Defining relations and the algebraic structure of the group SL_2 over integral Hamilton quaternions*

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In the present paper, we study the group $SL_2(\mathbb{H}_\mathbb{Z})$ of (2×2)-matrices with reduced norm 1 over the noncommutative ring of Hamilton quaternions with integral coefficients. In section 1, we obtain a presentation for $SL_2(\mathbb{H}_\mathbb{Z})$ in terms of generators and defining relations. In section 2, we exhibit a subgroup of finite index of $SL_2(\mathbb{H}_\mathbb{Z})$ with a nonabelian free quotient of rank 3. It is well known that the group $SL_2(\mathbb{Z})$ has the same property. In fact, the commutator subgroup of $SL_2(\mathbb{Z})$ is a free group of rank 2 and has index 12. The normal subgroup generated in $SL_2(\mathbb{Z})$ by the element

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \tag{1}$$

coincides with $SL_2(\mathbb{Z})$ if $m = 1$, is of finite index if $1 < m \leq 5$ and has infinite index otherwise. In $SL_2(\mathbb{H}_\mathbb{Z})$, the normal closure of the element (1) also coincides with the whole group if $m = 1$. But the normal closure of (1) in $SL_2(\mathbb{H}_\mathbb{Z})$ turns out to be of infinite index if m = 2.

We have tried to keep the exposition of the paper as elementary as possible.

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1 Defining relations for $SL_2(\mathbb{H}_\mathbb{Z})$

Since the ring $\mathbb{H}_\mathbb{Z}$ is Euclidean it follows easily that $SL_2(\mathbb{H}_\mathbb{Z})$ is generated by the elementary unipotent matrices:

$$\begin{aligned} q_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \\ q_3 &= \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad q_4 = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \\ q_5 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad q_6 = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \\ q_7 &= \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix}, \quad q_8 = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}. \end{aligned} \tag{2}$$

It is easy to check that the generators (2) can be expressed in terms of the matrices

$$\begin{aligned} a &= \begin{pmatrix} k & 0 \\ 0 & -j \end{pmatrix}, \quad b = \begin{pmatrix} i & 0 \\ 0 & k \end{pmatrix}, \\ c &= \begin{pmatrix} k & 1 \\ 0 & -k \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{3}$$

as follows:

$$\begin{aligned} q_1 &= dbdacdadb, \quad && q_2 = bq_1b^{-1}, \\ q_3 &= aq_1{}^{-1}a^{-1}, && q_4 = baq_1{}^{-1}a^{-1}b^{-1}, \\ q_5 &= dq_1{}^{-1}d^{-1}, && q_6 = dq_2{}^{-1}d^{-1}, \\ q_7 &= dq_3{}^{-1}d^{-1}, && q_8 = dq_4{}^{-1}d^{-1}. \end{aligned} \tag{4}$$

So we shall use the elements (3) as generators for $SL_2(\mathbb{H}_\mathbb{Z})$. Using the following well known representation of the quaternions:

$$\begin{aligned} i &\to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ k &\to \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad 1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

we represent any element

$$x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}_\mathbb{Q} \tag{5}$$

as a (2×2)-matrix over $\mathbb{Q}[i]$:

$$\begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \quad (x_j \in \mathbb{Q}).$$

Hence any element $A \in SL_2(\mathbb{H}_{\mathbb{Q}})$ has a representation as a unitary (4×4)-matrix in $SL_4(\mathbb{Q}[i])$:

$$A = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 & y_1 + iy_2 & y_3 + iy_4 \\ -x_3 + ix_4 & x_1 - ix_2 & -y_3 + iy_4 & y_1 - iy_2 \\ z_1 + iz_2 & z_3 + iz_4 & t_1 + it_2 & t_3 + it_4 \\ -z_3 + iz_4 & z_1 - iz_2 & -t_3 + it_4 & t_1 - it_2 \end{pmatrix} \quad (x_j, y_j, z_j, t_j \in \mathbb{Q}). \tag{6}$$

The integral matrices (6) with $\det A = 1$ form the unitary group $SU_4(\mathbb{Z}[i])$ isomorphic to $SL_2(\mathbb{H}_{\mathbb{Z}})$. Thus we have the following

Proposition 1 *The group $SU_4(\mathbb{Z}[i]) \simeq SL_2(\mathbb{H}_{\mathbb{Z}})$ is generated by the matrices*

$$\begin{gathered} a' = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b' = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \\ c' = \begin{pmatrix} 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad d' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{7}$$

In order to describe defining relations for $SL_2(\mathbb{H}_{\mathbb{Z}})$ we shall use the exceptional isomorphism between 6-dimensional orthogonal groups and 4-dimensional linear groups. In our context, this exceptional isomorphism can be described by an exact sequence of group homomorphisms

$$1 \to \mathbb{Z}_2 \to SU_4(\mathbb{Q}[i]) \to O_6(f, \mathbb{Q}) \to \mathbb{Q}^*/(\mathbb{Q}^*)^2 \to 1. \tag{8}$$

For the geometric background, the reader is referred to [2, §7].

To describe in detail the homomorphism $SU_4(\mathbb{Q}[i]) \to O_6(f, \mathbb{Q})$ in (8), we start out with the quadratic form

$$F = \xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6. \tag{9}$$

We parametrize the set of solutions of the equation

$$\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6 = 0 \tag{10}$$

as follows. Introduce variables u_i, v_i $(i = 1, 2, 3, 4)$ and the Plücker coordinates

$$\begin{aligned} &\xi_1 = -u_1v_4 + u_4v_1, \quad &&\xi_2 = -u_2v_3 + u_3v_2, \quad &&\xi_3 = -u_1v_3 + u_3v_1, \\ &\xi_4 = u_2v_4 - u_4v_2, &&\xi_5 = u_1v_2 - u_2v_1, &&\xi_6 = u_3v_4 - u_4v_3. \end{aligned}$$

In these variables, the equation (10) is an identity.

For any (4×4)-matrix $A = (a_{ij})$ over $\mathbb{Q}[i]$ with $\det A \neq 0$ consider the following two simultaneous transformations

$$\begin{pmatrix} u_1' \\ \vdots \\ u_4' \end{pmatrix} = A \begin{pmatrix} u_1 \\ \vdots \\ u_4 \end{pmatrix}, \qquad \begin{pmatrix} v_1' \\ \vdots \\ v_4' \end{pmatrix} = A \begin{pmatrix} v_1 \\ \vdots \\ v_4 \end{pmatrix}. \tag{11}$$

The corresponding Plücker coordinates ξ_i are transformed accordingly by a matrix B:

$$\begin{pmatrix} \xi_1' \\ \vdots \\ \xi_6' \end{pmatrix} = B \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_6 \end{pmatrix}. \tag{12}$$

The entries b_{ij} of B are computed as follows. We have

$$\begin{aligned} u_1' &= a_{11}u_1 + a_{12}u_2 + a_{13}u_3 + a_{14}u_4, \\ u_4' &= a_{41}u_1 + a_{42}u_2 + a_{43}u_3 + a_{44}u_4, \\ v_1' &= a_{11}v_1 + a_{12}v_2 + a_{13}v_3 + a_{14}v_4, \\ v_4' &= a_{41}v_1 + a_{42}v_2 + a_{43}v_3 + a_{44}v_4, \end{aligned}$$

$$\begin{aligned} \xi_1' &= -u_1'v_4' + u_4'v_1' \\ &= (a_{11}a_{44} - a_{14}a_{41})\xi_1 + (a_{12}a_{43} - a_{13}a_{42})\xi_2 + (a_{11}a_{43} - a_{13}a_{41})\xi_3 \\ &\quad + (a_{14}a_{42} - a_{12}a_{44})\xi_4 + (a_{12}a_{41} - a_{11}a_{42})\xi_5 + (a_{14}a_{43} - a_{13}a_{44})\xi_6 \\ &= b_{11}\xi_1 + b_{12}\xi_2 + b_{13}\xi_3 + b_{14}\xi_4 + b_{15}\xi_5 + b_{16}\xi_6. \end{aligned}$$

The computation of all other entries is similar.

It is easy to check that the form F is invariant under the linear transformation B if $\det A = 1$. One can check also that the mapping

$$\phi : \; SL_4(\mathbb{Q}[i]) \to O_6(F, \mathbb{Q}[i]), \quad A \mapsto B, \tag{13}$$

is a group homomorphism where $O_6(F, \mathbb{Q}[i])$ is the group of all linear transformations B with coefficients in the field $\mathbb{Q}[i]$ which leave invariant F.

Now we want images of matrices in $SU_4(\mathbb{Q}[i])$ under the mapping (13) to be matrices with real coefficients. For this purpose, introduce new variables $\eta_1, \eta_2, \ldots, \eta_6$:

$$\begin{aligned} \xi_1 &= -\eta_1 + i\eta_2, \\ \xi_2 &= \eta_1 + i\eta_2, \\ \xi_3 &= \eta_3 - i\eta_4, \\ \xi_4 &= -\eta_3 - i\eta_4, \\ \xi_5 &= \eta_5, \\ \xi_6 &= \eta_6. \end{aligned}$$

In matrix form, we have

$$\begin{pmatrix}\xi_1\\ \vdots\\ \xi_6\end{pmatrix} = T\begin{pmatrix}\eta_1\\ \vdots\\ \eta_6\end{pmatrix}, \quad \text{where} \quad T = \begin{pmatrix}-1 & i & 0 & 0 & 0 & 0\\ 1 & i & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -i & 0 & 0\\ 0 & 0 & -1 & -i & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

In the variables η_j the homomorphism (13) can be expressed as follows:

$$\phi: \ SL_4(\mathbb{Q}[i]) \to O_6(f, \mathbb{Q}[i]), \quad A \mapsto T^{-1}BT \tag{14}$$

where F is replaced by the form

$$f = -\eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 + \eta_5\eta_6. \tag{15}$$

One can easily check that the images of matrices (6) under the mapping (14) are matrices with real entries. Denoting

$$C = (c_{ij}) = T^{-1}BT \tag{16}$$

we write out the formulas for c_{ij} in terms of x_j, y_j, z_j, t_j from (6):

$$\begin{aligned}
c_{11} &= x_1t_1 + x_2t_2 + x_3t_3 + x_4t_4 + y_1z_1 + y_2z_2 + y_3z_3 + y_4z_4,\\
c_{12} &= -x_1t_2 + x_2t_1 + x_3t_4 - x_4t_3 + y_1z_2 - y_2z_1 - y_3z_4 + y_4z_3,\\
c_{13} &= x_1t_3 + x_2t_4 - x_3t_1 - x_4t_2 - y_1z_3 - y_2z_4 + y_3z_1 + y_4z_2,\\
c_{14} &= -x_1t_4 + x_2t_3 - x_3t_2 + x_4t_1 + y_1z_4 - y_2z_3 + y_3z_2 - y_4z_4,\\
c_{15} &= x_1z_1 + x_2z_2 + x_3z_3 + x_4z_4,\\
c_{16} &= y_1t_1 + y_2t_2 + y_3t_3 + y_4t_4,
\end{aligned} \tag{17}$$

$$\begin{aligned}
c_{21} &= x_1t_2 - x_2t_1 + x_3t_4 - x_4t_3 + y_1z_2 - y_2z_1 + y_3z_4 - y_4z_3,\\
c_{22} &= x_1t_1 + x_2t_2 - x_3t_3 - x_4t_4 - y_1z_1 - y_2z_2 + y_3z_3 + y_4z_4,\\
c_{23} &= x_1t_4 - x_2t_3 - x_3t_2 + x_4t_1 - y_1z_4 + y_2z_3 + y_3z_2 - y_4z_1,\\
c_{24} &= x_1t_3 + x_2t_4 + x_3t_1 + x_4t_2 - y_1z_3 - y_2z_4 - y_3z_1 - y_4z_2,\\
c_{25} &= x_1z_2 - x_2z_1 + x_3z_4 - x_4z_3,\\
c_{26} &= y_1t_2 - y_2t_1 + y_3t_4 - y_4t_3,
\end{aligned} \tag{18}$$

$$\begin{aligned}
c_{31} &= -x_1t_3 + x_2t_4 + x_3t_1 - x_4t_2 - y_1z_3 + y_2z_4 + y_3z_1 - y_4z_2,\\
c_{32} &= -x_1t_4 - x_2t_3 - x_3t_2 - x_4t_1 + y_1z_4 + y_2z_3 + y_3z_2 + y_4z_1,\\
c_{33} &= x_1t_1 - x_2t_2 + x_3t_3 - x_4t_4 - y_1z_1 + y_2z_2 - y_3z_3 + y_4z_4,\\
c_{34} &= x_1t_2 + x_2t_1 - x_3t_4 - x_4t_3 - y_1z_2 - y_2z_1 + y_3z_4 + y_4z_3,\\
c_{35} &= -x_1z_3 + x_2z_4 + x_3z_1 - x_4z_2,\\
c_{36} &= -y_1t_3 + y_2t_4 + y_3t_1 - y_4t_2,
\end{aligned} \tag{19}$$

$$
\begin{aligned}
c_{41} &= x_1t_4 + x_2t_3 - x_3t_2 - x_4t_1 + y_1z_4 + y_2z_3 - y_3z_2 - y_4z_1, \\
c_{42} &= -x_1t_3 + x_2t_4 - x_3t_1 + x_4t_2 + y_1z_3 - y_2z_4 + y_3z_1 - y_4z_2, \\
c_{43} &= -x_1t_2 - x_2t_1 - x_3t_4 - x_4t_3 + y_1z_2 + y_2z_1 + y_3z_4 + y_4z_3, \\
c_{44} &= x_1t_1 - x_2t_2 - x_3t_3 + x_4t_4 - y_1z_1 + y_2z_2 + y_3z_3 - y_4z_4, \\
c_{45} &= x_1z_4 + x_2z_3 - x_3z_2 - x_4z_1, \\
c_{46} &= y_1t_4 + y_2t_3 - y_3t_2 - y_4t_1,
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
c_{51} &= 2x_1y_1 + 2x_2y_2 + 2x_3y_3 + 2x_4y_4, \\
c_{52} &= -2x_1y_2 + 2x_2y_1 + 2x_3y_4 - 2x_4y_3, \\
c_{53} &= 2x_1y_3 + 2x_2y_4 - 2x_3y_1 - 2x_4y_2, \\
c_{54} &= -2x_1y_4 + 2x_2y_3 - 2x_3y_2 + 2x_4y_1, \\
c_{55} &= x_1^2 + x_2^2 + x_3^2 + x_4^2, \\
c_{56} &= y_1^2 + y_2^2 + y_3^2 + y_4^2,
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
c_{61} &= 2z_1t_1 + 2z_2t_2 + 2z_3t_3 + 2z_4t_4, \\
c_{62} &= -2z_1t_2 + 2z_2t_1 + 2z_3t_4 - 2z_4t_3, \\
c_{63} &= 2z_1t_3 + 2z_2t_4 - 2z_3t_1 - 2z_4t_2, \\
c_{64} &= -2z_1t_4 + 2z_2t_3 - 2z_3t_2 + 2z_4t_1, \\
c_{65} &= z_1^2 + z_2^2 + z_3^2 + z_4^2, \\
c_{66} &= t_1^2 + t_2^2 + t_3^2 + t_4^2.
\end{aligned} \tag{22}
$$

Restricting the homomorphism (14) to the subgroup $SU_4(\mathbb{Q}[i]) \simeq SL_2(\mathbb{H}_\mathbb{Q})$ of $SL_4(\mathbb{Q}[i])$ we obtain

$$\phi : SU_4(\mathbb{Q}[i]) \to O_6(f, \mathbb{Q}), \quad A \mapsto C. \tag{23}$$

Observe that the c_{kl}'s are quadratic polynomials in x_j, y_j, z_j, t_j with integral coefficients. Hence restricting these variables to rational integers we obtain from (23) the homomorphism

$$\phi : \; SU_4(\mathbb{Z}[i]) \to O_6(f, \mathbb{Z}). \tag{24}$$

Using the formulas (17)–(22) it is easy to verify the following proposition.

Proposition 2 *The kernel of ϕ in (24) is $\pm I$ and the images of the generators (7) under ϕ are the following matrices*

$$\phi(a') = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

$$\phi(b') = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{26}$$

$$\phi(c') = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

$$\phi(d') = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{28}$$

Now we need to describe defining relations for the group $O_6(f, \mathbb{Z})$.

Proposition 3 *The group $O_6(f, \mathbb{Z})$ has the following presentation*

$$\begin{aligned} O_6(f,\mathbb{Z}) = \langle z_1,\ z_2,\ z_3,\ z_4,\ z_5,\ z_6;\ & z_i^2 = 1 \quad (1 \le i \le 6), \\ & (z_1z_3)^2 = (z_1z_4)^2 = (z_1z_5)^2 = (z_1z_6)^2 = 1, \\ & (z_2z_4)^2 = (z_2z_5)^2 = (z_2z_6)^2 = 1, \\ & (z_3z_5)^2 = (z_3z_6)^2 = (z_4z_6)^2 = 1, \\ & (z_2z_3)^3 = (z_3z_4)^3 = (z_5z_6)^3 = 1, \\ & (z_1z_2)^4 = (z_4z_5)^4 = 1\rangle. \end{aligned} \tag{29}$$

Proof. Consider the hyperbolic 5-space H^5 defined over $\mathbb{Q}$ by the quadratic form (15). A point $P \in H^5$ is a homogeneous vector with six entries such that $f(P) > 0$. A hyperplane is described by a homogeneous vector σ with $f(\sigma) < 0$. Consider the simplex in H^5 spanned by the vertices

$$P_1 = \begin{pmatrix} 1\\1\\1\\1\\2\\2 \end{pmatrix}_0, \qquad P_2 = \begin{pmatrix} 0\\1\\1\\1\\2\\2 \end{pmatrix}_1, \qquad P_3 = \begin{pmatrix} 0\\0\\1\\1\\2\\2 \end{pmatrix}_2,$$

$$P_4 = \begin{pmatrix} 0\\0\\0\\1\\2\\2 \end{pmatrix}_3, \qquad P_5 = \begin{pmatrix} 0\\0\\0\\0\\1\\1 \end{pmatrix}_1, \qquad P_6 = \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix}_0$$

where the subscript denotes the value of $f(P)$. The square class of $f(x)$ is the spinorial norm of the reflection in a vector x which may be a point or a hyperplane.

Consider the following six hyperplanes

$$\begin{aligned} \tau_1 &= (1,0,0,0,0,0)_{-1}, & \tau_2 &= (1,-1,0,0,0,0)_{-2}, \\ \tau_3 &= (0,1,-1,0,0,0)_{-2}, & \tau_4 &= (0,0,1,-1,0,0)_{-2}, \\ \tau_5 &= (0,0,0,1,-1,0)_{-1}, & \tau_6 &= (0,0,0,0,1,-1)_{-1}. \end{aligned} \tag{30}$$

It is easy to verify that for any $1 \le j \le 6$ the five hyperplanes τ_i, $i \ne j$, meet in the vertex P_j.

Using the well known formula

$$\sigma : x \mapsto -x + 2\frac{f(x,\alpha)}{f(\alpha,\alpha)}\alpha.$$

for the reflection σ in a hyperplane α compute matrices z_i corresponding to reflections in the hyperplanes τ_i:

$$z_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$z_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad z_4 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$z_5 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad z_6 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

In particular, it follows that the reflections in hyperplanes τ_j are matrices with integral entries, and hence they are elements of $O_6(f, \mathbb{Z})$. Moreover, the simplex bounded by the hyperplanes τ_j is a fundamental domain for $O_6(f, \mathbb{Z})$, and hence $O_6(f, \mathbb{Z})$ is a Coxeter group.

The well known formula

$$\cos \alpha_{i,j} = \frac{f(\tau_i, \tau_j)}{\sqrt{f(\tau_i) f(\tau_j)}}$$

allows to compute the angle between any two hyperplanes τ_i and τ_j. The result of this computation is described by the Coxeter diagram

(31)

The proposition 3 is proved.

A simple calculation shows that the following equations hold in $O_6(f, \mathbb{Z})$

$$\begin{aligned} \phi(a') &= z_4 z_2, \quad \phi(b') = z_3 z_4 z_2 z_3, \\ \phi(c') &= z_5 z_1, \quad \phi(a') = z_6 z_1. \end{aligned} \tag{32}$$

Let us denote

$$V = \langle z_4 z_2,\ z_3 z_4 z_2 z_3,\ z_5 z_1,\ z_6 z_1 \rangle_{O_6(f,\mathbb{Z})}. \tag{33}$$

According to proposition 2 and equations (32) the kernel of ϕ in (24) is $\pm I$ and the image of $SL_2(\mathbb{H}_\mathbb{Z})$ under (24) is the subgroup V. Hence

$$V \simeq SL_2(\mathbb{H}_\mathbb{Z})/\{\pm I\} \simeq PSL_2(\mathbb{H}_\mathbb{Z}).$$

Proposition 4 *The group $PSL_2(\mathbb{H}_\mathbb{Z})$ has the presentation*

$$\begin{aligned} PSL_2(\mathbb{H}_\mathbb{Z}) = \langle a,\ b,\ c,\ d;\quad & a^2 = b^2 = c^2 = d^2 = 1, \\ & (ab)^2 = (ad)^4 = (bd)^4 = (abd)^4 = 1, \\ & (ac)^4 = (bc)^4 = (bdadc)^4 = (cd)^3 = 1, \\ & c(ad)^2 = (ad)^2 c, \quad c(bd)^2 = (bd)^2 c \rangle. \end{aligned} \tag{34}$$

Proof. We shall find the required presentation of V applying the Reidemeister-Schreier method (see [3, p.91]) to subgroups of $O_6(f,\mathbb{Z})$ in three steps.

Step 1. The presentation for the normal subgroup

$$V_1 = N\langle z_2,\ z_3,\ z_4,\ z_1z_5,\ z_1z_6\rangle_{O_6(f,\mathbb{Z})}$$

(coset representatives $\{1, z_1\}$).

Generators:

$$\begin{aligned} s_1 &= z_2, & s_2 &= z_1z_2z_1, \\ s_3 &= z_3, & s_4 &= z_1z_3z_1 = s_3, \\ s_5 &= z_4, & s_6 &= z_1z_4z_1 = s_5, \\ s_7 &= z_5z_1, & s_8 &= z_1z_5 = s_7, \\ s_9 &= z_6z_1, & s_{10} &= z_1z_6 = s_9, \end{aligned}$$

Defining relations:

$$s_1^2 = s_2^2 = s_3^2 = s_5^2 = s_7^2 = s_9^2 = 1, \tag{35}$$
$$(s_1s_5)^2 = 1, \quad (s_2s_6)^2 = 1, \tag{36}$$
$$s_1s_7s_2s_8 = 1 \ \Rightarrow\ s_2 = s_7s_1s_7, \tag{37}$$
$$s_1s_9s_2s_{10} = 1 \ \Rightarrow\ s_2 = s_9s_1s_9, \tag{38}$$
$$s_3s_7s_4s_8 = 1 \ \Leftrightarrow\ (s_3s_7)^2 = 1, \tag{39}$$
$$s_3s_9s_4s_{10} = 1 \ \Leftrightarrow\ (s_3s_9)^2 = 1, \tag{40}$$
$$s_5s_9s_6s_{10} = 1 \ \Leftrightarrow\ (s_5s_9)^2 = 1, \tag{41}$$
$$(s_1s_3)^3 = 1, \quad (s_2s_3)^3 = 1, \tag{42}$$
$$(s_3s_5)^3 = 1, \tag{43}$$
$$(s_7s_{10})^3 = 1 \ \Rightarrow\ (s_7s_9)^3 = 1, \tag{44}$$
$$(s_2s_1)^2 = 1 \ \Rightarrow\ (s_1s_7)^4 = 1, \tag{45}$$
$$(s_5s_7s_6s_8)^2 = 1 \ \Rightarrow\ (s_5s_7)^4 = 1. \tag{46}$$

Obtain from (38) and (37)

$$s_7s_1s_7s_9s_1s_9 = 1. \tag{47}$$

Notice that the second relation (36) is redundant:

$$(s_2s_6)^2 = 1 \ \Leftrightarrow\ (s_9s_1s_9s_5)^2 = 1 \ \Leftrightarrow\ (s_1s_5)^2 = 1.$$

Clearly, the second relation (42) is redundant as well. Hence V_1 is presented by the generators s_1, s_3, s_5, s_7, s_9 and the defining relations (39)–(47), (35), (36).

Step 2. The normal subgroup

$$V_2 = N\langle s_1s_3,\ s_1s_5,\ s_7,\ s_9\rangle_{V_1}$$

(coset representatives $\{1, s_1\}$).

Generators:

$$\begin{aligned} v_1 &= s_3 s_1, & v_2 &= s_1 s_3, \\ v_3 &= s_5 s_1, & v_4 &= s_1 s_5, \\ v_5 &= s_7, & v_6 &= s_1 s_7 s_1, \\ v_7 &= s_9, & v_8 &= s_1 s_9 s_1, \end{aligned}$$

Relations:

$$v_1 v_2 = 1 \;\Rightarrow\; v_2 = v_1^{-1},$$
$$v_3 v_4 = 1 \;\Rightarrow\; v_4 = v_3^{-1},$$
$$v_3^2 = 1, \quad v_5^2 = 1, \quad v_7^2 = 1, \tag{48}$$
$$v_1^3 = 1, \tag{49}$$
$$v_1 v_6 v_2 v_5 = 1 \;\Rightarrow\; v_6 = v_1^{-1} v_5 v_1,$$
$$v_1 v_8 v_2 v_7 = 1 \;\Rightarrow\; v_8 = v_1^{-1} v_7 v_1, \tag{50}$$
$$v_3 v_8 v_4 v_7 = 1 \;\Rightarrow\; v_8 = v_3 v_7 v_3, \tag{51}$$
$$(v_1 v_4)^3 = 1 \;\Rightarrow\; (v_1 v_3)^3 = 1, \tag{52}$$
$$(v_5 v_7)^3 = 1, \tag{53}$$
$$(v_6 v_5)^2 = 1 \;\Rightarrow\; (v_1^{-1} v_5 v_1 v_5)^2 = 1, \tag{54}$$
$$(v_3 v_6 v_4 v_5)^2 = 1 \;\Rightarrow\; (v_3 v_1^{-1} v_5 v_1 v_3 v_5)^2 = 1, \tag{55}$$
$$v_5 v_6 v_8 v_7 = 1 \;\Rightarrow\; v_5 v_1^{-1} v_5 v_7 v_1 v_7 = 1, \tag{56}$$
$$\tag{57}$$

From (50) and (51) we conclude

$$v_1^{-1} v_7 v_1 v_3 v_7 v_3 = 1. \tag{58}$$

After eliminating the generators v_{2i} the subgroup H_1 is presented by the generators v_1, v_3, v_5, v_7 and the defining relations (48), (49) and (52)–(58).

Step 3. The normal subgroup

$$V_3 = N\langle v_3,\ v_5,\ v_7,\ v_1 v_3 v_1^{-1}\rangle_{V_2}.$$

(coset representatives $\{1, v_1, v_1^{-1}\}$).

Generators:

$$\begin{aligned} a &= v_3 = z_4 z_2, & a_1 &= v_1 v_3 v_1^{-1} = b, & a_2 &= v_1^{-1} v_3 v_1, \\ c &= v_5 = z_5 z_1, & c_1 &= v_1 v_5 v_1^{-1}, & c_2 &= v_1^{-1} v_5 v_1, \\ d &= v_7 = z_6 z_1, & d_1 &= v_1 v_7 v_1^{-1}, & d_2 &= v_1^{-1} v_7 v_1, \end{aligned}$$

Defining relations:

$$a^2 = c^2 = d^2 = 1, \quad b^2 = a_1^2 = a_2^2 = 1, \quad c_1^2 = c_2^2 = 1, \quad d_1^2 = d_2^2 = 1, \tag{59}$$

$$a_1aa_2 = 1 \ \Rightarrow \ a_2 = ab, \tag{60}$$
$$(cd)^3 = 1, \tag{61}$$
$$(c_2c)^2 = 1, \tag{62}$$
$$(cc_1)^2 = 1, \tag{63}$$
$$(c_1c_2)^2 = 1, \tag{64}$$
$$(ac_2ac)^2 = 1, \tag{65}$$
$$(a_1ca_1c_1)^2 = 1, \tag{66}$$
$$(a_2c_1a_2c_2)^2 = 1, \tag{67}$$
$$cc_2d_2d = 1, \tag{68}$$
$$c_1cdd_1 = 1, \tag{69}$$
$$c_2c_1d_1d_2 = 1, \tag{70}$$
$$d_2ada = 1, \tag{71}$$
$$da_1d_1a_1 = 1, \tag{72}$$
$$d_1a_2d_2a_2 = 1. \tag{73}$$

Now we can eliminate the generators a_i, c_i, d_i obtaining

$$(59)\ \&\ (60) \ \Rightarrow \ (ab)^2 = 1, \tag{74}$$
$$(71) \ \Leftrightarrow \ d_2 = ada, \tag{75}$$
$$(72) \ \Leftrightarrow \ d_1 = bdb, \tag{76}$$
$$(68) \ \Leftrightarrow \ cc_2 = (da)^2 \ \Leftrightarrow \ c_2 = c(da)^2, \tag{77}$$
$$(69) \ \Leftrightarrow \ c_1c = (bd)^2 \ \Leftrightarrow \ c_1 = (bd)^2c, \tag{78}$$
$$(62)\ \&\ (77) \ \Rightarrow \ (ad)^4 = 1 \ \Leftrightarrow \ (ad)^2 = (da)^2, \tag{79}$$
$$(63)\ \&\ (78) \ \Rightarrow \ (db)^4 = 1 \ \Leftrightarrow \ (bd)^2 = (db)^2, \tag{80}$$
$$(70) \ \Leftrightarrow \ c_2c_1 = adabdb \ \Leftrightarrow \ c(adb)^2c = (adb)^2, \tag{81}$$
$$(64)\ \&\ (81) \ \Rightarrow \ (adb)^4 = 1 \ \Leftrightarrow \ (abd)^4 = 1, \tag{82}$$
$$(59)\ \&\ (77)\ \&\ (79) \ \Rightarrow \ (ad)^2c = c(ad)^2, \tag{83}$$
$$(59)\ \&\ (78)\ \&\ (80) \ \Rightarrow \ (bd)^2c = c(bd)^2. \tag{84}$$

By (79) and (83), $(da)^2$ commutes with a and c. Deduce from (65) and (77):

$$1 = (ac_2ac)^2 = (ac(da)^2ac)^2 = (da)^4(ac)^4 = (ac)^4 = 1.$$

Similarly obtain from (66)

$$(bc)^4 = 1.$$

Conclude from (77), (78) and (67):

$$(ab(bd)^2cab(ad)^2c)^2 = (adbdcbdadc)^2 = 1.$$

Then using the fact that $(bda)^2$ commutes with c and d obtain

$$1 = (bda)^4(adbdcbdadc)^2 = (bdadcbdadc)(bda)^2(adbdcbdadc) = (bdadc)^4 = 1.$$

Notice that the relation (81) follows from (83) and (84). Hence after eliminating the generators a_2, c_1, c_2, d_1 and d_2 we obtain that V_3 is generated by a, b, c, d and has the defining relations (34). Clearly $b = z_3 z_4 z_2 z_3$ and by (33) the two subgroups V_3 and V coincide. The proof of Proposition 2 is complete.

Denoting the generator $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of the center of $SL_2(\mathbb{H}_{\mathbb{Z}})$ by z we obtain from proposition 4

Theorem 1 *The group $SL_2(\mathbb{H}_{\mathbb{Z}})$ has the presentation*

$$\begin{aligned} \langle a,\ b,\ c,\ d;\ \ & a^2 = b^2 = c^2 = d^2 = (ab)^2 = z, \\ & (ad)^4 = (bd)^4 = (abd)^4 = z, \\ & (ac)^4 = (bc)^4 = (cd)^3 = (bdadc)^4 = z^2 = 1, \\ & (c(ad)^2)^2 = (c(bd)^2)^2 = z\rangle, \end{aligned}$$

where a, b, c, d are given in (3).

2 Finite index subgroups of $SL_2(\mathbb{H}_{\mathbb{Z}})$

Theorem 2 *The group $SL_2(\mathbb{H}_{\mathbb{Z}})$ contains a subgroup G of index $3 \cdot 2^9 = 1536$ which has a nonabelian free quotient group of rank 3. The same is true for the group $PSL_2(\mathbb{H}_{\mathbb{Z}})$.*

Proof. Notice that if a quotient group G_1 of a group G has a subgroup of a finite index n with nonabelian free quotient group,then the group G also has a subgroup of the same index n with nonabelian free quotient group. Thus it suffices to prove that $PSL_2(\mathbb{H}_{\mathbb{Z}})$ contains a subgroup of index $n = 3 \cdot 2^9$ with nonabelian free quotient. We shall find a required subgroup of V of index $3 \cdot 2^9$ by applying the Reidemeister-Schreier method to V in several steps.

We start out with the presentation of $PSL_2(\mathbb{H}_{\mathbb{Z}})$ given in Proposition 4:

$$\langle a,\ b,\ c,\ d;$$

$$a^2 = b^2 = c^2 = d^2 = (ab)^2 = 1, \tag{85}$$

$$(ad)^4 = 1, \tag{86}$$

$$(bd)^4 = 1, \tag{87}$$

$$(abd)^4 = 1, \tag{88}$$

$$(ac)^4 = 1, \tag{89}$$

$$(bc)^4 = 1, \tag{90}$$

$$(bdadc)^4 = 1, \tag{91}$$

$$(cd)^3 = 1, \tag{92}$$

$$c(ad)^2 = (ad)^2 c, \tag{93}$$

$$c(bd)^2 = (bd)^2 c\ \rangle. \tag{94}$$

Step 1. The presentation for the normal subgroup

$$V_1 = N\langle ab,\ c,\ d\rangle_V$$

(coset representatives $\{1, a\}$).

Generators:

$$\begin{aligned} x_1 &= ba, \quad x_2 = c, \quad x_3 = aca, \\ y_1 &= ab, \quad x_4 = d, \quad x_5 = ada. \end{aligned}$$

Relations:

$$\begin{aligned}
(85):&\quad x_i^2 = 1 \quad (i = 1, 2, 3, 4, 5), && (95)\\
&\quad y_1 x_1 = 1 \ \Rightarrow\ y_1 = x_1^{-1} = x_1, &&\\
(86):&\quad (x_5 x_4)^2 = 1, && (96)\\
(87):&\quad (x_1 x_5 x_1 x_4)^2 = 1, && (97)\\
(88):&\quad (x_1 x_4)^4 = 1, && (98)\\
&\quad (x_1 x_5)^4 = 1, && (99)\\
(89):&\quad (x_3 x_2)^2 = 1, && (100)\\
(90):&\quad (x_1 x_3 x_1 x_2)^2 = 1, && (101)\\
(91):&\quad (x_1 x_5 x_4 x_2)^4 = 1, && (102)\\
&\quad (x_1 x_4 x_5 x_3)^4 = 1, && (103)\\
(92):&\quad (x_2 x_4)^3 = 1, && (104)\\
&\quad (x_3 x_5)^3 = 1, && (105)\\
(93):&\quad (x_5 x_4 x_2)^2 = 1, && (106)\\
&\quad (x_4 x_5 x_3)^2 = 1, && (107)\\
(94):&\quad (x_1 x_5 x_1 x_4 x_2)^2 = 1, && (108)\\
&\quad (x_1 x_4 x_1 x_5 x_3)^2 = 1. && (109)
\end{aligned}$$

Step 2. The presentation for the normal subgroup

$$V_2 = N\langle x_2,\ x_3,\ x_4,\ x_5\rangle_{V_1}$$

(coset representatives $\{1, x_1\}$).

Generators:

$$\begin{aligned}
y_1 &= x_2, & y_2 &= x_1 x_2 x_1,\\
y_3 &= x_3, & y_4 &= x_1 x_3 x_1,\\
y_5 &= x_4, & y_6 &= x_1 x_4 x_1,\\
y_7 &= x_5, & y_8 &= x_1 x_5 x_1.
\end{aligned}$$

Relations:

$$\begin{aligned}
(95):&\quad y_i^2 = 1 \quad (1 \le i \le 8), && (110)\\
(96):&\quad (y_7 y_5)^2 = 1, && (111)\\
&\quad (y_8 y_6)^2 = 1, && (112)\\
(97):&\quad (y_8 y_5)^2 = 1, && (113)
\end{aligned}$$

$$
\begin{aligned}
& (y_7y_6)^2 = 1, && (114)\\
(98): \quad & (y_6y_5)^2 = 1, && (115)\\
(99): \quad & (y_8y_7)^2 = 1, && (116)\\
(100): \quad & (y_1y_3)^2 = 1, && (117)\\
& (y_2y_4)^2 = 1, && (118)\\
(101): \quad & (y_4y_1)^2 = 1, && (119)\\
& (y_3y_2)^2 = 1, && (120)\\
(102): \quad & (y_8y_6y_2y_7y_5y_1)^2 = 1, && (121)\\
(103): \quad & (y_6y_8y_4y_5y_7y_3)^2 = 1, && (122)\\
(104): \quad & (y_1y_5)^3 = 1, && (123)\\
& (y_2y_6)^3 = 1, && (124)\\
(105): \quad & (y_3y_7)^3 = 1, && (125)\\
& (y_4y_8)^3 = 1, && (126)\\
(106): \quad & (y_7y_5y_1)^2 = 1, && (127)\\
& (y_8y_6y_2)^2 = 1, && (128)\\
(107): \quad & (y_5y_7y_3)^2 = 1, && (129)\\
& (y_6y_8y_4)^2 = 1, && (130)\\
(108): \quad & (y_8y_5y_1)^2 = 1, && (131)\\
& (y_7y_6y_2)^2 = 1, && (132)\\
(109): \quad & (y_6y_7y_3)^2 = 1, && (133)\\
& (y_5y_8y_4)^2 = 1. && (134)
\end{aligned}
$$

Using relations (110), (115), (116) and (127)–(134) transform the relations (121) and (122) to

$$
\begin{aligned}
(121) \;\Leftrightarrow\; & y_2y_6y_5y_1y_8y_7y_8y_7y_2y_6y_5y_1 = 1 \;\Leftrightarrow\; (y_2y_6y_5y_1)^2 = 1, && (135)\\
(122) \;\Leftrightarrow\; & y_4y_8y_7y_3y_6y_5y_6y_5y_4y_8y_7y_3 = 1 \;\Leftrightarrow\; (y_4y_8y_7y_3)^2 = 1. && (136)
\end{aligned}
$$

Step 3. The presentation for the normal subgroup

$$V_3 = N\langle y_1,\ y_2,\ y_3,\ y_5,\ y_6,\ y_7,\ y_4y_8\rangle_{V_2}$$

(coset representatives $\{1, y_8\}$).

Generators:

$$
\begin{aligned}
& z_i = y_i, && z'_i = y_8y_iy_8 \quad (i = 1,2,3,5,6,7),\\
& z_4 = y_4y_8, && z'_4 = y_8y_4,
\end{aligned}
$$

Relations:

$$
\begin{aligned}
(110): \quad & z_i^2 = 1, \quad {z'_i}^2 = 1 \quad (i = 1,2,3,5,6,7), && (137)\\
& z_4z'_4 = 1 \;\Rightarrow\; z'_4 = z_4^{-1}, &&\\
(111): \quad & (z_7z_5)^2 = 1, && (138)
\end{aligned}
$$

$$(z_7'z_5')^2 = 1 \;\Leftrightarrow\; (z_7z_5)^2 = 1 \quad \mathrm{mod}\ (140),(143),$$

$$(112): \quad z_6'z_6 = 1 \;\Leftrightarrow\; z_6' = z_6, \tag{139}$$

$$(113): \quad z_5'z_5 = 1 \;\Leftrightarrow\; z_5' = z_5, \tag{140}$$

$$(114): \quad (z_7z_6)^2 = 1, \tag{141}$$

$$(z_7'z_6')^2 = 1 \;\Leftrightarrow\; (z_7z_6)^2 = 1 \quad \mathrm{mod}\ (139),(143),$$

$$(115): \quad (z_6z_5)^2 = 1, \tag{142}$$

$$(116): \quad z_7'z_7 = 1 \;\Leftrightarrow\; z_7' = z_7, \tag{143}$$

$$(117): \quad (z_1z_3)^2 = 1, \tag{144}$$

$$(z_1'z_3')^2 = 1 \;\Leftrightarrow\; (z_5z_1z_5z_3')^2 = 1 \quad \mathrm{mod}\ (160), \tag{145}$$

$$(118): \quad z_2z_4z_2'z_4' = 1 \;\Leftrightarrow\; z_2' = z_4^{-1}z_2z_4, \tag{146}$$

$$(119): \quad z_4z_1'z_4'z_1 = 1 \;\Leftrightarrow\; z_1' = z_4^{-1}z_1z_4, \tag{147}$$

$$(120): \quad (z_3z_2)^2 = 1, \tag{148}$$

$$(z_3'z_2')^2 = 1 \;\Leftrightarrow\; (z_6z_2z_6z_3')^2 = 1 \quad \mathrm{mod}\ (156), \tag{149}$$

$$(123): \quad (z_1z_5)^3 = 1, \tag{150}$$

$$(z_1'z_5)^3 = 1 \;\Leftrightarrow\; (z_1z_5)^3 = 1 \quad \mathrm{mod}\ (160),$$

$$(124): \quad (z_2z_6)^3 = 1, \tag{151}$$

$$(z_2'z_6)^3 = 1 \;\Leftrightarrow\; (z_2z_6)^3 = 1 \quad \mathrm{mod}\ (156),$$

$$(125): \quad (z_3z_7)^3 = 1, \tag{152}$$

$$(z_3'z_7)^3 = 1, \tag{153}$$

$$(126): \quad z_4^3 = 1, \tag{154}$$

$$(127): \quad (z_7z_5z_1)^2 = 1, \tag{155}$$

$$(z_7z_5z_1')^2 = 1 \;\Leftrightarrow\; (z_7z_5z_1)^2 = 1 \quad \mathrm{mod}\ (138),$$

$$(128): \quad z_6z_2z_6z_2' = 1 \;\Leftrightarrow\; z_2' = z_6z_2z_6, \tag{156}$$

$$(129): \quad (z_5z_7z_3)^2 = 1, \tag{157}$$

$$(z_5z_7z_3')^2 = 1, \tag{158}$$

$$(130): \quad (z_6z_4')^2 = 1 \;\Leftrightarrow\; (z_6z_4)^2 = 1, \tag{159}$$

$$(131): \quad z_5z_1'z_5z_1 = 1 \;\Leftrightarrow\; z_1' = z_5z_1z_5, \tag{160}$$

$$(132): \quad (z_7z_6z_2)^2 = 1, \tag{161}$$

$$(z_7z_6z_2')^2 = 1 \;\Leftrightarrow\; (z_7z_6z_2)^2 = 1 \quad \mathrm{mod}\ (156),$$

$$(133): \quad (z_6z_7z_3)^2 = 1, \tag{162}$$

$$(z_6z_7z_3')^2 = 1, \tag{163}$$

$$(134): \quad (z_5z_4')^2 = 1 \;\Leftrightarrow\; (z_5z_4)^2 = 1, \tag{164}$$

$$(135): \quad (z_2z_6z_5z_1)^2 = 1, \tag{165}$$

$$(z_2'z_6z_5z_1')^2 = 1 \;\Leftrightarrow\; (z_2z_6z_5z_1)^2 = 1 \quad \mathrm{mod}\ (156),(160),$$

$$(136): \quad (z_4z_7z_3)^2 = 1, \tag{166}$$

$$(z_4'z_7z_3')^2 = 1 \;\Leftrightarrow\; (z_3'z_7z_4)^2 = 1. \tag{167}$$

From relations (146) and (156) we get

$$z_4^{-1} z_2 z_4 z_6 z_2 z_6 = 1 \tag{168}$$

and from relations (147) and (160):

$$z_4^{-1} z_1 z_4 z_5 z_1 z_5 = 1. \tag{169}$$

Hence the group V_3 is presented by the generators z_i ($1 \le i \le 7$), z_3' and the defining relations (137), (138), (141), (142), (144), (145), (148)–(155), (157)–(159), (161)–(169).

Step 4. The presentation for the normal subgroup

$$V_4 = N\langle z_1,\ z_2,\ z_4,\ z_5,\ z_6,\ z_3 z_7,\ z_3' z_7\rangle_{V_3}$$

(coset representatives $\{1, z_7\}$).

Generators:

$$\begin{array}{lll} u_i = z_i, & v_i = z_7 z_i z_7 & (i = 1, 2, 4, 5, 6), \\ u_3 = z_3 z_7, & v_3 = z_7 z_3, & \\ u_7 = z_3' z_7, & v_7 = z_7 z_3'. & \end{array}$$

Relations:

$$\begin{array}{lll}
(137): & u_i^2 = 1, \quad v_i^2 = 1 \quad (i = 1, 2, 5, 6), & (170) \\
 & u_3 v_3 = 1 \ \Rightarrow\ v_3 = u_3^{-1}, & \\
 & u_7 v_7 = 1 \ \Rightarrow\ v_7 = u_7^{-1}, & \\
(138): & v_5 u_5 = 1 \ \Rightarrow\ v_5 = u_5, & \\
(141): & v_6 u_6 = 1 \ \Rightarrow\ v_6 = u_6, & \\
(142): & (u_6 u_5)^2 = 1, & (171) \\
(144): & u_1 u_3 v_1 v_3 = 1 \ \Rightarrow\ v_1 = u_3^{-1} u_1 u_3, & (172) \\
(145): & u_5 u_1 u_5 u_7 u_5 v_1 u_5 v_7 = 1 \ \Leftrightarrow\ u_5 u_1 u_5 u_7 u_1 u_7^{-1} = 1 \quad \mathrm{mod}\ (181), & (173) \\
(148): & u_3 v_2 v_3 u_2 = 1 \ \Leftrightarrow\ v_2 = u_3^{-1} u_2 u_3, & (174) \\
(149): & u_6 u_2 u_6 u_7 u_6 v_2 u_6 v_7 = 1 \ \Leftrightarrow\ u_6 u_2 u_6 u_7 u_2 u_7^{-1} = 1 \quad \mathrm{mod}\ (186), & (175) \\
(150): & (u_1 u_5)^3 = 1, & (176) \\
 & (v_1 v_5)^3 = 1 \ \Leftrightarrow\ (u_1 u_5)^3 = 1 \quad \mathrm{mod}\ (181), & \\
(151): & (u_2 u_6)^3 = 1, & (177) \\
 & (v_2 v_6)^2 = 1 \ \Leftrightarrow\ (u_2 u_6)^3 = 1 \quad \mathrm{mod}\ (186), & \\
(152): & u_3^3 = v_3^3 = 1, & (178) \\
(153): & u_7^3 = v_7^3 = 1, & (179) \\
(154): & u_4^3 = v_4^3 = 1, & (180) \\
(155): & v_5 v_1 u_5 u_1 = 1 \ \Leftrightarrow\ v_1 = u_5 u_1 u_5, & (181) \\
(157): & (u_5 v_3)^2 = 1 \ \Leftrightarrow\ (u_3 u_5)^2 = 1, & (182) \\
 & (v_5 u_3)^2 = 1 \ \Leftrightarrow\ (u_3 u_5)^2 = 1, &
\end{array}$$

$$
\begin{aligned}
(158): \quad & (u_5v_7)^2 = 1 \;\Leftrightarrow\; (u_7u_5)^2 = 1, && (183)\\
& (v_5u_7)^2 = 1 \;\Leftrightarrow\; (u_7u_5)^2 = 1, &&\\
(159): \quad & (u_6u_4)^2 = 1, && (184)\\
& (u_6v_4)^2 = 1, && (185)\\
(161): \quad & v_6v_2u_6u_2 = 1 \;\Leftrightarrow\; v_2 = u_6u_2u_6, && (186)\\
(162): \quad & (u_6v_3)^2 = 1 \;\Leftrightarrow\; (u_6u_3)^2 = 1, && (187)\\
& (v_6u_3)^2 = 1 \;\Leftrightarrow\; (u_6u_3)^2 = 1, &&\\
(163): \quad & (u_6v_7)^2 = 1 \;\Leftrightarrow\; (u_6u_7)^2 = 1, && (188)\\
& (v_6u_7)^2 = 1 \;\Leftrightarrow\; (u_6u_7)^2 = 1, &&\\
(164): \quad & (u_5u_4)^2 = 1, && (189)\\
& (u_5v_4)^2 = 1, && (190)\\
(165): \quad & (u_2u_6u_5u_1)^2 = 1, && (191)\\
& (v_2v_6v_5v_1)^2 = 1 \;\Leftrightarrow\; (u_2u_6u_5u_1)^2 = 1 \quad \mathrm{mod}\ (181), (186), &&\\
(166): \quad & (u_4v_3)^2 = 1 \;\Leftrightarrow\; (u_4u_3^{-1})^2 = 1, && (192)\\
& (v_4u_3)^2 = 1, && (193)\\
(167): \quad & (u_7u_4)^2 = 1, && (194)\\
& (v_7v_4)^2 = 1 \;\Leftrightarrow\; (u_7^{-1}v_4)^2 = 1, && (195)\\
(168): \quad & u_4^{-1}u_2u_4u_6u_2u_6 = 1, && (196)\\
& v_4^{-1}v_2v_4u_6v_2u_6 = 1 \;\Leftrightarrow\; v_4^{-1}u_6u_2u_6v_4u_2 = 1 \quad \mathrm{mod}\ (186), && (197)\\
(169): \quad & u_4^{-1}u_1u_4u_5u_1u_5 = 1, && (198)\\
& v_4^{-1}v_1v_4u_5v_1u_5 = 1 \;\Leftrightarrow\; v_4^{-1}u_5u_1u_5v_4u_1 = 1 \quad \mathrm{mod}\ (181). && (199)
\end{aligned}
$$

From relations (172) and (181) we get

$$u_3^{-1}u_1u_3u_5u_1u_5 = 1 \tag{200}$$

and from relations (174) and (186):

$$u_3^{-1}u_2u_3u_6u_2u_6 = 1. \tag{201}$$

Finally, for V_4 we obtain the generators u_i $(1 \le i \le 7)$, v_4 and the defining relations (170), (171), (173), (175)–(180), (182)–(185), (187)–(201).

Step 5. The presentation for the normal subgroup

$$V_5 = N\langle u_1u_5,\ u_2,\ u_3,\ u_4,\ u_6,\ u_7,\ v_4\rangle_{V_4}$$

(coset representatives $\{1, u_5\}$).

Generators:

$$
\begin{aligned}
a_1 &= u_1u_5, & b_1 &= u_5u_1,\\
a_i &= u_i, & b_i &= u_5u_iu_5 \quad (i = 2,3,4,6,7),\\
a_8 &= v_4, & b_8 &= u_5v_4u_5
\end{aligned}
$$

Relations:

$$
\begin{aligned}
(170):\quad & a_i^2=1,\quad b_i^2=1\quad (i=2,6),\\
& a_1b_1=1 \ \Leftrightarrow\ b_1=a_1^{-1},\\
(171):\quad & a_6b_6=1 \ \Leftrightarrow\ b_6=a_6,\\
(173):\quad & b_1b_7b_1a_7^{-1}=1,\quad a_1a_7a_1b_7^{-1}=1,\\
(175):\quad & a_6a_2a_6a_7a_2a_7^{-1}=1,\quad b_6b_2b_6b_7b_2b_7^{-1}=1,\\
(176):\quad & a_1^3=1,\\
(177):\quad & (a_2a_6)^3=1,\quad (b_2b_6)^3=1,\\
(178):\quad & a_3^3=b_3^3=1,\\
(179):\quad & a_7^3=b_7^3=1,\\
(180):\quad & a_4^3=b_4^3=1,\quad a_8^3=b_8^3=1,\\
(182):\quad & a_3b_3=1 \ \Leftrightarrow\ b_3=a_3^{-1},\\
(183):\quad & a_7b_7=1 \ \Leftrightarrow\ b_7=a_7^{-1},\\
(184):\quad & (a_6a_4)^2=1,\quad (b_6b_4)^2=1,\\
(185):\quad & (a_6a_8)^2=1,\quad (b_6b_8)^2=1,\\
(187):\quad & (a_6a_3)^2=1,\quad (b_6b_3)^2=1,\\
(188):\quad & (a_6a_7)^2=1,\quad (b_6b_7)^2=1,\\
(189):\quad & b_4a_4=1 \ \Leftrightarrow\ b_4=a_4^{-1},\\
(190):\quad & b_8a_8=1 \ \Leftrightarrow\ b_8=a_8^{-1},\\
(191):\quad & (a_2a_6b_1)^2=1,\quad (b_2b_6a_1)^2=1,\\
(192):\quad & (a_4a_3^{-1})^2=1,\quad (b_4b_3^{-1})^2=1,\\
(193):\quad & (a_8a_3)^2=1,\quad (b_8b_3)^2=1,\\
(194):\quad & (a_7a_4)^2=1,\quad (b_7b_4)^2=1,\\
(195):\quad & (a_7^{-1}a_8)^2=1,\quad (b_7^{-1}b_8)^2=1,\\
(196):\quad & a_4^{-1}a_2a_4a_6a_2a_6=1,\quad b_4^{-1}b_2b_4b_6b_2b_6=1,\\
(197):\quad & a_8^{-1}a_6a_2a_6a_8a_2=1,\quad b_8^{-1}b_6b_2b_6b_8b_2=1,\\
(198):\quad & a_4^{-1}a_1b_4a_1=1,\quad b_4^{-1}b_1a_4b_1=1,\\
(199):\quad & a_8^{-1}b_1b_8b_1=1,\quad b_8^{-1}a_1a_8a_1=1,\\
(200):\quad & a_3^{-1}a_1b_3a_1=1,\quad b_3^{-1}b_1a_3b_1=1,\\
(201):\quad & a_3^{-1}a_2a_3a_6a_2a_6=1,\quad b_3^{-1}b_2b_3b_6b_2b_6=1.
\end{aligned}
$$

Eliminating the generators b_i $(i=1,3,4,6,7,8)$ we get the following defining relations for V_5:

$$a_2^2=b_2^2=a_6^2=1,\quad a_1^3=a_3^3=a_4^3=a_7^3=a_8^3=1, \tag{202}$$

$$(a_1a_7)^2=1, \tag{203}$$

$$a_7a_2a_7^{-1}a_6a_2a_6=1, \tag{204}$$

$$a_7^{-1}b_2a_7a_6b_2a_6=1, \tag{205}$$

$$(a_2a_6)^3 = 1, \tag{206}$$
$$(b_2a_6)^3 = 1, \tag{207}$$
$$(a_4a_6)^2 = 1, \tag{208}$$
$$(a_6a_8)^2 = 1, \tag{209}$$
$$(a_3a_6)^2 = 1, \tag{210}$$
$$(a_6a_7)^2 = 1, \tag{211}$$
$$(a_1a_6a_2)^2 = 1, \tag{212}$$
$$(b_2a_6a_1)^2 = 1, \tag{213}$$
$$(a_4a_3^{-1})^2 = 1, \tag{214}$$
$$(a_8a_3)^2 = 1, \tag{215}$$
$$(a_7a_4)^2 = 1, \tag{216}$$
$$(a_7^{-1}a_8)^2 = 1, \tag{217}$$
$$a_4^{-1}a_2a_4a_6a_2a_6 = 1, \tag{218}$$
$$a_4b_2a_4^{-1}a_6b_2a_6 = 1, \tag{219}$$
$$a_8a_2a_8^{-1}a_6a_2a_6 = 1, \tag{220}$$
$$a_8^{-1}b_2a_8a_6b_2a_6 = 1, \tag{221}$$
$$(a_4^{-1}a_1)^2 = 1, \tag{222}$$
$$(a_8a_1)^2 = 1, \tag{223}$$
$$(a_3^{-1}a_1)^2 = 1, \tag{224}$$
$$a_3^{-1}a_2a_3a_6a_2a_6 = 1, \tag{225}$$
$$a_3b_2a_3^{-1}a_6b_2a_6 = 1. \tag{226}$$

Step 6. The presentation for the normal subgroup

$$V_6 = N\langle a_2a_6,\ a_1,\ a_3,\ a_4\rangle_{V_5}$$

(coset representatives $\{1, a_6\}$).

Generators:

$$\begin{array}{lll} c_i = a_i, & d_i = a_6a_ia_6 & (i = 1,3,4,7,8), \\ c_2 = b_2a_6, & d_2 = a_6b_2, & \\ c_6 = a_2a_6, & d_6 = a_6a_2, & \end{array}$$

Relations:

$$\begin{array}{ll} (202): & c_6d_6 = 1 \ \Leftrightarrow\ d_6 = c_6^{-1}, \\ & c_2d_2 = 1 \ \Leftrightarrow\ d_2 = c_2^{-1}, \\ & c_i^3 = d_i^3 = 1 \quad (i = 1,3,4,7,8), \\ (203): & (c_1c_7)^2 = 1, \quad (d_1d_7)^2 = 1, \\ (204): & c_7c_6d_7^{-1}c_6 = 1, \quad d_7d_6c_7^{-1}d_6 = 1, \\ (205): & c_7^{-1}c_2d_7c_2 = 1, \quad d_7^{-1}d_2c_7d_2 = 1, \end{array}$$

$$
\begin{aligned}
(206):&\quad c_6^3 = d_6^3 = 1,\\
(207):&\quad c_2^3 = d_2^3 = 1,\\
(208):&\quad c_4d_4 = 1 \;\Leftrightarrow\; d_4 = c_4^{-1},\\
(209):&\quad c_8d_8 = 1 \;\Leftrightarrow\; d_8 = c_8^{-1},\\
(210):&\quad c_3d_3 = 1 \;\Leftrightarrow\; d_3 = c_3^{-1},\\
(211):&\quad c_7d_7 = 1 \;\Leftrightarrow\; d_7 = c_7^{-1},\\
(212):&\quad (c_1d_6)^2 = 1, \quad (d_1c_6)^2 = 1,\\
(213):&\quad (c_2c_1)^2 = 1, \quad (d_2d_1)^2 = 1,\\
(214):&\quad (c_4c_3^{-1})^2 = 1, \quad (d_4d_3^{-1})^2 = 1,\\
(215):&\quad (c_8c_3)^2 = 1, \quad (d_8d_3)^2 = 1,\\
(216):&\quad (c_7c_4)^2 = 1, \quad (d_7d_4)^2 = 1,\\
(217):&\quad (c_7^{-1}c_8)^2 = 1, \quad (d_7^{-1}d_8)^2 = 1,\\
(218):&\quad c_4^{-1}c_6d_4c_6 = 1, \quad d_4^{-1}d_6c_4d_6 = 1,\\
(219):&\quad c_4c_2d_4^{-1}c_2 = 1, \quad d_4d_2c_4^{-1}d_2 = 1,\\
(220):&\quad c_8c_6d_8^{-1}c_6 = 1, \quad d_8d_6c_8^{-1}d_2 = 1,\\
(221):&\quad c_8^{-1}c_2d_8c_2 = 1, \quad d_8^{-1}d_2c_8d_2 = 1,\\
(222):&\quad (c_4^{-1}c_1)^2 = 1, \quad (d_4^{-1}d_1)^2 = 1,\\
(223):&\quad (c_8c_1)^2 = 1, \quad (d_8d_1)^2 = 1,\\
(224):&\quad (c_3^{-1}c_1)^2 = 1, \quad (d_3^{-1}d_1)^2 = 1,\\
(225):&\quad c_3^{-1}c_6d_3c_6 = 1, \quad d_3^{-1}d_6c_3d_6 = 1,\\
(226):&\quad c_3c_2d_3^{-1}c_2 = 1, \quad d_3d_2c_3^{-1}d_2 = 1.
\end{aligned}
$$

Eliminating the generators d_i $(i = 2, 3, 4, 6, 7, 8)$ we get the following defining relations for V_6:

$$
\begin{gathered}
d_1^3 = 1, \quad c_i^3 = 1 \quad (i = 1, 2, 3, 4, 6, 7, 8),\\
(c_1c_7)^2 = (d_1c_7^{-1})^2 = (c_7c_6)^2 = (c_7^{-1}c_2)^2 = 1,\\
(c_1c_6^{-1})^2 = (d_1c_6)^2 = (c_1c_2)^2 = (d_1c_2^{-1})^2 = 1,\\
(c_4c_3^{-1})^2 = (c_3c_8)^2 = (c_4c_7)^2 = (c_8c_7^{-1})^2 = 1,\\
(c_4^{-1}c_6)^2 = (c_4c_2)^2 = (c_8c_6)^2 = (c_8^{-1}c_2)^2 = 1,\\
(c_1c_4^{-1})^2 = (d_1c_4)^2 = (c_1c_8)^2 = (d_1c_8^{-1})^2 = 1,\\
(c_1c_3^{-1})^2 = (d_1c_3)^2 = (c_3^{-1}c_6)^2 = (c_3c_2)^2 = 1.
\end{gathered}
$$

Step 7. The presentation for the normal subgroup

$$V_7 = N\langle d_1c_1,\ d_1c_3,\ d_1c_4,\ d_1c_6,\ d_1c_2^{-1},\ d_1c_7^{-1},\ d_1c_8^{-1}\rangle_{V_6}$$

(coset representatives $\{1, d_1, d_1^{-1}\}$).

Generators:

$$\begin{array}{lll} e_1 = c_1 d_1, & g_1 = d_1 c_1, & h_1 = d_1^{-1} c_1 d_1^{-1}, \\ e_2 = c_2 d_1^{-1}, & g_2 = d_1 c_2 d_1, & h_2 = d_1^{-1} c_2, \\ e_3 = c_3 d_1, & g_3 = d_1 c_3, & h_3 = d_1^{-1} c_3 d_1^{-1}, \\ e_4 = c_4 d_1, & g_4 = d_1 c_4, & h_4 = d_1^{-1} c_4 d_1^{-1}, \\ e_6 = c_6 d_1, & g_6 = d_1 c_6, & h_6 = d_1^{-1} c_6 d_1^{-1}, \\ e_7 = c_7 d_1^{-1}, & g_7 = d_1 c_7 d_1, & h_7 = d_1^{-1} c_7, \\ e_8 = c_8 d_1^{-1}, & g_8 = d_1 c_8 d_1, & h_8 = d_1^{-1} c_8, \end{array}$$

Relations:

$$\begin{gathered} e_i h_i g_i = 1 \ (i = 1,3,4,6), \quad e_i g_i h_i = 1 \ (i = 2,7,8), \\ e_i^2 = g_i^2 = h_i^2 = 1 \quad (i = 2,3,4,6,7,8), \\ (e_1 h_7)^2 = (g_1 e_7)^2 = (h_1 g_7)^2 = 1, \\ (e_7 g_6)^2 = (g_7 h_6)^2 = (h_7 e_6)^2 = 1, \\ (h_7^{-1} h_2)^2 = (e_7^{-1} e_2)^2 = (g_7^{-1} g_2)^2 = 1, \\ (e_1 e_6^{-1})^2 = (g_1 g_6^{-1})^2 = (h_1 h_6^{-1})^2 = 1, \\ (e_1 h_2)^2 = (g_1 e_2)^2 = (h_1 g_2)^2 = 1, \\ (e_4 e_3^{-1})^2 = (g_4 g_3^{-1})^2 = (h_4 h_3^{-1})^2 = 1, \\ (e_3 h_8)^2 = (g_3 e_8)^2 = (h_3 g_8)^2 = 1, \\ (e_4 h_7)^2 = (g_4 e_7)^2 = (h_4 g_7)^2 = 1, \\ (e_8 e_7^{-1})^2 = (g_8 g_7^{-1})^2 = (h_8 h_7^{-1})^2 = 1, \\ (g_4^{-1} g_6)^2 = (h_4^{-1} h_6)^2 = (e_4^{-1} e_6)^2 = 1, \\ (e_4 h_2)^2 = (g_4 e_2)^2 = (h_4 g_2)^2 = 1, \\ (e_8 g_6)^2 = (g_8 h_6)^2 = (h_8 e_6)^2 = 1, \\ (h_8^{-1} h_2)^2 = (e_8^{-1} e_2)^2 = (g_8^{-1} g_2)^2 = 1, \\ (e_1 e_4^{-1})^2 = (g_1 g_4^{-1})^2 = (h_1 h_4^{-1})^2 = 1, \\ (e_1 h_8)^2 = (g_1 e_8)^2 = (h_1 g_8)^2 = 1, \\ (e_1 e_3^{-1})^2 = (g_1 g_3^{-1})^2 = (h_1 h_3^{-1})^2 = 1, \\ (g_3^{-1} g_6)^2 = (h_3^{-1} h_6)^2 = (e_3^{-1} e_6)^2 = 1, \\ (e_3 h_2)^2 = (g_3 e_2)^2 = (h_3 g_2)^2 = 1. \end{gathered}$$

Now let us consider the following quotient group

$$H = V_7 / \{e_3 = e_4 = e_6 = h_2 = h_7 = h_8,\ e_2 = e_7 = e_8,\ h_3 = h_4 = h_6\}.$$

Eliminating the generators g_i we obtain the presentation

$$\begin{aligned} H = \langle e_1, h_1, e_2, e_3, h_3;\ & e_2^2 = e_3^2 = h_3^2 = 1, \\ & (e_2 e_3)^2 = (e_3 h_3)^2 = 1, \\ & (e_1 e_3)^2 = (h_1 h_3)^2 = 1, \\ & (e_1 h_1 e_2)^2 = (e_1 h_1 e_3 h_3)^2 = 1, \\ & (e_2 e_3 h_1)^2 = (e_2 e_3 h_3)^2 = 1 \rangle. \end{aligned}$$

Applying the Reidemeister-Schreier method to the chain of subgroups of H

$$H \supset H_1 \supset H_2 \supset H_3$$

where

$$\begin{gathered} H_1 = N\langle h_1, e_1, e_2, e_3 \rangle_H, \quad |H : H_1| = 2, \\ H_2 = N\langle h_1, e_1, e_2 \rangle_{H_1}, \quad |H_1 : H_2| = 2, \\ H_3 = N\langle h_1, e_1 \rangle_{H_2}, \quad |H_2 : H_3| = 2 \end{gathered}$$

we successively obtain the presentations

$$\begin{aligned} H_1 = \langle h_1,\ e_1,\ e_2,\ e_3;\ & e_2^2 = e_3^2 = 1,\ (e_2e_3)^2 = (e_1e_3)^2 = 1, \\ & (e_1h_1e_2)^2 = (e_2e_3h_1)^2 = 1\rangle \end{aligned}$$

$$H_2 = \langle h_1,\ e_1,\ e_2;\ e_2^2 = 1,\ (e_1h_1e_2)^2 = 1\rangle,$$

$$H_3 = \langle x,\ y,\ z,\ v;\ xyzv = 1\rangle$$

where

$$x = e_1,\ y = h_1,\ z = e_2e_1e_2,\ v = e_2h_1e_2.$$

The free group H_3 of rank 3 is a subgroup of index 2^3 of H which in turn is a quotient of the subgroup V_7 of V of index $3 \cdot 2^6$. The proof of Theorem 2 is complete.

Conjecture (J.Mennicke): Let f be a quadratic form defined over $\mathbb{Q}$ with ortogonal group acting on hyperbolic space of dimension ≥ 2 and let G be the group of units of f. Then G contains a subgroup of finite index with a nonabelian free quotient group.

Of course, a study of this conjecture needs much stronger tools than have been used in this paper.[1]

Consider $m \in \mathbb{N}$ and let Q_m denote the normal closure in $SL_2(\mathbb{H}_\mathbb{Z})$ of the element

$$q_1^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{H}_\mathbb{Z}).$$

As the elements (2) generate $SL_2(\mathbb{H}_\mathbb{Z})$ it follows from (4) that

$$SL_2(\mathbb{H}_\mathbb{Z}) = Q_1.$$

Notice that in $SL_2(\mathbb{Z})$, the index of the corresponding normal subgroup Q_m is finite for $m \leq 5$, and infinite otherwise. Here we prove the following

[1]This conjecture was formulated in September 1995 when a preliminary version of this paper was appered. After we had finished the paper we received the preprint by Lubotzky [4] where he proves some general results related to the conjecture. In particular, he proved that any finite volume reflection hyperbolic group acting on a hyperbolic space H^n of dimension $n \geq 2$ contains a subgroup of finite index with nonabelian free quotient group (see Corrollary 3.6 in [4]).

Proposition 5 *The quotient group*

$$SL_2(\mathbb{H}_\mathbb{Z})/Q_2 = SL_2(\mathbb{H}_\mathbb{Z})/\{q_1^2 = 1\}$$

is infinite.

Proof. It is sufficient to prove the infiniteness of $PSL_2(\mathbb{H}_\mathbb{Z})/Q_2$. By (4), a presentation for $PSL_2(\mathbb{H}_\mathbb{Z})/Q_2$ is obtained by adding the relation $(dbdacdadb)^2 = 1$ to the presentation of $PSL_2(\mathbb{H}_\mathbb{Z})$. Modulo the defining relations of $PSL_2(\mathbb{H}_\mathbb{Z})$, we have

$$\begin{aligned}(dbdacdadb)^2 = 1 \;&\Leftrightarrow\; (cdadbdbda)^2 = 1\\ &\Leftrightarrow\; (cdabdba)^2 = 1\\ &\Leftrightarrow\; (c(abd)^2)^2 = 1.\end{aligned}$$

It is easy to check that the elements $(ad)^2$, $(bd)^2$ and $(abd)^2$ of $PSL_2(\mathbb{H}_\mathbb{Z})/Q_2$ generate an abelian normal subgroup. We shall prove that the quotient group

$$G = PSL_2(\mathbb{H}_\mathbb{Z})/\{q_1^2 = 1,\ (ad)^2 = (bd)^2 = (abd)^2 = 1\}$$

is infinite. Adding the relations $(ad)^2 = (bd)^2 = (abd)^2 = 1$ to the presentation of $PSL_2(\mathbb{H}_\mathbb{Z})/Q_2$ and eliminating redundant relations we obtain a presentation for G:

$$\begin{aligned}G = \langle a,\ b,\ c,\ d;\ \ & a^2 = b^2 = c^2 = d^2 = 1,\\ & (ab)^2 = (ad)^2 = (bd)^2 = 1,\\ & (ac)^4 = (bc)^4 = (cd)^3 = (abc)^4 = 1\rangle.\end{aligned} \tag{227}$$

Now consider the Euclidean space $\mathbb{R}^4 = \{(t_1, t_2, t_3, t_4)\}$ and let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ be reflections of $\mathbb{R}^4$ in 2-planes $\{t_3 = t_4 = 0\}$, $\{t_2 = t_4 = 0\}$ and hyperplanes $\{t_1 = t_2 = t_3 = 0\}$, $\{t_1 + t_2 + t_3 + t_4 = 1\}$, respectively. In the homogeneous coordinates $(z_1, z_2, z_3, z_4, z_5)$, $t_i = \frac{z_i}{z_5}$ $(i = 1, 2, 3, 4)$, the reflections $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ may be written as matrices

$$\bar{a} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{c} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1\\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1\\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{d} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

An easy computation shows that $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ satisfy the relations (227). Hence $\bar{G} = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \rangle$ is a quotient group of G. It is not difficult to see that $\bar{G}$ is infinite. For instance,

$$(\bar{a}\bar{c}\bar{b}\bar{c}\bar{d})^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -8 \\ 0 & 1 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and hence $\bar{a}\bar{c}\bar{b}\bar{c}\bar{d}$ has infinite order. Proposition 5 is proved.

One can also prove that $SL_2(\mathbb{H}_{\mathbb{Z}})/Q_2$ contains a subgroup of finite index which is a free abelian group of rank 3. Hence it does not contain a subgroup of finite index with a nonabelian free quotient.

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