A Visualization of the Isometry Group Action on the Fomenko–Matveev–Weeks Manifold

Alexander MEDNYKH[a,1] and Andrei VESNIN[a,b,2]

[a] *Sobolev Institute of Mathematics, SB RAS*
Novosibirsk-90, 630090, Russia

[b] *Department of Mathematics, Pusan National University,*
Pusan 609-735, Korea

Abstract

The smallest known three-dimensional closed orientable hyperbolic manifold $\mathcal{M}_1$, which volume is equal to $0.94\ldots$, was obtained independently by A. Fomenko and S. Matveev and by J. Weeks. It is known, that the isometry group of the manifold $\mathcal{M}_1$ is isomorphic to the dihedral group $\mathbb{D}_6$ of order 12. We study the lattice of the action of the isometry group on the manifold $\mathcal{M}_1$ and obtain all orbifolds which arised as the quotient spaces.

1. Introduction

In the present paper we study the properties of the smallest known three-dimensional closed orientable hyperbolic manifold $\mathcal{M}_1$, which was constructed independently by A. Fomenko and S. Matveev [7] and by J. Weeks [13].

A three-dimensional hyperbolic manifold can be defined as a quotient space $M = \mathbb{H}^3/\Gamma$, where Γ is a discrete group of isometries of the three-dimensional Lobachevsky space $\mathbb{H}^3$, acting without fixed points. The concept of the volume in $\mathbb{H}^3$ is carried over naturally to M. Further we will consider three-dimensional orientable hyperbolic manifolds of finite volume.

The structure of the set of volumes of three-dimensional hyperbolic manifolds was described by W. Thurston and T. Jørgensen. According to the Thurston–Jørgensen theorem [12], volumes of three-dimensional hyperbolic manifolds form a well-ordered

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[1]E-mail address: *mednykh@math.nsc.ru*.

[2]E-mail address: *vesnin@math.nsc.ru* and *vesnin@arirang.math.pusan.ac.kr*.

non-discrete subset of the real line of the order type ω^ω. In particular, there exists a closed manifold with the smallest volume.

The first manifold which pretended to be the smallest closed manifold was constructed by W. Thurston [12]. This manifold was obtained by $(5, -1)$ Dehn surgery on the figure-eight knot, and its volume is equal to $0.98\ldots$. But several years later the another pretender appeared.

In [13] J. Weeks calculated volumes of hyperbolic manifolds, obtained by Dehn surgeries on hyperbolic knots and links of the small order. It was possible due to his nice computer program *SnapPea* [11]. As the result of the calculations, the closed hyperbolic manifold $\mathcal{M}_1$ with the volume equals to $0.94\ldots$ was founded. The manifold $\mathcal{M}_1$ was obtained by Dehn surgeries with parameters $(5, -2)$ and $(5, -1)$ on both components of the Whitehead link. At the same time, numerous computer calculations of volumes were given by A. Fomenko and S. Matveev [7]. The manifold $\mathcal{M}_1$ also was appeared in their list (but it was obtained by another way). Further the manifold $\mathcal{M}_1$ will be referred to as the *Fomenko–Matveev–Weeks manifold*.

Moreover, A. Fomenko and S. Matveev [7] conjectured the structure of the initial segment of the set of volumes of three-dimensional hyperbolic manifolds. In [5] C. Hodgson and J. Weeks refined the smallest manifolds computing a lot of volumes by SnapPea program.

In [9] E. Molnar constructed the fundamental polyhedron in $\mathbb{H}^3$ for the fundamental group of the manifold $\mathcal{M}_1$ and remarked, that the isometry group of this manifold contains the dihedral group $\mathbb{D}_6$ of order 12. Using computer calculation, it was shown in [5], that the isometry group of $\mathcal{M}_1$ has exactly 12 elements.

The aim of the present paper is to describe the lattice of the action of the isometry group $\mathrm{Isom}(\mathcal{M}_1)$ on the manifold $\mathcal{M}_1$. We will obtain all orbifolds which arise as quotient spaces of $\mathcal{M}_1$ by the action of the subgroups of $\mathrm{Isom}(\mathcal{M}_1)$. In particular, we will describe the manifold $\mathcal{M}_1$ as the two-fold covering of the 3-sphere, branched over the knot 9_{49} and as the cyclic three-fold covering of the 3-sphere, branched over the two-bridge knot 5_2.

2. The lattice of the isometry group's action

Let us consider the fundamental group $\Gamma = \pi_1(\mathcal{M}_1)$ of the Fomenko-Matveev-Weeks manifold $\mathcal{M}_1$. According to [9], the group Γ has the following presentation:

$$\Gamma = \langle\, a, b \mid abab^2a^{-2}b^2 = baba^2b^{-2}a^2 = 1 \,\rangle. \tag{1}$$

It was observed by E. Molnar [9], that the group Γ has the following three automorphisms

$$\left\{ \begin{array}{l} a \to a^{-1}, \\ b \to b^{-1}, \end{array} \right. \qquad \left\{ \begin{array}{l} a \to b, \\ b \to a, \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} a \to a, \\ b \to a^{-1}b^{-1}. \end{array} \right. \tag{2}$$

By the Mostow rigidity theorem, there exist isometries r, s and t of the Lobachevsky space $\mathbb{H}^3$ such that

$$\begin{cases} r\,a\,r^{-1} \;=\; a^{-1}, \\ r\,b\,r^{-1} \;=\; b^{-1}, \end{cases} \qquad \begin{cases} s\,a\,s^{-1} \;=\; b, \\ s\,b\,s^{-1} \;=\; a, \end{cases} \qquad \text{and} \qquad \begin{cases} t\,a\,t^{-1} \;=\; a, \\ t\,b\,t^{-1} \;=\; a^{-1}b^{-1}. \end{cases} \tag{3}$$

From (3) we get the properties of the products of the isometries r, s and t:

$$\begin{cases} r^2\,a\,r^{-2} \;=\; a, \\ r^2\,b\,r^{-2} \;=\; b, \end{cases} \qquad \begin{cases} s^2\,a\,s^{-2} \;=\; a, \\ s^2\,b\,s^{-2} \;=\; b, \end{cases} \qquad \begin{cases} t^2\,a\,t^{-2} \;=\; a \;=\; a^{-1}aa, \\ t^2\,b\,t^{-2} \;=\; a^{-1}ba, \end{cases}$$

$$\begin{cases} (rs)\,a\,(rs)^{-1} \;=\; b^{-1}, \\ (rs)\,b\,(rs)^{-1} \;=\; a^{-1}, \end{cases} \qquad \begin{cases} (rs)^2\,a\,(rs)^{-2} \;=\; a, \\ (rs)^2\,b\,(rs)^{-2} \;=\; b, \end{cases}$$

$$\begin{cases} (rt)\,a\,(rt)^{-1} \;=\; a^{-1}, \\ (rt)\,b\,(rt)^{-1} \;=\; ab, \end{cases} \qquad \begin{cases} (rt)^2\,a\,(rt)^{-2} \;=\; a, \\ (rt)^2\,b\,(rt)^{-2} \;=\; b, \end{cases}$$

$$\begin{cases} (st)\,a\,(st)^{-1} \;=\; b, \\ (st)\,b\,(st)^{-1} \;=\; b^{-1}a^{-1}, \end{cases} \qquad \begin{cases} (st)^2\,a\,(st)^{-2} \;=\; b^{-1}a^{-1}, \\ (st)^2\,b\,(st)^{-2} \;=\; a, \end{cases} \qquad \begin{cases} (st)^3\,a\,(st)^{-3} \;=\; a, \\ (st)^3\,b\,(st)^{-3} \;=\; b. \end{cases}$$

Because the centralizer of any discrete cocompact subgroup of the group $\mathrm{Isom}(\mathbb{H}^3)$ is trivial, we have $\mathrm{Inn}(\Gamma) \;\cong\; \Gamma$. Whence, using the above formulae, we get:

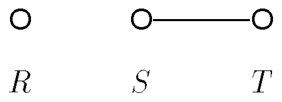
$$r^2 \;=\; s^2 \;=\; (rs)^2 \;=\; (rt)^2 \;=\; (st)^3 \;=\; 1, \quad t^2 \;=\; a^{-1}. \tag{4}$$

We recall, that the group of isometries of a hyperbolic 3-manifold is canonically isomorphic to the group of outer automorphisms of its fundamental group. Therefore,

$$\mathrm{Isom}(\mathcal{M}_1) \;\cong\; N(\Gamma)\,/\,\Gamma,$$

where $N(\Gamma)$ is the normalizer of the group Γ in the isometry group $\mathrm{Isom}(\mathbb{H}^3)$.

Let us consider the cosets $R = r\Gamma$, $S = s\Gamma$ and $T = t\Gamma$ in the group $N(\Gamma)/\Gamma$, and denote by G the subgroup of $N(\Gamma)/\Gamma$ generated by R, S and T. According to (4), the group G is a Coxeter group with the following Coxeter diagram:



Therefore, the group G is isomorphic to the dihedral group $\mathbb{D}_6$ of order 12. Hence, the group $\mathrm{Isom}(\mathcal{M}_1)$ contains the subgroup of order 12.

By the other side, it was shown by C. Hodgson and J. Weeks [5] using computer program SnapPea [11], that the group $\mathrm{Isom}(\mathcal{M}_1)$ consists of 12 elements exactly. Therefore,

$$\mathrm{Isom}(\mathcal{M}_1) \;\cong\; G \;=\; \langle\, R,\, S,\, T \,\rangle \;\cong\; \mathbb{D}_6, \tag{5}$$

and

$$N(\Gamma) \;\cong\; \langle\, a,\, b,\, r,\, s,\, t \,\rangle, \tag{6}$$

where relations between generators a, b, r, s, and t are given by (1), (3) and (4).

The aim of the present paper is to describe the action of the group $\mathrm{Isom}(\mathcal{M}_1)$ on the manifold $\mathcal{M}_1$.

Theorem. *The lattice of the action of the isometry group Isom($\mathcal{M}_1$) on the manifold $\mathcal{M}_1$ is presented in Fig. 1,*

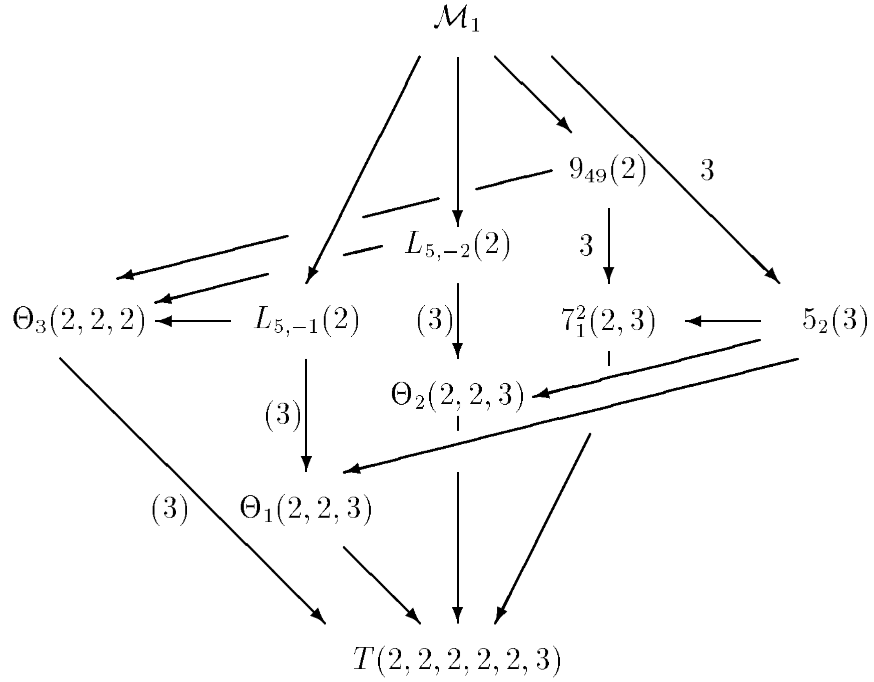


Fig. 1. The lattice of the action.

where $9_{49}(2)$, $5_2(3)$, $7_1^2(2,3)$, $\Theta_1(2,2,3)$, $\Theta_2(2,2,3)$, $\Theta_3(2,2,2)$, and $T(2,2,2,2,2,3)$ are three-dimensional hyperbolic orbifolds whose underlying space is the 3-sphere, and whose singular sets are pictured in Figures 3 – 9; $L_{5,-1}(2)$ and $L_{5,-2}(2)$ are orbifolds whose underlying spaces are the lens spaces $L_{5,-1}$ and $L_{5,-2}$. In this diagram notations $A \longrightarrow B$, $A \xrightarrow{3} B$ and $A \xrightarrow{(3)} B$ correspond to a two-fold covering, to a regular three-fold covering and to an irregular three-fold covering, respectively.

3. The lattice of subgroups of the group G

In this section we consider the subgroup structure of the group $G \cong \mathrm{Isom}(\mathcal{M}_1)$. We observe, that the group $G = \langle R, S, T \rangle \cong \mathbb{D}_6$ has exactly 10 subgroups, up to conjugation in G. These subgroups are the following: the trivial group $\langle \mathbf{1} \rangle$; the three subgroups $\langle R \rangle$, $\langle S \rangle$ and $\langle T \rangle$ of order 2; the subgroup $\langle ST \rangle$ of order 3; the subgroup $\langle R, S \rangle$ of order 4; the three subgroup $\langle S, T \rangle$, $\langle RS, RT \rangle$ and $\langle R, RT \rangle$ of order 6; and the group $G = \langle R, S, T \rangle$ of order 12. By direct considerations we see that these subgroups of the group G form the lattice with respect to set inclusion, which is pictured in Fig. 2. In this diagram notations $A \longrightarrow B$, $A \xrightarrow{3} B$ and $A \xrightarrow{(3)} B$ correspond respectively to the case when A is a subgroup of index 2 in B; to the case when A is a normal subgroup of index 3 in B; and to the case when A is a subgroup of index 3 in B, but not normal.

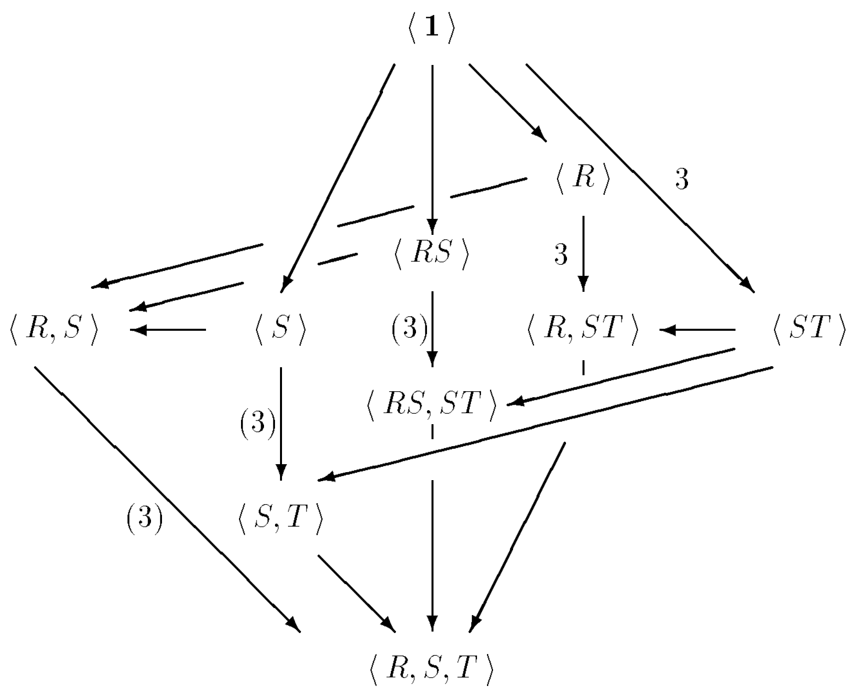


Fig. 2. The lattice of subgroups.

We will formulate the above-mentioned property of the group G as the following

Proposition 1. *The lattice of subgroups of the group G up to conjugation is presented in Fig. 2.*

Recall the well-known close relations between subgroups of the isometry group G of a manifold $M = \mathbb{H}^3/\Gamma$ and its quotient spaces.

Denote by φ the canonical epimorphism

$$\varphi \,:\, N(\Gamma) \longrightarrow N(\Gamma)\,/\,\Gamma \,\cong\, G,$$

defined by

$$\varphi : r \to R, \quad s \to S, \quad t \to T.$$

We note that the exact sequence defined by the epimorphism φ:

$$\mathbf{1} \longrightarrow \Gamma \longrightarrow N(\Gamma) \xrightarrow{\varphi} G \longrightarrow \mathbf{1}$$

does not split.

Proposition 2. *Assume that M, Γ, and G are as above.*

(1) Let $H < G$ and $\Gamma_H = \varphi^{-1}(H)$. Then the orbifold M/H is isometric to the quotient space $\mathbb{H}^3/\Gamma_H$ and its orbifold group is $\pi_1(M/H) \cong \Gamma_H$.

(2) Let $H_1 < H_2 < G$. Then the orbifold covering $M/H_1 \to M/H_2$, induced by the group inclusion $H_1 < H_2$, is regular if and only if $H_1 \triangleleft H_2$.

(3) Let Γ_H be a group such that $\Gamma < \Gamma_H < N(\Gamma)$ and $H = \varphi(\Gamma_H)$. Then the orbifold $\mathbb{H}^3/\Gamma_H$ is isometric to the orbifold M/H.

4. The orbifolds arising under the group G action

4A. The orbifold $\mathbf{9_{49}(2)}$

We recall [7], [13] that the Fomenko-Matveev-Weeks manifold $\mathcal{M}_1$ can be obtained by the Dehn surgeries on the components of the Whitehead link with surgery parameters $(5,-1)$ and $(5,-2)$. The covering properties of the manifolds obtained by Dehn surgeries on the Whitehead link were studied in [8], where these manifolds were described as the two-fold branched coverings of the 3-sphere. In particular, it was shown in [8] that the manifold $\mathcal{M}_1$ is the two-fold covering of the 3-sphere, branched over the knot 9_{49} (see Fig. 3).

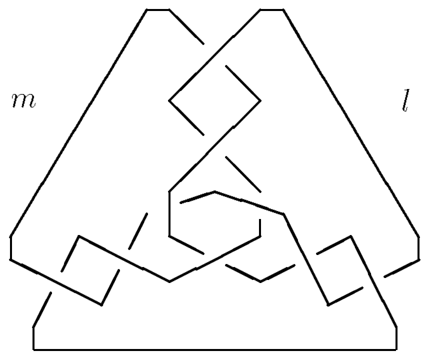


Fig. 3. The knot 9_{49}.

Denote by $9_{49}(2)$ the orbifold whose underlying space is the 3-sphere and whose singular set is the knot 9_{49} with branch index 2. In this case can say that the manifold $\mathcal{M}_1$ is the two-fold covering of the orbifold $9_{49}(2)$. As one can see from the diagram in Fig. 2, there are three subgroups of order 2 in the group G. Therefore, the covering $\mathcal{M}_1 \xrightarrow{2} 9_{49}(2)$ can be induced either by the involution r, by the involution s, or by the involution rs. In these cases we will get either the orbifold $\mathcal{M}_1/\langle R \rangle = \mathbb{H}^3/\langle \Gamma, r \rangle$, the orbifold $\mathcal{M}_1/\langle S \rangle = \mathbb{H}^3/\langle \Gamma, s \rangle$, or the orbifold $\mathcal{M}_1/\langle RS \rangle = \mathbb{H}^3/\langle \Gamma, rs \rangle$.

In virtue of the Armstrong theorem [1], the fundamental group of the underlying space of a orbifold $\mathbb{H}^3/\Delta$ is isomorphic to the factor group Δ/Δ_0, where Δ_0 is the normal subgroup of Δ, generated by all elements of finite order. Now we will find the fundamental groups of the underlying spaces of three above orbifolds. Firstly we consider the underlying space of the orbifold $\mathcal{M}_1/\langle R \rangle$ and its fundamental group $\langle \Gamma, r \rangle / \mathrm{Ncl}(r)$, where $\mathrm{Ncl}(r)$ denotes the normal closure of r in the group $\langle \Gamma, r \rangle$. Analogously, we use notations $\mathrm{Ncl}(s)$ and $\mathrm{Ncl}(rs)$ for normal closures of elements s and rs in groups $\langle \Gamma, s \rangle$ and $\langle \Gamma, rs \rangle$, respectively. From (1) and (3) we have

$$\begin{aligned} \langle \Gamma, r \rangle \,/\, \mathrm{Ncl}(r) \;=\; \langle\, a, b, r \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,& \\ r\,a\,r^{-1} \;=\; a^{-1}, \quad r\,b\,r^{-1} \;=\; b^{-1}, \quad r \;=\; 1 \,\rangle& \\ =\; \langle\, a, b \;\mid\; a^2 \;=\; b^2 \;=\; 1, \quad aba \;=\; bab \;=1 \,\rangle \;=\; \langle\, \mathbf{1} \,\rangle.& \end{aligned} \tag{7}$$

Analogously, we find

$$\begin{aligned} \langle \Gamma, s \rangle \,/\, \mathrm{Ncl}(s) \;=\; \langle\, a, b, s \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,& \\ s\,a\,s^{-1} \;=\; b, \quad s\,b\,s^{-1} \;=\; a, \quad s \;=\; 1 \,\rangle& \\ =\; \langle\, a \;\mid\; a^5 \;=\; 1 \,\rangle \;=\; \mathbb{Z}_5.& \end{aligned} \tag{8}$$

By the same arguments, we get

$$\begin{aligned} \langle \Gamma, rs \rangle \,/\, \mathrm{Ncl}(rs) \;=\; \langle\, a, b, rs \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,& \\ (rs)\,a\,(rs)^{-1} \;=\; b^{-1}, \quad (rs)\,b\,(rs)^{-1} \;=\; a^{-1}, \quad rs \;=\; 1 \,\rangle& \\ =\; \langle\, a \;\mid\; a^5 \;=\; 1 \,\rangle \;=\; \mathbb{Z}_5.& \end{aligned} \tag{9}$$

Because only one of the groups (7), (8) and (9) is trivial, we get, that the covering $\mathcal{M}_1 \xrightarrow{2} 9_{49}(2)$ is induced by the involution r.

Let us demonstrate the same by the direct calculation. More exactly, let us show that the orbifold groups of the orbifold $\mathcal{M}_1/\langle R \rangle$ and of the orbifold $9_{49}(2)$ are isomorphic. Indeed, from (1) and (3) we get

$$\begin{aligned} \pi_1(\mathcal{M}_1/\langle R \rangle) \;=\; \langle \Gamma, r \rangle \;=\; \langle\, a, b, r \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,& \\ r\,a\,r^{-1} \;=\; a^{-1}, \quad r\,b\,r^{-1} \;=\; b^{-1}, \quad r^2 \;=\; 1 \,\rangle& \end{aligned} \tag{10}$$

If we denote $l = ar$ and $m = rb$, then from (10) we get

$$\begin{aligned} \pi_1(\mathcal{M}_1/\langle R \rangle) \;=\; \langle\, l, m, r \;\mid\; l^2 \;=\; m^2 \;=\; r^2 \;=1,& \\ (lm)^2\,(lr)^2\,(mr)^2\,lr \;=\; 1, \quad (rl)^2\,(rm)^2\,(lm)^2\,rm \;=\; 1 \,\rangle,& \end{aligned} \tag{11}$$

what is the group of the orbifold $9_{49}(2)$, where l, m and r correspond to the notations in Fig. 3. Because both compact orbifolds $\mathcal{M}_1/\langle R \rangle$ and $9_{49}(2)$ are hyperbolic and their groups are isomorphic, they are isometric by the rigidity theorem. Thus, the following statement is proved.

Lemma 1. *In the above notations we have*

$$\langle \Gamma, r \rangle \;\cong\; \pi_1(9_{49}(2))$$

and

$$\mathcal{M}_1/\langle\, R\,\rangle \;\cong\; 9_{49}(2).$$

4B. The orbifold $5_2(3)$

The group $G = \mathrm{Isom}(\mathcal{M}_1)$ contains only one subgroup of order three, which is generated by the element ST. Let us consider the orbifold group of the corresponding quotient space:

$$\pi_1(\mathcal{M}_1/\langle\, ST\,\rangle) \;=\; \langle\, \Gamma, st\,\rangle \;==\; \langle\, a, b, st \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,$$

$$(st)\,a\,(st)^{-1} \;=\; b, \quad (st)\,b\,(st)^{-1} \;=\; b^{-1}a^{-1}, \quad (st)^3 = 1\,\rangle$$

$$=\; \langle\, a, b, u \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,$$

$$u\,a\,u^{-1} \;=\; b, \quad u\,b\,u^{-1} \;=\; b^{-1}a^{-1}, \quad u^3 = 1\,\rangle,$$

where $u = st$. Denote $v = ua$, then $a = u^{-1}v$ and $b = vu^{-1}$. Hence,

$$\pi_1(\mathcal{M}_1/\langle\, ST\,\rangle) \;=\; \langle\, u, v \;\mid\; uv^{-1}uv^{-1}u^{-1}vu^{-1}v^{-1}uv^{-1}uvu^{-1}v \;=\; 1,$$

$$u^3 \;=\; v^3 \;=\; 1\,\rangle. \tag{12}$$

By direct calculations, using the Wirtinger algorithm, we see that the group (12) is the orbifold group of the orbifold whose underlying space is the 3-sphere and the singular set is the knot, pictured in Fig. 4, with branch index equals three.

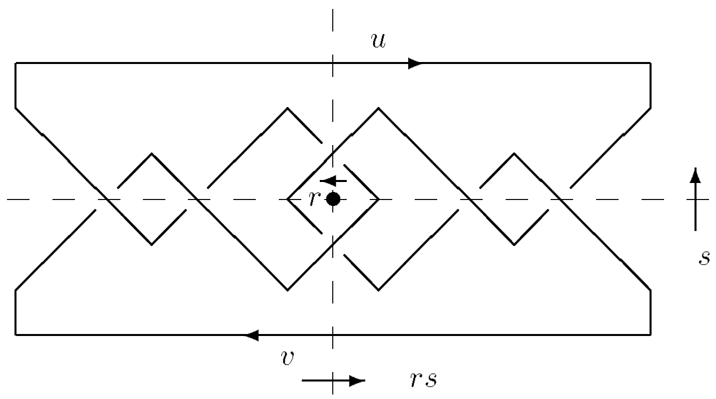


Fig. 4. The knot 5_2.

Moreover, generators u and v from (12) correspond to the loops in Fig. 4. The knot diagram in Fig. 4 is non-alternative. Using Reidemeister moves, one can simplify this diagram to the diagram of the knot 5_2. Let us denote by $5_2(3)$ the orbifold with the 3-sphere as its underlying space and as the knot 5_2 with the branch index 3 as its singular set. Because the orbifold $5_2(3)$ is hyperbolic [12], summarizing the above considerations, we get

Lemma 2. *In the above notations we have*

$$\langle\, \Gamma, st\, \rangle \;\cong\; \pi_1(5_2(3))$$

and

$$\mathcal{M}_1/\langle\, ST\, \rangle \;\cong\; 5_2(3).$$

4C. The isometries of the orbifold $5_2(3)$

It was shown in [4] and [6]that the symmetry group of the complement $S^3 \setminus 5_2$ of the knot 5_2 in the 3-sphere consists of four elements, and all non-trivial symmetries are involutions. We will show that these involutions induce the involutions of the orbifold $5_2(3)$, which lifts to the involutions r, s and rs on the universal covering space $\mathbb{H}^3$. We recall that the group $\pi_1(5_2(3))$ of the orbifold $5_2(3)$ is isomorphic to the group $\pi_1(\mathcal{M}_1/\langle\, ST\, \rangle)$ and by (12) it has the following presentations:

$$\pi_1(5_2(3)) \;=\; \langle\, u, v, \mid\; uv^{-1}uv^{-1}u^{-1}vu^{-1}v^{-1}uv^{-1}uvu^{-1}v \;=\; 1, \quad u^3 \;=\; v^3 \;= 1\, \rangle, \tag{13}$$

where $u = st$ and $v = ua = sta$. By (4) we have $t^2 = a^{-1}$, so we get $v = st^{-1}$. Considering the action of the involutions r, s and rs on the generators u and v, we will get:

$$r\,u\,r^{-1} \;=\; r\,st\,r^{-1} \;=\; st^{-1} \;=\; v, \quad r\,v\,r^{-1} \;=\; r\,st^{-1}\,r^{-1} \;=\; st \;=\; u, \tag{14}$$

$$s\,u\,s^{-1} \;=\; s\,st\,s^{-1} \;=\; ts^{-1} \;=\; v^{-1}, \quad s\,v\,s^{-1} \;=\; s\,st^{-1}\,s^{-1} \;=\; t^{-1}s^{-1} \;=\; u^{-1}, \tag{15}$$

$$(rs)\,u\,(rs)^{-1} \;=\; r\,v^{-1}\,r^{-1} \;=\; u^{-1}, \quad (rs)\,v\,(rs)^{-1} \;=\; r\,u^{-1}\,r^{-1} \;=\; v^{-1}, \tag{16}$$

where we used (4). Denote isometries of $5_2(3)$ defined by (14), (15) and (16) also by r, s and rs in the correspondence to the notations of the isometries of $\mathbb{H}^3$. The axes of the involutions s and rs are pictured in Fig. 4, and the axe of the involution r is perpendicular to both above axes.

We will consider the quotient spaces of the orbifold $5_2(3)$ by the isometries r, s and rs, and denote these orbifolds by

$$7_1^2(2,3) \;=\; 5_2(3)\,/\,\langle\, r\,\rangle, \tag{17}$$

$$\Theta_1(2,2,3) \;=\; 5_2(3)\,/\,\langle\, s\,\rangle, \tag{18}$$

and

$$\Theta_2(2,2,3) \;=\; 5_2(3)\,/\,\langle\, rs\,\rangle. \tag{19}$$

4D. The orbifold $7_1^2(2,3)$

By the definition, the orbifold $7_1^2(2,3)$ is the quotient space of the orbifold $5_2(3)$ by the action of the involution r. Therefore, the underlying space of $7_1^2(2,3)$ is the 3-sphere. Because the axis of the involution r and the singular set of $5_2(3)$ are disjoint, the singular set of the quotient space is a two-component link, one component of which has the branch index 2, and other has the branch index 3. One can check directly, that this two-component singular set is the two-component link with seven cross-points which is noted by 7_1^2 in the table of knots and links [10]. This explains the notation $7_1^2(2,3)$ used for this orbifold.

Lemma 3. *In the above notations we have*

$$\langle\, \Gamma, r, st\,\rangle \;\cong\; \pi_1(7_1^2(2,3))$$

and

$$\mathcal{M}_1/\langle\, R, ST\,\rangle \;\cong\; 7_1^2(2,3).$$

Proof. According to the action of the automorphisms r and st on the group Γ, we have:

$$\begin{aligned}
\langle\, \Gamma, r, st\,\rangle \;&=\; \langle\, a,b,r,st \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,\\
&\qquad (st)\,a\,(st)^{-1} \;=\; b, \quad (st)\,b\,(st)^{-1} \;=\; b^{-1}a^{-1}, \quad (st)^3 \;=\; 1,\\
&\qquad r\,a\,r^{-1} \;=\; a^{-1}, \quad r\,b\,r^{-1} \;=\; b^{-1}, \quad r^2 \;=\; 1\,\rangle\\
&=\; \langle\, u,v,r \;\mid\; uv^{-1}uv^{-1}u^{-1}vu^{-1}v^{-1}uv^{-1}uvu^{-1}v \;=\; 1, \quad u^3 \;=\; v^3 \;=\; 1,\\
&\qquad r\,u\,r^{-1} \;=\; v, \quad r\,v\,r^{-1} \;=\; u, \quad r^2 \;=\; 1\,\rangle,
\end{aligned}$$

where $u = st$, $v = ua = st^{-1}$. Whence

$$\langle\, \Gamma, r, st\,\rangle \;=\; \langle\, u, r \;\mid\; uru^{-1}ruru^{-1}ru^{-1}ruru^{-1}ru^{-1}ruru^{-1}rururu^{-1}rur \;=\; 1,\,\rangle. \tag{20}$$

As one can see by direct calculations, the group $\langle\, \Gamma, r, st\,\rangle$ is isomorphic to the group of the orbifold $7_1^2(2,3)$, where generators u and r correspond to the loops in Fig. 5.

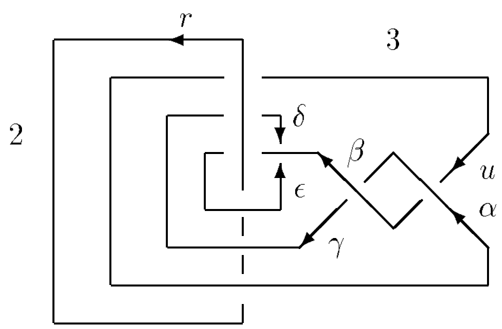


Fig. 5. The link 7_1^2.

Indeed, by the Wirtinger algorithm, $\alpha = ru^{-1}r$, $\beta = \alpha^{-1}u\alpha$, $\gamma = \beta^{-1}\alpha\beta$, $\delta = r\gamma r$, $\epsilon = r\beta r$, and from the relation $\epsilon^{-1} = \beta^{-1}\delta\beta$ we will get the non-trivial relation from the presentation (20) for the group $\langle\, \Gamma, r, st \,\rangle$. $\square$

Corollary. *The following diagram of the regular orbifold coverings is commutative:*

$$\begin{tikzcd}
& \mathcal{M}_1 \arrow[dl, "2"'] \arrow[dr, "3"] & \\
9_{49}(2) \arrow[dr, "3"'] & & 5_2(3) \arrow[dl, "2"] \\
& 7_1^2(2,3) &
\end{tikzcd}$$

Proof. We recall, that by virtue of lemmata 1 – 3, we have the following descriptions of the orbifolds: $9_{49}(2) = \mathcal{M}_1/\langle\, R \,\rangle$, $5_2(3) = \mathcal{M}_1/\langle\, ST \,\rangle$ and $7_1^2(2,3) = \mathcal{M}_1/\langle\, R, ST \,\rangle$. The commutativety of the above diagram follows from the fact that the groups $\langle\, R \,\rangle \cong \mathbb{Z}_2$ and $\langle\, ST \,\rangle \cong \mathbb{Z}_3$ are normal subgroups of the group $\langle\, R, ST \,\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ of indices 3 and 2 respectively. $\square$

4E. The orbifold $\boldsymbol{\Theta_1(2,2,3)}$

Let us consider the orbifold $\Theta_1(2,2,3) = 5_2(3)/\langle\, s \,\rangle$.

Lemma 4. *In the above notations we have*

$$\langle\, \Gamma, s, t \,\rangle \;\cong\; \pi_1(\Theta_1(2,2,3))$$

and

$$\mathcal{M}_1/\langle\, S, T \,\rangle \;\cong\; \Theta_1(2,2,3).$$

Proof. The proof of the lemma follows from the definition of the orbifold $\Theta_1(2,2,3)$ and Proposition 2. □

Because the underlying space of the orbifold $5_2(3)$ is the 3-sphere and the isometry s is an involution, we get that the underlying space of the orbifold $\Theta_1(2,2,3)$ is the 3-sphere also. The singular set Θ_1 of $\Theta_1(2,2,3)$ is pictured in Fig. 6.

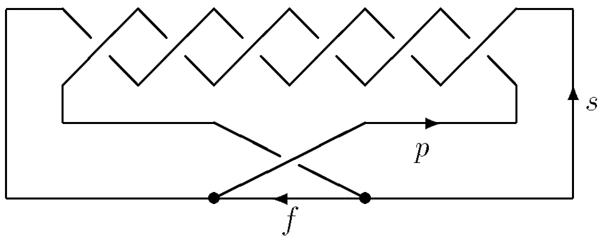


Fig. 6. The Θ_1-curve.

From (1), (3) and (4) we get the following presentation:

$$\begin{aligned} \langle\, \Gamma, s, t \,\rangle \;&=\; \langle\, a, b, s, t \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1, \\ & s^2 \;=\; (st)^3 \;=\; 1, \quad t^2 \;=\; a^{-1}, \\ & s\,a\,s^{-1} \;=\; b, \quad s\,b\,s^{-1} \;=\; a, \quad t\,a\,t^{-1} \;=\; a. \quad t\,b\,t^{-1} \;=\; a^{-1}b^{-1} \,\rangle \\ &=\; \langle\, p, s \;\mid\; p^3 \;=\; s^2 \;=\; \left[(sp^{-1})^3\,(sp)^4\right]^2 \;=\; 1 \,\rangle, \end{aligned} \tag{21}$$

where $p = st$. The elements s, p and $f = (sp^{-1})^3\,(sp)^4$ correspond to the loops in $\Theta_1(2,2,3)$ which are shown in Fig. 6. We remark that the singular set of the orbifold $\Theta_1(2,2,3)$ is a Θ-curve in the terminology of [14], that is a spatial Θ-graph with 2 vertices and 3 edges. If we delete the edge, corresponding to the loop f, then we will get the torus knot 7_1. Hence, we can regard the singular set of the orbifold $\Theta_1(2,2,3)$ as the knot 7_1 with a bridge. The relations $p^2 = 1$, $s^3 = 1$, and $f^2 = 1$ in the group of the orbifold $\Theta_1(2,2,3)$ correspond to the loops around the three edges of the spatial Θ-graph.

By the other hand, as we see from the presentation (21), the group $\langle\, \Gamma, s, t \,\rangle$ is a generalized triangle group in the terminology of [2]. We recall that some connections between generalized triangle groups and 3-orbifolds with a two-bridge knot with a bridge as the singular set, were studied in [3].

4F. The orbifold $\Theta_2(2,2,3)$

Let us consider the orbifold $\Theta_2(2,2,3) = 5_2(3)/\langle\, rs \,\rangle$.

Lemma 5. *In the above notations we have*

$$\langle\, \Gamma, rs, st \,\rangle \;\cong\; \pi_1(\Theta_2(2,2,3))$$

and

$$\mathcal{M}_1/\langle\, RS, ST\,\rangle \;\cong\; \Theta_2(2,2,3).$$

Proof. The proof of the lemma follows from the definition of the orbifold $\Theta_2(2,2,3)$ and Proposition 2. □

Because the underlying space of the orbifold $5_2(3)$ is the 3-sphere and the isometry rs is an involution, we get that the underlying space of the orbifold $\Theta_2(2,2,3)$ is the 3-sphere also. The singular set Θ_2 of $\Theta_2(2,2,3)$ is pictured in Fig. 7.

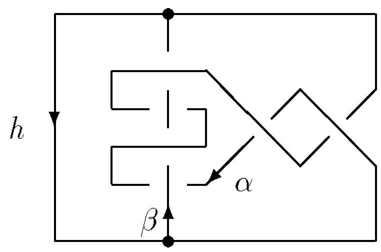


Fig. 7. The Θ_2-curve.

From (1), (3) and (4) we get the following presentation:

$$\begin{aligned}\langle\, \Gamma, rs, st\,\rangle \;=\; \langle\, a, b, rs, st \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,\\ (rs)\,a\,(rs)^{-1} \;=\; b^{-1}, \quad (rs)\,b\,(rs)^{-1} \;=\; a^{-1}, \quad (rs)^2 \;=\; 1,\\ (st)\,a\,(st)^{-1} \;=\; b, \quad (st)\,b\,(st)^{-1} \;=\; b^{-1}a^{-1}, \quad (st)^3 \;=\; 1\,\rangle.\end{aligned} \tag{22}$$

In respect to generators α and β pictured in Fig. 7, this group has presentation:

$$\langle\, \Gamma, rs, st\,\rangle \;=\; \langle\, \alpha, \beta \;\mid\; \alpha^3 \;=\; \beta^2 \;=\; \left[(\alpha\beta)^2\,(\alpha^{-1}\beta)\,(\alpha\beta)^2\,(\alpha^{-1}\beta)^2\right]^2 \;=\; 1\,\rangle, \tag{23}$$

where $h = (\alpha\beta)^2\,(\alpha^{-1}\beta)\,(\alpha\beta)^2\,(\alpha^{-1}\beta)^2$. We remark that the singular set of the orbifold $\Theta_2(2,2,3)$ is the two-bridge knot 5_2 with a bridge; the orbifold group with the presentation (23) is a generalized triangle group; and the relations $\alpha^3 = 1$, $\beta^2 = 1$ and $h^2 = 1$ correspond to the loops around the edges of the spatial Θ-graph.

4G. The orbifold $\Theta_3(2,2,2)$

According to [5] and [6], the symmetry group of the knot 9_{49} is the dihedral group $\mathbb{D}_3$ of order 6. This group consists of three elements which form the cyclic group of order 3 and of the three mutually conjugated involutions. We will show that one of these involutions is induced by the involution rs of the space $\mathbb{H}^3$.

Indeed, from (1), (3) and (4), using the relations $l = ar$ and $m = rb$ from subsection 4A, we have:

$$(rs)\, l\, (rs)^{-1} \;=\; m, \quad (rs)\, r\, (rs)^{-1} \;=\; r, \quad (rs)\, m\, (rs)^{-1} \;=\; l,$$

$$l^2 \;=\; m^2 \;=\; r^2 \;=\; (rs)^2 \;=\; 1.$$

Therefore, the isometry rs induces an isometry of the orbifold $9_{49}(2)$ which interchange the arcs corresponding to l and m, and preserves the arc corresponding to r invariant (see Fig. 3).

The quotient space of the orbifold $9_{49}(2) = \mathbb{H}^3/\langle\, \Gamma, r\,\rangle$ by the action of the isometry rs is the orbifold $\Theta_3(2,2,2)$ whose underlying space is the 3-sphere and whose singular set, pictured in Fig. 8, is the Θ-curve, each arc of which has branch index 2 (that is such called π-orbifold).

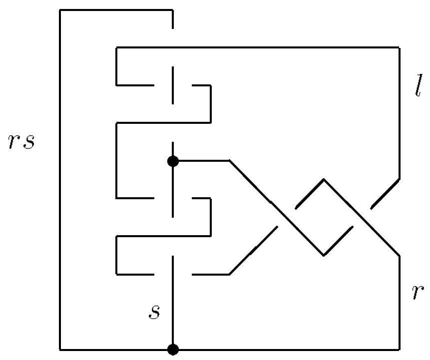


Fig. 8. The Θ_3-curve.

Because of (1), (3) and (4), using that

$$\Theta_3(2,2,2) \;=\; 9_{49}(2)\,/\,\langle\, rs\,\rangle \;=\; \mathbb{H}^3\,/\,\langle\, \Gamma, r, rs\,\rangle \;=\; \mathbb{H}^3\,/\,\langle\, \Gamma, r, s\,\rangle,$$

the group $\pi_1(\Theta_3(2,2,2))$ is isomorphic to the group $\langle\, \Gamma, r, s\,\rangle$ and has the following presentation:

$$\langle\, \Gamma, r, s\,\rangle \;=\; \langle\, a, b, r, s \;\mid\; abab^2a^{-2}b^2 \;=\; baba^2b^{-2}a^2 \;=\; 1,$$

$$r^2 \;=\; s^2 \;=\; (rs)^2 \;=\; 1, \quad r\,a\,r^{-1} \;=\; a^{-1}, \quad r\,b\,r^{-1} \;=\; b^{-1}, \quad s\,a\,s^{-1} \;=\; b, \quad s\,b\,s^{-1} \;=\; a\,\rangle$$

$$=\; \langle\, l, r, s \;\mid\; l^2 \;=\; r^2 \;=\; s^2 \;=\; (rs)^2 \;=\; (srl)^2(rlsl)r(lslr)(lrs)^2 l \;=\; 1\,\rangle, \tag{24}$$

where generators $l = ar$, r and s correspond to Fig. 8.

Summarizing above considerations, we get

Lemma 6. *In the above notations we have*

$$\langle\, \Gamma, r, s\,\rangle \;\cong\; \pi_1(\Theta_3(2,2,2))$$

and

$$\mathcal{M}_1/\langle\, R, S\,\rangle \;\cong\; \Theta_3(2,2,2).$$

4H. The orbifolds $\mathbf{L_{5,-1}(2)}$ and $\mathbf{L_{5,-2}(2)}$

It is clear that the group $\pi_1(\Theta_3(2,2,2)) = \langle\, \Gamma, r, s\,\rangle$ with the presentation (24) has three subgroups of index 2, which can be derived as kernels of the three epimorphisms φ_s, φ_{ls} and φ_{lrs} of the group $\langle\, \Gamma, r, s\,\rangle$ to $\mathbb{Z}_2 = \langle\, \alpha \mid \alpha^2 = 1\,\rangle$. These epimorphisms are defined by

$$\varphi_s(s) = \alpha, \quad \varphi_s(l) = \varphi_s(r) = 1, \tag{25}$$

$$\varphi_{lr}(s) = 1, \quad \varphi_{lr}(l) = \varphi_{lr}(r) = \alpha, \tag{26}$$

and

$$\varphi_{lrs}(l) = \varphi_{lrs}(r) = \varphi_{lrs}(s) = \alpha. \tag{27}$$

By the Reidemeister–Schreier method, using formulae (1), (3) and (4), we have:

$$\mathrm{Ker}\varphi_s \cong \langle\, l, r, sls\,\rangle \cong \langle\, ar, r, br\,\rangle \cong \langle\, \Gamma, r\,\rangle, \tag{28}$$

$$\mathrm{Ker}\varphi_{lr} \cong \langle\, lr, s\,\rangle \cong \langle\, a, s\,\rangle \cong \langle\, \Gamma, s\,\rangle, \tag{29}$$

and

$$\mathrm{Ker}\varphi_{lrs} \cong \langle\, lr, rs\,\rangle \cong \langle\, a, rs\,\rangle \cong \langle\, \Gamma, rs\,\rangle. \tag{30}$$

The singular set Θ_3 of the orbifold $\Theta_3(2,2,2)$ is pictured in Fig. 8. Similar to [14], we can correspond to the Θ-curve Θ_3 three constituent knots K_1, K_2 and K_3. The first knot K_1, which we will get after deleting from Θ_3 the edge corresponding to r (see Fig. 8), is a trivial knot $\mathbf{0}$. The second knot K_2, corresponding to the deleting of the edge s, is the torus knot 5_1, that is the two-bridge knot $K(-5/1)$. The third knot K_3, corresponding to the deleting of the edge rs, is the figure-eight knot 4_1, that is the two-bridge knot $K(-5/2)$.

The corresponding orbifolds $\mathbb{H}^3/\mathrm{Ker}\varphi_s = \mathbb{H}^3/\langle\, \Gamma, r\,\rangle$, $\mathbb{H}^3/\mathrm{Ker}\varphi_{lr} = \mathbb{H}^3/\langle\, \Gamma, s\,\rangle$ and $\mathbb{H}^3/\mathrm{Ker}\varphi_{lrs} = \mathbb{H}^3/\langle\, \Gamma, rs\,\rangle$ can be obtained as the two-fold coverings of the orbifold $\Theta_3(2,2,2)$ branched over the constituent knots $K_1 = \mathbf{0}$, $K_2 = -5/1$ and $K_3 = -5/2$, respectively. Therefore, the underlying space of the orbifold $\mathbb{H}^3/\langle\, \Gamma, r\,\rangle$ is the 3-sphere, the underlying space of the orbifold $\mathbb{H}^3/\langle\, \Gamma, s\,\rangle$ is the lens space $L_{5,-1}$ and the underlying space of the orbifold $\mathbb{H}^3/\langle\, \Gamma, rs\,\rangle$ is the lens space $L_{5,-2}$.

The singular set of the orbifold $\mathbb{H}^3/\langle\, \Gamma, r\,\rangle$ is the knot 9_{49} with branch index 2. Hence $\mathbb{H}^3/\mathrm{Ker}\varphi_s = 9_{49}(2)$ (see Lemma 1) and this orbifold was discussed in subsection 4A. The singular sets of the orbifolds $\mathbb{H}^3/\langle\, \Gamma, s\,\rangle$ and $\mathbb{H}^3/\langle\, \Gamma, rs\,\rangle$ have branch index 2, so we denote these orbifolds by $L_{5,-1}(2)$ and $L_{5,-2}(2)$, respectively. M. Sakuma kindly informed the authors, that using his approach, one can describe the singular sets of the orbifolds

$L_{5,-1}(2)$ and $L_{5,-2}(2)$ by lifting them on the 5-fold coverings of $L_{5,-1}$ and $L_{5,-2}$ which are already the 3-sphere.

Using Proposition 2, from above discussions we get the following statements.

Lemma 7. *In the above notations we have*

$$\langle\, \Gamma, s \,\rangle \;\cong\; \pi_1(L_{5,-1}(2))$$

and

$$\mathcal{M}_1/\langle\, S \,\rangle \;\cong\; L_{5,-1}(2).$$

Lemma 8. *In the above notations we have*

$$\langle\, \Gamma, rs \,\rangle \;\cong\; \pi_1(L_{5,-2}(2))$$

and

$$\mathcal{M}_1/\langle\, RS \,\rangle \;\cong\; L_{5,-2}(2).$$

4I. The orbifold $\mathbf{T(2,2,2,2,2,2,3)}$

We remark that the orbifold $\Theta_1(2,2,3)$ has the non-trivial symmetry of order two. This is the involution which interchanges the two vertices of the spatial Θ-curve Θ_1 shown in Fig. 6.

We will prove that one of the liftings of this involution on the universal covering $\mathbb{H}^3$ is the isometry r. Indeed, from (1), (3), (4) and (21), using the notation $p = st$, we get

$$rpr^{-1} \;=\; st^{-1} \;=\; sp^{-1}s^{-1}, \quad r^2 \;=\; 1. \tag{31}$$

Whence, the group $\langle\, \Gamma, s, t \,\rangle$ with the presentation (21) is invariant under the conjugation by r. Therefore r induces the isometry of the orbifold $\Theta_1(2,2,3)$ with the quotient space

$$T(2,2,2,2,2,3) \;=\; \Theta_1(2,2,3)\,/\,\langle\, r \,\rangle \;=\; \langle\, \Gamma, r, s, t \,\rangle.$$

Because r is the involution and the underlying space of the orbifold $\Theta_1(2,2,3)$ is the 3-sphere, the underlying space of the quotient orbifold is also the 3-sphere. The singular set of the quotient orbifold is the knotted 1-skeleton of the tetrahedron, that is a spatial full graph on four vertices K_4 (see Fig. 9). One of its edges has the branch index 3, and all other have the branch indices 2. This explains the notation $T(2,2,2,2,2,3)$ for this quotient orbifold.

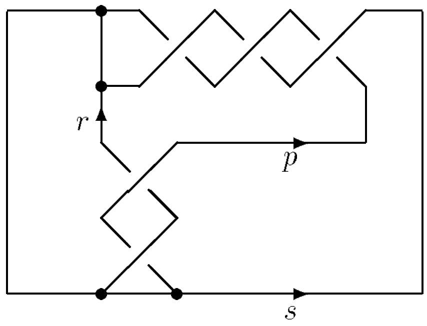


Fig. 9. The spatial graph K_4.

According to the definition of the orbifold $T(2,2,2,2,2,3)$ and by Proposition 2 we get

Lemma 9. *In the above notations we have*

$$\langle\, \Gamma, r, s, t \,\rangle \;\cong\; \pi_1(T(2,2,2,2,2,3))$$

and

$$\mathcal{M}_1/\langle\, R, S, T \,\rangle \;\cong\; T(2,2,2,2,2,3).$$

Using (4), (21), and (31), one can derive the following presentation for the fundamental group of the orbifold $T(2,2,2,2,2,3)$:

$$\langle\, \Gamma, r, s, t \,\rangle \;=\; \langle\, p, r, s \;|\; s^2 \;=\; r^2 \;=\; p^3 \;=\; \left[(sp)^4(sp^{-1})^3\right]^2 \;=\; 1,$$

$$rsr^{-1} \;=\; s. \quad rpr^{-1} \;=\; sp^{-1}s^{-1} \,\rangle.$$

where the generators p, r and s of the group $\pi_1(T(2,2,2,2,2,3))$ are correspond to Fig. 9.

4J. The proof of the theorem

The proof of the theorem is the immediate consequence of Proposition 1 and Lemmata 1 – 9.

References

[1] M. A. Armstrong, *The fundamental group of the orbit space of a discontinuous group,* Proc. Camb. Phil. Soc., **64** (1968), 299–301.

[2] G. Baumslag, J. Morgan, and P. Shalen, *Generalized triangle groups,* Math. Proc. Camb. Phil. Soc., **102** (1987), 25–31.

[3] M. Hagelberg and A. Vesnin, *On the One Family of Hyperbolic Dihedral $\Theta(p/q)$-Orbifolds,* Vychislitelnye sistemy, Novosibirsk, **155** (1996), 15–36 (in Russian).

[4] S. Henry and J. Weeks, *Symmetry groups of hyperbolic knots and links,* Journal of Knot Theory and Its Ramifications, **1** (1992), 185–201.

[5] C. Hodgson and J. Weeks, *Symmetries, isometries and length spectra of closed hyperbolic three-manifolds,* Experimental Mathematics, **3** (1994), 101–113.

[6] K. Kodama and M. Sakuma, *Symmetry groups of prime knots up to 10 crossings,* Knots 90 (ed. A. Kawauchi), de Gruyter, 1992, 323–340.

[7] S. V. Matveev and A. T. Fomenko, *Constant energy surfaces of Hamiltonian systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds,* Russian Math. Surveys, **43** (1988), 1, 3–24.

[8] A. Mednykh and A. Vesnin, *Hyperbolic 3-manifolds as 2-fold coverings according to Montesinos,* preprint, Universität Bielefeld, Bielefeld, 1995, no. **95-010**, 23 pp.

[9] E. Molnar, *On isometries of space forms,* in: Differential Geometry and its Applications (Eger, 1989), North-Holland, Amsterdam, 1992. 509–534.

[10] D. Rolfsen, *Knots and links,* Publish or Perish Inc., Berkely Ca., 1976.

[11] *SnapPea, Version 5/18/92,* A program of J. Weeks for Macintosh to compute hyperbolic structures on 3-manifolds, available by ftp from *geom.umn.edu*.

[12] W. Thurston, *The geometry and topology of 3-manifolds,* Lecture Notes, Princeton University 1980.

[13] J. Weeks, *Hyperbolic structures on 3-manifolds,* Princeton Univ. Ph. D. Thesis, 1985.

[14] K. Wolcott, *The knotting of theta curves and other graphs in S^3,* Geometry and Topology, 1987, Vol. 105, 325–346.