On the generalized Fitting group of locally finite, finitary groups

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Abstract

Let G be a locally finite, finitary group and $F^*(G)$ the group generated by the Hirsch-Plotkin radical of G and the components of G. Our main theorem asserts that $C_G(F^*(G)) \leq F^*(G)$.

1 Introduction

The main purpose of this paper is to extend the concept of the generalized Fitting group from finite groups to locally finite, finitary groups. Recall that a group G is called locally finite if every finite subset of G lies in a finite subgroup. G is called finitary if there exist a field K and a faithful KG-module V so that $[V, g]$ is finite dimensional for all $g \in G$. G is quasi-simple if it is perfect, and $G/Z(G)$ is simple. A component of G is a non-trivial quasi-simple, subnormal subgroup of G. The layer $E(G)$ is defined as the group generated by the components of G. It is easy to see that distinct components of a group commute. Thus $E(G)/Z(E(G))$ is semisimple, that is the (restricted) direct product of simple groups.

Assume for the moment that G is finite. Then the Fitting group $F(G)$ is the largest nilpotent normal subgroup of G and the generalized Fitting group $F^*(G)$ is defined as $F(G)E(G)$. A well known theorem (see for example [As]) asserts that $C_G(F^*(G)) \leq F^*(G)$. This means that the group $G/Z(F(G))$ is a subgroup of the automorphism of $F^*(G)$. Thus a finite group G can be well described in terms of $F^*(G)$. As the strucure of $F^*(G)$ is fairly simple compared with the structure of an arbitray finite group, this explains the importance of the generalized Fitting group in the theory of finite groups.

An infinite group G does not necessarily have a largest nilpotent normal subgroup, but it does have a unique maximal locally nilpotent, normal subgroup, the Hirsch-Plotkin radical (see [Ro, page 58]. We denote the Hirsch Plotkin radical by $\mathcal{LN}(G)$ and define the generalized Fitting group $F^*(G)$ of G to be the product $\mathcal{LS}(G)E(G)$. The reader should notice that the generalized Fitting group of a group can be trivial. Indeed this is the case for free groups of rank at least two and even for some infinite, locally finite, groups. Surprisingly the situation is much better for locally finite, finitary groups. Indeed, we will prove:

Theorem A *Let G be a locally finite, finitary group. Then $C_G(F^*(G)) \leq F^*(G)$.*

Similar but different results have been obtained independently by V.V Belyeav [Be, Theorem A] and R.E Phillips [Ph].

As I was kindly reminded by W. Gaschütz, for finite groups the generalized Fitting group can also be described as the set of elements which induce inner automorphism on each chief-factor of the group. 5.11 below implies that same is true for locally finite, finitary groups.

In section 4 we prove that the elements in locally finite, finitary groups have a purely group theoretical property we call *bounded.* (See the end of the introduction for the relevant definitions). In section 5 we prove a structure theorem for arbitray locally finite bounded groups modulo their largest locally solvable normal subgroup. In section 6 we use this structure theorem to prove Theorem A. Finally in section 7 we use the methods and results developed in the earlier sections to show that any locally finite finitary simple group has a Kegel cover which is unipotent by quasi-simple. This last result also has been proved independently in [1, Proposition 4.4] and [Be, Theorem B].

Our results depend on the classification of finite simple groups, but only in form of the Schreier conjecture. That is we assume that the outer automorphism group of every finite simple group is solvable.

We finish the introduction with a list of some of the notation and conventions used throughout this paper.

Let H be group and $\mathcal{P}$ a group theoretical property. H is called locally $\mathcal{P}$ if every finite subset of H lies in a subgroup of H which property $\mathcal{P}$.

F is a finite group.

G is a locally finite group.

$q(F)$ is the number of non-abelian composition factors in a given composition series for F.

Let $X \subseteq Aut(F)$. Then $q(F,X) = \sum_{i=1}^{n} q([F_i/F_{i-1}, X])$, where

$$1 = F_0 < F_1 < \ldots < F_{n-1} \leq F_n < F$$

is a maximal X-invariant subnormal series for F.

Let X a finite subset of G and $n \geq 0$.

$\mathcal{H} = \{H \leq G | H \text{ finite}\}$.

$\mathcal{H}(X) = \{H \in \mathcal{H} | X \subseteq H\}$,

X is n-bounded if there exists $H_X \in \mathcal{H}(X)$ such that $q(F,X) \leq n$ for all $F \in \mathcal{H}(H_X)$ and X is bounded if X is n-bounded for some n.

B_n is the set of n-bounded elements, B_∞ is the set of all bounded elements in G, and G is called bounded if $G = B_\infty$.

$\mathcal{LS}(G)$ is the subgroup of G generated by all the locally solvable normal subgrous of G. Note that $\mathcal{LS}(G)$ itself is locally solvable and so $\mathcal{LS}(G)$ is the largest locally solvable normal subgroup of G.

G^∞ is the group generated by all the perfect finite subgroups of G, i.e G^∞ is the smallest normal subgroup of G such that G/G^∞ is locally solvable.

$S^*(G) = X^\infty$, where $X/\mathcal{LS}(G)$ is group generated by the simple subnormal subgroups of $G/\mathcal{LS}(G)$. Note that $X/\mathcal{LS}(G)$ is the largest semisimple normal subgroup of $G/\mathcal{LS}(G)$.

$S_0^*(G) = 1$ and $S_{n+1}^*(G)$ is defined inductively by $S_{n+1}^*(G)/S_n^*(G) = S^*(G/S_n^*(G))$.

$S_\infty^*(G) = \bigcup_{i=0}^{\infty} S_i^*(G)$.

Put $S^0_*(F) = F^\infty$ and inductively for all $i > 0$ let $S^i_*(F)$ be smallest normal subgroup of $S^{i-1}_*(F)$ such that $S^{i-1}_*(F)/S^i_*(F)$ is solvable by semisimple. Note that if $S^{i-1}_*(F) \neq 1$, then $S^i_*(F) < S^{i-1}_*(F)$. Let $s(F)$ be the smallest non-negative integer s such that $S^s_*(F) = 1$. If F is solvable, put $S_*(F) = 1$, otherwise let $S_*(F) = S^{s(F)-1}_*(F)$.

Note that $S_*(F)$ is a perfect, solvable by semisimple, normal subgroup of F and so $S_*(F) \leq S^*(F)$. To comprehend the above definitio the reader might verify the following example (the example also shows that $S_*(F)$ is in general a proper subgroup of $S^*(F)$).

Let $F = C_2 \times SL_2(5) \times Sym(5) \wr Alt(5)$. Then
$S^*(F) = SL_2(5) \times Alt(5)^5$.
$F' = S^1_*(F) = S^*_2(F) = S^*_\infty(F) = SL_2(5) \times Alt(5)^5.2^4.Alt(5)$.
$S^2_*(F) = S_*(F) = Alt(5)^5$.
$s(F) = 2$.

Acknowledgement: I would like to thank the Universität Bielefeld and the Universität Kiel for their hospitality.

2 Preliminaries

Lemma 2.1 *Let H be a group with $H = E(H)$ and N a normal subgroup of H. Then $H = C_H(N)N$.*

Proof: Let L be a component of H. If $[N, L] \leq Z(H)$, then $[N, L, L] = 1$ $[N, L] = 1$ and $L \leq C_H(N)$ If $[N, L] \not\leq Z(H)$, then as $L/Z(L)$ is simple, $L = [N, L]Z(L)$ and as L is perfect, $L = [N, L] \leq N$. Thus in both cases, $L \leq C_H(N)N$.

Lemma 2.2 *Let N be normal in F and $i \geq 0$, then $S^i_*(F)N/N = S^i_*(F/N)$.*

Proof: Clearly $S^0_*(F)N/N = S^0_*(F/N)$. Suppose $i > 0$ and put $H = S^{i-1}_*(F)$. By induction $HN/N = S^{i-1}_*(F/N)$. Let P be the inverse image of $S^i_*(F/N)$ in F. Then

$$HN/P \cong S^{i-1}_*(F/N)/S^i_*(F/N)$$

and so HN/P is solvable by semisimple. Since $N \leq P \leq HN$ we have $HP = HN$. Thus

$$H/H \cap P \cong HP/P = HN/P$$

and we conclude that $H/H \cap P$ is semisimple by solvable. Therefore $S^i_*(F) \leq H \cap P \leq P$. On the other hand

$$HN/S^i_*(F)N \cong H/H \cap S^i_*(F)N = H/(H \cap N)S^i_*(F)$$

is solvable by semisimple and so $P \leq S^i_*(F)N$ and $P = (S^i_*(F) \cap P)N = S^i_*(F)N$.

Lemma 2.3 *Let $H \leq F$ with $S^*(F) \leq H$. Then $S^*(H) = S^*(F)$.*

Proof: We assume without loss that $\mathcal{LS}(F)=1$. Put $E=S^*(F)$. Since $\mathcal{LS}(F)=1$, $E=F^*(F)$ and E is direct product of the simple subnormal subgroups of F. Since $C_F(F^*(F))\leq F^*(F)$ (see [As]) we get $C_F(E)=1$. Put $K=S^*(H)$ and note that $E\leq K$. Since $[\mathcal{LS}(K),E]\leq \mathcal{LS}(E)=1$ we get $\mathcal{LS}(K)=1$ and K is semisimple. Thus by 2.1 $K=C_K(E)E=E$.

Lemma 2.4 *Let $x\in K\leq F$, and put $s=s(\langle x^K\rangle)$. If $s\geq 1$ and $s(\langle x^T\rangle)\leq s$ for every subgroup T of F with $K\leq T$, then $S_*(\langle x^K\rangle)\leq S^*(F)$.*

Proof: Put $Y=S_*(\langle x^K\rangle)$. Since Y is perfect it suffices to show $Y\leq S^*(F)\mathcal{LS}(F)$. Hence we may assume that $\mathcal{LS}(F)=1$. Put $L=\langle x^{KS^*(F)}\rangle$ and $N=L\cap S^*(F)$. Since $[x,S^*(F)]\leq N$, $L=\langle x^K\rangle N$. Thus by (2.2), $YN=S_*^{s-1}(L)N$. In particular, $YS^*(F)=YNS^*(F)=S_*^{s-1}(L)S^*(F)$. Since $S_*^{s}(L)=1$, $S_*^{s-1}(L)$ is solvable by semisimple and since $S_*^{s-1}(L)$ and $S^*(F)$ normalize each other we conclude that also $YS^*(F)$ is solvable by semisimple. Note that both Y and $S^*(F)$ are perfect and so $YS^*(F)=S^*(YS^*(F))$. Furthermore, by 2.3 $S^*(YS^*(F))=S^*(F)$ and so $Y\leq S^*(F)$.

Lemma 2.5 *Suppose that $G=G^\infty$ and G is locally solvable by simple. Then $G=\langle x^G\rangle$ for all $x\in G\setminus \mathcal{LS}(G)$.*

Proof: As $G/\mathcal{LS}(G)$ is simple, $G=\langle x^G\rangle\mathcal{LS}(G)$. Thus $G/\langle x^G\rangle$ is locally solvable and $G=G^\infty\leq\langle x^G\rangle$.

Lemma 2.6 *Let L be a non-trivial 1-bounded subgroup of G and P a finite, perfect, solvable by simple subgroup of G containing L. Then P is 1-bounded.*

Proof: Put $H_P=\langle P,H_L\rangle$, let $F\in\mathcal{H}(H_P)$ and $\mathcal{F}$ be a maximal P-invariant subnormal series for F. Then L acts non-trivial on at most one of the non-abelian factors of $\mathcal{F}$. By 2.5, $P=\langle L^P\rangle$ and so the same is true for P. Let W be a non-abelian factor of $\mathcal{F}$ not centralized by L. Then clearly L and so also P normalizes the components of W. By maximality of $\mathcal{F}$ we conclude that W is simple and so $q(F,P)\leq 1$.

Lemma 2.7 *Let F be a non-abelian finite group and M a normal subgroup of F minimal by inclusion with respect to $[M,F]\neq 1$.*

(a) *Suppose that $C_F(M)=Z(F)$ and $Z(F)$ is cyclic. Then one of the following holds:*

  (a.1) *$M=E(M)$ and F acts transitively on the components of M.*

  (a.2) *M is p-group for some prime p, $M=[M,F]$, $F/Z(F)M$ acts faithfully and irreducibly on $M/C_M(F)$. Moreover, $M\cong \mathrm{Ext}(p^{1+2m})$ or $p=2$ and $M\cong C_4\circ \mathrm{Ext}(2^{1+2m})$. In particular, $F/Z(F)M$ is isomorphic to a subgroup of $Sp_{2m}(p)$.*

(b) *Suppose that F is a primitive, tensor indecomposable subgroup of $GL_n(K)$, where K is an algebraicly closed field. Then* (b.a) *$C_F(M) = Z(F)$ and $Z(F)$ is cyclic.*

(b.b) *(a.1) or (a.2) holds.*

(b.c) *M acts irreducible on K^n and in case (a.2) $n = p^m$.*

Proof: (a) Since $Z(M) \leq Z(F)$, $Z(M) \neq M$ and $Z(M) = C_M(F)$ is the unique maximal F invariant subgroup of M. In particular, $M/Z(M)$ is semisimple. If $M/Z(M)$ is not abelian, (1) holds. Thus we may assume that $M/Z(M)$ is an elementary abelian p-group for some prime p. Clearly F acts irreducible on $M/Z(M)$. If $M \neq [M, F]$, then $[M, F] \leq Z(M)$ and so $M/Z(M)$ is cyclic and $M = Z(M)$, a contradiction. Thus $M = [M, F]$. As M has class two and $M/Z(M)$ is elementary abelian, M' is elementary abelian. As $Z(M)$ is cyclic, this implies $|M'| = p$. Moreover, M/M' is elementary abelian and so $M = Z(M)E$ for some E with $E \cap Z(M) = Z(E) = M' = E'$. Thus E is extra-special. If p is odd, M has exponent p and so $|Z(M)| = p$ and $E = M$. If $p = 2$ we get that M has exponent four and so either $Z(M) \cong C_4$ or C_2. The commutatormap from $M/Z(M) \times M/Z(M) \to M'$ defines a non-degenerated symplectic form on $M/Z(M)$. Finally, an automorphism of M which centralizes $M/Z(M)$ and $Z(M)$ is inner and so $C_F(M/Z(M)) = C_F(M)M = Z(F)M$ and all parts of (a) are established.

(b) Put $V = K^n$. As F is primitive, V is the direct sum of isomorphic irreducible KM-modules. In particular, $Z(M)$ acts by scalars on V, $Z(M) \leq Z(F)$ and so M is not abelian. Futhermore, if Q does not act irreducible, then $N_{GL_n(K)}(M)$ does not act tensor indecomposable on V, a contradiction. Thus M acts irreducible on V and $(b.a)$ holds. In particular, the assumptions of (a) and therefore (a.1) or (a.2) holds. If (a.2) holds, then $M = Z(M)E$ with E extra-special of order p^{1+2m}. As M is irreducible, E is irreducible on V. Thus by [Go, 5.5.5], $n = p^m$.

Lemma 2.8 *Let p be a prime and m a postive integer and put $n = p^m$.*

(a) *$2m < n$ unless $p = 2$ and $m \leq 2$.*

(b) *$4m - 2 \leq n$ unless $p = 2$ and $m \leq 3$.*

(c) *$n(n-1)3^{2m-1} \leq 3^{n-1}$ unless $n = 2, 3, 4$ or 8.*

(d) *$n^2(n^2-1)3^{4m-1} \leq 3^{2n-1}$ unless $n = 2, 3, 4$ or 8.*

Proof: This is readily verified.

Lemma 2.9 (a) *Let F be a subgroup of $Sp_4(2)$. Then $q(F) \leq 1$.*

(b) *Let F be a subgroup of $Sp_6(2)$. Then $q(F) \leq 2$.*

(c) *Let $(p, m) \in \{(2,1), (2,2), (2,3), (3,1)\}$ and $n = p^m$. Then $GL_m(p)$ has no solvable subgroup of order larger than $3^{n-1}/n$.*

(d) *Let $(p, m) \in \{(2,1), (2,2), (2,3), (3,1)\}$ and $n = p^m$. Then $Sp_{2m}(p)$ has no solvable subgroup of order larger than $3^{2n-1}/n^2$.*

Proof: (a) If $q(F) > 1$, F has at least two non-abelian composition factors and so

$$|F| \leq 60 \cdot 60 = 3,600 > 720 = |Sp_4(2)|.$$

(b) Note that $|Sp_6(2)| = 2^9 \cdot 3^4 \cdot 5 \cdot 7 = 1,451,520$ and so $|Sp_6(2)|/60^3 < 7$. As $Sp_6(2)$ has no proper subgroup of index less than 7, $q(F) \leq 2$. We remark that, as can be see from the list of maximal subgroups of $Sp_6(2)$, $q(F) \leq 1$, but we do not need this fact.

(c) If $n = 2, 3$ or 4, then $|GL_m(p)| \leq 3^{n-1}/n$. So suppose $n = 8$. The largest solvable subgroup of $GL_3(2)$ has order 24, and $24 \leq 3^{8-1}/8$.

(d) If $n = 2$ or 3, $|Sp_{2m}(p)| \leq 3^{2n-1}/n^2$. If $n = 4$, then $3^{2n-1}/n^2 = 3^7/16 > |Sp_4(2)|/6$. The only subgroup of $Sp_4(2) \cong Sym(6)$ of index less than 6 are $Alt(6)$ and $Sym(6)$. If $n = 8$, then $3^{2n-1}/n^2 > |Sp_6(2)|/6$. As $Sp_6(2)$ is simple, it does not have a proper subgroup of index less then 6.

3 On $F^*(G)$ for locally finite groups

Lemma 3.1 (a) *The locally solvable chief-factors of G are elementary abelian p-groups.*

(b) *Let $\mathcal{C}$ be a chief-series for G. Then $\mathcal{LN}(G)$ is the largest subgroup of G centralizing all the factors of $\mathcal{C}$.*

Proof: [KW, 1.B.4, 1.B.10]

Lemma 3.2 *Suppose D is a normal subgroup of G and $\mathcal{D}$ is a chief-series for G on D such that D centralizes all the abelian factors of $\mathcal{C}$. Then*

(a) $[\mathcal{LS}(D), D^\infty] = 1$

(b) $\mathcal{LS}(D) = \mathcal{LN}(D)$.

(c) $S^*(D) = E(D)$

Proof: Note that by 3.1a the factors of $\mathcal{D} \cap \mathcal{LS}(D)$ are abelian and so centralized by D. Let H be a finite subgroup of D and T a finite subgroup of $\mathcal{LS}(D)$ and put $F = \langle T^H \rangle$. Then F is finite and H centralizes the factors of $\mathcal{D} \cap F$. Hence $[F, H, i] = 1$ for some $i \geq 1$.

If H is perfect, we conclude that H centralizes F. Thus (a) holds.

If $H = T = F \leq \mathcal{LS}(D)$, we conclude that H is nilpotent and so (b) holds.

Finally by (a), $S^*(D) \cap \mathcal{LS}(D) \leq Z(S^*(D))$ and so (c) holds.

Lemma 3.3 *$F^*(G)$ induces inner automorphism on each chief-factor of G.*

Proof: Let X/Y be a chief-factor of G. As $F^*(G)Y/Y \leq F^*(G/Y)$, we may assume that $Y = 1$.

By 3.1b, $[X, \mathcal{LN}(G)] = 1$ and so we may assume that $[X, E(G)] \neq 1$. Then $X = [X, E(G)] \leq E(G)$ and so by 2.1 $E(G) = XC_{E(G)}(X)$.

Example 3.4 *There exists a locally finite group G with $G \neq F^*(G)$ such that G induces inner automorphism on all of it chief-factors.*

Proof: Let H be a finite, perfect, simple group, I an infinite set, $H_i = H$ for all $\in I$, E the restricted cartesian product of the $H_i, i \in I$, viewed as a normal subgroup of the full cartesian product. Pick $1 \neq h \in H$ and put $h_i = h$ and $h^* = (h_i)_{i \in I}$. Finally but $G = E\langle h^* \rangle$. Then it is easy to check that $E = F^*(G)$ and G induces inner automorphism on all of its chief-factors.

4 A bound for $q(F, x)$ in finitary groups

Definition 4.1 (a) *Let $x \in GL_K(V)$. Then* $\deg(x) = \deg_V(x) = \dim[V, x]$ *and*

$$\mathrm{pdeg}(x) = \mathrm{pdeg}(xZ(GL_K(V)) = \min_{0 \neq \lambda \in K} \deg(\lambda \cdot x).$$

(b) *A $\mathcal{P}KG$-module is a K-vector space V together with a homomorphism*

$$\phi : G \to PGL_K(V).$$

Lemma 4.2 *Let K be an algebraicly closed field, $x \in PGL_n(K)$ and $y \in PGL_m(K)$. Then* $\mathrm{pdeg}(x \otimes y) \geq m \cdot \mathrm{pdeg}(x)$. *In particular, if $n, m \geq 2$, then* $\mathrm{pdeg}(x) + \mathrm{pdeg}(y) \leq \mathrm{pdeg}(x \otimes y)$.

Proof: Let $0 = V_0 \leq V_1 < V_2 < \ldots V_{m-1} < V_m = V$ be a maximal chain of y-invariant subspaces in V. As K is algebraicly closed, y acts on each of the V_i/V_{i-1} as a scalar. Thus

$$\mathrm{pdeg}(x \otimes y) \geq \sum_{i=1}^{m} \mathrm{pdeg}_{K^n \otimes V_i / K^n \otimes V_{i-1}}(x \otimes y) = \sum_{i=1}^{m} \mathrm{pdeg}_{K^n}(x) = m \cdot \mathrm{pdeg}(x).$$

To prove the second claim we may assume that $\mathrm{pdeg}(y) \leq \mathrm{pdeg}(x)$. Then

$$\mathrm{pdeg}(x \otimes y) \geq 2 \cdot \mathrm{pdeg}(x) \geq \mathrm{pdeg}(x) + \mathrm{pdeg}(y).$$

Lemma 4.3 (a) *If $F \leq PGL_n(K)$, then $q(F) \leq n - 1$.*

(b) *If $F \leq PGL_n(K)$ and $x \in F$, then $q(F, x) \leq 2 \cdot \mathrm{pdeg}(x)$.*

Proof: We assume without loss that K is algebraicly closed. Put $V = K^n$ and $d = \mathrm{pdeg}(x)$. We will prove (a) and (b) by induction on n, and for (b) also by induction on d. Moreover, in (b) we assume without loss that (a) is true and that x is contained in no proper subnormal subgroup of F. In particular, $F = \langle x^F \rangle$.

Suppose that U is a proper F-submodule in V. Then $C_F(V/U) \cap C_F(U)$ is abelian and so

$$\begin{aligned} q(F) = q(F/C_F(V/U)) + q(C_F(V/U)C_F(U)/C_F(U)) \leq \\ \leq (\dim V/U - 1) + (\dim U - 1) = n - 2 < n - 1 \end{aligned}$$

Similarly, $q(F,x) \leq 2d$.

So we may assume that F acts irreducibly on V. If F is not primitive on V, let Δ be a system of imprimitivity for F on V and put $k = |\Delta|$ and $m = n/k$.

Then $q(F) = q(F/C_F(\Delta)) + q(C_F(\Delta))$. Note that $F/C_F(\Delta)$ is isomorphic to a subgroup of $Sym(k)$ and $Sym(k)$ is isomorphic to a subgroup of $GL_{k-1}(K)$. Thus by induction $q(F/C_F(\Delta)) \leq k-2$. Moreover, $C_F(\Delta)$ is contained in the direct product of k subgroups of $GL_m(K)$ and hence by induction, $q(C_F(\Delta)) \leq k(m-1)$ and $q(F) \leq (k-2) + k(m-1) = km - 2 < n - 1$.

Put $t = \mathrm{pdeg}_\Delta(x)$, $\Gamma = C_\Delta(x)$, $U = \sum \Gamma$ and $W = \sum \Delta \setminus \Gamma$. Let l be the number of non-trivial orbits for x on Δ. As each non-trivial orbit for x in Δ has at least length 2:

(1) $2l \leq t$

Let $X_1, X_2, ... X_s$ be an orbit for x on Γ and $X = \sum_{i=1}^{s} X_i$. Then $X = X_1[X,x]$ and so $\dim[X,x] \geq (s-1)m$. It follows that $\mathrm{pdeg}_W(x) \geq (t-l)m$. Thus $(t-l)m + \mathrm{pdeg}_U(x) \leq \mathrm{pdeg}_W(x) + \mathrm{pdeg}_U(x) = d$ and so :

(2) $\mathrm{pdeg}_U(x) \leq d - tm + lm$

Note that the even permutation module for $Sym(k)$ gives rise to a faithful $F/C_F(\Delta)$-module Z with $\dim Z < n$ and $\mathrm{pdeg}_Z(x) \leq t - l$. Thus by induction :

(3) $q(F/C_F(\Delta), x) \leq 2(t-l)$

Since $C_F(\Delta)C_F(W)/C_F(W)$ acts faithfully on W, it is contained in the direct product of at most t copies of $GL_m(K)$. Thus (a) implies

(4) $q(C_F(\Delta)C_F(W)/C_F(W)) \leq t(m-1)$

Since $F = \langle x^F \rangle$, x does not fix all the elements of Δ and so U is a proper subspace of V. Futhermore, $C_F(W) \cap C_F(\Delta)$ acts faithfully on U and so by induction:

(5) $q(C_F(W) \cap C_F(\Delta), x) \leq 2 \cdot \mathrm{pdeg}_U(x)$

Note that

$$q(F,x) \leq q(F/C_F(\Delta), x) + q(C_F(\Delta)C_F(W)/C_F(W)) + q(C_F(\Delta) \cap C_F(W), x)$$

and so by (3), (4) and (5):

$$q(F,x) \leq 2(t-l) + t(m-1) + 2 \cdot \mathrm{pdeg}_U(x)$$

Hence by (2)

$$q(F,x) \leq 2(t-l) + t(m-1) + 2d - 2tm + 2lm = 2d - t(m-1) + 2l(m-1) = 2d - (t-2l)(m-1)$$

and so by (1), $q(F,x) \leq 2d$.

So we may assume that F acts primitively on V. Suppose that $V \cong U \otimes_K W$ for some at least 2-dimensional $\mathcal{P}KF$-modules U and W. Then

$$q(F) \leq q(F/C_F(U)) + q(F/C_F(W)) \leq (\dim U - 1) + (\dim W - 1)) \leq$$
$$\leq \dim U \cdot \dim W - 3 = n - 3 < n - 1.$$

Moreover, by (4.2) $\mathrm{pdeg}_U(x) + \mathrm{pdeg}_W(x) \leq \mathrm{pdeg}(x)$ and so

$$q(F,x) \leq q(F/C_F(U),x) + q(F/C_F(W),x) \leq 2 \cdot \mathrm{pdeg}_U(x) + 2 \cdot \mathrm{pdeg}_W(x) \leq 2d.$$

So we may assume that F acts tensor indecomposable on V. Since K is algebraicly closed and F acts primitively on V we conclude that each normal subgroup of F either acts as scalars or acts irreducibly on V. Let N be a normal subgroup of F minimal with respect to $N \not\leq Z(F)$. Then N is irreducible and so $C_F(N)$ act as scalars on V. In particular, $C_F(N) = Z(F)$.

Suppose first that N is not solvable and let Δ be the set of components of N. Then $N = \langle \Delta \rangle$. Put $e = |\Delta|$. Since the outer automorphism group of every finite simple group is solvable, $C_F(\Delta)/N$ is solvable. Hence $q(F) = q(F/C_F(\Delta)) + q(N)$ and so

(6) $q(F,x) \leq q(F) = q(F/C_F(\Delta)) + e.$

If $e = 1$, then $F = C_F(\Delta)$ and so by (6) $q(F) = 1 \leq n-1$ and $q(F,x) \leq 1 \leq 2 \cdot \mathrm{pdeg}(x)$. So we may assume that $e \geq 2$. As $F/C_F(\Delta) \leq \mathrm{Sym}(\Delta) \leq GL_{e-1}(K)$ we conclude by induction that $q(F/C_F(\Delta)) \leq e-2$. Thus by (6):

(7) $q(F,x) \leq q(F) \leq 2(e-1) \leq 2^{e-1}$

Since N acts irreducibly on V, $V \cong \bigotimes_{L \in \Delta} V_L$ as a KN-module, where V_L is an irreducible $\mathcal{P}KL$-module. In particular, $\dim V \geq 2^e$ and so by (7) $q(F) \leq n-1$.

Let $L \in \Delta$ with $[L,x] \neq 1$ and let Λ be the set of all irreducible $C_N(L)$-submodules in V. Then N acts non trivally on Λ. Suppose that x normalizes all elements of Λ. Since $L \leq [L,x]$, $N = [L,x]C_N(L)$ and so N acts trivally on Λ, a contradiction. Pick $W \in \Lambda$ with $W \neq W^x$. Since $C_N(LL^x)$ normalizes W and W^x, $C_N(LL^x)$ acts on $W/W \cap W^x$. Therefore $\dim W/W \cap W^x \geq 2^{e-2}$ and so $2^{e-2} \leq d$. Thus by (7) $q(F,x) \leq 2d$.

Suppose next that N is solvable. Then by 2.7 $\tilde{N} := N/C_N(F)$ is an elementary abelian group of order p^{2m} for some prime p and some positive integer m, $N \cong \mathrm{Ext}(p^{1+2m})$ or $N \cong C_4 \circ \mathrm{Ext}(2^{1+2m})$, $n = p^m$ and $C_F(\tilde{N}) = Z(F)N$ is solvable. Also $F/C_F(\tilde{N})$ is isomorphic to a subgroup of $Sp_{2m}(p) \leq GL_{2m}(p)$. By 2.8a either $2m < p^m = n$, or $p = 2$ and $m \geq 2$. In the first case by induction, $q(F) = q(F/C_F(\tilde{N})) \leq m-1 \leq n-1$. In the second case 2.9a implies $q(F) \leq 1 \leq n-1$.

Let $a \in F$, then $[x,a] = x^{-1}x^a$ has pdeg at most $2d$. If $[N,x] \leq Z(F)$, $\langle x^F \rangle$ is solvable and $q(F,x) = 0$. So we may assume that $[N,x] \not\leq Z(F)$. Thus $N \setminus Z(F)$ contains an element

y with $\mathrm{pdeg}(y) \leq 2d$. Then $1 \neq [y, N] \leq Z(F)$ and $Z(F)$ contains a non-trivial element z with $\mathrm{pdeg}(z) \leq 4d$. Since F acts irreducible on V, $V = [V, z]$ and so $p^m \leq n \leq 4d$. By 2.8b $2(2m-1) \leq p^m$ or $p = 2$ and $m \leq 3$. In the first case by (a)

$$q(F, x) \leq q(F) \leq q(F/C_F(\tilde{N})) \leq 2m - 1 \leq p^m/2 \leq 2d.$$

In the second case, $F/C_F(\tilde{N})$ is isomorphic to a subgroup of $Sp_6(2)$ and so by 2.9b $q(F) \leq 2 \leq 2d$. This completes the proof of Lemma 4.3.

5 The structure of B_∞

Lemma 5.1 (a) *B_∞ is a normal subgroup of G.*

(b) *Every finite subgroup of B_∞ is bounded.*

Proof: Let X and Y be n- and m-bounded subsets of G, respectively. Put $Z = \langle X, Y\rangle$ and $H = \langle H_X, H_Y\rangle$. Let $F \in \mathcal{H}(H)$ and $1 = F_0 \leq F_1 ... \leq F_{n-1} \leq F_n = F$ a chief-series for F. Then $[F_i/F_{i-1}, Z] = [F_i/F_{i-1}, X][F_i/F_{i-1}, Y]$ and so $q(F, Z) \leq q(F, X) + q(F, Y) \leq n + m$. Thus Z is $(n+m)$-bounded. In particular, any subgroup of G generated by finitely many bounded elements is bounded. Thus (a) and (b) follow.

Lemma 5.2 *Let $x \in B_n$ and $K \in \mathcal{H}(H_x)$. Put $F = \langle x^K\rangle$ and $s = s(F)$.*

(a) *$s \leq n$.*

(b) *$q(F/S_*(F), x) < n$*

Proof: Put $R_i = S_*^i(F)/S_*^{i+1}(F)$. As F is normal in K, $q(F, x) \leq q(K, x) \leq n$. Since $q(F, x) = \sum_{i=1}^s q(R_i, x)$ it suffices to show that $q(R_i, x) \geq 1$ for all $1 \leq i \leq s$. Suppose that $q(R_i, x) = 0$ for some $1 \leq i \leq s$ and put $\overline{R_i} = R_i/\mathcal{LS}(R_i)$. Then $\overline{R_i}$ is a non trivial, perfect, semisimple group with $q(\overline{R_i}, x) = 0$. Thus x centralizes $\overline{R_i}$. As K acts on $\overline{R_i}$ and $F = \langle x^K\rangle$ we conclude that F centralizes $\overline{R_i}$. But then $\overline{R_i}$ is abelian, a contradiction.

Let B be a normal subgroup of G contained in B_∞.

Let $x \in B$. Then by 5.2a there exist $d \geq 0$ and $K_x \in \mathcal{H}(H_x)$ so that $s(\langle x^{K_x}\rangle) = d$ and $s(\langle x^K\rangle) \leq d$ for all $K \in \mathcal{H}(K_x)$. Let $d = d(x)$ be a minimal such d. Note that $x \in \mathcal{LS}(B)$ if and only if $d(x) = 0$.

Let R be the union of the finite and perfect subgroups U of B such that there exists $K_U \in \mathcal{H}(U)$ with $U \leq S^*(F)$ for all $F \in \mathcal{H}(K_U)$. Note here that R is a characteristcic subgroup of B.

$S = \langle L \leq B | L$ is perfect and 1-bounded $\rangle$.

$T = \langle S_*(\langle x^K\rangle) \mid x \in B, K \in \mathcal{H}(K_x)\rangle$.

Lemma 5.3 *$T \leq R$ and if B is not locally solvable, $R \neq 1$.*

Proof: Let $x \in B$ and $F \in \mathcal{H}(K_x)$. By choice of d and K_x and by 2.4 we have $S_*(\langle x^K \rangle) \leq S^*(F)$. Thus $S_*(\langle x^K \rangle) \leq R$ and so $T \leq R$.

If B is not locally solvable, there indeed exists $x \in B$ with $d(x) \neq 0$. Hence $T \neq 1$ and $R \neq 1$.

Proposition 5.4 (a) *Let L be perfect and 1-bounded. Then for all $F \in \mathcal{H}(H_L), L \leq S^*(F)$ and $\langle L^{\langle L^F \rangle} \rangle$ is solvable by simple.*

(b) *Let L be perfect and 1-bounded. Then $\langle L^G \rangle$ is locally solvable by semisimple and $\langle L^{\langle L^G \rangle} \rangle$ is locally solvable by simple.*

(c) *$S \leq R$ and S is locally solvable by semisimple.*

Proof: (a) Put $\overline{F} = F/\mathcal{LS}(F)$ and $E = S^*(F)$. Then $C_{\overline{F}}(E) = 1$ and so $[\overline{E}, \overline{L}] \neq 1$. Since $q(F, L) = 1$, we conclude that $q(F/E, L) = 0$ and since L is perfect, $L \leq E$. Similarly, $[\overline{E}, L]$ is simple and $\overline{L} \leq [\overline{E}, L]$. Hence $\langle \overline{L}^{\langle L^F \rangle} \rangle = [\overline{E}, L]$ and (a) is proved

(b) By (a) $Y = \langle L^{\langle L^G \rangle} \rangle$ has a local system of perfect, solvable by simple, finite subgroups. Hence 2.5 implies $Y = \langle y^Y \rangle$ for all $y \in Y \setminus \mathcal{LS}(Y)$ and so Y is locally solvable by simple. Thus $\langle L^G \rangle$ is locally solvable by semisimple.

(c) By (a) $L \leq R$ and so $S \leq R$. By (b), S is locally solvable by semisimple.

Lemma 5.5 *$S = R$, in particular, if B is not locally solvable, $S \neq 1$.*

Proof: Let U be a perfect finite subgroup of B such that there exists $K_U \in \mathcal{H}$ with $U \leq S^*(F)$ for all $F \in \mathcal{H}(K_U)$. Let d be minimal with respect to U being d-bounded and put $H = \langle K_U, H_U \rangle$. By the minimality of d there exists $F \in \mathcal{H}(H)$ with $q(F, U) = d$. Put $L = \langle U^{\langle U^F \rangle} \rangle$. Then clearly $q(L, U) = q(F, U) = d$. Moreover, since $U \leq S^*(F)$, it is easy to see that $L/\mathcal{LS}(L)$ is the direct product of d simple, perfect subgroups. Let $L_1, L_2, \ldots L_k$ be the non trivial, minimal perfect, normal subgroups of $\langle U^F \rangle$ and choose notation so that $L = L_1 L_2 \ldots L_d$.

We will now prove that for all i, L_i is 1-bounded, where we choose $H_{L_i} = F$. Let $P \in \mathcal{H}(F)$, $Q = \langle U^P \rangle$ and $\overline{P} = P/\mathcal{LS}(Q)$. Since $U \leq S^*(P)$, $\overline{Q}$ is semisimple. Also $L = \langle L^U \rangle \leq Q$. Let $\overline{Q}_1, \overline{Q}_2, \ldots, \overline{Q}_l$ be the components of $\overline{Q}$. Then $\overline{Q}$ is the direct product of the $\overline{Q}_j$'s. Let $\overline{L}_{ij}$ be the projection of $\overline{L}_i$ onto $\overline{Q}_j$ and let L_{ij} be the inverse image of $\overline{L_{ij}}$ in P. Let M be the group generated by the L_{ij}'s. Since L_{ij} is a solvable by simple and is normal in M, M is solvable by semisimple. Note that F normalizes M and so $q(M, U) \leq q(FM, U) \leq d$.

Suppose that for some $1 \leq i \leq d$ and some j, $[L_{ij}, U] \leq \mathcal{LS}(M)$. Then $L = \langle U^L \rangle$ and $[L_{ij}, L] \leq \mathcal{LS}(M)$. Hence $[\overline{L}_{ij}, \overline{L}_{ij}]$ is solvable. Since $\overline{L}_{ij}$ is perfect, $\overline{L}_{ij} = 1$.

Suppose that for some $1 \leq i < t \leq d$, and some j, $L_{ij}\mathcal{LS}(M) = L_{tj}\mathcal{LS}(M)$. Since $[L_i, L_t]$ is solvable we conclude $[\overline{L}_{ij}, \overline{L}_{ij}]$ is solvable and again $\overline{L}_{ij} = 1$.

For $1 \leq i \leq d$ put $p(i) = |\{j \mid 1 \leq j \leq l, \overline{L}_{ij} \neq 1\}|$. Since $p(i) \geq 1$ we conclude from the last two paragraphs that

$$d \leq q(M,U) \leq \sum_{i=1}^{d} p(i) \leq d.$$

Hence for all $1 \leq i \leq d$, $p(i) = 1$ and $q(P, L_i) = q(Q, L_i) = 1$. Thus L_i is 1-bounded, $L \leq S$ and since $U \leq L, U \leq S$ and $R \leq S$. By 5.4(c), $S \leq R$ and so $S = R$.

$S \neq 1$ follows now from 5.3.

Lemma 5.6 *For all $n \geq 0$, $\langle B_n \cap B\rangle^\infty \leq \langle x \in B \mid d(x) \leq n\rangle^\infty \leq S^*_n(B)$. In particular, $B^\infty = S^*_\infty(B)$.*

Proof: The first containment follows from 5.2(a).

Let $x \in B$ with $d(x) \leq n$. If $n = 0$, $\langle x^G\rangle$ is locally solvable and so $\langle x \in B \mid d(x) \leq 0\rangle^\infty = 1$. So suppose that $n \geq 1$. By 5.3 and 5.5, $T \leq S$. As $d(x) \leq n$, $S^{n-1}_*(\langle x^K\rangle) \leq S_*(\langle x^K\rangle) \leq T \leq S$ for all $T \in \mathcal{H}(K_x)$. But this implies that in G/S, $d(xS) \leq n-1$. Thus the second containment follows by induction on n.

Theorem 5.7 (a) $C_B(S/\mathcal{LS}(S)) = \mathcal{LS}(B)$.

(b) $S = S^*(B)$.

Proof: We may assume without loss that $\mathcal{LS}(B) = 1$. Then S is semisimple and $Z(S) = 1$.

Put $C = C_B(S)$ and assume that $C \neq 1$. Then by 5.5 applied to C instead of B, C has a non-trivial perfect 1-bounded subgroup U. Thus $U \leq S \cap C = Z(S) = 1$, a contradiction, which proves (a).

Since $S^*(B)$ is semisimple, $S^*(B) = C_{S^*(B)}(S)S = S$.

Proposition 5.8 (a) *Let $x \in B_n$. Then at most n of the components of $S/\mathcal{LS}(S)$ are not centralized by x.*

(b) *B acts finitarily on the set of components of $S/\mathcal{LS}(S)$.*

Proof: Without loss $\mathcal{LS}(B) = 1$. Let L be component of B not centralized by $\langle x^{H_x}\rangle$. Since x^{H_x} is finite it is easy to see that there exists a non-trivial, finite perfect, $N_{H_x}(L)$-invariant subgroup A_L of L with the following property:

(1) Let $h \in H_x$ such that x^h normalizes but does not centralize L. Then $[A_L, x^h]$ is not solvable.

Choose the A_L's in such a way that $A_L^h = A_{L^h}$ for all $h \in H_x$. If x does not normalize L it is easy to see that $A_L = A'_L \leq [A_L, x]$. Together with (1) we conclude

(2) If x does not centralize L then $[A_L, x] \cap L$ is not solvable.

Let Δ be finite set of components of B none of which is centralized by x. Put $\Gamma = \{L^h | L \in \Delta, h \in H_x\}$ and $A = \langle A_L \mid L \in \Gamma\rangle = \prod_{L\in\Gamma} A_L$. Then A is a finite subgroup of B normalized by H_x. By (2), none of groups $[A_L, x] \cap L$ for $L \in \Delta$ is solvable and thus

$$|\Delta| \leq q(A, x) \leq q(AH_x, x) \leq n.$$

This implies (a). (b) follows immediately from (a).

Lemma 5.9 *Let M be a component of B, $L \leq M$ and F a finite subgroup of $N_G(M)$. If L is perfect and 1-bounded, then L is contained in a perfect, 1-bounded, F-invariant subgroup of M.*

Proof: Note that $M = \langle L^{\langle L^M\rangle}\rangle$ and so $\langle L^F\rangle \leq \langle L^{\langle L^T\rangle}\rangle$ for some $T \in \mathcal{H}$. Put $K = \langle F, H_L, T\rangle$, $\overline{K} = K/\mathcal{LS}(K)$ and $P = \langle L^{\langle L^K\rangle}\rangle$. Then P is perfect and $\langle L^F\rangle \leq P$. Furthermore, as L is 1-bounded, $\overline{P}$ is a component of $\overline{K}$. Since F normalizes a non-trivial subgroup of $\overline{P}$, (namely $\overline{\langle L^F\rangle}$) we conclude that F normalizes $\overline{P}$ and so also $P\mathcal{LS}(K)$. As P is perfect and subnormal in $P\mathcal{LS}(K)$, $P = (P\mathcal{LS}(K))^\infty$ and so P is F-invariant. Moreover, by 2.6, P is 1-bounded.

Proposition 5.10 *All chief-factors for G on B are semisimple as groups.*

Proof: Let Y/X be chief-factor for G on B.

If Y/X is locally solvable, Y/X is semisimple by 3.1(a).

If Y/X is not locally solvable, then $\mathcal{LS}(Y/X) = 1$. Thus by 5.7, $Y/X = S^*(Y/X) = E(Y/X)$ and again Y/X is semisimple.

We are now able to improve 3.2 for bounded groups:

Proposition 5.11 *Let $\mathcal{C}$ be a chief-series for G on B. Then $F^*(B)$ is the largest subgroup of B inducing inner automorphism on all the factors of $\mathcal{C}$.*

Proof: Let D largest subgroup of B inducing inner automorphism on all the factors of $\mathcal{C}$. Since $F^*(B) \leq F^*(G)$, 3.3implies that $F^*(B) \leq D$.

Let $\mathcal{D} = D \cap \mathcal{C}$. Then D also induces inner automorphisms on all the factors of $\mathcal{D}$. In particular D centralizes the abelian factors of $\mathcal{D}$ and so by 3.2 $\mathcal{LS}(D) = \mathcal{LN}(D)$ and $S^*(D) = E(D)$. Let $X = C_D(E(D)$. Since $S^*(D) = E(D)$, 5.7 applied to D in place of B, yields $X \leq \mathcal{LS}(D)$. Thus $X \leq \mathcal{LS}(D) = \mathcal{LN}D \leq F^*(D)$.

Let $d \in D$. By 5.8 d centralizes all put finitely many componets of D. As d induces inner autmorphisms on $\mathcal{D} \cap E(D)$, d induces inner automorphism on each component of D. Thus $d \in XE(B) \leq F^*(D) \leq F^*(B)$. Thus $D \leq F^*(B)$ and the proposition is proved.

6 Proof of Theorem A

In this section we prove Theorem A. For this let G be a locally finite, finitary group. Then by 4.3, G is bounded. Put $B = C_G(\mathcal{LN}(G))$. Then $\mathcal{LN}(B) = Z(B)$.

Suppose that $\mathcal{LS}(B) \not\leq Z(B)$ and let $b \in \mathcal{LS}(B) \setminus Z(B)$. Put $D = \langle b^G \rangle$. Then by [MPP, Proposition 1], D is solvable and we can choose $E \leq D$ with $E \not\leq Z(B)$, but $E' \leq Z(B)$. Then $E' \leq Z(E)$, E is nilpotent and $E \leq \mathcal{LN}(B) \leq Z(B)$, a contradcition.

Hence $\mathcal{LS}(B) = Z(B)$ and so $S^*(B) = E(B) \leq E(G)$. Thus by 5.7, $C_B(E(B)) = C_B(S^*(B)) \leq \mathcal{LS}(B) \leq Z(B)$ and so

$$C_G(F^*(G)) = C_G(E(G)\mathcal{LN}(G) \leq C_B(E(B)) \leq Z(B) \leq F^*(G).$$

and Theorem A is proved.

7 On Kegel covers for simple, locally finite, finitary groups

In this chapter we investigate Kegel covers of simple, locally finite, finitary groups. Recall that a Kegel covers of the locally finite groups G is a set of pairs $\{(H_i, M_i) \mid i \in I\}$ such that

(i) For all $i \in I$, H_i is a finite subgroup of G and M_i is a maximal normal subgroup of H_i.

(ii) For all finite subgroup F of G there exists $i \in I$ with $F \leq H_i$ and $F \cap M_i = 1$.

We remark that every locally finite, simple group has a Kegel cover. See [KW] for a proof.

Lemma 7.1 (a) *Let $F \leq Sym(n)$. Then $|\mathcal{LS}(F)| \leq 3^{n-1}$.*

(b) *Let $F \leq GL_n(K)$. Then F has a unipotent by abelian normal subgroup U with $|\mathcal{LS}(F)/U| \leq 3^{2n-1}$.*

(c) *Let $F \leq GL_n(K)$ and $x \in F \setminus \mathcal{LS}(F)$ with $F = \langle x^F \rangle$. Put*

$$t = \max\{|F/\mathcal{LS}(F)| \cdot \mathrm{pdeg}(x), 4 \cdot \mathrm{pdeg}(x)\}.$$

(c.a) *The sum of the dimensions of the non-trivial composition factors for F on K^n is at most t.*

(c.b) *F has a unipotent by abelian normal subgroup U with $|\mathcal{LS}(F)/U| \leq 3^{2t-1}$.*

(c.c) *Any abelian normal subgroup of $F/\mathrm{Unip}(F)$ has rank at most t.*

Proof: Note that (c.b) follows from (c.a) and (b). Furthermore any abelian subgroup of $GL_n(K)$ has, modulo its unipotent part, rank at most n and so (d.c) follows from (c.a).

(a),(b) and (c.a) are proved simultanously by induction on n. We argue similary as in (4.3). Put $V = K^n$, $\Omega = \{1, 2, \ldots, n\}$, $S = \mathcal{LS}(F)$ and $l = |F/S|$. For (b) we assume without loss that K is algebraicly closed. As in (4.3) we may assume that G acts transitively on Ω and irrreducibly on V, respectively.

If G acts imprimitively, let Δ be a maximal system of imprimitivity for G on Ω and V, respectively. Put $d = |\Delta|$ and $k = n/d$.

In (a) we get

$$|S| \leq |\mathcal{LS}(C_F(\Delta)||\mathcal{LS}(F/C_F(\Delta))| \leq (3^{k-1})^d 3^{d-1} = 3^{n-1}.$$

In (b) let $W \in \Delta$. By induction $N_F(W)$ has a normal subgroup X containing $C_F(W)$ such that $X/C_F(W)$ is unipotent by abelian and $|\mathcal{LS}(N_F(W))C_F(W)/X| \leq 3^{2k-1}$. Put $U = \bigcap_{g \in F} X^g$. Then U is unipotent by abelian and

$$|\mathcal{LS}(F)/U| \leq |\mathcal{LS}(C_F(\Delta))/U||\mathcal{LS}(F/C_F(\Delta))| \leq (3^{2k-1})^d 3^{d-1} = 3^{2n-1}.$$

In (c) note that $k \cdot \deg_\Delta(x) \leq 2 \cdot \mathrm{pdeg}(x)$. Assume first that $S \not\leq C_F(\Delta)$ and let A be a minimal normal subgroup of $SC_F(\Delta)/C_F(\Delta)$. Then A is abelian and either $xC_F(\Delta) \in A$ or $[A, x] \neq 1$. In both cases A contains an element $1 \neq y$ with $\deg_\Delta(y) \leq 2 \cdot \deg_\Delta(x) \leq \frac{4}{k} \cdot \mathrm{pdeg}(x)$. Since A acts regularly on Δ we conclude that $d \leq \frac{4}{k} \cdot \mathrm{pdeg}(x)$. Thus $n = dk \leq 4 \cdot \mathrm{pdeg}(x) \leq t$. Assume next that $S \leq C_F(\Delta)$. Since F acts transitively on Δ, $d \leq |F/C_F(\Delta)| \leq |F/S| = l$, and we conclude that $n \leq lk \leq l \cdot \mathrm{pdeg}(x) \leq t$.

So we may assume that G acts primitively.

In case (b) suppose that $V \cong X \otimes_K Y$ for some at least 2-dimensional $\mathcal{P}KF$-modules X and Y.Then a finite central extension of G acts faithfully on $X \oplus Y$ and $\dim(X \oplus Y) \leq n$. Thus by the reducible case, (b) holds. So we may assume in case (b) that V is tensor-indecomposable.

Let M be normal subgroup of F in S minimal with respect to $[M, F] \neq 1$.

In (a), M is an elementary abelian p-group acting regularly on Ω. So $n = |M| = p^m$ for some m and $|F/M|$ is isomorphic to a subgroup of $GL_m(p)$. Moreover, F/M acts irreducible on M and so has no non-trivial unipotent normal subgroup. Note that every abelian p' subgroup of $GL_m(p)$ has order at most $p^m - 1 = n - 1$. Clearly $m < n$. Thus by induction and (b), $|S/M| \leq (n-1)3^{2m-1}$ and so $|S| \leq n(n-1)3^{2m-1}$. By 2.8c we conclude that either $n(n-1)3^{2m-1} \leq 3^{n-1}$ or $n \in \{2, 3, 4, 8\}$. In the first case $|S| \leq 3^{n-1}$. In the second case, 2.9c implies $|S/M| \leq 3^{n-1}/n$ and again $|S| \leq 3^{n-1}$.

In (b) since F is primitive, M is not abelian. Since V is tensor indecomposable and K is algebraicly closed, M acts irreducibly on V. Thus $C_F(M) = Z(F)$ is cyclic and we can apply 2.7. We conclude that $M \cong \mathrm{Ext}(p^{1+2m})$ or $M \cong C_4 \circ \mathrm{Ext}(2^{1+2m})$ for some m that $n = p^m$ and that $F/MZ(F)$ acts faithfully $M/Z(M)$. Clearly $m < n$. As every abelian p' subgroup of $GL_{2m}(p)$ has order at most $p^{2m} - 1 = n^2 - 1$, we get by induction that $|S/MZ(F)| \leq (n^2 - 1)3^{4m-1}$. Thus $|S/Z(F)| \leq n^2(n^2 - 1)3^{4m-1}$. By 2.8 either

$n^2(n^2-1)3^{4m-1} \leq 3^{2n-1}$ or $n \in \{2,3,4,8\}$. In the first case $|S/Z(F)| \leq 3^{2n-1}$. In the second case 2.9d implies that $|S/Z(F)M| \leq 3^{2n-1}/p^{2m}$ and so again $|S/Z(F)| \leq 3^{2n-1}$.

In (c), M is not abelian and $M' \leq Z(F)$. As in 4.3 $Z(F)$ contains a non-trivial element of pdeg at most $4 \cdot \mathrm{pdeg}(x)$ and thus, $n \leq 4 \cdot \mathrm{pdeg}(x) \leq t$.

Theorem 7.2 *Let G be a simple, locally finite, finitary group and L a perfect 1-bounded subgroup of G. Then there exists a finite subgroup H of G containing L such that $\langle L^{\langle L^F \rangle}\rangle$ is unipotent by quasisimple for all finite subgroups F of G containing H.*

Proof: For $F \in \mathcal{H}(H_L)$ define $\tilde{F} = \langle L^{\langle L^F \rangle}\rangle$. For $F \in \mathcal{H}$ let $U(F)$ be the set of unipotent by abelian normal subgroups of F, $u(F) = \min\{|\mathcal{LS}(F)/U| | U \in U(F)\}$, $U_+(F) = \{U \in U(F) || \mathcal{LS}(F)/U| = u(F)\}$, $r(F) = \min\{\mathrm{rank}(U/\mathrm{Unip}(U)) | U \in U_+(F)\}$ and $U_*(F) = \{U \in U_+(F) | \mathrm{rank}(U/\mathrm{Unip}(U)) = r(F)\}$. Replacing L be $\tilde{F}$ for some $F \in \mathcal{H}(H_L)$ we may assume that L is solvable by simple. Let E be a perfect, solvable by simple, finite subgroup of G with $E = \mathcal{LS}(E)L$ and let $x \in L \setminus \mathcal{LS}(L)$. By 2.5, $E = \langle x^E \rangle$ and so by (7.1), $u(E)$ and $r(E)$ are bounded by a function of $|L/\mathcal{LS}(L)|$ and $\mathrm{pdeg}(x)$. Hence we can choose E such that first $u(E)$ and then $r(E)$ is maximal. By 2.6, E is 1-bounded. Thus we may and do assume that $L = E$.

As G has a Kegel cover there exists a subgroup $\hat{R}$ of G and a maximal normal subgroup M of $\hat{R}$ so that $H_L \leq \hat{R}$ and $L \cap M = 1$. Put $R = \langle L^{\hat{R}} \rangle$. As L is perfect and 1-bounded, R is perfect, $RM = \hat{R}$ and $R \cap M = \mathcal{LS}(R)$. In particular, $L \cap \mathcal{LS}(R) = 1$ and R is solvable by simple.

Put $H = \langle R, H_L \rangle$ and let $F \in \mathcal{H}(H)$. To complete the proof of the theorem we will show that $\tilde{F}$ is unipotent by quasi-simple. Note first that $\tilde{F}$ is solvable by simple. As $R \leq \tilde{F}$ we have $L \cap \mathcal{LS}(\tilde{F}) \leq L \cap \mathcal{LS}(R) = 1$. Put $C = [\mathcal{LS}(\tilde{F}), L]$, $D = CL$ and $\overline{D} = D/C$. Then $D = \langle L^{\mathcal{LS}(\tilde{F})} \rangle$, D is perfect, $L \cap C = 1$, $\overline{D} \cong L$ and $\mathcal{LS}(D) = \mathcal{LS}(L)C$. Hence by maximal choice of $u(E)$ $(= u(L))$, $u(D) \leq u(L)$. Let $U \in U_*(D)$. Then

$$|\mathcal{LS}(D)/U| = u(D) \leq u(L) = u(\overline{D}) \leq |\mathcal{LS}(\overline{D})/\overline{U}| = |\mathcal{LS}(D)/UC| \leq |\mathcal{LS}(D)/U|.$$

Thus $C \leq U$, $u(D) = u(L)$,

$$|\mathcal{LS}(L)/U \cap L| = |\mathcal{LS}(L)U/U| = |\mathcal{LS}(D)/U| = u(L)$$

and $U \cap L \in U_+(L)$.

Suppose first that U is unipotent. Then $[\mathcal{LS}(\tilde{F}), L]$ is a unipotent normal subgroup of $\mathcal{LS}(\tilde{F})$ and so $[\mathcal{LS}(\tilde{F}), L] \leq \mathrm{Unip}(\tilde{F})$. Since $\tilde{F} = \langle L^{\tilde{F}} \rangle$ we conclude $[\mathcal{LS}(\tilde{F}), \tilde{F}] \leq \mathrm{Unip}(\tilde{F})$ and $\tilde{F}$ is unipotent by quasisimple.

Suppose next that U is not unipotent. Then $r(D) \neq 0$ and by maximality of $r(E)$ $(= r(L))$, $r(L) \neq 0$. That is, $U \cap L$ is not unipotent. Let p be a prime and A an elementary abelian p-subgroup of $U \cap L$ with $\mathrm{rank}(A) = \mathrm{rank}(U \cap L/\mathrm{Unip}(U \cap L))$. Since $U \cap L \in U_+(L)$, $\mathrm{rank}(A) \geq r(L)$. For $X \leq L\mathcal{LS}(\tilde{F})$, put $X^* = X\mathrm{Unip}(U)/\mathrm{Unip}(U)$. As $[\mathcal{LS}(\tilde{F}), A, A] \leq$

$[C,A] \leq [U,U] \leq \mathrm{Unip}(U)$ we conclude that $[\mathcal{LS}(\tilde{F}),A]^*A^* = \langle A^{\mathcal{LS}(\tilde{F})}\rangle^*$ is an elementary abelian p-group. Since A is a p- and $\mathrm{Unip}(C)$ is a p'-group, $\mathrm{Unip}(U) \cap AC \leq C$. Hence $\mathrm{Unip}(U)C \cap A = C \cap A \leq \mathcal{LS}(\tilde{F}) \cap L = 1$. Thus $C^* \cap A^* = 1$ and in particular $[\mathcal{LS}(\tilde{F}),A]^* \cap A^* = 1$. As $\mathrm{rank}(U^*) = r(D) \leq r(L) \leq \mathrm{rank}(A^*)$, we get $[\mathcal{LS}(\tilde{F}),A]^* = 1$ and $[\mathcal{LS}(\tilde{F}),A] \leq \mathrm{Unip}(U)$. Since $\tilde{F} = \langle A^{\tilde{F}}\rangle$, we conclude $[\mathcal{LS}(\tilde{F}),\tilde{F}] \leq \mathrm{Unip}(\tilde{F})$ and $\tilde{F}$ is unipotent by quasisimple.

Theorem 7.3 *Let G be a perfect, simple, locally finite, finitary group.*

(a) *Let $\{(H_i, M_i) \mid i \in I\}$ be a Kegel cover for G with H_i/M_i perfecr for all i. Let R_i the unique minimal normal supplement to M_i in H_i. Then there exists a finite subgroup H in G so that*

$$\{(R_i, R_i \cap M_i) \mid i \in I, H \leq H_i, H \cap M_i = 1\}$$

is Kegel cover for G and such that all these R_i's are unipotent by quasisimple.

(b) *G has a Kegel cover $\{(H_i, M_i) \mid i \in I\}$ such that the H_i's are unipotent by quasisimple.*

Proof: Clearly (a) implies (b).

To prove (a) note that by 4.3 and 5.5, G has a finite perfect and 1-bounded subgroup L. Let H be as in 7.2. Without loss, $H \leq H_i$ and $L \cap M_i = 1$ for all $i \in I$. Put $H_i^* = \langle L_i^H\rangle$. It is easy to see that $\{(H_i^*, H_i^* \cap M_i) \mid i \in I\}$ is a Kegel cover for G. By 7.2, H_i^* is perfect and unipotent by simple. In particular, H_i^* has no proper normal supplement to $\mathcal{LS}(H_i^*)$. Since $H_i^* M_i = H_i$, $H_i^* \cap M_i = \mathcal{LS}(H_i^*)$. Thus $H_i^* = R_i$ and the theorem is proved.

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