \in \mathbf{R}^d$ and $t \in [s,T]$.

(ii) Next, we check that we may (trivially) generalize Lemma 7.2 in Ikeda and Watanabe (loc. cit.), Ch. VI, to our equations with time—and ω—dependent coefficients, which implies for any multiindex $\mathbf{j}$ with $|\mathbf{j}| \le m$), any $N \in \mathbf{N}$, $p \ge 2$ and any $\tilde{r}_{M,L} \in \{\overline{r}_{M,L}, \check{r}_{M,L}\}$.

$$\sup_{|r|\le N} \sup_{L} E \sup_{s\le t\le T} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r)|^p < \infty \tag{10.44}$$

The same estimates (without L) hold for $\overline{r}_M(\cdot,r)$ and $\check{r}_M(\cdot,r)$ and their partial derivatives up to order m by (10.36). Hence, we truncate all involved processes using stopping times $\sigma_{k,L}$ and σ_k such that for $\mathbf{j}$ with $|\mathbf{j}| \le m$ and $(\tilde{r}_{M,L}, \tilde{r}_M) \in \{(\overline{r}_{M,L}, \overline{r}_M), (\check{r}_{M,L}, \check{r}_M)\}$

$$\sup_{s\le t\le T\wedge\sigma_{k,L}} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r)| + \sup_{s\le t\le T\wedge\sigma_k} |\partial^{\mathbf{j}} \tilde{r}_M(t,r)| \le K < \infty, \tag{10.45}$$

whence

$$\left.\begin{array}{l} \sup_{s\le t\le T\wedge\sigma_{k,L}\wedge\sigma_k} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r) - \partial^{\mathbf{j}} \tilde{r}_M(t,r)|^p \\ \le (2K)^{p-2} \cdot \sup_{s\le t\le T\wedge\sigma_{k,L}\wedge\sigma_k} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r) - \partial^{\mathbf{j}} \tilde{r}_M(t,r)|^2. \end{array}\right\} \tag{10.46}$$

The estimates (10.44) and (10.34) imply that for any $N \in \mathbf{N}$

$$\left.\begin{array}{l} \sup_{|r|\le N} E \sup_{T\wedge\sigma_{k,L}\wedge\sigma_k \le t\le T} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r) - \partial^{\mathbf{j}} \tilde{r}_M(t,r)|^p \\ \quad \le \frac{2}{K} \sup_{|r|\le N} E \sup_{s\le t\le T} (|\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r)| + |\partial^{\mathbf{j}} \tilde{r}_M(t,r)|)|)^{p+1} \\ \quad \to 0, \text{ as } K \to \infty \end{array}\right\} \tag{10.47}$$

(cf. Ikeda and Watanabe (loc. cit.), Ch. V, Proof of Lemma 2.1 therein).

(iii) Now we check that Theorem 7.2, Ch. VI, in Ikeda and Watanabe (loc. cit.) generalizes to our equations, which implies by (10.46) and (10.47) for any multiindex $\mathbf{j}$ with $|\mathbf{j}| \le m$, $N \in \mathbf{N}$ and $p \ge 2$

$$\sup_{|r|\le N} E \sup_{s\le t\le T} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r) - \partial^{\mathbf{j}} \tilde{r}_M(t,r)|^p \to 0, \text{ as } L \to \infty, \tag{10.48}$$

which, of course, implies for any $N \in \mathbf{N}$, $|\mathbf{j}| \le m$ and $p \ge 2$

$$E \int_{|r|\le N} \sup_{s\le t\le T} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r) - \partial^{\mathbf{j}} \tilde{r}_M(t,r)|^p dr \to 0, \text{ as } L \to \infty. \tag{10.49}$$

(iv) Choose $p > d\vee 2$ (where "$\vee$" denotes "maximum"). Then (10.49) implies by the Sobolev imbedding theorem (cf., e.g., Triebel (1978)) that for any $N \in \mathbf{N}$ and $|\mathbf{j}| \le m-1$

$$E \sup_{|r|\le L} \sup_{s\le t\le T} |\partial^{\mathbf{j}} \tilde{r}_{M,L}(t,r) - \partial^{\mathbf{j}} \tilde{r}_M(t,r)| \to 0, \text{ as } L \to \infty. \tag{10.50}$$

(v) (10.41) follows now from steps (i)-(iv) and (10.37). ∎

Lemma 10.9

Assume (2.20), $m \ge 1$, and let $p \ge 2$. Then, for any $(\tilde{r}_M, \tilde{r}) \in \{(\overline{r}_M, \overline{r}), (\check{r}_M, \check{r})\}$ and any $L < \infty$

$$\sup_{|q|\le L} E \sup_{s\le t\le T} |\tilde{r}_M(t,q) - \tilde{r}(t,q)|^p \to \infty, \text{ as } M \to \infty. \tag{10.51}$$

Proof

(i) Assume without loss of generality $(\tilde{r}_M, \tilde{r}) = (\overline{r}_M, \overline{r})$. By the Burkholder-Davis-Gundy and Hölder inequalities

$$\left.\begin{array}{l} E\sup_{s\le t\le T}|\overline{r}(t,q)-\overline{r}_M(t,q)|^p \\ \le c_{T,p}\{\int_s^T E|F(\overline{r}(u,q),u)-F(\overline{r}_M(u,q),u)|^p du \\ \quad +\int_s^T E\{\sum_{n=1}^M|\tilde{\sigma}_n(\overline{r}(u,q),u)-\tilde{\sigma}_n(\overline{r}_M(u,q),u)|^2\}^{\frac{p}{2}}du \\ \quad +\int_s^T E\{\sum_{n=M+1}^\infty|\tilde{\sigma}_n(\overline{r}(u,q),u)|^2\}^{\frac{p}{2}}du\} \\ =: c_{T,p}\{\sum_{i=1}^3 A_{i,M}(T)\}. \end{array}\right\} \tag{10.52}$$

(ii)

$$\begin{aligned} A_{2,M}(T) &\le \int_s^T E\{\sum_{n=1}^\infty|\tilde{\sigma}_n(\overline{r}(u,q),u)-\tilde{\sigma}_n(\overline{r}_M(u,q),u)|^2\}^{\frac{p}{2}}du \\ &\le c_{\mathcal{J},p}\int_s^T E|\overline{r}(u,q)-\overline{r}_M(u,q)|^p du \end{aligned}$$

by assumption (1.14) on $\tilde{\mathcal{J}}(r,p,u)$.

(iii) Since $\sum_{n=1}^\infty|\tilde{\sigma}_n(\overline{r}(u,q),u)|^2 = \sum_{k,\ell=1}^d\int\tilde{\mathcal{J}}_{k\ell}^2(\overline{r}(u,q),\tilde{p},u)^2d\tilde{p} < \infty$ uniformly in u, ω and q the monotone convergence theorem implies that for any $\varepsilon > 0$ there is an $M(\varepsilon)$ such that for all $M \ge M(\varepsilon)$

$$A_{3,M}(T) \le \varepsilon.$$

(iv)

$$A_{1,M}(T) \le c_{F,p}\int_s^T E|\overline{r}(u,q)-\overline{r}_M(u,q)|^p du$$

by the Lipschitz condition on F.

(v) Hence, for all $M \ge M(\varepsilon)$

$$E\sup_{s\le t\le T}|\overline{r}(t,q)-\overline{r}_M(t,q)|^p \le \tilde{c}_{T,F,\mathcal{J},p}\varepsilon. \tag{10.53}$$

■

Let $\overline{\varphi}(t,q)$ and $\check{\varphi}(t,q)$ be the solutions of (10.23) associated with the forward flow $\overline{r}(t,q)$ and backward flow $\check{r}(t,q)$, respectively.

Lemma 10.10

Assume (2.20), $m \ge 1$, and let $p \ge 2$. Then for any $(\tilde{\varphi}_M, \tilde{\varphi}) \in \{(\overline{\varphi}_M, \overline{\varphi}), (\check{\varphi}_M, \check{\varphi})\}$ and any $L < \infty$

$$\sup_{|q|\le L} E\sup_{s\le t\le T}|\tilde{\varphi}_M(t,q)-\tilde{\varphi}(t,q)|^p \to \infty, \text{ as } M \to \infty. \tag{10.54}$$

Proof

(i) Assume without loss of generality $(\tilde{\varphi}_M, \tilde{\varphi}) = (\overline{\varphi}_M, \overline{\varphi})$. Set $\psi_M := \overline{\varphi} - \overline{\varphi}_M$. Then

$$
\begin{aligned}
\psi_M(t,q) &= \int_s^t \delta F(\overline{r}(u,q),u)\psi_M(u,q)du + \sum_{\ell=1}^d \int_s^t \int \delta \tilde{\mathcal{J}}_{M,\cdot\ell}(\overline{r}(u,q),\tilde{p},u)\psi_M(u,q)w_\ell(d\tilde{p},du) \\
&+ \int_s^t (\delta F(\overline{r}_M(u,q),u) - \delta F(\overline{r}(u,q),u))(\psi_M(u,q) - \overline{\varphi}(u,q))dq \\
&+ \sum_{\ell=1}^d \int_s^t \int (\delta \tilde{\mathcal{J}}_{M,\cdot\ell}(\overline{r}_M(u,q),\tilde{p},u) - \delta \tilde{\mathcal{J}}_{M,\cdot\ell}(\overline{r}(u,q),\tilde{p},u))(\psi_M(u,q) - \overline{\varphi}(u,q))w_\ell(d\tilde{p},du) \\
&+ \sum_{\ell=1}^d \sum_{n=M+1}^\infty \int_s^t \delta\tilde{\sigma}_{n,\cdot\ell}(\overline{r}(u,q),u)d\beta_{n,\ell}(u) \\
&= \sum_{i=1}^5 B_{i,M}(t).
\end{aligned}
$$

(ii) By the assumptions on $\tilde{\mathcal{J}}$ we obtain as in the previous proof of the previous lemma (step iv) (cf. also the proof of Lemma 10.7) that for any $p \geq 2$ and $\varepsilon > 0$

$$
\sup_q E \sup_{s \leq t \leq T} |B_{5,M}(t,q)|^p \leq \varepsilon
$$

provided $M \geq M(\varepsilon)$.

(iii) We easily see that for any $p \geq 2$ and $L < \infty$

$$
\sup_M \sup_{|q| \leq L} E \sup_{s \leq t \leq T} |\overline{\varphi}_M(t,q)|^p < \infty. \tag{10.55}
$$

Therefore, the boundedness and continuity of all coefficients in addition to Lemma (10.9) imply (10.54). ∎

Corollary 10.11

Assume (2.20) with $m \geq 1$. Then, for any $(\tilde{r}_M, \tilde{r}) \in \{(\overline{r}_M, \overline{r}), (\check{r}_M, \check{r})\}$, $L < \infty$ and $|\mathbf{j}| \leq m-1$

$$
E \sup_{|q| \leq L} \sup_{s \leq t \leq T} |\partial^{\mathbf{j}} \tilde{r}_M(t,q) - \partial^{\mathbf{j}} \tilde{r}(t,q)| \to 0, \text{ as } M \to \infty. \tag{10.56}
$$

Moreover, for any multiindex $\mathbf{j}$ with $|\mathbf{j}| \leq m-1$

$$
\partial^{\mathbf{j}} \tilde{r} \in L_0(\Omega; C(\mathbf{R}^d \times [s,T]; \mathbf{R}^d)). \tag{10.57}
$$

Proof

(10.56) follows in the same way as (10.50). (10.57) follows from (10.56) and (10.37). ∎

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