On inverse spectral theory for self–adjoint extensions: mixed types of spectra

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Abstract

Let H be a symmetric operator in a separable Hilbert space $\mathcal{H}$. Suppose that H has some gap J. We shall investigate the question about what spectral properties the self–adjoint extensions of H can have inside the gap J and provide methods how to construct self–adjoint extensions of H with prescribed spectral properties inside J.

Under some weak assumptions about the operator H which are satisfied, e. g., provided the deficiency indices of H are infinite and the operator $(H-\lambda)^{-1}$ is compact for one regular point λ of H, we shall show that for every (auxiliary) self–adjoint operator M' in the Hilbert space $\mathcal{H}$ and every open subset J_0 of the gap J of H there exists a self–adjoint extension $\tilde{H}$ of H such that inside J the self–adjoint extension $\tilde{H}$ of H has the same absolutely continuous and the same point spectrum as the given operator M' and the singular continuous spectrum of $\tilde{H}$ in J equals the closure of J_0 in J. Moreover we shall present a method how to construct such a self–adjoint extension $\tilde{H}$. Via our methods it is possible to construct new kinds of self–adjoint realizations of the Laplacian on a bounded domain Ω in $\mathbb{R}^d$, $d>1$, with spectral properties very different from the spectral properties of the self–adjoint realizations known before.

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1 Introduction

Spectral properties of self–adjoint realizations of the Laplacian on a bounded domain Ω in $\mathbb{R}^d$ (cf. Example 6.3 below for the definition) have been investigated for a very long time. Originally these investigations had been motivated by the important role of such operators in a wide variety of models in acoustics, electromagnetism and thermodynamics. In the present century the interest gradually shifted to fascinating new research areas. In non–relativistic quantum mechanics self–adjoint realizations of the Laplacian on a bounded domain Ω are used as Hamiltonians of "one–particle–systems" where the "quantum mechanical particle" is constrained to be inside the domain Ω for all times. In probability theory such operators occur as infinitesimal generator of Markov processes describing the motion of a Brownian particle where the underlying state space is the closure of Ω.

While in the classical theories one has mainly concentrated on "nice" domains, i. e. domains with smooth boundary, and a very restricted class of self–adjoint realizations of the Laplacian characterized by special boundary conditions, e. g. the Dirichlet or the Neumann boundary condition, in the new areas one is also interested in domains with irregular boundaries and a much larger class of self–adjoint realizations. In order to treat such more general cases one has used new techniques from the theory of pseudodifferential operators (cf., e. g., [19]), the theory of Dirichlet forms (cf., e. g. [14]) and operator theory (cf., e. g. [4]).

However, for a long time even this larger class of self–adjoint realizations has been very narrow with respect to spectral properties. More precisely for a long time all the known self–adjoint realizations of the Laplacian on a bounded domain Ω had, up to at most one isolated eigenvalue with infinite multiplicity, a purely discrete spectrum.

This situation changed only very recently. R. Hempel, Th. Kriecherbauer, P. Plankensteiner, L. A. Seco and B. Simon have given bounded domains Ω such that the Neumann Laplacian $-\Delta_N^\Omega$ on Ω has "unusual spectral properties". In [17] one has given, among others, bounded domains Ω in $\mathbb{R}^d$ such that the spectrum of the Neumann Laplacian on Ω equals $[0, \infty)$. Various additional results concerning the "shape of the spectrum of the Neumann Laplacian" have been presented in [16]. B. Simon has given bounded domains

Ω such that the Neumann Laplacian on Ω has non–empty absolutely continuous spectrum (cf. [21]). Using essentially the results of [21], R. Hempel and R. Weder have made new discoveries in quantum mechanical scattering theory ("completeness of wave operators under loss of local compactness", cf. [18]).

In [5], [9], [7], [8] and [10] we have, partially joint with J. Weidmann, introduced new classes of self–adjoint realizations of the Laplacian on a bounded domain Ω in $\mathbb{R}^d$, $d > 1$, with "unusual spectral properties", e. g. dense point spectrum, non–empty absolutely continuous and singular continuous spectrum. While R. Hempel, Th. Kriecherbauer, P. Plankensteiner, L. A. Seco, B. Simon and R. Weder have considered a fixed kind of self–adjoint realization, namely the Neumann Laplacian, and have investigated the dependence of the spectral properties on the domain Ω, in our approach the domain Ω is fixed and we have investigated the dependence of the spectral properties on the choice of the self–adjoint realization. It is a well known classical result by H. Weyl that the Neumann Laplacian on a bounded domain Ω with smooth boundary has a purely discrete spectrum. Thus, by necessity, R. Hempel et al. have considered domains Ω with irregular boundary while in our approach Ω might be any non–empty bounded domain in $\mathbb{R}^d$, $d > 1$, for instance it might be a ball.

Both in order to have very simple and clear concepts and in order to enlarge the possible range of applications we have worked within a rather general operator theoretical framework. More precisely in our papers the following two things have been given:

- A symmetric operator H in an infinite–dimensional Hilbert space $\mathcal{H}$.
- Certain "spectral properties" (we shall be more precise later).

We have investigated the following two closely related questions:

- Does there exist a self-adjoint extension of H with the given spectral properties ?
- How to construct a self-adjoint extension of H with the given spectral properties ?

These questions are so general that one cannot hope to get affirmative answers in general. Therefore we have made some assumptions about H and the "spectral properties".

We have assumed that there exist a and b, $-\infty \leq a < b < \infty$, such that the open interval $J := (a, b)$ is a *gap* of H, i. e. such that

$$\begin{array}{rcll} (Hf, f) & \geq & b\|f\|^2, \quad f \in D(H), & \text{if} \quad a = -\infty, \\ \|(H - \frac{a+b}{2})f\| & \geq & \frac{b-a}{2}\|f\|, \quad f \in D(H), & \text{if} \quad -\infty < a. \end{array} \tag{1}$$

It is well known that a real number λ is a *regular point* of H, i. e., that the operator $H - \lambda$ is continuously invertible, if and only if λ belongs to some gap J of H and that the gap condition (1) is equivalent to the fact that there exists at least one self–adjoint extension H' of H such that the open interval J is a gap in the spectrum of H', i. e. is contained in the resolvent set of H'. Let z be any regular point of H. Since H has a self–adjoint extension the dimension $n(H, z)$ of the space $N(H^* - z)$ does not depend on the special choice of the regular point z of H and, in particular, the deficiency indices $n(H, i)$ and $n(H, -i)$ of H coincide. Here H^* denotes the adjoint of H and $D(M)$, $R(M)$ and $N(M)$ the domain, the range and the kernel of the operator M, respectively. This dimension, which measures the degree of freedom one has in the construction of self–adjoint extensions, will be called the *deficiency number* of H.

Moreover, we have only considered spectral properties inside the union $\Gamma_r(H)$ of gaps of H and mainly concentrated on the "local spectral properties". At the **first** very rough **level** this means that a closed subset S of J, i.e. the intersection S of J and some closed subset of $\mathbb{R}$, is given and that one wants to know whether there exist other self-adjoint extensions $\tilde{H}$ of H such that

$$\sigma(\tilde{H}) \cap J = S \ ,$$

and how to construct such extensions. Here $\sigma(M)$, $\sigma_{ac}(M)$, $\sigma_{sc}(M)$ and $\sigma_{pp}(M)$ denote the spectrum, the absolutely continuous, the singular continuous and the pure point spectrum of the operator M, respectively.

We have given complete answers to these questions in [5]. When the deficiency number of H is infinite we have shown that for every closed subset S of J there exists a self-adjoint extension $\tilde{H}$ of H such that

$$\sigma(\tilde{H}) \cap J = S \ .$$

Moreover we have given the construction of such an extension $\tilde{H}$. Trivially, the spectrum of every operator is a closed set. Thus at least at the first level inside the set $\Gamma_r(H)$ the "local spectral properties" of the self-adjoint extensions of H do not depend on H provided the deficiency number of H is infinite.

Of course even at the first level the "global spectral properties" of the self-adjoint extensions of H strongly depend on H. 1947 M. G. Krein has posed the question about how the spectral properties of the self–adjoint extensions of H in one gap of H are "coupled with" the spectral properties in the other gaps ([20]). It was only in 1991 that V. A. Derkach and M. M. Malamud have gotten a breakthrough concerning this very hard problem (cf. [12]). We hope in future work to get new results concerning the global spectral properties of self–adjoint extensions via a combination of the methods by V. A. Derkach and M. M. Malamud and our methods. Trivially inside the set $\mathbb{R} \setminus \Gamma_r(H)$ even the local spectral properties of the self–adjoint extensions of H strongly depend on H.

At the **second level** the supports S_{pp}, S_{ac} and S_{sc} of a pure point, absolutely continuous and singular continuous, respectively, positive Radon measure on the gap J of H are given and one wants to know whether there exists a self-adjoint extension $\tilde{H}$ of H such that

$$\sigma_{pp}(\tilde{H}) \cap J = S_{pp}, \quad \sigma_{ac}(\tilde{H}) \cap J = S_{ac}, \quad \sigma_{sc}(\tilde{H}) \cap J = S_{sc},$$

and how to construct such an extension.

The point spectrum $\sigma_{pp}(M)$ of a self-adjoint operator M equals the closure of the set $\sigma_p(M)$ of eigenvalues of M. Thus in general the set $\sigma_{pp}(M)$ only gives rough information about the positions of the eigenvalues of M and, of course, no information about their multiplicities. Similarly, the sets $\sigma_{ac}(M)$ and $\sigma_{sc}(M)$ only give rough information about the absolutely continuous and singular continuous spectral measures of M. Thus at a third level one may ask more detailed questions about these measures.

For the formulation and the discussion of these deeper questions it is convenient to introduce some notation. For a self-adjoint operator M and a

Borel set B we denote by M_B the restriction of M to $D(M) \cap R(1_B(M))$, i. e.

$$M_B := M_{|D(M)\cap R(1_B(M))},$$

where $1_B(x) = 1$ for $x \in B$ and $1_B(x) = 0$ otherwise. By the Spectral Theorem, the Hilbert space $R(1_B(M))$ is a reducing subspace of M and M_B a self-adjoint operator in $R(1_B(M))$. In general "the spectral properties" of M and M_B "inside B" are very different. For instance, we may have $\sigma(M) \supset B$ and $\sigma(M_B) = \emptyset$. However, when B is open then "inside B the operators M and M_B have the same spectral properties", i.e. the spectral measures with respect to M and M_B coincide on the Borel σ-algebra of B. In particular, for open sets B we have that

$$\begin{gathered} \sigma(M) \cap B = \sigma(M_B) \cap B, \quad \sigma_{ac}(M) \cap B = \sigma_{ac}(M_B) \cap B, \\ \sigma_{sc}(M) \cap B = \sigma_{sc}(M_B) \cap B, \end{gathered} \tag{2}$$

and inside B the operators M and M_B have the same eigenvalues with the same multiplicities. All these facts easily follow from the Spectral Theorem.

At the **third level** one may ask the following questions. Suppose that an (auxiliary) self-adjoint operator M' in $\mathcal{H}$ is given. Does there exist a self-adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq M'_J,$$

i. e. such that $\tilde{H}_J$ is unitarily equivalent to M'_J, and how can such an extension $\tilde{H}$ be constructed?

Let S_{pp}, S_{ac} and S_{sc} be the support of some pure point, absolutely continuous and singular continuous positive Radon measure on J, respectively. Trivially there exists a self-adjoint operator M' in $\mathcal{H}$ such that

$$\sigma_{pp}(M') \cap J = S_{pp},\ \sigma_{ac}(M') \cap J = S_{ac},\ \sigma_{sc}(M') \cap J = S_{sc}.$$

If $\tilde{H}$ is a self-adjoint extension of H such that

$$\tilde{H}_J \simeq M'_J$$

then, by (2),

$$\sigma_{pp}(\tilde{H}) \cap J = S_{pp},\ \sigma_{ac}(\tilde{H}) \cap J = S_{ac},\ \sigma_{sc}(\tilde{H}) \cap J = S_{sc}.$$

Thus if one can answer the mentioned questions at the third level then one can answer these questions also at the second level, but the third level is much deeper than the second one.

In the special case when the deficiency number of H is finite M.G. Krein has completely answered the mentioned questions even at the deepest third level, cf. [20]. In particular he has proved the following

Theorem 1.1 (M.G. Krein, [20], Theorem 22 and Theorem 23) *Let H be a symmetric operator in a Hilbert space $\mathcal{H}$. Suppose that H has a gap J and the deficiency number n of H is finite. Let M' be a self-adjoint operator in $\mathcal{H}$. There exists a self-adjoint extension $\tilde{H}$ of H such that*

$$\tilde{H}_J \simeq M'_J$$

if and only if

$$dim\ R(1_J(M')) \leq n\ .$$

Due to this classical result by M. G. Krein the following conjecture seems to be natural:

Conjecture 1.2 *Let H be a symmetric operator in the separable Hilbert space $\mathcal{H}$. Suppose that H has a gap J and the deficiency number of H is infinite. Then for every self-adjoint operator M' in $\mathcal{H}$ there exists a self–adjoint extension $\tilde{H}$ of H such that*

$$\tilde{H}_J \simeq M'_J\ .$$

In a joint work with J. Weidmann ([9]) we have shown that the conjecture is true under the additional assumption that the auxiliary self–adjoint operator M' has a pure point spectrum inside the gap J of H.

In [8] we have shown that for every open subset J_0 of J there exists a self-adjoint extension $\hat{H}$ of H such that the singular continuous spectrum of $\hat{H}$ inside J equals the closure of J_0 in J, i. e.

$$\sigma_{sc}(\hat{H}) \cap J = \overline{J_0} \cap J, \tag{3}$$

provided the deficiency number of H is infinite. Under the additional hypothesis that the operator H is significantly deficient in the sense of [7], cf. the Definition 4.4 below, we have shown that for every self–adjoint operator M' in the separable Hilbert space $\mathcal{H}$ there exists a self-adjoint extension $\hat{H}$ of H such that

$$\hat{H}_{acJ} \simeq M'_{acJ}, \tag{4}$$

cf. [7]. Here M_{ac}, M_{sc} and M_{pp} denote the absolutely continuous, the singular continuous and the pure point part of the operator M, respectively.

However we have not been able to control simultaneously several parts of the spectrum. For instance it has been an open question whether the self–adjoint operator $\hat{H}$ in (3) and (4) can be chosen such that in addition

$$\sigma_p(\hat{H}) \cap J = \emptyset$$

and

$$\sigma_p(\hat{H}) \cap J = \emptyset = \sigma_{sc}(\hat{H}) \cap J,$$

respectively.

In the present paper we shall give a new result about the structure of symmetric extensions, cf. Lemma 2.1. It easily follows from this result that for every self–adjoint extension $\hat{H}$ of H there exist self–adjoint extensions $\tilde{H}^{(1)}$ and $\tilde{H}^{(2)}$ of H such that

$$\tilde{H}^{(1)}_J \simeq \hat{H}_{scJ}, \quad \tilde{H}^{(2)}_J \simeq \hat{H}_{acJ},$$

cf. the proofs of the Proposition 4.10 and the Corollary 5.2 below. Thus, in particular, we easily get from this positive answers to the mentioned open questions.

Under the assumption that H is the infinite orthogonal sum of symmetric operators with strictly positive deficiency numbers and the underlying Hilbert space $\mathcal{H}$ is complex we have given a method to construct a self–adjoint extension $\hat{H}$ of H such that

$$\hat{H} \simeq P \oplus G \tag{5}$$

for some self–adjoint operator G where P denotes the one–dimensional momentum operator, cf. [7]. This is a first result which at the second level just tells us that there exists a self–adjoint extension $\hat{H}$ of H such that

$$\sigma_{ac}(\hat{H}) \cap J = J.$$

Lemma 2.1 can also be used in order to improve this result considerably. We shall show that for every self–adjoint operator M' in the separable Hilbert space $\mathcal{H}$ there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq M'_{acJ},$$

cf. Corollary 4.13.

In the present paper we shall also introduce new results about the representation of self–adjoint extensions, cf. section 4.1.3. These new results make it possible to construct self–adjoint extensions of the symmetric operator H with arbitrarily prescribed absolutely continuous spectrum inside a gap J of H provided the operator H is weakly significantly deficient in the sense of the Definition 4.8 below. The price one has to pay for this greater generality is that the construction of self–adjoint extensions of H with prescribed absolutely continuous spectrum inside a gap J of H becomes more complicated. In the appendix we shall give a symmetric operator H which is weakly significantly deficient but not significantly deficient.

Our main results are contained in section 6. By the help of Lemma 2.1 and an additional trick, presented in the proofs of the Theorems 6.1 and 6.2, the problem to construct a self–adjoint extension $\tilde{H}$ of H with prescribed absolutely continuous, singular continuous and point spectrum inside the gap J of H can be reduced to the much more simple independent problems to construct one self–adjoint extension with prescribed absolutely continuous, another self–adjoint extension with prescribed singular continuous and a third self–adjoint extension with prescribed point spectrum inside the gap J of H. This makes it possible to derive fairly strong results concerning mixed types of spectra. For instance we shall show that the Conjecture 1.2 is true provided

$$\sigma_{sc}(M') \cap J = \emptyset$$

and the operator H is weakly significantly deficient in the sense of the definition 4.8 or H is the infinite orthogonal sum of symmetric operators with strictly positive deficiency numbers and the underlying Hilbert space $\mathcal{H}$ is complex. This result will be contained in Theorem 6.2. It is, however, not the main goal of the present paper to prove pure existence results but rather to provide methods to construct self–adjoint extensions with prescribed spectral properties.

Via Lemma 2.1 and the ideas presented in section 4.1.3 it is not only possible to get new results but it is also possible to leave away many superfluous steps in previous constructions. For convenience of the reader and in order to make this paper to some extend self–contained we shall present shortened versions of some of our previous constructions from [9] and [7] in the sections 3, 4.1.1 and 4.1.2. For simplicity occasionally we shall assume that the underlying Hilbert space is separable even if there is an obvious extension of our results to the general case.

2 An equivalent formulation of the problem

Let $\tilde{H}$ be a self–adjoint extension of H. By the Spectral Theorem, $\tilde{H}$ can be decomposed into the orthogonal sum of $\tilde{H}_J$ and $\tilde{H}_{\mathbb{R}\setminus J}$, i.e.

$$\tilde{H} = \tilde{H}_J \oplus \tilde{H}_{\mathbb{R}\setminus J}.$$

Since $\tilde{H}$ is a symmetric extension of H it is a restriction of the adjoint H^* of H. Thus the operator

$$H^*_{|D(H)+D(\tilde{H}_J)}$$

is a restriction of $\tilde{H}$ and consequently it can be decomposed into the orthogonal sum of $\tilde{H}_J$ and some symmetric operator G_0 in the Hilbert space $R(1_{\mathbb{R}\setminus J}(\tilde{H})) = R(1_J(\tilde{H}))^\perp$, i.e.

$$H^*_{|D(H)+D(\tilde{H}_J)} = \tilde{H}_J \oplus G_0 \ .$$

Since G_0 is a restriction of $\tilde{H}_{\mathbb{R}\setminus J}$ the gap J of H is also a gap of G_0.

Let M' be an auxiliary self–adjoint operator in $\mathcal{H}$. If there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq M'_J$$

then, by the preceeding considerations, there also exists a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ and a self–adjoint operator M in $\mathcal{H}_0$ such that $D(M) \subset D(H^*)$, $Mf = H^*f$ for all $f \in D(M)$,

$$\sigma(M) \subset \bar{J}, \quad M \simeq M'_J,$$

and

$$H^*_{|D(H)+D(M)} = M \oplus G_0$$

for some symmetric operator G_0 with gap J in the Hilbert space $\mathcal{H}_0^\perp$. For instance we may put

$$\mathcal{H}_0 := R(1_J(\tilde{H})), \quad M := \tilde{H}_J \ .$$

In the following it will be very useful that the existence of $\mathcal{H}_0$ and M with the mentioned properties is not only necessary but also sufficient in order that there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq M'_J.$$

More precisely we have

Lemma 2.1 *Let H be a symmetric operator in the Hilbert space $\mathcal{H}$. Suppose that H has a gap J. Let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}$ and M a self–adjoint operator in $\mathcal{H}_0$ such that M is a restriction of the adjoint H^* of H and $\sigma(M) \subset \bar{J}$. Then*

$$H_M := H^*_{|D(H)+D(M)} = M \oplus G_0 \tag{6}$$

for some symmetric operator G_0 with gap J in the Hilbert space $\mathcal{H}_0^\perp$. In particular, H has a self–adjoint extension $\tilde{H}$ such that

$$\tilde{H}_J = M_J.$$

Proof. We have only to show that there exists a symmetric operator G_0 with gap J such that (6) holds. In fact, since J is a gap of G_0 there exists a self–adjoint operator G in $\mathcal{H}_0^\perp$ such that $\sigma(G) \cap J = \emptyset$ and G is an extension of G_0. Then obviously $\tilde{H} := M \oplus G$ is a self–adjoint extension of H and satisfies $\tilde{H}_J = M_J$.

The existence of such an operator G_0 is an easy consequence of the following lemma which might be of interest in itself.

Lemma 2.2 *Let H be a symmetric operator in the Hilbert space $\mathcal{H}$. Let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}$ and M a self–adjoint operator in $\mathcal{H}_0$ such that M is a restriction of H^*. Then*

$$H_M := H^*_{\upharpoonright D(H)+D(M)} = M \oplus G_0$$

for some symmetric operator G_0 in $\mathcal{H}_0^\perp$.

In fact, suppose that Lemma 2.2 is true. Then we have only to show that under the additional hypothesis of Lemma 2.1 J is a gap of G_0. We shall give the proof of this assertion in the case when $J = (-\infty, b)$ for some real number b. The proof in the other case is virtually the same.

Assume that

$$(G_0 f, f) < b \, \| \, f \, \|^2 \tag{7}$$

for some $f \in D(G_0)$. We choose $g \in D(H)$ and $h \in D(M)$ such that

$$f = g + h.$$

Then we have

$$\begin{aligned} (Hg, g) &= (H_M(f-h), f-h) \\ &= (G_0 f, f) + (Mh, h) \\ &< b\,(\,(f, f) + (h, h)\,) \\ &= b \, \| \, f - h \, \|^2 = b \, \| \, g \, \|^2. \end{aligned} \tag{8}$$

In the second and the fourth step we have used that $H_M = M \oplus G_0$ and in the third step the assumption (7) and the hypothesis that $\sigma(M) \subset \bar{J}$ and consequently $(Mh, h) \le b \, \| \, h \, \|^2$ for every $h \in D(M)$. (8) is a contradiction to the hypothesis that $(-\infty, b)$ is a gap of H. Thus we have shown that $(G_0 f, f) \ge b \, \| \, f \, \|^2$ for every $f \in D(G_0)$, i.e. that $(-\infty, b)$ is a gap of G_0, provided $(-\infty, b)$ is a gap of H. □

Proof of Lemma 2.2

Let $f, \tilde{f} \in D(H)$ and $g, \tilde{g} \in D(M)$. We have

$$\begin{aligned} (H_M(f+g), \tilde{f} + \tilde{g}) &= (Hf, \tilde{f} + \tilde{g}) + (H^* g, \tilde{f}) + (Mg, \tilde{g}) \\ &= (f, H^*(\tilde{f} + \tilde{g})) + (g, H\tilde{f}) + (g, M\tilde{g}) \\ &= (f + g, H_M(\tilde{f} + \tilde{g})). \end{aligned}$$

Thus H_M is a symmetric operator in the Hilbert space $\mathcal{H}$.

Let $f \in D(H_M)$. For every $n \in \mathbb{Z}$ let

$$P_n := 1_{[n,n+1)}(M)P_{\mathcal{H}_0}.$$

Since M is a self–adjoint operator in the Hilbert space $\mathcal{H}_0$ it follows from the Spectral Theorem that for every $n \in \mathbb{Z}$ the operator P_n is an orthogonal projection in $\mathcal{H}$ onto the closed subspace $R(P_n)$ of $\mathcal{H}$,

$$R(P_n) \subset D(M), \quad n \in \mathbb{Z}, \tag{9}$$

$$R(P_n) \perp R(P_m), \quad n \neq m, \tag{10}$$

$$\sum_{n\in\mathbb{Z}} P_n = P_{\mathcal{H}_0}, \tag{11}$$

$$P_n M g = M P_n g, \quad g \in D(M), n \in \mathbb{Z}. \tag{12}$$

Thus we have

$$\begin{aligned}(P_n H_M f, g) &= (H_M f, g) = (f, Mg) \\ &= (P_n f, Mg) = (M P_n f, g)\end{aligned}$$

for every $g \in R(P_n)$. In the second step we have used (9) and the facts that H_M is symmetric and M a restriction of H_M. In the third step we have used (12) and in the last step again (9). Thus we have

$$P_n H_M f = M P_n f, \quad n \in \mathbb{Z}. \tag{13}$$

Since, by (13), (10) and (11),

$$(M P_n f, M P_k f) = (P_n H_M f, P_k H_M f) = 0, \quad k \neq n,$$

and

$$\sum_{n\in\mathbb{Z}} \| M P_n f \|^2 = \sum_{n\in\mathbb{Z}} \| P_n H_M f \|^2 = \| P_{\mathcal{H}_0} H_M f \|^2 < \infty,$$

the sequence $\{M\sum_{n=-N}^{N} P_n f\}_{N\in\mathbb{N}}$ converges in $\mathcal{H}_0$. Since M is closed and, by (11), $\lim_{N\to\infty}\sum_{n=-N}^{N} P_n f = P_{\mathcal{H}_0} f$ it follows that

$$P_{\mathcal{H}_0} f \in D(M) \tag{14}$$

and

$$MP_{\mathcal{H}_0} f = \lim_{N\to\infty}\sum_{n=-N}^{N} MP_n f = \sum_{n\in\mathbb{Z}} P_n H_M f = P_{\mathcal{H}_0} H_M f. \tag{15}$$

Here again we have used (13) and (11). By (14) and (15),

$$H_M = M \oplus G_0,$$

where

$$G_0 := H_{M|\ D(H_M)\cap\mathcal{H}_0^{\perp}}.$$

G_0 is a symmetric operator in the Hilbert space $\mathcal{H}_0^{\perp}$ since H_M is a symmetric operator in $\mathcal{H}$. $\square$

Remark: Note that the last assertion in Lemma 2.1 is not only an existence result: If G_0 is a symmetric operator with gap J then one can give explicitly a self–adjoint extension G of G_0 preserving the gap J, e.g. the Friedrichs and the Krein extension of G_0 when J is unbounded and bounded, respectively. Thus in the situation of Lemma 2.1 one can give explicitly a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J = M_J$$

provided one knows the operator M explicitly.

3 Point spectra

Since the set of eigenvalues of a self–adjoint operator in a separable Hilbert space is countable (i.e. empty, finite or countable infinite) and the multiplicity of an eigenvalue is also countable the following proposition provides a complete answer to the question about which kinds of point spectra the self–adjoint extensions of H can have inside the gap J of H.

Proposition 3.1 ([9], Theorem 2.2) *Let H be a symmetric operator in the separable Hilbert space $\mathcal{H}$. Suppose that H has a gap J. Let $\{\lambda_n\}_{n=1}^N$ be a (finite or infinite) sequence in J. Then there exists a self–adjoint extension $\tilde{H}$ of H such that*

$$\sigma_p(\tilde{H}) \cap J = \{\lambda_n : n \in \mathbb{N},\ 1 \le n \le N\}$$

and for every eigenvalue λ of $\tilde{H}$ in J the multiplicity of λ equals the number of times it occurs in the sequence $\{\lambda_n\}_{n=1}^N$ if and only if N is less than or equal to the deficiency number of H. In this case $\tilde{H}$ can be chosen such that it has a pure point spectrum inside the gap J.

Proof: First we shall prove the "only-if-part". Trivially the assertion of this part is true provided the deficiency number of H is infinite. Thus we may assume that the deficiency number of H is finite. But then the "only-if-part" follows from Krein's Theorem, i.e. Theorem 1.1.

Suppose now that N is less than or equal to the deficiency number of H. Then we can choose, by induction, an orthonormal system $\{e_n\}_{n=1}^N$ such that

$$e_n \in N(H^* - \lambda_n),\ n \in \mathbb{N},\ 1 \le n \le N,$$

due to the well known fact that the dimension of $N(H^* - \lambda)$ equals the deficiency number of H for every regular point λ of H and, in particular, for every $\lambda \in J$.

Let

$$\mathcal{H}_{00} := \operatorname{span}\{e_n : n \in \mathbb{N},\ 1 \le n \le N\},$$

and

$$\mathcal{H}_0 := \overline{\mathcal{H}_{00}},$$

$$M_0 := H^*_{|\mathcal{H}_{00}}.$$

By construction, $\{e_n\}_{n=1}^N$ is an orthonormal base of the Hilbert space $\mathcal{H}_0$ and for every $n \in \mathbb{N}$, $1 \le n \le N$, e_n is an eigenvector of M_0 corresponding to the real eigenvalue λ_n. Thus M_0 can and will be regarded as an operator in the Hilbert space $\mathcal{H}_0$, its closure M is a self–adjoint operator in $\mathcal{H}_0$, has a pure point spectrum,

$$\sigma_p(M) = \{\lambda_n : n \in \mathbb{N},\ 1 \le n \le N\}$$

and for every eigenvalue λ of M the multiplicity of λ equals the number of times it occurs in the sequence $\{\lambda_n\}_{n=1}^N$. M is a restriction of H^* since the adjoint of any operator is closed. Thus, by Lemma 2.1, H has a self–adjoint extension $\tilde{H}$ such that

$$\tilde{H}_J = M,$$

i.e. such that $\tilde{H}$ has the required properties. □

Remarks: (i) Proposition 3.1 is not only an existence result, but the proof of the proposition yields a method how to construct a self–adjoint extension of H with the given spectral properties.

(ii) Trivially,

$$\sigma(M) \cap J = \overline{\sigma_p(M)} \cap J$$

for every self–adjoint operator M with pure point spectrum inside J. Thus, by Krein's Theorem, there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\sigma(\tilde{H}) \cap J = S$$

if and only if S is a subset of J with a number of elements less than or equal to n, provided the deficiency number n of H is finite. For every closed subset S of J there exists a self–adjoint extension $\tilde{H}$ such that

$$\sigma(\tilde{H}) \cap J = S$$

provided the deficiency number of $\tilde{H}$ is infinite since S has a countable dense subset.

The proof of Proposition 3.1 yields a method how to construct such a self–adjoint extension. Thus at the first level we have given complete answers to the questions mentioned in the introduction of this chapter.

(iii) By Proposition 3.1, the Conjecture 1.2 is true, provided

$$\sigma_{ac}(M') \cap J = \emptyset = \sigma_{sc}(M') \cap J \ .$$

4 Absolutely continuous spectra

Let H be a symmetric operator in a Hilbert space $\mathcal{H}$. Suppose that H has a gap J and the deficiency number of H is infinite. It is an open question whether there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\emptyset \neq \sigma_{ac}(\tilde{H}) \cap J.$$

However given any self-adjoint operator M' in a separable Hilbert space one can easily construct a self–adjoint extension $\tilde{H}$ of H such that even

$$\tilde{H}_{acJ} \simeq M'_{acJ}$$

provided that the operator $(H-\lambda)^{-1}$ is nuclear, or, in other words, belongs to the trace class, for one and therefore every $\lambda \in J$. We shall present the corresponding construction in the following subsection 4.1.1. As an illuminating example we shall consider the special case where $\mathcal{H} = L^2(D)$, with

$$D = \bigcup_{n\in\mathbb{N}} (a_n, b_n) \tag{16}$$

for some sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ of pairwise disjoint non–empty bounded open intervals (a_n, b_n) such that

$$\sum_{n=1}^{\infty} |b_n - a_n| < \infty, \tag{17}$$

and $H = P_D$, where for every non–empty open subset D of $\mathbb{R}$ the operator P_D in $L^2(D)$ is defined by

$$\begin{array}{rcl} D(P_D) & := & C_0^\infty(D), \\ P_D f & := & -if', \quad f \in D(P_D). \end{array} \tag{18}$$

Here $C_0^\infty(D)$ denotes the space of infinitely differentiable functions with compact support in D. The condition that the operator $(H-\lambda)^{-1}$ belongs to the trace class can be considerably weakened. Actually it suffices to require that the operator H is *significantly deficient* in the sense of the Definition 4.4. Using the results from the following subsection 4.1.1 we shall present the corresponding construction in subsection 4.1.2. We shall also show that the operator H is significantly deficient provided that the deficiency number of H is infinite and the operator $(H-\lambda)^{-1}$ is compact for one and therefore every $\lambda \in J$. Thus, in particular, the operator P_D, defined by (18), is significantly deficient, provided that D is the union of pairwise disjoint non–empty bounded open intervals (a_n, b_n) and

$$b_n - a_n \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

Another important example for a significantly deficient operator is the minimal Laplacian on a bounded domain Ω in $\mathbb{R}^d$, $d > 1$, cf. Example 6.3.

The condition that H is significantly deficient can be replaced by the strictly weaker one that H is weakly significantly deficient in the sense of the Definition 4.8 below, cf. subsection 4.1.3.

"Significantly deficient" is an abbreviation for "significantly different from an operator with finite deficiency number". In fact one can easily show that a symmetric operator H with finite deficiency number and, more generally, the orthogonal sum of arbitrarily many pairwise unitarily equivalent symmetric operators with finite deficiency number are not weakly significantly deficient. In particular, the operator P_D is not weakly significantly deficient if all the intervals (a_n, b_n) have the same length. In subsection 4.2 we shall present a method to construct self–adjoint extensions with prescribed absolutely continuous spectrum inside the gap J of the symmetric operator H which is very different from the methods used in the previous subsection 4.1 and which works provided the underlying Hilbert space $\mathcal{H}$ is complex and the operator H is the orthogonal sum of infinitely many operators with strictly positive deficiency numbers. Of course this hypothesis is satisfied when $H = P_D$ and D is **any** infinite union of pairwise disjoint non–empty open intervals.

4.1 The trace class method

4.1.1 Operators with nuclear inverse

Without loss of generality we may assume that the symmetric operator H is closed and zero belongs to the gap J of H. Then H is continuously invertible and the range $R(H)$ of H is a closed subspace of the underlying Hilbert space $\mathcal{H}$. Thus

$$\mathcal{H} = R(H) \oplus R(H)^\perp.$$

According to this decomposition the inverse H^{-1} of H can be represented by a block matrix:

$$H^{-1} = \begin{pmatrix} A \\ B \end{pmatrix} \;:\; R(H) \to \begin{array}{c} R(H) \\ \oplus \\ R(H)^\perp \end{array}.$$

In other words,

$$A := P_{R(H)}H^{-1},\ B := P_{R(H)^\perp}H^{-1}, \tag{19}$$

where $P_{\mathcal{L}}$ denotes the orthogonal projection in $\mathcal{H}$ onto the subspace $\mathcal{L}$. A and B can and will be regarded as an operator in the Hilbert space $R(H)$ and as an operator from the Hilbert space $R(H)$ to the Hilbert space $R(H)^\perp$, respectively. We shall use the following trivial consequence of the results from [6], section 2.

Lemma 4.1 *Let $-\infty \le a < 0 < b < \infty$ and $J = (a, b)$. Let H be a closed symmetric operator in a Hilbert space $\mathcal{H}$ such that J is a gap of H and let A and B be defined by (19). Then A is a bounded invertible self–adjoint operator in the Hilbert space $R(H)$ and*

$$\sigma(A^{-1}) \cap J = \emptyset \tag{20}$$

and B is an everywhere defined bounded operator from $R(H)$ to $R(H)^\perp$ such that $R(B)$ is dense in $R(H)^\perp$ and

$$R(B^*) \cap R(A) = \{0\}. \tag{21}$$

Conversely let $\mathcal{R}$ and $\mathcal{N}$ be Hilbert spaces, A a bounded invertible self–adjoint operator in $\mathcal{R}$ such that

$$\frac{1}{a} \le A \le \frac{1}{b}$$

and B an everywhere defined bounded operator from $\mathcal{R}$ to $\mathcal{N}$ such that $R(B)$ is dense in $\mathcal{N}$ and (21) holds. Then the operator L in the Hilbert space $\mathcal{H} := \mathcal{R} \oplus \mathcal{N}$, defined by

$$\begin{aligned} D(L) &:= \mathcal{R}, \\ Lf &:= Af \oplus Bf, \quad f \in D(L), \end{aligned}$$

is the inverse of a closed symmetric operator H and J is a gap of H.

Let Q be any self–adjoint operator in the Hilbert space $R(H)^\perp$. Q is our "free parameter". Obviously the operator

$$\begin{pmatrix} A & B^* \\ B & Q \end{pmatrix} : \begin{array}{c} R(H) \\ \oplus \\ D(Q) \end{array} \rightarrow \begin{array}{c} R(H) \\ \oplus \\ R(H)^\perp \end{array}$$

is an extension of H^{-1}. Since it is the sum of the self–adjoint operator $A \oplus Q$ and the bounded self–adjoint operator

$$\begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix} : \begin{array}{c} R(H) \\ \oplus \\ R(H)^\perp \end{array} \rightarrow \begin{array}{c} R(H) \\ \oplus \\ R(H)^\perp \end{array}$$

it is self–adjoint. Since it is an extension of H^{-1} its range contains $R(H^{-1}) = D(H)$. Thus it is a self–adjoint operator with dense range. Consequently it is invertible and its inverse $\tilde{H}$ is a self–adjoint extension of H.

Suppose now that the operator B belongs to the trace class. Then $\tilde{H}^{-1}$ equals the sum of $A \oplus Q$ plus a self–adjoint trace class operator. Thus, by the Kato–Rosenblum Theorem, we have

$$\left(\tilde{H}^{-1}\right)_{ac} \simeq (A \oplus Q)_{ac} \ . \tag{22}$$

Up to now our "parameter" Q has been "free". In the following we shall also require that Q is invertible. It easily follows from the Spectral Theorem that

$$(M^{-1})_{ac} = (M_{ac})^{-1}$$

for every invertible self–adjoint operator M and that

$$(M_1 \oplus M_2)_{ac} = (M_1)_{ac} \oplus (M_2)_{ac}$$

for all self–adjoint operators M_1 and M_2. Thus (22) implies that

$$\tilde{H}_{ac} \simeq A^{-1}_{ac} \oplus Q^{-1}_{ac} \ .$$

Since $\sigma(A^{-1}) \cap J = \emptyset$ this implies that

$$\tilde{H}_{ac\ J} \simeq Q^{-1}_{ac\ J}.$$

Thus we have proved the following

Lemma 4.2 *Let H be a closed symmetric operator in the Hilbert space $\mathcal{H}$. Suppose that H has a gap J and $0 \in J$. Let*

$$A := P_{R(H)} H^{-1}\,, B := P_{R(H)^\perp} H^{-1},$$

and Q an invertible self–adjoint operator in the Hilbert space $R(H)^\perp$. Suppose that B belongs to the trace class. Then the operator

$$\begin{pmatrix} A & B^* \\ B & Q \end{pmatrix} : \begin{matrix} R(H) \\ \oplus \\ D(Q) \end{matrix} \to \begin{matrix} R(H) \\ \oplus \\ R(H)^\perp \end{matrix}$$

is invertible, its inverse $\tilde{H}$ is a self–adjoint extension of H and

$$\tilde{H}_{acJ} \simeq Q^{-1}_{acJ}.$$

Remark: Suppose that B belongs to the trace class and $R(H)^\perp$ is infinite dimensional. Let M' be an auxiliary self–adjoint operator in a separable Hilbert space. Then we can choose an invertible self–adjoint operator Q in $R(H)^\perp$ such that

$$Q^{-1}_{acJ} \simeq M'_{acJ}.$$

Thus the previous lemma provides a method to construct a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_{acJ} \simeq M'_{acJ}$$

provided the deficiency number of H is infinite and B belongs to the trace class.

Example 4.3 Let $-\infty < a < b < \infty$. It is well known and easily verified that $\tilde{H}$ is a self–adjoint extension of the operator $P_{(a,b)}$, defined by (18), if and only if $\tilde{H} = P_{a,b,\theta}$ for some $\theta \in \mathbb{R}$ where

$$\begin{aligned} D(P_{a,b,\theta}) &:= \{f \in H^1(a,b) : f(b-) = e^{i\theta} f(a+)\}, \\ P_{a,b,\theta} f &:= -if', \quad f \in D(P_{a,b,\theta}). \end{aligned}$$

Obviously $P_{a,b,\theta} = P_{a,b,\theta'}$ if and only if $\theta = \theta' \bmod 2\pi$. Moreover for every $\theta \in \mathbb{R}$ the operator $P_{a,b,\theta}$ has a purely discrete spectrum, all its eigenvalues are simple and

$$\sigma(P_{a,b,\theta}) = \{\frac{\theta + 2\pi n}{b-a} : n \in \mathbb{Z}\}.$$

Thus the non–empty open interval J is a gap of the operator $P_{(a,b)}$ if and only if the length $|J|$ of J is less than or equal to $2\pi/(b-a)$. In particular, the real axis is covered by gaps of $P_{(a,b)}$.

Let $\lambda \in \mathbb{R}$. The operator $\overline{P_{(a,b)}} - \lambda$ is a restriction of the invertible self–adjoint operator $P_{a,b,\lambda-\pi} - \lambda$. Thus it is also invertible and

$$\|(\overline{P_{(a,b)}} - \lambda)^{-1}\| \leq \|(P_{a,b,\lambda-\pi} - \lambda)^{-1}\| = \frac{b-a}{\pi}.$$

Here $\|\cdot\|$ denotes the operator norm. It is well known and easily seen that

$$R(\overline{P_{(a,b)}} - \lambda)^{\perp} = [e_{a,b,\lambda}]$$

where $e_{a,b,\lambda} : (a,b) \longrightarrow \mathbb{C}$ is defined by

$$e_{a,b,\lambda}(x) := \frac{1}{\sqrt{b-a}} e^{i\lambda x}, \quad x \in (a,b),$$

and $[\ldots]$ denotes the closure of the span of the set $\{\ldots\}$. Thus the operator

$$B_{a,b,\lambda} := P_{R(\overline{P_{(a,b)}}-\lambda)^{\perp}} (\overline{P_{(a,b)}} - \lambda)^{-1}$$

is a rank one operator and

$$\|B_{a,b,\lambda}\|_1 = \|B_{a,b,\lambda}\| \leq \frac{b-a}{\pi}. \tag{23}$$

Here $\|\cdot\|_1$ denotes the trace norm.

Let D be the union of pairwise disjoint non–empty bounded open intervals (a_n, b_n), $n \in \mathbb{N}$. Then obviously the closure of the operator P_D, defined by (18), can be decomposed as

$$\overline{P_D} = \bigoplus_{n\in\mathbb{N}} \overline{P_{(a_n,b_n)}}$$

where we have identified $L^2(D)$ and $\bigoplus_{n\in\mathbb{N}} L^2(a_n, b_n)$ in the canonical way. Thus P_D has a gap if and only if $\sup_{n\in\mathbb{N}} |b_n - a_n| < \infty$ and in this case the non–empty open interval J is a gap of P_D if and only if

$$|J| \leq \frac{2\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|}$$

and the real axis is covered by gaps of P_D.

Assume now that $\sup_{n\in\mathbb{N}} |b_n - a_n| < \infty$. Let $\lambda \in \mathbb{R}$. Then the interval

$$J := (\frac{-\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|}, \frac{\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|})$$

is a gap of the operator

$$H := \overline{P_D} - \lambda.$$

It is easily verified that

$$R(H)^{\perp} = \bigoplus_{n=1}^{\infty} [e_{a_n,b_n,\lambda}],$$

$$B := P_{R(H)^{\perp}} H^{-1} = \bigoplus_{n=1}^{\infty} B_{a_n,b_n,\lambda}.$$

This implies that the deficiency number of H is infinite since it equals the dimension of $R(H)^{\perp}$ and, by (23), that the operator B belongs to the trace class provided

$$\sum_{n=1}^{\infty} |b_n - a_n| < \infty. \tag{24}$$

Thus Lemma 4.2 can be used in order to construct self–adjoint extensions of the operator P_D with prescribed absolutely continuous spectrum inside the gap

$$(\lambda - \frac{\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|}, \lambda + \frac{\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|})$$

of P_D provided that (24) holds.

4.1.2 Significantly deficient operators

So far we have described a method how to construct self–adjoint extensions with given absolutely continuous spectra inside the gap J which works provided the operator B belongs to the trace class. When the operator B does not belong to the trace class then one has to modify the above considerations.

Let P be an orthogonal projection in the Hilbert space $R(H)^{\perp}$. Let 0 be the zero–operator in the Hilbert space $R(1-P) = R(H)^{\perp} \ominus R(P) \subset R(H)^{\perp}$. Obviously the opertor 0 satisfies the hypothesis of Lemma 2.1 and, by this lemma,

$$H_0 := H^*_{|D(H)+R(1-P)} = 0 \oplus G_0 \tag{25}$$

for some symmetric operator G_0 in $R(1-P)^{\perp} = R(H) \oplus R(P)$ with gap J. Obviously $R(G_0) = R(H_0) = R(H)$. By (25), we have

$$P_{R(1-P)^{\perp}} H^{-1} f = Af \oplus PBf \in D(G_0)$$

and

$$\begin{aligned} G_0(Af \oplus PBf) &= H_0(Af \oplus PBf \oplus (1-P)Bf) \\ &= HH^{-1}f = f \end{aligned}$$

for every $f \in R(H) = R(G_0)$. Thus G_0^{-1} can be represented in the form

$$G_0^{-1} = \begin{pmatrix} A \\ PB \end{pmatrix} : R(G_0) \longrightarrow \begin{matrix} R(G_0) \\ \oplus \\ R(P) \end{matrix} .$$

Thus in Lemma 4.2 we can replace H and B by G_0 and PB, respectively, provided the operator PB belongs to the trace class. Thus this lemma provides a method to construct self–adjoint extensions G of G_0 with arbitrarily prescribed absolutely continuous spectrum inside the gap J of H provided PB belongs to the trace class and $R(P)$ is infinite dimensional. Since for every self–adjoint extension G of G_0 the operator $0 \oplus G$ is a self–adjoint extension of H and obviously $(0 \oplus G)_{ac\,J} = G_{ac\,J}$ we also have a method to construct self–adjoint extensions of H with arbitrarily prescribed absolutely continuous spectrum inside the gap J of H in this case.

Thus we are led to the question about when there exists an orthogonal projection P in $R(H)^{\perp}$ such that the operator PB belongs to the trace class and $R(P)$ is infinite dimensional. If $R(H)^{\perp}$ is infinite dimensional and P is an orthogonal projection in this Hilbert space such that $\dim R(P) = \infty$ and the operator PB is compact then obviously we have $R(B) \neq R(H)^{\perp}$.

Conversely let us assume that $R(B) \neq R(H)^{\perp}$. Then $R(BB^*) \neq R(H)^{\perp}$ and therefore also $R(\sqrt{BB^*}) = R(|\, B^* \,|) \neq R(H)^{\perp}$. Thus 0 belongs to the spectrum of the self–adjoint operator $|\, B^* \,|$ in $R(H)^{\perp}$. On the other hand 0 is not an eigenvalue of $|\, B^* \,|$ since $R(B)$ is dense in $R(H)^{\perp}$ and therefore $N(B^*) = N(|\, B^* \,|) = \{0\}$. Thus 0 belongs to the essential spectrum of $|\, B^* \,|$. Thus we can choose an orthonormal system $\{e_n\}_{n\in\mathbb{N}}$ in $R(H)^{\perp}$ such that

$$\sum_{n=1}^{\infty} \|\, |B^*|\, e_n \,\| = \sum_{n=1}^{\infty} \|\, B^* e_n \,\| < \infty.$$

Thus the operator $B^*_{|\mathcal{Q}}$ from $\mathcal{Q} := [e_n : n \in \mathbb{N}]$ to $R(H)$ belongs to the trace class. Let P be the orthogonal projection in $R(H)^{\perp}$ onto $\mathcal{Q}$. Obviously $(PB)^* = B^*_{|Q}$ and PB also belongs to the trace class. Thus we have shown that there exists an orthogonal projection such that PB belongs to the trace class and $R(P)$ is infinite dimensional if and only if $R(B) \neq R(H)^{\perp} = N(H^*)$. Of course this is equivalent to $P_{N(H^*)}D(H) \neq N(H^*)$. By [10], Proposition 9, we have

$$P_{N(H^*-z)}D(\bar{H}) \neq N(H^* - z) \tag{26}$$

for every regular point z of H provided H is a symmetric operator, has at least one real regular point and (26) holds for at least one regular point z of H. Thus we have shown that there exists an orthogonal projection P with the required properties if and only if the operator H is significantly deficient in the sense of the following definition 4.4 and we have proved the following Proposition 4.5 and Corollary 4.6.

Definition 4.4 *A symmetric operator H is* **significantly deficient** *if and only if it has a real regular point and*

$$P_{N(H^*-z)}D(\bar{H}) \neq N(H^* - z)$$

for every regular point z of H.

Proposition 4.5 *(i) Let H be a closed symmetric operator in the Hilbert space $\mathcal{H}$. Let J be a gap of H and $0 \in J$. Let $A := P_{R(H)}H^{-1}$, $B := P_{R(H)^{\perp}}H^{-1}$ and P an orthogonal projection in $R(H)^{\perp}$ such that the operator*

PB belongs to the trace class and let 0 be the zero–operator in the Hilbert space $R(1-P)$. Then for every invertible self–adjoint operator Q in the Hilbert space $R(P)$ the operator

$$\tilde{L} := \begin{pmatrix} A & (PB)^* \\ PB & Q \end{pmatrix} : \begin{matrix} R(H) \\ \oplus \\ D(Q) \end{matrix} \longrightarrow \begin{matrix} R(H) \\ \oplus \\ R(P) \end{matrix}$$

is invertible and the operator

$$\tilde{H} := 0 \oplus \tilde{L}^{-1}$$

is a self–adjoint extension of H such that

$$\tilde{H}_{acJ} \simeq Q^{-1}_{acJ}.$$

(ii) There exists an orthogonal projection P in the Hilbert space $R(H)^\perp$ such that the operator PB belongs to the trace class and $R(P)$ is infinite dimensional if and only if the operator H is significantly deficient in the sense of the Definition 4.4.

Corollary 4.6 *Let H be a significantly deficient symmetric operator and let J be a gap of H. Then for every self–adjoint operator M' in a separable Hilbert space there exists a self–adjoint extension $\tilde{H}$ of H such that*

$$\tilde{H}_{acJ} \simeq M'_{acJ}.$$

Example 4.7 In the following we shall use the notation and the results from the Example 4.3. Let D be the union of pairwise disjoint non–empty bounded open intervals (a_n, b_n) such that

$$b_n - a_n \longrightarrow 0, \quad n \to \infty.$$

Let $\lambda \in \mathbb{R}$ and $H := \overline{P_D} - \lambda$. Then the open interval

$$J = \left(-\frac{\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|}, \frac{\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|}\right)$$

is a gap of H. The operator

$$\hat{H} := \bigoplus_{n=1}^{\infty}(P_{a_n,b_n,\lambda-\pi} - \lambda)$$

is a self–adjoint extension of H, has a purely discrete spectrum and

$$\sigma(\hat{H}) = \{\frac{\pi + 2\pi k}{b_n - a_n} : k \in \mathbb{Z}, n \in \mathbb{N}\}.$$

Thus the operator $\hat{H}$ is continuously invertible and its inverse $\hat{H}^{-1}$ is compact. Thus H is also continuously invertible and its inverse H^{-1} is compact. Since the deficiency number of H is infinite it follows that H is significantly deficient. Thus our preceding results can be used in order to construct self–adjoint extensions of P_D with arbitrarily prescribed absolutely continuous spectrum inside the gap

$$J = (\lambda - \frac{\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|}, \lambda + \frac{\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|})$$

of P_D provided

$$b_n - a_n \longrightarrow 0, \quad n \to \infty.$$

Trivially this condition can be weakened. It suffices to require that

$$b_{n_j} - a_{n_j} \longrightarrow 0, \quad j \to \infty,$$

for some subsequence $\{n_j\}_{j\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$.

4.1.3 Weakly significantly deficient operators

In this subsection let H be a symmetric operator in a Hilbert space $\mathcal{H}$ which is weakly significantly deficient in the sense of the following

Definition 4.8 A symmetric operator H is *weakly significantly deficient* if there exist a real regular point ν of H and a real number λ which is not an eigenvalue of the operator

$$A := P_{R(\bar{H}-\nu)}(\bar{H} - \nu)^{-1}$$

such that

$$R(B(A-\lambda)) \neq N(H^*-\nu)$$

where

$$B := P_{N(H^*-\nu)}(\bar{H}-\nu)^{-1}.$$

We may assume that the operator H is closed and $\nu = 0$. $R(A-\lambda)$ is dense in the Hilbert space $R(H)$ since A is self–adjoint and λ is not an eigenvalue of A. Since B is bounded and $R(B)$ is dense in $R(H)^{\perp}$ this implies that $R(B(A-\lambda))$ is dense in $R(H)^{\perp}$. Thus we can replace B by $B(A-\lambda)$ in the considerations at the beginning of the previous subsection 4.1.2 and get that there exists an orthogonal projection P in the Hilbert space $R(H)^{\perp}$ such that

$$\dim R(P) = \infty$$

and the operator $PB(A-\lambda)$ belongs to the trace class. Let 0 be the zero operator in $R(1-P)$. We have shown in the previous subsection 4.1.2 that $H_0 := H^*_{|D(H)+R(1-P)}$ can be decomposed as

$$H_0 = 0 \oplus G_0 \tag{27}$$

for some continuously invertible symmetric operator G_0 in $R(H) \oplus R(P)$ such that

$$G_0^{-1} = \begin{pmatrix} A \\ PB \end{pmatrix} : R(H) \longrightarrow \begin{array}{c} R(H) \\ \oplus \\ R(P) \end{array}.$$

Let Q be any invertible self–adjoint operator in the Hilbert space $R(P)$. By the considerations in the previous subsection, G_0 has an invertible self–adjoint extension G such that

$$G^{-1} - \lambda = \begin{pmatrix} A-\lambda & B^*P \\ PB & Q-\lambda \end{pmatrix} : \begin{array}{c} R(H) \\ \oplus \\ D(Q) \end{array} \longrightarrow \begin{array}{c} R(H) \\ \oplus \\ R(P) \end{array}. \tag{28}$$

The following simple lemma will play a key role in the investigation of the absolutely continuous spectrum of the operator G.

Lemma 4.9 *Let K_2 be a bounded self–adjoint operator and K_1 a self–adjoint operator such that $K_2K_1 1_\Delta(K_1)$ belongs to the trace class for every bounded interval Δ. Then*

$$(K_1 + K_2)_{ac} = K_1' \oplus R$$

for some self–adjoint operators K_1' and R such that

$$K_1' \simeq K_{1ac}.$$

Proof: For every bounded interval Δ let

$$\mathcal{I}_\Delta := K_1 1_\Delta(K_1).$$

Then

$$(K_1 + K_2)\mathcal{I}_\Delta - \mathcal{I}_\Delta K_1 = K_2 K_1 1_\Delta(K_1)$$

belongs to the trace class for every bounded interval Δ and the conclusion of the lemma follows from Pearson's Theorem. □

The hypothesis of the lemma is satisfied by

$$K_1 := \begin{pmatrix} 0 & B^*P \\ PB & Q-\lambda \end{pmatrix}, \quad K_2 := \begin{pmatrix} A-\lambda & 0 \\ 0 & 0 \end{pmatrix},$$

since, by our choice of the orthogonal projection P, the operator

$$\overline{K_2K_1} = \begin{pmatrix} 0 & (A-\lambda)B^*P \\ 0 & 0 \end{pmatrix}$$

belongs to the trace class. By (28) and Lemma 4.9, this implies that

$$(G^{-1} - \lambda)_{ac} = K_1' \oplus R \tag{29}$$

for some self–adjoint operators K_1' and R such that

$$K_1' \simeq K_{1ac}. \tag{30}$$

Since

$$K_1 = 0 \oplus K$$

for some self–adjoint operator K and the zero operator 0 on the kernel $N(PB)$ we may asume that PB is invertible. Under this assumption the polar decomposition of B^*P yields that

$$PB = |\, B^*P \,|\, V$$

for some unitary transformation $V : R(H) \longrightarrow R(P)$. There exists a positive number μ in the essential spectrum of the operator $|\, B^*P \,|$ since $|\, B^*P \,|$ is a positive, self–adjoint and bounded operator in an infinite dimensional Hilbert space. Thus we can choose an orthonormal system $\{e_n\}_{n\in\mathbb{N}}$ in $R(P)$ such that

$$\begin{array}{ll} \sum_{n=1}^{\infty} \|\, |\, B^*P \,|^2 \, e_n \,\|^{1/2} < \infty, & \text{if} \quad \mu = 0, \\ \sum_{n=1}^{\infty} \|\, (|\, B^*P \,| - \mu) \, e_n \,\| < \infty, & \text{if} \quad \mu > 0. \end{array} \tag{31}$$

Let P_1 be the orthogonal projection in $R(H)^\perp$ onto $[e_n : n \in \mathbb{N}]$. We have

$$\|\, |\, B^*P \,|^2 e_n - \mu^2 e_n \,\| \geq \|\, |\, B^*P_1 \,|^2 e_n - \mu^2 e_n \,\|.$$

Thus $\|\, |B^*P_1\,| e_n \,\|^2 \leq \|\, |\, B^*P_1 \,|^2 e_n \,\| \leq \|\, |\, B^*P e_n \,|^2 e_n \,\|$, if $\mu = 0$, and

$$\|\, (|\, B^*P_1 \,| - \mu) \, e_n \,\| \leq \frac{1}{\mu} \|\, |\, B^*P \,| + \mu \,\| \, \|\, (|\, B^*P \,| - \mu) \, e_n \,\|,$$

if $\mu > 0$. Thus, by (31),

$$\sum_{n=1}^{\infty} \|\, (|\, B^*P_1 \,| - \mu) \, e_n \,\| < \infty.$$

Since $R(P_1) \subset R(P)$ along with $PB(A - \lambda)$ also $P_1 B(A - \lambda)$ belongs to the trace class. Thus we may assume that $P = P_1$ and get that there exists an orthogonal projection P and a real number μ such that

$$\dim R(P) = \infty$$

and both $PB(A - \lambda)$ and $|\, B^*P \,| - \mu$ belong to the trace class.

Let

$$K_1'' := \begin{pmatrix} 0 & \mu V^* \\ \mu V & Q - \lambda \end{pmatrix}.$$

By (30) and since $|B^*P| - \mu$ is nuclear and $PB = |B^*P|V$ we have that

$$K_1' \simeq K_{1ac}''. \tag{32}$$

Let M_1 be any self–adjoint operator in any infinite dimensional separable Hilbert space such that the spectrum of M_1 is purely absolutely continuous. We shall show that the operator Q can be chosen such that there exist self–adjoint operators M and R'' such that

$$K_1'' = K_{1ac}'' = M \oplus R'' \quad \text{and} \quad M \simeq M_1. \tag{33}$$

This is obvious if $\mu = 0$. Suppose that $\mu \neq 0$. Then there exists a bounded self–adjoint operator X in $R(H)$ such that X has purely absolutely continuous spectrum, the operator norm of X is less than or equal to 1 and

$$\mu X^{-1}(1 - X^2)^{1/2} \simeq M_1, \tag{34}$$

as is easily verified. We put

$$F(t) := \mu \left(\frac{\sqrt{1-t^2}}{t} - \frac{t}{\sqrt{1-t^2}} \right), \quad t \in \mathbb{R} \setminus \{0, \pm 1\},$$

and put

$$Q - \lambda = VF(X)V^*.$$

Obviously the operator

$$U := \begin{pmatrix} X & \sqrt{1-X^2}V^* \\ V\sqrt{1-X^2} & VXV^* \end{pmatrix}$$

is unitary. A straightforward computation yields that

$$U^* K_1'' U = \begin{pmatrix} \mu X^{-1}\sqrt{1-X^2} & 0 \\ 0 & -\mu VX(1-X^2)^{-1/2}V^* \end{pmatrix}$$

and, by (34), (33) holds.

Let J be any gap of the operator H; it is not necessary that 0 belongs to J. Let M' be any self–adjoint operator in any separable Hilbert space. By (29), (32) and (33), the operator G_0, defined by (27), has a self–adjoint extension G such that

$$G \simeq M'_{acJ} \oplus R'$$

for some self–adjoint operator R'. Thus there exist a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ and a self–adjoint operator M in $\mathcal{H}_0$ such that M is a restriction of G and

$$M \simeq M'_{acJ}.$$

By (27), the operator $0 \oplus G$ is an extension of H and therefore a restriction of the adjoint H^* of H. Thus M is also a restriction of H^* . Thus, by Lemma 2.1, there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq M'_{acJ}$$

and we have proved the following

Proposition 4.10 *Let H be a symmetric operator which is weakly significantly deficient in the sense of the Definition 4.8. Then for every gap J of H and for every self–adjoint operator M' in a separable Hilbert space there exists a self–adjoint extension $\tilde{H}$ of H such that*

$$\tilde{H}_J \simeq M'_{acJ}.$$

4.2 Infinite orthogonal sums

There are very interesting examples of symmetric operators H which are not weakly significantly symmetric but have self–adjoint extensions with non–empty absolutely continuous spectrum, e. g., the restriction of the one–dimensional momentum operator P to the space $C_0^\infty(\mathbb{R} \setminus \mathbb{Z})$.

In the following we shall describe another method to construct self–adjoint extensions with prescribed absolutely continuous spectrum which can be used in this example and many other situations. In order to apply this special method one essentially needs that the underlying Hilbert space $\mathcal{H}$ is complex. It is remarkable that in the following we shall not require that the symmetric operator H has a real regular point.

Proposition 4.11 ([7], Theorem 3.1) *Let H be a symmetric operator in the complex Hilbert space $\mathcal{H}$. Suppose that H can be represented in the form*

$$H = H_1 \oplus H_2$$

for some symmetric operators H_1 and H_2 where both H_1 and H_2 have self–adjoint extensions and infinite deficiency numbers. Then H has a self–adjoint extension $\hat{H}$ such that

$$\hat{H} \simeq P \oplus G \tag{35}$$

for some self–adjoint operator G where P denotes the one–dimensional momentum operator, i. e. P is the operator in $L^2(\mathbb{R})$ given by

$$\begin{aligned} D(P) &:= H^1(\mathbb{R}), \\ Pf &:= -if', \quad f \in D(P). \end{aligned}$$

The previous proposition is only a first result and at the second level it just tells us that H has some self–adjoint extension $\hat{H}$ such that $\sigma_{ac}(\hat{H}) \cap J = J$. The question whether there exists a self–adjoint extension $\tilde{H}$ of H such that $\emptyset \neq \sigma_{ac}(\tilde{H}) \cap J \neq J$ is left open. However, by a combination of the previous proposition and Lemma 2.1 we shall get an affirmative result.

First let us introduce some notation. Let $\rho \in L^{1,+}_{loc}(\mathbb{R})$, i.e. $\rho : \mathbb{R} \to [0, \infty)$ is locally integrable with respect to the Lebesgue measure dx. We shall denote by Q_ρ the operator of multiplication by the identity function in $L^2(\mathbb{R}, \rho dx)$, i.e.

$$\begin{aligned} D(Q_\rho) &:= \{f \in L^2(\mathbb{R}, \rho dx) : \int_{\mathbb{R}} x^2 |f(x)|^2 \rho(x) dx < \infty\}, \\ (Q_\rho f)(x) &:= x f(x) \quad \rho dx - \text{a.e}, \ f \in D(Q_\rho). \end{aligned}$$

If $\rho \equiv 1$ then we also simply write Q instead of Q_ρ.

Then obviously

$$Uf := \sqrt{\rho} f, \quad f \in L^2(\mathbb{R}, \rho dx) \ ,$$

defines a partial isometry form $L^2(\mathbb{R}, \rho dx)$ to $L^2(\mathbb{R})$, the range $R(U)$ of U is a closed subspace of $L^2(\mathbb{R})$ and U is a unitary transformation from $L^2(\mathbb{R}, \rho dx)$ onto $R(U)$. One can easily verify that $R(U)$ is a reducing subspace of Q. Obviously

$$U^{-1} Q_{|D(Q) \cap R(U)} U = Q_\rho \ .$$

Since $P = V^*QV$ for some unitary transformation V, it follows that

$$\hat{\mathcal{H}}_0 := V^* R(U)$$

is a reducing subspace of the momentum operator P and the self–adjoint operator $\hat{M} := P_{|D(P)\cap\hat{H}_0}$ in $\hat{\mathcal{H}}_0$ is unitarily equivalent to the operator Q_ρ.

Under the hypothesis of Proposition 4.11 H has a self–adjoint extension H_{aux} such that

$$H_{aux} \simeq P \oplus G$$

for some self–adjoint operator G. By the previous considerations, this implies that there exists a reducing subspace $\mathcal{H}_0$ of H_{aux} such that

$$M := H_{aux|D(H_{aux})\cap\mathcal{H}_0} \simeq Q_\rho \ .$$

Obviously M is a self–adjoint operator in the Hilbert space $\mathcal{H}_0$ and a restriction of H^*.

Suppose now in addition that $\rho = 0$ $dx-$a.e. on $\mathbb{R} \setminus J$. Then $\sigma(Q_\rho) = \sigma(M) \subset \overline{J}$ and, by Lemma 2.1, there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq Q_\rho \ .$$

Thus we have proved the following

Corollary 4.12 *Let H be a symmetric operator with gap J in the complex Hilbert space $\mathcal{H}$. Suppose that H can be decomposed into the orthogonal sum of two operators with infinite deficiency numbers. Then for every $\rho \in L^{1,+}_{loc}(\mathbb{R})$, $\rho = 0 \quad dx-$a.e. on $\mathbb{R} \setminus J$, there exists a self–adjoint extension $\tilde{H}$ of H such that*

$$\tilde{H}_J \simeq Q_\rho.$$

It easily follows from the previous corollary that inside the gap J of H every kind of absolutely continuous spectrum "can be generated" by a suitably chosen self–adjoint extension of H, provided H is the orthogonal sum of infinitely many symmetric operators with strictly positive deficiency numbers.

Corollary 4.13 *Let H be a symmetric operator with gap J in the complex Hilbert space $\mathcal{H}$. Suppose that*

$$H = \bigoplus_{n=1}^{\infty} H_n$$

for some symmetric operators H_n, $n \in \mathbb{N}$, with strictly positive deficiency numbers. Then for every self–adjoint operator M' in a separable Hilbert space there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq M'_{acJ}.$$

Proof: It easily follows from the Spectral Theorem that

$$M'_{ac\ J} \simeq \bigoplus_{n=1}^{\infty} Q_{\rho_n}$$

for suitably chosen $\rho_n \in L^{1,+}_{loc}$, $\rho_n = 0 \;\; dx - a.e.$ on $\mathbb{R} \setminus J$, $n \in \mathbb{N}$. Since $\mathbb{N}$ can be decomposed into infinitely many infinite sets the operator H can be decomposed as

$$H = \bigoplus_{n=1}^{\infty} H^{(n)}$$

where for every $n \in \mathbb{N}$ the operator $H^{(n)}$ is the orthogonal sum of two operators with infinite deficiency numbers.

By Corollary 4.12, for every $n \in \mathbb{N}$ there exists a self–adjoint extension $\tilde{H}_n$ of $H^{(n)}$ such that

$$\tilde{H}_{nJ} \simeq Q_{\rho_n} \; .$$

Then

$$\tilde{H} := \bigoplus_{n=1}^{\infty} \tilde{H}_n$$

is a self–adjoint extension of H and

$$\tilde{H}_J = \bigoplus_{n=1}^{\infty} \tilde{H}_{n\,J} \simeq \bigoplus_{n=1}^{\infty} Q_{\rho_n} \simeq M'_{acJ}.$$

□

Example 4.14 In the following we shall continue Example 4.7. Let D be the union of pairwise disjoint non–empty bounded open intervals (a_n, b_n), $n \in \mathbb{N}$. If

$$\sup_{n\in\mathbb{N}} |b_n - a_n| = \infty$$

then the operator P_D has no gap, i. e.

$$\sigma(\tilde{H}) = \mathbb{R}$$

for every self–adjoint extension $\tilde{H}$ of P_D. Of course the type of the spectrum (absolutely continuous, pure point etc.) really occurring still strongly depends on the special choice of the self–adjoint extension. $\tilde{H}$ has a pure point spectrum provided $\tilde{H}$ is of the form

$$\tilde{H} = \bigoplus_{n=1}^{\infty} P_{a_n,b_n,\theta_n}.$$

On the other hand if D is dense in $\mathbb{R}$ then the one–dimensional momentum operator P is another self–adjoint extension of P_D with a purely absolutely continuous spectrum.

It is outside the scope of the present paper to investigate the spectral properties of the self–adjoint extensions of a symmetric operator outside the union of the gaps of the symmetric operator. Thus let us assume that

$$\sup_{n\in\mathbb{N}} |b_n - a_n| < \infty.$$

Then every non–empty open interval J such that

$$|J| \leq \frac{2\pi}{\sup_{n\in\mathbb{N}} |b_n - a_n|}$$

is a gap of P_D and, via the methods described resp. cited in this subsection, for every self–adjoint operator M' in $L^2(D)$ one can construct a self–adjoint extension $\tilde{H}$ of P_D such that

$$\tilde{H}_J \simeq M'_{acJ}.$$

5 Singular continuous spectra

Let H be a symmetric operator in some Hilbert space $\mathcal{H}$. Suppose that H has a gap J and the deficiency number of H is infinite. Let J_0 be an open subset of J. In [8] we have presented a method to construct self–adjoint extensions $\hat{H}$ of H such that

$$\sigma_{sc}(\hat{H}) \cap J = \overline{J_0} \cap J.$$

The basic idea of this construction is as follows. Via the results in section 3 one can give a self–adjoint extension H_{aux} of H with dense pure point spectrum in J_0. Using essentially classical results by N. Aronszajn and W. Donoghue one can get a self–adjoint extension $\hat{H}$ of H with the required properties by a rank–one perturbation of the resolvent of H_{aux}. We refer to [11] and [15] for recent remarkable extensions of the mentioned classical results. In particular, we have proved the following

Proposition 5.1 ([8], Theorem 1) *Let H be a symmetric operator in some Hilbert space $\mathcal{H}$. Suppose that the operator H has some gap J and its deficiency number is infinite. Let J_0 be an open subset of J. Then H has a self–adjoint extension $\tilde{H}$ such that $\sigma_{sc}(\tilde{H}) \cap J = \overline{J_0} \cap J$.*

Let J_0 be any non–empty open subset of J and $\hat{H}$ be as in the previous proposition. By the Spectral Theorem, there exists a family $\{\mu_i\}_{i\in I}$ of singular continuous positive Radon measures μ_i on $\mathbb{R}$ such that

$$\hat{H}_{sc} \simeq \bigoplus_{i\in I} Q_i$$

where Q_i denotes the operator of multiplication by the identity function in $L^2(\mathbb{R}, \mu_i)$ for every $i \in I$. Since the singular continuous spectrum of $\hat{H}$ equals the closure of the union of the topological supports of the measures μ_i, $i \in I$, there exists an $i_0 \in I$ such that $\mu_{i_0}(J_0) > 0$. Since μ_{i_0} is a singular continuous positive Radon measure this implies that there exists a compact subset C of J_0 such that $\mu_{i_0}(C) > 0$ and the Lebesgue measure of C equals zero. Obviously the set

$$\{f \in L^2(\mathbb{R}, \mu_{i_0}) : \; f = 0 \quad \text{on} \quad \mathbb{R} \setminus C \quad \mu_{i_0}\text{–a.e.}\}$$

is a reducing subspace of Q_{i_0}, the restriction $Q_{i_0}^C$ of Q_{i_0} to this space is unitarily equivalent to some self–adjoint operator M in some closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that M is a restriction of $\hat{H}$ and

$$\emptyset \neq \sigma_{sc}(Q_{i_o}^C) \subset C$$

and therefore also

$$\emptyset \neq \sigma_{sc}(M) \subset C.$$

Obviously M is a restriction of the adjoint H^* of H. Thus, by Lemma 2.1, there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_J \simeq M$$

and therefore

$$\emptyset \neq \sigma_{sc}(\tilde{H}) \cap J \subset C.$$

Thus we have proved one part of the following

Corollary 5.2 *Let H be a symmetric operator in the Hilbert space $\mathcal{H}$. Suppose H has a gap J and the deficiency number of H is infinite. Let J_0 be a non–empty open subset of J. Then there exist self–adjoint extensions $\check{H}$ and $\tilde{H}$ of H and a non–empty compact subset C_0 of J_0 with Lebesgue measure zero such that $\check{H}$ and $\tilde{H}$ have a purely singular continuous spectrum in the gap J of H,*

$$\sigma_{sc}(\check{H}) \cap J = \overline{J_0} \cap J$$

and

$$\sigma_{sc}(\tilde{H}) \cap J = C_0.$$

Proof: By the considerations preceeding the statement of the corollary we have only to prove the existence of the operator $\check{H}$. By Proposition 5.1, there exists a self–adjoint extension $\hat{H}$ of H such that

$$\sigma_{sc}(\hat{H}) \cap J = \overline{J_0} \cap J.$$

We put

$$\mathcal{H}_0 := R(1_J(\hat{H}_{sc})),$$

$$M := \hat{H}_{sc\ J}\ .$$

Obviously M is a self–adjoint operator in the Hilbert space $\mathcal{H}_0$, $\sigma(M) \subset \overline{J}$ and M is a restriction of H^*. Thus by Lemma 2.1, H has a self–adjoint extension $\check{H}$ such that

$$\check{H}_J = M = \hat{H}_{scJ},$$

and, in particular,

$$\sigma_{sc}(\check{H}) \cap J = \overline{J_0} \cap J, \quad \sigma_{ac}(\check{H}) \cap J = \sigma_{pp}(\check{H}) \cap J = \emptyset.$$

□

6 Mixed spectra

In this section we shall combine our results on the point, singular continuous and absolutely continuous spectra.

Theorem 6.1 *Let H be a symmetric operator in the Hilbert space $\mathcal{H}$. Suppose that H has a gap J and its deficiency number is infinite. Then for every open subset J_0 of J and every, finite or infinite, sequence $\{\lambda_n\}_{n=1}^N$ in J there exists a self–adjoint extension $\tilde{H}$ of H with the following properties.*

(i)

$$\sigma_p(\tilde{H}) \cap J = \{\lambda_n : n \in \mathbb{N}, \quad n \leq N\}$$

and for every eigenvalue λ of $\tilde{H}$ in J the multiplicity of λ equals the number of times it occurs in the sequence $\{\lambda_n\}_{n=1}^N$.

(ii) $\sigma_{sc}(\tilde{H}) \cap J = \overline{J_0} \cap J$.

(iii) $\sigma_{ac}(\tilde{H}) \cap J = \emptyset$.

Proof: Let λ_0 be any point in J and, if $N < \infty$, let $\lambda_n := \lambda_0$ for all $n \in \mathbb{N}$, $n > 2N$. Since for every regular point λ of H and, in particular, for $\lambda = \lambda_0$, the dimension of $N(H^* - \lambda)$ equals the deficiency number of H, we can choose, by induction, an orthonormal system $\{e_n\}_{n \in \mathbb{N}}$ such that

$$e_n \in N(H^* - \lambda_n), \quad n \in \mathbb{N},$$

and

$$e_j \in N(H^* - \lambda_0), \quad j = 2n-1, \quad n \in \mathbb{N}.$$

As in the proof of Proposition 3.1 we can show that there exist self–adjoint operators M and $\hat{M}$ in the Hilbert space

$$\begin{aligned}
\mathcal{H}_0 &:= [e_{2n} : n \in \mathbb{N}, n \leq N], \\
\hat{\mathcal{H}}_0 &:= [e_n : n \in \mathbb{N}],
\end{aligned}$$

respectively, such that

$$\begin{aligned}
e_{2n} &\in N(M - \lambda_n), \qquad n \in \mathbb{N}, \qquad n \leq N, \\
e_{2n} &\in N(\hat{M} - \lambda_n), \qquad n \in \mathbb{N}, \\
e_j &\in N(\hat{M} - \lambda_0), \quad j = 2n-1, \quad n \in \mathbb{N},
\end{aligned}$$

and M and $\hat{M}$ are restrictions of the adjoint H^* of H. Obviously M is also a restriction of the operator $\hat{M}$ and therefore

$$M \subset H_{\hat{M}}, \quad H_M \subset H_{\hat{M}},$$

where the operators H_M and $H_{\hat{M}}$ are defined by (6). Moreover the self–adjoint operator M has a pure point spectrum,

$$\sigma_p(M) = \{\lambda_n : n \in \mathbb{N}, \quad n \leq N\}$$

and for every eigenvalue λ of M the multiplicity of λ equals the number of times it occurs in the sequence $\{\lambda_n\}_{n=1}^N$.

By Lemma 2.1, the operator H_M can be decomposed as

$$H_M = M \oplus G_0$$

where G_0 is a symmetric operator in the Hilbert space $\mathcal{H}_0^\perp$ and J is also a gap of G_0. We have only to show that the deficiency number of G_0 is infinite. In fact, then Corollary 5.2 yields that there exists a self–adjoint operator G in $\mathcal{H}_0^\perp$ such that G is an extension of G_0 and

(i) $\sigma_{sc}(G) \cap J = \overline{J_0} \cap J$.

(ii) $\sigma_{ac}(G) \cap J = \sigma_{pp}(G) \cap J = \emptyset$.

Then obviously the operator

$$\tilde{H} := M \oplus G$$

is a self–adjoint extension of H with the required spectral properties.

By Lemma 2.1, the operator $H_{\hat{M}}$ can be decomposed as

$$H_{\hat{M}} = \hat{M} \oplus \hat{G}_0$$

where $\hat{G}_0$ is a symmetric operator in the Hilbert space $\hat{\mathcal{H}}_0^{\perp}$ and J is also a gap of $\hat{G}_0$. Since the symmetric operator $\hat{G}_0$ has a gap it has a self–adjoint extension $\hat{G}$. Let

$$\hat{H} := \hat{M} \oplus \hat{G}.$$

Since $M \subset \hat{H} = \hat{H}^*$ we have $\hat{H}_M = \hat{H}$. Since M is a self–adjoint operator in $\mathcal{H}_0$ and $\sigma(M) \subset \bar{J}$ it follows from Lemma 2.1, that

$$\hat{H} = M \oplus G'$$

for some self–adjoint operator G' in the Hilbert space $\mathcal{H}_0^{\perp}$. Since

$$\hat{H} e_j = \hat{M} e_j = G' e_j = \lambda_0 e_j, \quad j = 2n-1, \quad n \in \mathbb{N},$$

the point λ_0 in the gap J of G_0 is an eigenvalue of G' with infinite multiplicity. Since obviously G' is a self–adjoint extension of G_0 this implies, by Krein's Theorem, i. e. Theorem 1.1, that the deficiency number of G_0 is infinite. □

In the following theorem one may choose for J_0 the empty set. Thus the theorem implies, in particular, that the Conjecture 1.2 is true provided the operator H is weakly significantly deficient or satisfies the hypothesis of the Corollary 4.13 and M' has no singular continuous spectrum in the gap J.

Theorem 6.2 *Let H be a symmetric operator with gap J. Suppose that H is weakly significantly deficient in the sense of Definition 4.8 or that the Hilbert space $\mathcal{H}$ is complex and*

$$H = \bigoplus_{n=1}^{\infty} H_n$$

for some symmetric operators H_n with strictly positive deficiency numbers. Then for every self–adjoint operator M' in a separable Hilber space and every open subset J_0 of J there exists a self–adjoint extension $\tilde{H}$ of H such that

$$\tilde{H}_{acJ} \simeq M'_{acJ}, \quad \tilde{H}_{ppJ} \simeq M'_{ppJ}$$

and

$$\sigma_{sc}(\tilde{H}) \cap J = \overline{J_0} \cap J.$$

Proof: If H is the orthogonal sum of infinitely many operators with strictly positive deficiency numbers then H can be decomposed as

$$H = H^{(1)} \oplus H^{(2)}$$

where the deficiency number of $H^{(1)}$ is infinite and $H^{(2)}$ is also the orthogonal sum of infinitely many operators with strictly positive deficiency numbers. By Theorem 6.1, the operator $H^{(1)}$ has a self–adjoint extension $\tilde{H}_1$ such that

$$\tilde{H}_{1ppJ} \simeq M'_{ppJ},$$

$$\sigma_{sc}(\tilde{H}_1) \cap J = \overline{J_0} \cap J$$

and

$$\sigma_{ac}(\tilde{H}_1) \cap J = \emptyset.$$

By Corollary 4.13, $H^{(2)}$ has a self–adjoint extension $\tilde{H}_2$ such that

$$\tilde{H}_{2J} \simeq M'_{acJ}.$$

Obviously $\tilde{H} := \tilde{H}_1 \oplus \tilde{H}_2$ is a self–adjoint extension of H and has the required spectral properties.

Suppose now that H is weakly significantly deficient. We choose any self–adjoint operator M'' in any separable Hilbert space such that $M'' = M''_{acJ} \neq 0$ and put

$$\tilde{M} := M'_{acJ} \oplus M''.$$

By Proposition 4.10, there exists a self–adjoint extension $\hat{H}$ of H such that

$$\hat{H}_{acJ} \simeq \tilde{M}.$$

Since $\tilde{M} = M'_{acJ} \oplus M''$ we have that

$$\hat{H}_{acJ} = M \oplus \hat{M}$$

for some self–adjoint operator M in some closed subspace $\mathcal{H}_0$ of $R(1_J(\hat{H}_{ac}))$ and some self–adjoint operator $\hat{M}$ in $R(1_J(\hat{H}_{ac})) \ominus \mathcal{H}_0$ with the properties that

$$M \simeq M'_{acJ}, \quad \hat{M} \simeq M'',$$

and, in particular,

$$\sigma(M) \subset \bar{J}, \quad \sigma_{ac}(\hat{M}) \cap J \neq \emptyset.$$

By Lemma 2.1, the operator $H_M := H^*_{|D(H)+D(M)}$ can be represented in the form

$$H_M = M \oplus G_0 \tag{36}$$

for some symmetric operator G_0 in the Hilbert space $\mathcal{H}_0^\perp$ where J is a gap of G_0.

By Theorem 6.1, there exists a self–adjoint operator G in $\mathcal{H}_0^\perp$ such that $G \supset G_0$ and

(i) $\sigma_{ac}(G) \cap J = \emptyset$,

(ii) $G_{ppJ} \simeq M'_{ppJ}$.

(iii) $\sigma_{sc}(G) \cap J = \overline{J_0} \cap J$

provided the deficiency number of G_0 is infinite. Then obviously $\tilde{H} := M \oplus G$ is a self–adjoint extension of H with the required properties. Thus we have only to show that the deficiency number of G_0 is infinite.

Since $M \subset \hat{H} = \hat{H}^*$ and therefore $\hat{H} = \hat{H}_M$ it follows from Lemma 2.1 that

$$\hat{H} = M \oplus \hat{G} \tag{37}$$

for some symmetric operator $\hat{G}$ in $\mathcal{H}_0^\perp$. $\hat{G}$ is self–adjoint since $\hat{H}$ is self–adjoint. Since $H_M \subset \hat{H}$ and by (36) and (37), we have

$$G_0 \subset \hat{G}. \tag{38}$$

By (37) and since $\hat{H} \supset M \oplus \hat{M}$, the self–adjoint operator $\hat{G}$ is an extension of $\hat{M}$. In particular, we have

$$\sigma_{ac}(\hat{G}) \cap J \neq \emptyset. \tag{39}$$

Since J is a gap of G_0 it follows from (38) and (39) that the deficiency number of G_0 is infinite. $\square$

Remark: In the previous two theorems and the following example the assertion

$$\sigma_{sc}(\tilde{H}) \cap J = \overline{J_0} \cap J$$

can be replaced by the following assertion: there exists a non–empty compact subset C_0 of J_0 with Lebesgue measure zero such that

$$\sigma_{sc}(\tilde{H}) \cap J = C_0.$$

This can easily be shown by using Corollary 5.2 and modifying the corresponding proofs in an obvious way.

Example 6.3 Let Ω be a non–empty open subset of $\mathbb{R}^d$. $\tilde{H}$ is a *self–adjoint realization of the Laplacian on* Ω if and only if $\tilde{H}$ is a self–adjoint operator in the Hilbert space $L^2(\Omega)$ and an extension of the *minimal Laplacian on* Ω, i. e. the operator $-\Delta^{\Omega}_{min}$ in $L^2(\Omega)$ defined by

$$\begin{aligned} D(-\Delta^{\Omega}_{min}) &:= C_0^{\infty}(\Omega), \\ -\Delta^{\Omega}_{min} f &:= -\Delta f, \quad f \in C_0^{\infty}(\Omega). \end{aligned}$$

Obviously

$$\overline{-\Delta^{\Omega}_{min}} = \bigoplus \overline{-\Delta^{\Omega_i}_{min}}$$

where the orthogonal sum is taken over all connection components Ω_i of Ω. It is well known and easily seen that the open interval $(-\infty, b)$ is a gap of the minimal Laplacian $-\Delta^{\Omega}_{min}$ on Ω for some positive real number b. b equals zero and $[0, \infty)$ does not contain any gap of $-\Delta^{\Omega}_{min}$ provided Ω contains balls with arbitrarily large diameter. On the other hand b can be chosen such that it is strictly positive and the interval $[b, \infty)$ can be covered by bounded gaps of the minimal Laplacian $-\Delta^{\Omega}_{min}$ on Ω provided the supremum over the diameters of the connection components of Ω is finite.

$-\Delta^{\Omega_i}_{min}$ has a self–adjoint extension with compact resolvent, for instance the Dirichlet Laplacian on Ω_i, provided Ω_i is bounded. Moreover the deficiency number of $-\Delta^{\Omega_i}_{min}$ is infinite provided Ω_i is non–empty and bounded and the dimension d of $\mathbb{R}^d$ is strictly larger than one. Thus the minimal Laplacian on the open subset Ω of $\mathbb{R}^d$, $d > 1$, is significantly deficient provided Ω has at least one non–empty bounded connection component. In this case for every gap J of $-\Delta^{\Omega}_{min}$, every self–adjoint operator M' in $L^2(\Omega)$ and every open subset J_0 of J there exists a self–adjoint realization $-\tilde{\Delta}^{\Omega}$ of the Laplacian on Ω such that

$$-\tilde{\Delta}^{\Omega}{}_{acJ} \simeq M'_{acJ}, \quad -\tilde{\Delta}^{\Omega}{}_{ppJ} \simeq M'_{ppJ},$$

and

$$\sigma_{sc}(-\tilde{\Delta}^{\Omega}) \cap J = \overline{J_0} \cap J.$$

Via the methods described resp. cited in this paper it is possible to construct such self–adjoint realizations.

In applications one is very strongly interested in the special case when $\Omega = \mathbb{R}^d \setminus N$ for some closed subset N of $\mathbb{R}^d$ with Lebesgue measure zero because then, after the canonical identification of the Hilbert spaces $L^2(\Omega)$ and $L^2(\mathbb{R}^d)$ and the operators $-\Delta^{\Omega}_{min}$ and $-\Delta_{|\,N^c}$, defined by

$$\begin{aligned} D(-\Delta_{|\,N^c}) &:= C_0^\infty(\mathbb{R}^d \setminus N), \\ -\Delta_{|\,N^c} f &:= -\Delta f, \quad f \in D(-\Delta_{|\,N^c}), \end{aligned}$$

the self–adjoint realizations of the Laplacian on Ω can be identified with Hamiltonians describing the interaction of a “quantum mechanical particle” with a singular potential localized inside the zero–measure set N. There is a huge literature about these so called singular Schrödinger operators. Here we only mention [1], [2], [3], [13] and also refer to the references given therein.

The deficiency number of the operator $-\Delta_{|\,N^c}$ is infinite if and only if the set N has infinitely many points and the $c_{2,2}$–capacity of N is strictly positive, cf. [5], section 1. Thus in this case for every sequence $\{\lambda_n\}$ in $(-\infty, 0)$ and every open subset J_0 of $(-\infty, 0)$ there exists a self-adjoint extension $\tilde{H}$ of the operator $-\Delta_{|\,N^c}$ such that $\lambda < 0$ is an eigenvalue of $\tilde{H}$ if and only if it belongs to the given sequence $\{\lambda_n\}$ and the multiplicity of an eigenvalue $\lambda < 0$ of $\tilde{H}$ equals the number of times it occurs in this sequence and $\sigma_{sc}(\tilde{H}) \cap (-\infty, 0) = \overline{J_0} \cap (-\infty, 0)$.

If in addition $\mathbb{R}^d \setminus N$ has a non–empty bounded connection component then the simple method described in section 4.1.2 can be used for the construction of self–adjoint extensions of $-\Delta_{|\,N^c}$ with arbitrarily prescribed absolutely continuous spectrum below zero.

If N satisfies "suitable symmetry conditions" then the operator $\overline{-\Delta_{|\,N^c}}$ can be decomposed into the orthogonal sum of infinitely many symmetric operators with strictly positive deficiency number, as is easily verified. Then our method presented in section 4.2 can be used for the construction of self–adjoint extensions of $-\Delta_{|\,N^c}$ with arbitrarily prescribed absolutely continuous spectrum below zero, where here it is not required that $\mathbb{R}^d \setminus N$ has a non–empty bounded connection component. As an interesting example we mention the set

$$N = \{(x_1, \dots, x_d) : \sum_{j=1}^{d-\alpha} |x_j|^2 = r^2\}$$

for some real number $r > 0$ and $\alpha \in \{1, 2, 3\}$, $\alpha < d$.

Appendix

We shall give symmetric operators which are weakly significantly deficient but not significantly deficient. We shall denote the closure of the minimal Laplacian on $\mathbb{R}$ by $-\Delta$. It is well known that $-\Delta$ is a positive self–adjoint operator in the Hilbert space $L^2(\mathbb{R})$ and every f in the domain $D(-\Delta) = H^2(\mathbb{R})$ of $-\Delta$ is continuously differentiable (here we do not distinguish between functions and their equivalence classes). We shall use the well known fact that there exist Borel measurable functions $q : \mathbb{R} \longrightarrow [1, \infty)$ which are nowhere locally square–integrable, i. e. satisfy

$$\int_{a-\varepsilon}^{a+\varepsilon} q(t)^2 dt = \infty, \quad a \in \mathbb{R}, \quad \varepsilon > 0. \tag{40}$$

For instance we may choose any $\alpha \in [1/2, \infty)$ and any sequence $\{c_n\}_{n\in\mathbb{N}}$ in $(0, \infty)$ and $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} c_n < \infty$$

and $\{t_n : n \in \mathbb{N}\}$ is dense in $\mathbb{R}$ and put

$$q(t) := \begin{cases} 1 + \sum_{n=1}^{\infty} c_n |\, t - t_n \,|^{-\alpha}, & \text{if the series converges,} \\ 1, & \text{otherwise.} \end{cases}$$

For notational brevity we shall denote the operator of multiplication by a function f in the Hilbert space $L^2(\mathbb{R})$ also by f.

Example: Let $q : \mathbb{R} \longrightarrow [1, \infty)$ be any Borel measurable function such that (40) holds. Then the operator

$$L := \begin{pmatrix} (-\Delta + 1)^{-1} & \frac{1}{q} \\ \frac{1}{q} & (-\Delta + 1)^{-1} \\ 0 & 1 \end{pmatrix} : \begin{array}{c} L^2(\mathbb{R}) \\ \oplus \\ L^2(\mathbb{R}) \end{array} \longrightarrow \begin{array}{c} L^2(\mathbb{R}) \\ \oplus \\ L^2(\mathbb{R}) \\ \oplus \\ L^2(\mathbb{R}) \end{array}$$

is the inverse of some closed symmetric operator H, 0 belongs to some gap J of H, and H is weakly significantly deficient but not significantly deficient.

Proof: Obviously

$$A := \begin{pmatrix} (-\Delta + 1)^{-1} & \frac{1}{q} \\ \frac{1}{q} & (-\Delta + 1)^{-1} \end{pmatrix}$$

is a bounded self–adjoint operator in the Hilbert space $\mathcal{R} := L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. By (40), there does not exist a continuous function $h \in D(q)$ such that $h \not\equiv 0$. Thus

$$(-\Delta + 1)^{-1} f + \frac{1}{q} g = 0$$

implies $f = g = 0$ and A is invertible.

Let

$$B := (0\,1) : \begin{array}{c} L^2(\mathbb{R}) \\ \oplus \\ L^2(\mathbb{R}) \end{array} \longrightarrow L^2(\mathbb{R}).$$

Apparently

$$R(B^*) = \begin{array}{c} \{0\} \\ \oplus \\ L^2(\mathbb{R}) \end{array} .$$

Thus $B^*h = A(f \oplus g)$ for some $f, g, h \in L^2(\mathbb{R})$ implies that

$$0 = (-\Delta + 1)^{-1} f + \frac{1}{q} g$$

and consequently $f = g = 0$ and $B^*h = 0$. Thus

$$R(B^*) \cap R(A) = \{0\}.$$

Thus, by Lemma 4.1,

$$L = \begin{pmatrix} A \\ B \end{pmatrix}$$

is the inverse of some closed symmetric operator H in the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and 0 belongs to some gap J of H. Since $R(B) = L^2(\mathbb{R})$ the operator H is not significantly deficient. Let h be any continuous function such that $h \in L^2(\mathbb{R}) \setminus H^2(\mathbb{R})$. Since for every $g \in L^2(\mathbb{R})$ the function $h - (-\Delta + 1)^{-1} g$ is continuous and not identically equal to zero, we have that

$$h \neq \frac{1}{q} f + (-\Delta + 1)^{-1} g$$

for every $f \oplus g \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and therefore $h \notin R(BA)$. Thus the operator H is weakly significantly deficient.

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