A short proof of Dirac's theorem on the number of edges in chromatically critical graphs

W. A. Deuber
Universität Bielefeld,
Bielefeld, Germany

A. V. Kostochka *
Novosibirsk State University,
630090 Novosibirsk, Russia

H. Sachs
Technische Universität Ilmenau,
D-98684 Ilmenau, Germany

Abstract

A short proof is presented for a theorem of G. A. Dirac (1974) giving a lower bound for the number of edges of a k-critical graph on a given number of vertices and describing all non-complete k-critical graphs with minimum edge excess.

1 Introduction

All graphs considered in this note are finite, undirected and have neither loops nor multiple edges. For a graph G, $V(G)$ is its vertex set and $E(G)$ is its edge set; $D_G(x)$ denotes the degree (valency) of $x \in V(G)$. Througout, k denotes an integer not smaller than 4. G is called k-critical if the chromatic number of G is k and each proper subgraph of G is $(k-1)$-colourable. Let $\mathcal{C}(k)$ denote the set of k-critical graphs. Clearly,

$$\min\{d_G(x) \mid x \in V(G),\, G \in \mathcal{C}(k)\} = k-1.$$

This definition gives rise to the definition of the *degree excess* $\epsilon(G)$ for $G \in \mathcal{C}(k)$:

$$\epsilon(G) := \sum_{x \in V(G)} (d_G(x) - (k-1)) = 2|E(G)| - (k-1)|V(G)|.$$

Let K_m denote the complete graph on m vertices. Making use of Brooks' theorem, we immediately obtain

$$\epsilon(G) = 0 \quad \text{if and only if} \quad G \simeq K_k.$$

Let $\mathcal{C}^0(k) = C(k) \setminus K_k$. G. A. Dirac [2] proved

*Research of this author was supported in part by the grant 96-01-01614 of the Russian Foundation for Fundamental Research, by the Network DIMANET of the European Union and SFB 343 "Diskrete Strukturen in der Mathematik".

Theorem 1 *$\epsilon(G) \geq k-3$ for every $G \in \mathcal{C}^0(k)$.*

Shorter proofs were given by Kronk and Mitchem [5] and Weinstein [7]. The bound is attained by all graphs in the family $\mathcal{D}(k)$ that is defined as follows. Each member D of $\mathcal{D}(k)$ has $2k-1$ vertices partitioned into three subsets: V', V'' and $\{v_1, v_2\}$ where $|V'| = k-2$, $|V''| = k-1$, the subgraphs of D induced by $V' \cup \{v_1\}$, $V' \cup \{v_2\}$, and V'' are complete graphs, each vertex in V'' is adjacent to exactly one of v_1 and v_2, each of v_1 and v_2 is adjacent to some vertex in V'', and there are no other edges.

In 1974, Dirac [3] extended Theorem 1 as follows.

Theorem 2 *For every $G \in \mathcal{C}^0(k) \setminus \mathcal{D}(k)$,*

$$\epsilon(G) \geq \begin{cases} 2, & \text{if } k = 4; \\ k-1, & \text{if } k > 4. \end{cases}$$

The proof is rather long. Mitchem [6] gave a shorter proof based on Kempe chains. The aim of the present note is to give a shorter proof of Theorem 2 based on a different idea.

2 Proof of Theorem 2

Lemma 1 (Dirac [1]) *Each edge cut in a k-critical graph has at least $k-1$ edges.* □

For a simple proof, see [4].

Let $G = (V, E) \in \mathcal{C}^0(k)$ and $x, v, w \in V$ be such that $xv, xw \in E$ and $vw \notin E$. We obtain $H = H(G; x, v, w)$ from G by amalgamating v and w into a new vertex $v * w$ (replacing double edges by single ones). Since the chromatic number of H is at least k, H contains a k-critical subgraph. The set of all k-critical subgraphs of H is denoted by $\mathcal{R}(G; x, v, w)$. Graph $H - v * w$ being a proper subgraph of G, its chromatic number is smaller than k, therefore

$$v * w \in V(G^*) \text{ for every } G^* \in \mathcal{R}(G; x, v, w). \tag{1}$$

If $d_G(x) = k-1$, then $d_H(x) = k-2$ thus

$$d_G(x) = k-1 \text{ implies } x \notin V(G^*) \text{ for every } G^* \in \mathcal{R}(G; x, v, w). \tag{2}$$

From now on, assume that $G = (V, E)$ is a counterexample to the theorem with the minimum number n of vertices. We denote by K_m^- the graph obtained from K_m by deleting one edge. For disjoint $A, B \subset V$, $E_G(A, B)$ is the set of edges in G joining A with B.

Lemma 2 *For each vertex $x \in V$ of degree $k-1$ and any pair $\{v, w\}$ of nonadjacent neighbours of x,*

$$\mathcal{R}(G; x, v, w) \subseteq \mathcal{D}(k) \cup \{K_k\}.$$

PROOF. Let $G^* \in \mathcal{R}(G; x, v, w)$, $V^* = V(G^*)$, $U = V \setminus V^*$. Then

$$2|E| = 2|E(G^*)| + \sum_{u \in U} d_G(u) + |E_G(U, V^*)| + 2|N_G(v) \cap N_G(w) \cap V^*|. \qquad (3)$$

Since, by (2), $U \neq \emptyset$, Lemma 1 gives $|E_G(U, V^*)| \geq k-1$. Thus, if G^* is neither K_k nor a graph in $\mathcal{D}(k)$, we deduce from (3) that

$$\epsilon(G) \geq \epsilon(G^*) + \sum_{u \in U} d_G(u) - (k-1)(|U|+1) + (k-1) \geq \epsilon(G^*),$$

a contradiction. ∇

Note that G should contain a vertex x of degree $k-1$, and some neighbours v and w of x are non-adjacent. Since $G^* - v * w$ is a subgraph of G and deleting any vertex from any graph in $\mathcal{D}(k)$ (as well as from K_k) results in a graph containing K_{k-1}, Lemma 2 implies that G contains a copy of K_{k-1}.

Let $W \subset V$ induce K_{k-1} in G. Since $\epsilon(G) \leq k-2$, some $w_1 \in W$ has $d_G(w_1) = k-1$. Let v be the vertex in $G - W$ adjacent to w_1, $W_1 = \{w \in W \mid (w, v) \in E\}$, $W_2 = W - W_1$. We choose such a W and w_1 so that to make $|W_1|$ as large as possible, and among those with the maximum $|W_1|$ we consider W with the sum of degrees of its vertices as small as possible.

Let also $w_2 \in W_2$ be such that

$$d_G(w_2) = \min\{d_G(w) \mid w \in W_2\}. \qquad (4)$$

Let $G^* \in \mathcal{R}(G; w_1, v, w_2)$, $Z = V(G^*) - v * w_2$, $U = V(G) \setminus (Z \cup \{v, w_2\})$, $W' = W \cap U$, $Q = U \setminus W'$. Denote $j = |W'|$.By (1) and (2), $v * w_2 \in V(G^*)$, $w_1 \in W'$. Hence $1 \leq j \leq k-2$. In view of Lemma 2, it is enough to consider the following cases.

CASE 1. $G^* \simeq K_k$ and $j = k-2$. Then Z induces K_{k-1} in G. Let v be adjacent to exactly s vertices in Z. Then w_2 is adjacent to $k-1-s$ vertices in Z, and, by the choice of w_2, $d_G(w) \geq k-2-s$ for each $w \in W_2$. On the other hand, by the choice of W, $s \leq |W_1|$ and if $s = |W_1|$,

$$\epsilon(G) \geq (d_G(v) - k + 1) + \sum_{z \in Z} (d_G(z) - k + 1) + \sum_{w \in W} (d_G(w) - k + 1) \geq$$

$$\geq (2s - k + 1) + 2|W_2|(k - 2 - s) = k - 3 + 2(k-2-s)^2.$$

Thus, in this case, if $s < k-2$, we have $\epsilon(G) \geq k-1$; otherwise, the subgraph of G induced by $Z \cup W \cup \{v\}$ belongs to $\mathcal{D}(k)$. Now, let $s < |W_1|$. Then

$$(d_G(v) - k + 1) + \sum_{w \in W} (d_G(w) - k + 1)$$

$$\geq (s+|W_1|-k+1)+|W_2|(k-2-s) \geq 2|W_1|-k+(k-1-|W_1|)^2.$$

If $|W_1| \leq k-4$, then the last expression is at least k. If $|W_1| = k-2$, then the subgraph of G induced by $Z \cup W \cup \{v\}$ contains a graph in $\mathcal{D}(k)$. Finally, let $|W_1| = k-3$. Then $(d_G(v)-k+1)+\sum_{w\in W}(d_G(w)-k+1) \geq k-2$, and $\epsilon(G) \leq k-2$ only if each vertex in $Z \cup W_1$ has degree $k-1$, and v is not adjacent to vertices in Q. But then the three neighbours in $V(G)\setminus W$ of the vertex $w_3 \in W_2 - w_2$ should belong to Q and $|E_G(Q, V(G)\setminus Q)| = 3$. By Lemma 1, this is possible only for $k=4$.

CASE 2. $Q \neq \emptyset$. By construction, $|E_G(W', V(G)\setminus U)| \geq j(k-1-j)+1$. By Lemma 1, $|E_G(Q, V(G)\setminus U)| \geq (k-1)-|E_G(Q,W')| \geq k-1-(j-1)-\sum_{w\in W'}(d_G(w)-k+1)$. Hence (3) implies that

$$\epsilon(G) \geq \epsilon(G^*)-(k-1)+|E_G(U, V(G)\setminus U)| + \sum_{w\in U}(d_G(w)-k+1)$$

$$\geq \epsilon(G^*)+(j+1)(k-1-j)+3-k.$$

For $1 \leq j \leq k-3$, we have $(j+1)(k-1-j)+3-k \geq k-1$. If $l = k-2$ and $G^* \in \mathcal{D}(k)$, then $\epsilon(G^*)+(j+1)(k-1-j)+3-k = k-1$. The case $l=k-2$ and $G^* \simeq K_k$ is a subcase of Case 1.

In view of Case 2, below $U = W'$ and

$$|E_G(W', V(G)\setminus W')| = j(k-j) + \sum_{w\in W'}(d_G(w)-k+1). \tag{5}$$

CASE 3. $G^* \in \mathcal{D}(k)$. By (3) and (5),

$$\epsilon(G) \geq \epsilon(G^*)-(k-1)+j(k-j) = j(k-j)-2.$$

For $2 \leq j \leq k-2$, $j(k-j)-2 \geq 2(k-3)$. Hence we need to consider only the case $W' = \{w_1\}$. Then (3) gives

$$\epsilon(G) \geq \epsilon(G^*)-(k-1)+(k-1)+2|N_G(v)\cap N_G(w)\cap Z| = (k-3)+2|N_G(v)\cap N_G(w)\cap Z|.$$

If $|N_G(v)\cap N_G(w)\cap Z| \geq 1$, we are done. In order to have $|N_G(v)\cap N_G(w)\cap Z| = 0$, we need $W_1 = \{w_1\}$. Then, by the choice of W, $\sum_{x\in X} d_G(x) \geq sum_{w\in W} d_G(w)$ for each $X \subset V(G)$ such that $G(X) \simeq K_{k-1}$. For each $H \in \mathcal{D}(k)$ and each clique C in H, $H-C$ contains a K_{k-1}. Hence $G^*-W-v*w_2$ contains a K_{k-1}, and, by the choice of W, $\epsilon(G) \geq 2\sum_{w\in W}(d_G(w)-k+1)$. But

$$\sum_{w\in W} d_G(w) \geq d_G(w_1)+d_G(w_2)+\sum_{w\in W-w_1-w_2}(d_{G^*}(w)+1) \geq (k-1)^2+k-3.$$

Thus, $\epsilon(G) \geq 2(k-3)$.

CASE 4. $G^* \simeq K_k$ and $j=1$. Then $|V(G)| = k+2$, and for nonadjacent vertices v and w_2 we have

$$|N_G(v)\cap N_G(w_2)| \geq |N_G(v)|+|N_G(w_2)|-k \geq k-2.$$

Consequently, by (3),

$$\epsilon(G) \geq 2|N_G(v) \cap N_G(w_2) \cap Z| \geq 2(|N_G(v) \cap N_G(w_2)| - 1) \geq 2(k-3).$$

CASE 5. $G^* \simeq K_k$ and $j = k-3$. In view of Case 4, we may assume $k - 3 \geq 2$. Let $\{w_3\} = W \cap Z$. By (3) and (5),

$$\epsilon(G) \geq 3(k-3) - (k-1) + 2 \sum_{w \in W'} (d_G(w) - k + 1) + 2|N_G(v) \cap N_G(w_2) \cap Z|.$$

For $k \geq 7$, already $3(k-3) - (k-1) \geq k-1$. Thus, we need to consider only the case when $k \in \{5, 6\}$ (then still $3(k-3) - (k-1) \geq k-3$),

$$\sum_{w \in W'} (d_G(w) - k + 1) = 0, \tag{6}$$

and $N_G(v) \cap N_G(w_2) \cap Z = \emptyset$. In particular, $(w_3, v) \notin E$. Consider a graph $G^\dagger \in \mathcal{R}(G; w_1, v, w_3)$. By (6), $W' \cap V(G^\dagger) = \emptyset$. If $w_2 \in V(G^\dagger)$, then $d_G(w_2) \geq d_{G^\dagger}(w_2) + j \geq 2k-4$, $d_G(w_3) \geq d_{G^*}(w_3) + j \geq 2k-4$, and $\epsilon(G) \geq d_G(w_2) + d_G(w_3) - 2(k-1) \geq 2(k-3)$, a contradiction. Thus, $w_2 \notin V(G^\dagger)$. But then $|V(G^\dagger)| < |V(G^*)| = k$, a contradiction, again.

CASE 6. $G^* \simeq K_k$ and $j = 2$. We may assume $W' = \{w_1, w_3\}$. By (3) and (5),

$$\epsilon(G) \geq 2(k-2) - (k-1) + 2(d_G(w_3) - k + 1) + 2|N_G(v) \cap N_G(w_2) \cap Z|.$$

Thus, in order to have $\epsilon(G) \leq k-2$, we need $d_G(w_3) = k-1$ and $N_G(v) \cap N_G(w_2) \cap Z = \emptyset$. Since $|N_G(v) \cup N_G(w_2)| \leq |V(G)| - 2 = k+1$, this is possible only if $k \in \{4, 5\}$. But in view of Cases 1 and 5, we may assume $2 \leq k-4$, which gives $k \geq 6$.

The only case not covered by above is the following.

CASE 7. $G^* \simeq K_k$ and $3 \leq j \leq k-4$. Then $k \geq 7$ and $j(k-j) - (k-1) \geq 3(k-3) - (k-1) = 2k - 8 \geq k-1$. Hence, by (3) and (5), we are done.

References

[1] G. A. Dirac, The structure of k-chromatic graphs, Fund. Math., 40(1953), 42-55.

[2] G. A. Dirac, A theorem of R.L. Brooks and a conjecture of H. Hadwiger, Proc. London Math. Soc. (3) 7 (1957), 161-195.

[3] G. A. Dirac, The number of edges in critical graphs, J. reine u. angew. Math. 268/269 (1974), 150-164.

[4] Lovász, Combinatorial problems and exercises, Akademiai Kiado, Budapest, 1979.

[5] H. V. Kronk and J. Mitchem, On Dirac's generalization of Brooks' theorem, Canad. J. Math., 24(1972), 805-807.

[6] J. Mitchem, A new proof of a theorem of Dirac on the number of edges in critical graphs, J. reine u. angew. Math. 299/300 (1978), 84-91.

[7] J. Weinstein, Excess in critical graphs, J. Combin. Th.(B), 18(1975), 24-31.