When $b \neq 1$ and $a > b$,

$$f_{\widetilde{E}(a,b,0)}(t) = \begin{cases} \widetilde{E}(a,b,a+b-t^3) & \arg t = 0, \\ \widetilde{E}(a-b,b,t^3-b) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(a,a+b,a+b-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(a,b,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(a+b,b,a+b+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}((a/b),b-(a/b),a+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(a,b,a+b+t^3) & \arg t = \pi, \\ \widetilde{E}(a-b,b,-t^3-b) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(a,a+b,a+b+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(a,b,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(a+b,b,a+b-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}((a/b),b-(a/b),a-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

For each $\widetilde{E}(a,b,0)$ of type IV, the map $f_{\widetilde{E}(a,b,0)}$ gives rise to a two-fold branched covering;

$$f_{\widetilde{E}(a,b,0)}(t) = f_{\widetilde{E}(a,b,0)}(-t).$$

Then $(f_{\widetilde{E}(a,b,0)}, D_\epsilon, \mathbb{Z}/2\mathbb{Z})$ gives rise to a local manifold cover of $\mathcal{ML}_{1,1}$ around $\widetilde{E}(a,b,0)$. In this way, the set $\mathcal{ML}_{1,1}$ is regarded as a complex V-manifold.

References

[K] Kodaira, K.: *Complex Manifolds and Deformation of Complex Structures*, Grund. Math. Wiss. 283, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, (1986).

[Mo] Morelli, R.: *A theory of polyhedra*, Adv. in Math. **97** (1993), 1-73.

[Mu] Mumford, D.: *Tata Lectures on Theta I*, Progress in Mathematics vol. 28, Birkhäuser, Boston-Basel-Stuttgart, (1983).

[S] Sah, C-H: *Hilbert's third problem: scissors congruence*, Research Notes in Mathematics 33, Pitman Advanced Publishing Program, San Francisco-London-Melbourne, (1979).

(Yoshitake HASHIMOTO) Department of Mathematics, Faculty of Science, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka 558, Japan

E-mail address: hashimot@sci.osaka-cu.ac.jp

(Kiyoshi OHBA) Department of Mathematics, Faculty of Science, Ochanomizu University, Otsuka 2-1-1, Bunkyo-ku, Tokyo 112, Japan

E-mail address: ohba@math.ocha.ac.jp

When $a = b = 1$,

$$f_{\widetilde{E}(1,1,0)}(t) = \begin{cases} \widetilde{E}(0,1,1-t^3) & \arg t = 0, \\ \widetilde{E}(0,1,t^3-1) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(1,2,2-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(1,1,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(2,1,2+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(0,1,1+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(0,1,1+t^3) & \arg t = \pi, \\ \widetilde{E}(0,1,-t^3-1) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(1,2,2+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(1,1,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(2,1,2-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(0,1,1-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

When $b = 1$ and $a > b$,

$$f_{\widetilde{E}(a,1,0)}(t) = \begin{cases} \widetilde{E}(a,1,a+1-t^3) & \arg t = 0, \\ \widetilde{E}(a-1,1,t^3-1) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(a,a+1,a+1-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(a,1,-t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(a+1,1,a+1+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(0,1,a+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(a,1,a+1+t^3) & \arg t = \pi, \\ \widetilde{E}(a-1,1,-t^3-1) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(a,a+1,a+1+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(a,1,t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(a+1,1,a+1-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(0,1,a-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

When $a < b$ and $a \neq 1$ (Figure 16),

$$f_{\widetilde{E}(a,b,0)}(t) = \begin{cases} \widetilde{E}(a, b, a+b-t^3) & \arg t = 0, \\ \widetilde{E}(a-(b/a), (b/a), t^3-b) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(a, a+b, a+b-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(a, b, -t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(a+b, b, a+b+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(a, b-a, a+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(a, b, a+b+t^3) & \arg t = \pi, \\ \widetilde{E}(a-(b/a), (b/a), -t^3-b) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(a, a+b, a+b+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(a, b, t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(a+b, b, a+b-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(a, b-a, a-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

When $a = 1$ and $a < b$,

$$f_{\widetilde{E}(1,b,0)}(t) = \begin{cases} \widetilde{E}(1, b, 1+b-t^3) & \arg t = 0, \\ \widetilde{E}(0, 1, t^3-b) & 0 < \arg t < \frac{1}{3}\pi, \\ \widetilde{E}(1, 1+b, 1+b-t^3) & \arg t = \frac{1}{3}\pi, \\ \widetilde{E}(1, b, -t^3) & \frac{1}{3}\pi < \arg t < \frac{2}{3}\pi, \\ \widetilde{E}(1+b, b, 1+b+t^3) & \arg t = \frac{2}{3}\pi, \\ \widetilde{E}(1, b-1, 1+t^3) & \frac{2}{3}\pi < \arg t < \pi, \\ \widetilde{E}(1, b, 1+b+t^3) & \arg t = \pi, \\ \widetilde{E}(0, 1, -t^3-b) & \pi < \arg t < \frac{4}{3}\pi, \\ \widetilde{E}(1, 1+b, 1+b+t^3) & \arg t = \frac{4}{3}\pi, \\ \widetilde{E}(1, b, t^3) & \frac{4}{3}\pi < \arg t < \frac{5}{3}\pi, \\ \widetilde{E}(1+b, b, 1+b-t^3) & \arg t = \frac{5}{3}\pi, \\ \widetilde{E}(1, b-1, 1-t^3) & \frac{5}{3}\pi < \arg t < 2\pi. \end{cases}$$

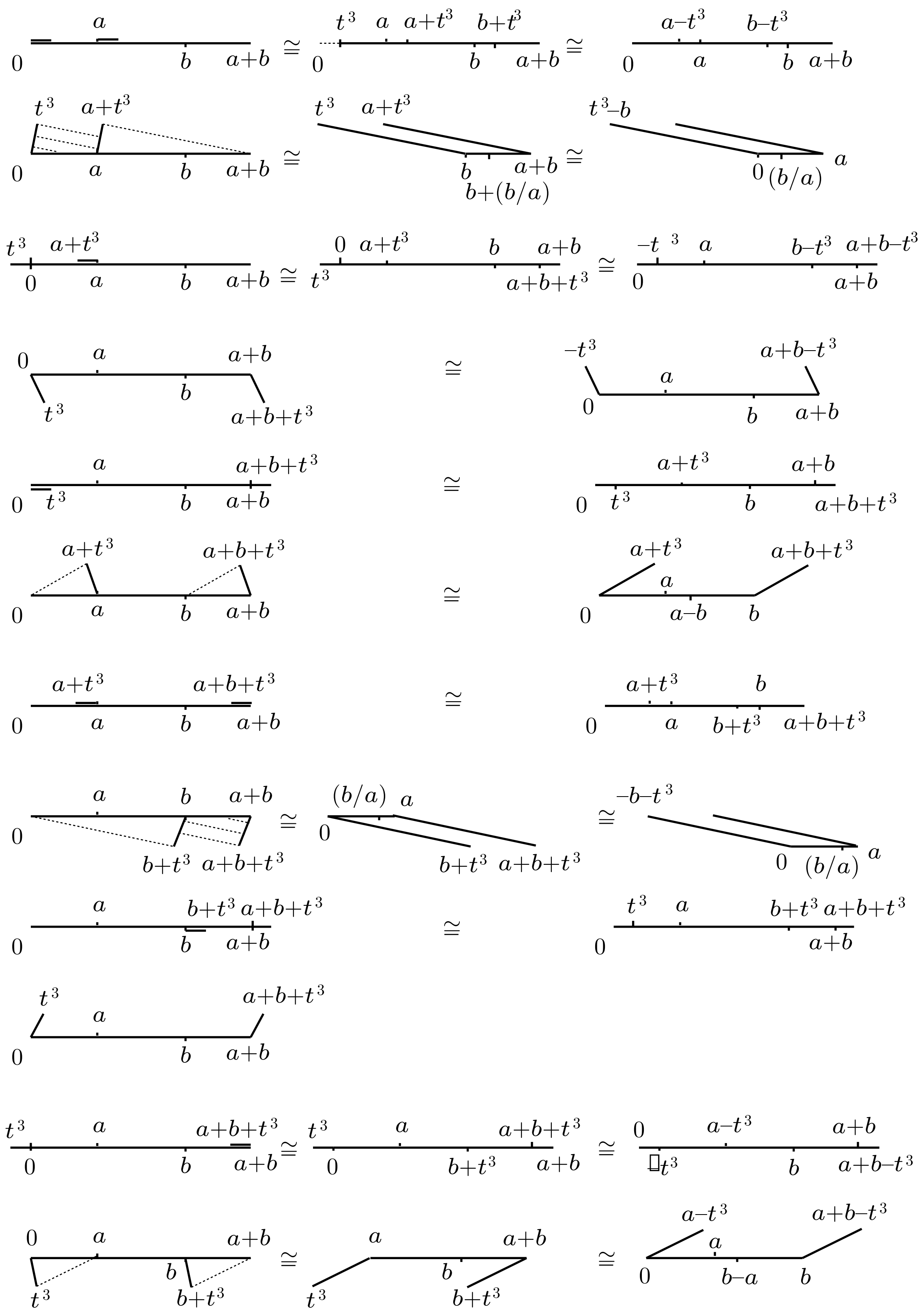


FIGURE 16. $\frac{2}{3}\pi$-deformation of $\widetilde{E}(a,b,0)$ of type IV ($a < b$ and $a \neq 1$)

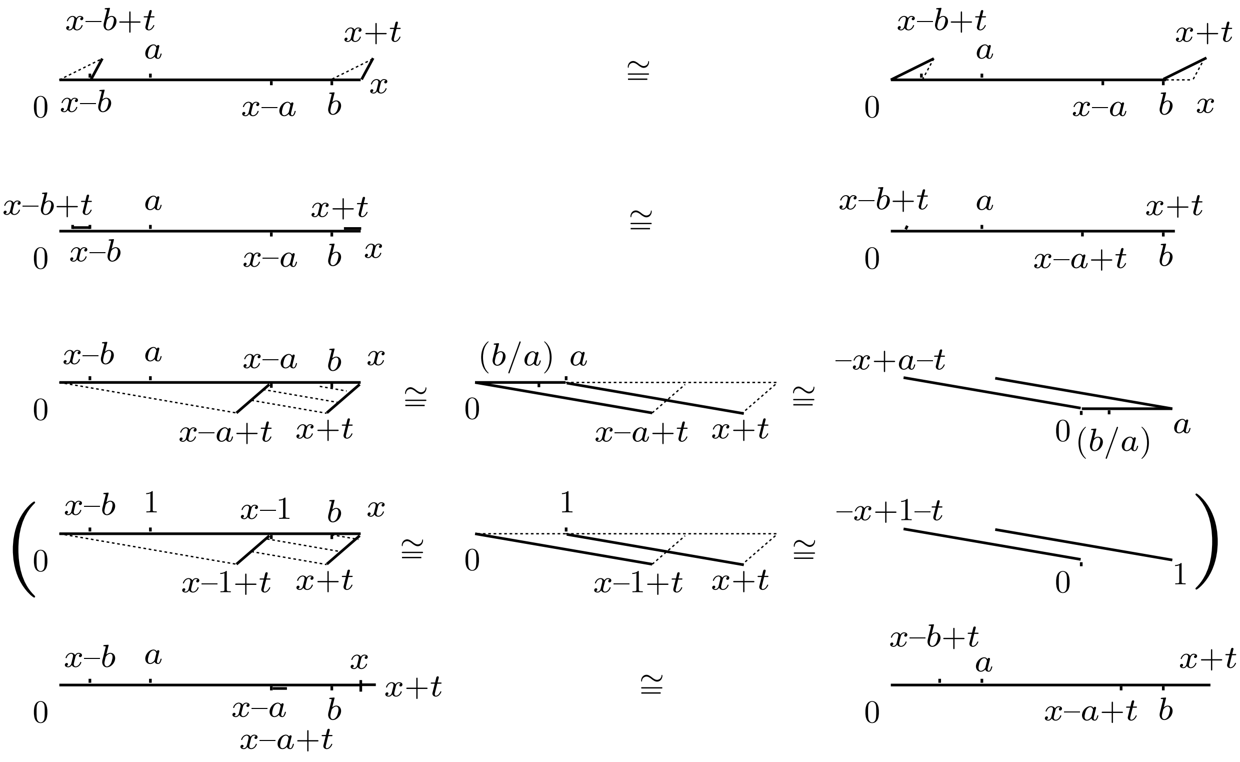


FIGURE 15. π-deformation of $\widetilde{E}(a,b,x)$ of type III ($a<b$)

To $\widetilde{E}(a,b,0)$ of type IV we apply a $\frac{2}{3}\pi$-deformation: Fix a local coordinate $z : \mathcal{U}_{p_0} \to \mathbb{C}$ around p_0, the unique zero of ω_0, such that

$$\zeta = \begin{cases} z^3 & 0 < \arg z < \frac{2}{3}\pi, \\ z^3 + a & \frac{2}{3}\pi < \arg z < \pi, \\ z^3 + a + b & \pi < \arg z < \frac{5}{3}\pi, \\ z^3 + b & \frac{5}{3}\pi < \arg z < 2\pi. \end{cases}$$

Then the segment s'_t is transferred to two parallel line segments with the same length t^3 in ζ-plane.

By cutting and pasting we obtain the following map $f_{\widetilde{E}(a,b,0)} : D_\epsilon \to \mathcal{ML}_{1,1}$:

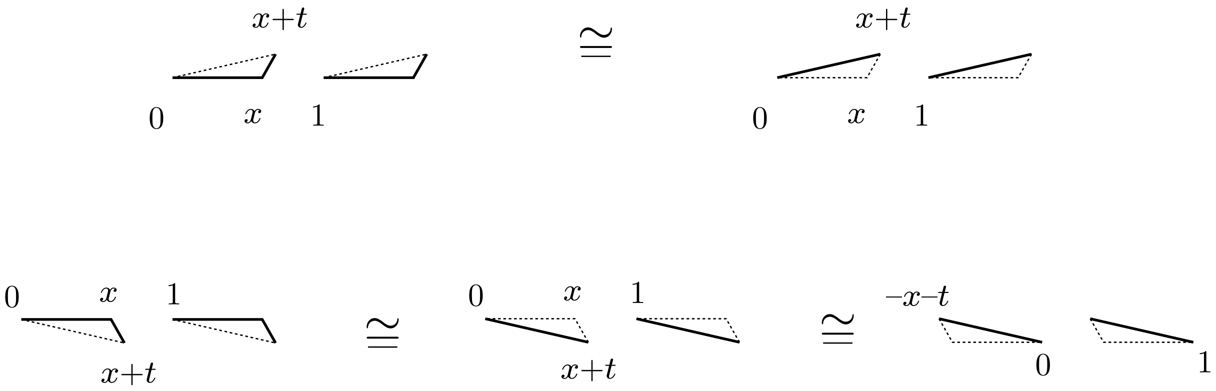


FIGURE 14. π-deformation of $\widetilde{E}(0,1,x \in (0,1))$ of type I

We consider the case when a is less than b. It is easy to see that the fiber F_t of this deformation is equivalent to $\widetilde{E}(a,b,x+t)$ when $\arg t = 0$ or π. When $0 < \arg t < \pi$, we see by cutting and pasting that F_t is equivalent to $\widetilde{E}(a, b-a, x-b+t)$. When $\pi < \arg t < 2\pi$ and $a \neq 1$, we see that F_t is equivalent to $\widetilde{E}(a-(b/a), (b/a), -x+a-t)$ where (b/a) is a non-negative integer such that $0 \leq (b/a) < a$ and $(b/a) \equiv b \bmod a$. Note that (b/a) is equal to 0 if and only if a is equal to 1, and then F_t is equivalent to $\widetilde{E}(0,1,-x+1-t)$ of type I. (See Figure 15.) Accordingly we obtain the following injective map $f_{\widetilde{E}(a,b,x)} : D_\epsilon \to \mathcal{ML}_{1,1}$ when a is less than b, and the inverse mapping of $f_{\widetilde{E}(a,b,x)}$ is a local coordinate of $\mathcal{ML}_{1,1}$ around $\widetilde{E}(a,b,x)$.

$$f_{\widetilde{E}(a,b,x)}(t) = \begin{cases} \widetilde{E}(a,b,x+t) & \arg t = 0 \text{ or } \pi, \\ \widetilde{E}(a,b-a,x-b+t) & 0 < \arg t < \pi, \\ \widetilde{E}(a-(b/a),(b/a),-x+a-t) & \pi < \arg t < 2\pi \ (a \neq 1), \\ \widetilde{E}(0,1,-x+1-t) & \pi < \arg t < 2\pi \ (a = 1). \end{cases}$$

Similarly we obtain the following injective map $f_{\widetilde{E}(a,b,x)} : D_\epsilon \to \mathcal{ML}_{1,1}$ when a is greater than b, and the inverse mapping of $f_{\widetilde{E}(a,b,x)}$ is a local coordinate of $\mathcal{ML}_{1,1}$ around $\widetilde{E}(a,b,x)$.

$$f_{\widetilde{E}(a,b,x)}(t) = \begin{cases} \widetilde{E}(a,b,x+t) & \arg t = 0 \text{ or } \pi, \\ \widetilde{E}(a-b,b,-x+a-t) & \pi < \arg t < 2\pi, \\ \widetilde{E}((a/b),b-(a/b),x-b+t) & \pi < \arg t < 2\pi. \end{cases}$$

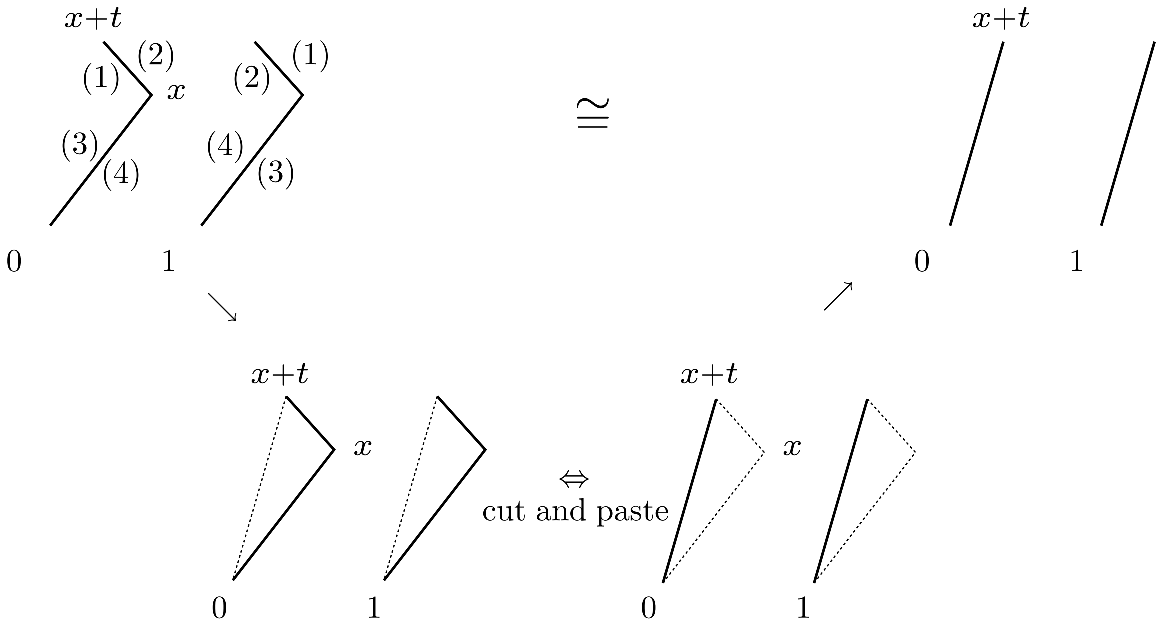


FIGURE 12. π-deformation of $\widetilde{E}(0,1,x \in \mathbf{H})$

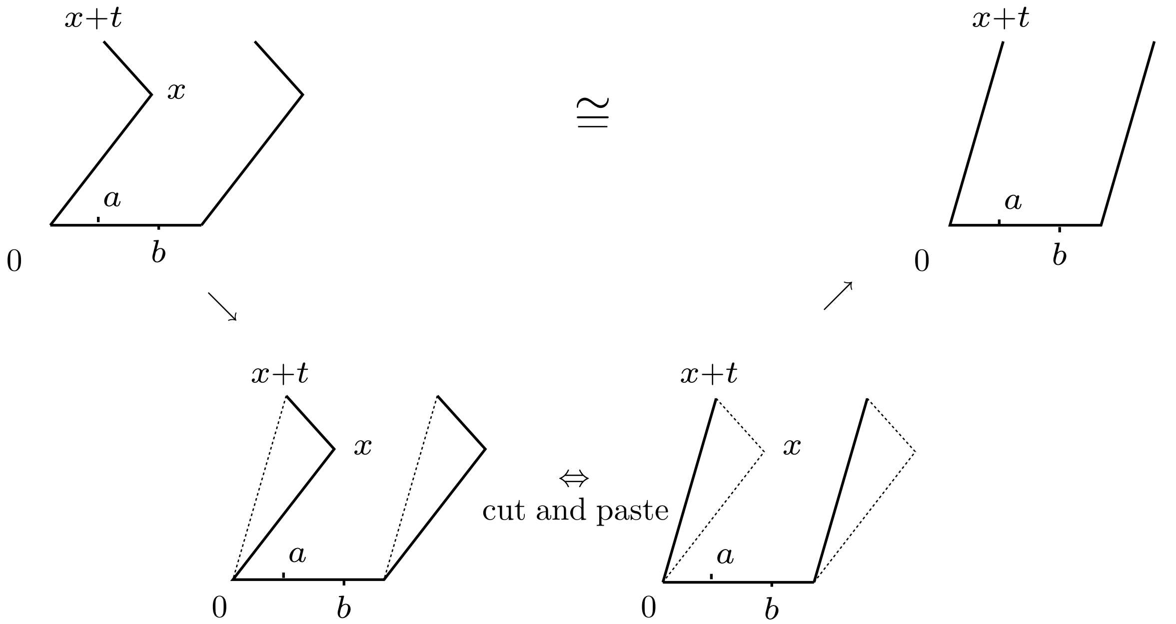


FIGURE 13. π-deformation of $\widetilde{E}(a,b,x)$ of type II

We also apply a π-deformation to $\widetilde{E}(a,b,x)$ of type III in the same way: Fix a local coordinate $z : \mathcal{U}_{p_1} \to \mathbb{C}$ around p_1 such that

$$\zeta = \begin{cases} z^2 + x & -\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi, \\ z^2 + x - a & \frac{1}{2}\pi < \arg z < \pi, \\ z^2 + x - b & \pi < \arg z < \frac{3}{2}\pi. \end{cases}$$

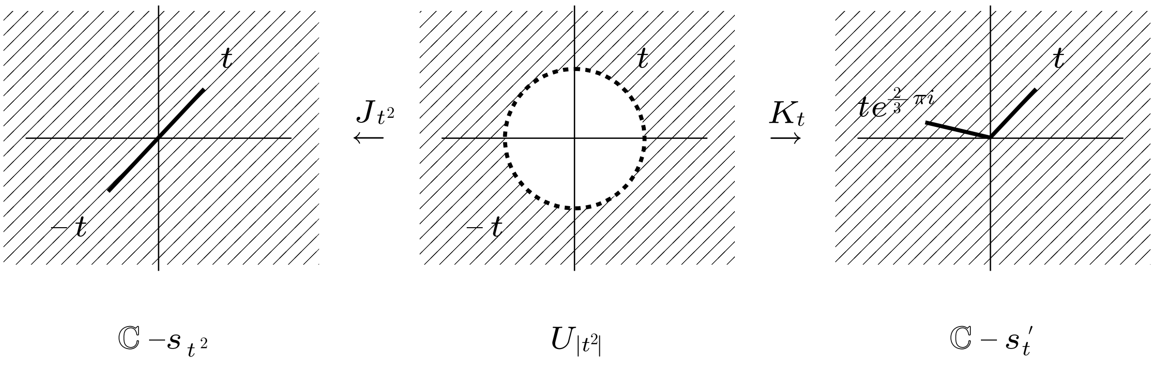


FIGURE 11

$-\ell\sqrt{t}$ on the other side of s_t $(0 < k, \ell \le 1)$.

$\frac{2}{3}\pi$*-deformation of* $(R, (\mathcal{U}_p, z))$*:* For a complex number $t \in D_\epsilon = \{t \in \mathbb{C}; |t| < \epsilon\}$ we paste $R - s'_t$ and $\{z \in \mathbb{C}; |z| < \epsilon\}$ by the attaching map

$$K_t \circ J_{t^2}^{-1} : \{z \in \mathbb{C}; |z| < \epsilon\} - s_{t^2} \longrightarrow \mathcal{U}_p - s'_t \subset R - s'_t.$$

Then we obtain a holomorphic family of Riemann surfaces on D_ϵ. Note that the fiber on the origin is R, and that the fiber R_t on t is obtained as follows: cut R along s'_t and paste by identifying kt and $kte^{\frac{2}{3}\pi i}$ on one side of s'_t and identifying ℓt and $-\ell te^{\frac{2}{3}\pi i}$ on the other side of s'_t $(0 < k, \ell \le 1)$.

We apply a π-deformation to each $\widetilde{E}(a, b, x)$ of type I, II or III.

Let p_1 be the point on $\widetilde{E}(0, 1, x)$ $(x \in \mathbf{H})$ coming from x and $x + 1$, and fix a local coordinate $z : \mathcal{U}_{p_1} \to \mathbb{C}$ around p_1 such that $\zeta = z^2 + x$ or $\zeta = z^2 + x + 1$, where ζ is the global coordinate of $\mathbb{C}$ as before. When we consider the coordinate $(\mathcal{U}_{p_1}, z)$, the line segment s_t is transferred onto two parallel line segments with the same length t in ζ-plane: one is from x to $x + t$ and the other is from $x + 1$ to $x + 1 + t$. The fiber F_t on $t \in D_\epsilon$ is obtained from $\widetilde{E}(0, 1, x)$ by P-cutting and P-pasting along the two line segments; F_t is equivalent to $\widetilde{E}(0, 1, x + t)$. (See Figure 12. The numbers (1), ... ,(4) indicate where to paste.) Accordingly we can regard $x \in \mathbf{H}$ as a local coordinate when $a = 0$ and $b = 1$.

Applying a π-deformation to $\widetilde{E}(a, b, x)$ of type II in the same way, we see that x is also a local coordinate if we fix a and b. (See Figure 13.)

We apply a π-deformation to $\widetilde{E}(0, 1, x \in (0, 1))$ of type I in the same way, and it follows that the fiber F_t is equivalent to $\widetilde{E}(0, 1, x+t)$ when $0 \le \arg t \le \pi$, and that F_t is equivalent to $\widetilde{E}(0, 1, -x - t)$ when $\pi < \arg t < 2\pi$ (we use the rotation of angle π). (See Figure 14.)

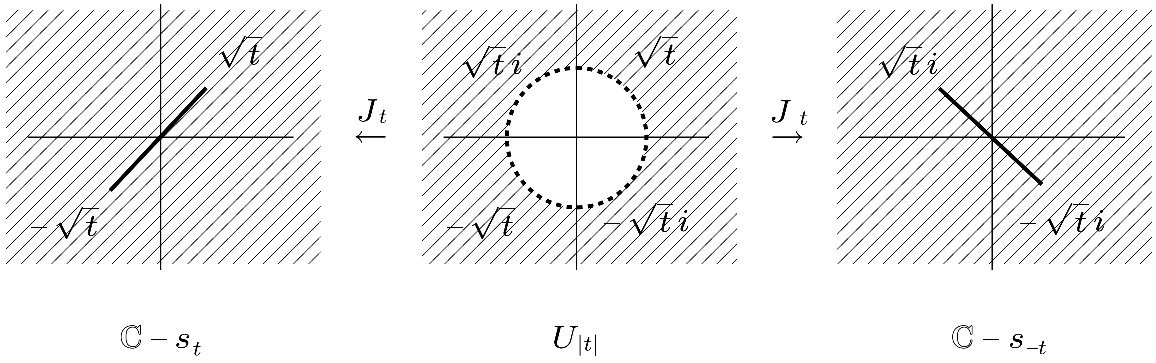


FIGURE 10

Hence we obtain a biholomorphic map $J_t \circ J_{-t}^{-1}$ from $\mathbb{C} - s_{-t}$ to $\mathbb{C} - s_t$. Note that the map $J_t \circ J_{-t}^{-1}$ is parametrized by t holomorphically. (See Figure 10.)

On the other hand, let s'_1 be the union of two line segments; the line segment from 0 to 1 and the line segment from 0 to $e^{\frac{2}{3}\pi i}$. We denote by K the biholomorphic map from $U_1 = \{z \in \mathbb{C}; |z| > 1\}$ to $\mathbb{C} - s'_1$ such that $K(1) = 0$. (There exists uniquely such a biholomorphic map due to Riemann's mapping theorem.) Let t be a complex number as above. We also denote by t the automorphism of $\mathbb{C}$ defined by multiplication by t. Set

$$s'_t = t(s'_1).$$

Then $K_t := t \circ K \circ t^{-1}$ is a biholomorphic map from $U_{|t|}$ to $\mathbb{C} - s'_t$. The map K_t is parametrized by t holomorphically even at $t = 0$. We consider the biholomorphic map $K_t \circ J_{t^2}^{-1}$ from $\mathbb{C} - s_{t^2}$ to $\mathbb{C} - s'_t$, which is also parametrized by t holomorphically. (See Figure 11.)

Let p be a point on a Riemann surface R, and fix a local coordinate $z : \mathcal{U}_p \to \mathbb{C}$ $(z(p) = 0)$ around p. The subset $\mathcal{U}_p$ of R is also regarded as a subset of $\mathbb{C}$, and then both sets s_t and s'_t are also subsets of $\mathcal{U}_p$ if t is sufficiently small. We can choose a sufficiently small positive number ϵ such that

$$(J_t \circ J_{-t}^{-1})(\{z \in \mathbb{C}; |z| < \epsilon\} - s_{-t}) \subset \mathcal{U}_p,$$
$$(K_t \circ J_{t^2}^{-1})(\{z \in \mathbb{C}; |z| < \epsilon\} - s_{t^2}) \subset \mathcal{U}_p.$$

for any $t \in U_p$.

π-deformation of $(R, (\mathcal{U}_p, z))$: For a complex number $t \in D_\epsilon = \{t \in \mathbb{C}; |t| < \epsilon\}$ we paste $R - s_t$ and $\{z \in \mathbb{C}; |z| < \epsilon\}$ by the attaching map

$$J_t \circ J_{-t}^{-1} : \{z \in \mathbb{C}; |z| < \epsilon\} - s_{-t} \longrightarrow \mathcal{U}_p - s_t \subset R - s_t.$$

Then we obtain a holomorphic family of Riemann surfaces on D_ϵ. Note that the fiber on the origin is R, and that the fiber R_t on t is obtained as follows: cut R along s_t and paste by identifying $k\sqrt{t}$ and $-k\sqrt{t}$ on one side of s_t and identifying $\ell\sqrt{t}$ and

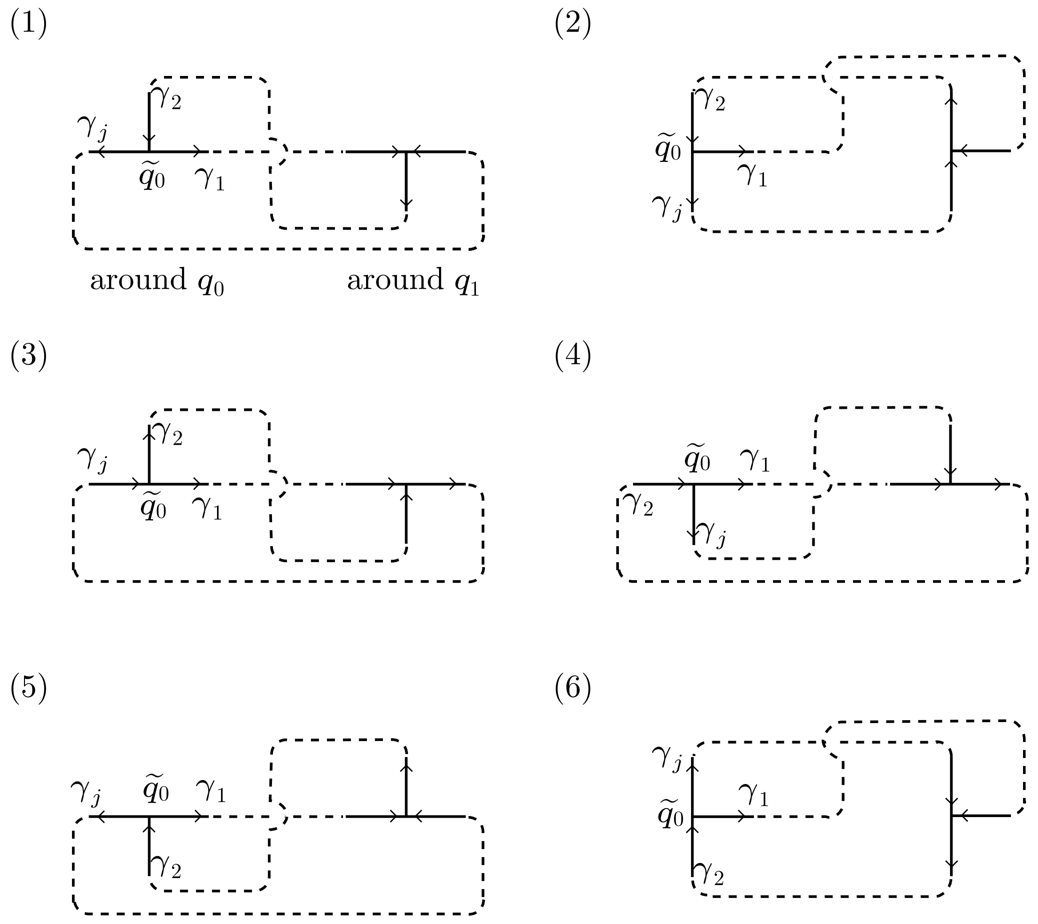


FIGURE 9

We will consider the complex structure of $\mathcal{ML}_{1,1}$ in the next section.

5. Complex structure of $\mathcal{ML}_{1,1}$

When we consider the forgetful map from the set $\mathcal{ML}_{1,1}$ of once punctured elliptic curves with Lagrangian sublattices to the moduli space $\mathcal{M}_{1,1}$ of once punctured elliptic curves, it should be a holomorphic map; $\mathcal{ML}_{1,1}$ should be a complex V-manifold whose complex structure is induced from $\mathcal{M}_{1,1}$ by the forgetful map.

We consider the complex structure of $\mathcal{ML}_{1,1}$ by using the description in Theorem 3: We give a local coordinate of $\mathcal{ML}_{1,1}$ around each once punctured elliptic curve with a Lagrangian sublattice $\widetilde{E}(a, b, x)$. For this purpose we first prepare two methods of deforming complex structures of Riemann surfaces.

Let t be a complex number and let s_t be the line segment between $\sqrt{t}$ and $-\sqrt{t}$ in $\mathbb{C}$. Set

$$U_{|t|} := \{z \in \mathbb{C}; |z| > |\sqrt{t}|\}.$$

Then the following map J_t is biholomorphic:

$$J_t : U_{|t|} \ni z \longmapsto \frac{1}{2}(z + \frac{t}{z}) \in \mathbb{C} - s_t.$$

In the case (1), we shall show that the homology classes represented by the simple loops (γ_1, γ_2) and (γ_j, γ_2) form a basis of $H_1(E;\mathbb{Z})$; If the homology classes $[(\gamma_1, \gamma_2)]$ and $-[(\gamma_j, \gamma_2)]$ represent the same homology class, then we get a loop $\gamma'_{(0)}$ representing a generator of Λ by removing γ_1, γ_2, and γ_j from $\gamma_{(0)}$ and placing $-\gamma_2$ instead, and $\gamma'_{(0)}$ is shorter than $\gamma_{(0)}$. This is a contradiction. Hence from Facts (1) and (2) it follows that the intersection number $[(\gamma_1, \gamma_2)] \cdot [(\gamma_j, \gamma_2)]$ is not equal to zero. On the other hand, (γ_1, γ_2) intersects (γ_j, γ_2) only at γ_2. Therefore the pair $([(\gamma_1, \gamma_2)], [(\gamma_j, \gamma_2)])$ is a basis of $H_1(E;\mathbb{Z})$. In the same way, we obtain the fact that the pair $([(\gamma_1, \gamma_2)], [(\gamma_1, \gamma_j)])$ is a basis of $H_1(E;\mathbb{Z})$ in the case (2), and that the pair $([(\gamma_1, \gamma_2)], [\gamma_j])$ is a basis of $H_1(E;\mathbb{Z})$ in the case (3).

Perform the P-cutting with E along γ_1, γ_2, and γ_j in each case, and then denote by $\bar{E}$ the resulting Riemann polygon. We obtain a holomorphic map Φ from $\bar{E}$ to $\mathbb{C}P_1$

$$\Phi(q) = \int_{\tilde{q}_0}^{q} \omega_\Lambda$$

when we choose a base point $\tilde{q}_0$ in $\bar{E}$, because we obtain a basis of $H_1(E;\mathbb{Z})$ from γ_1, γ_2, and γ_j.

In the case (1), let $\tilde{q}_0$ be the point in the boundary $\partial\bar{E}$ which is the initial point of γ_1 and is also the end point of γ_2. Then we get the image $\Phi(\bar{E})$ as in the middle figure in Figure 8. It is now easy to modify $\bar{E}$ by P-cutting and P-pasting from $\Phi(\bar{E})$ such that we can obtain a Riemann polygon of type II. (See Figure 8.) Note that a' and b' are relatively prime integers where b' is the integral of ω_Λ along (γ_j, γ_2).

In the similar way, we obtain a Riemann polygon of type II in the cases (2) and (3), and omit the explanation.

(ii-b) When the integral of ω_Λ along γ_1 is a real number, there also exists an element γ_j in $\gamma_{(0)}$ different from γ_1 and γ_2. We obtain the fact that γ_j joins q_0 with q_1 as follows: Suppose γ_j is a Λ-edge from q_0 to q_0 itself. We may assume

$$(\int_{\gamma_1+\gamma_2} \omega_\Lambda) \cdot (\int_{\gamma_j} \omega_\Lambda) < 0.$$

The simple loop γ_j intersects the simple loop (γ_1, γ_2) at q_0 with intersection number zero because the directions of γ_1, γ_2, and γ_j can be described as in the following figure. Hence it follows from Fact (2) that the homology class $[\gamma_1 + \gamma_2 + \gamma_j]$ vanishes. This contradicts the definition of $\gamma_{(0)}$.

Now we may assume that γ_j is an edge from q_0 to q_1. Then the two homology classes $[(\gamma_1, \gamma_2)]$ and $[(\gamma_j, \gamma_2)]$ form a basis of $H_1(E;\mathbb{Z})$. We can also describe the directions of the three Λ-edges γ_1, γ_2, and γ_j around q_0 and q_1 as in Figure 9.

Perform the P-cutting with E along γ_1, γ_2, and γ_j, and denote by $\bar{E}$ the resulting Riemann polygon. We choose the point $\tilde{q}_0$ at the vertex on $\partial\bar{E}$ where the boundary is smooth. (See Figure 9.) We choose the sign of ω_Λ such that Φ maps each point to a positive number around $\tilde{q}_0$. Then the map Φ gives rise to an isomorphism between (E, p, Λ) and $\widetilde{E}(a, b, x)$ of type III for some (a, b, x). We have completed the proof of Theorem 3. ☐

We denote by $\mathcal{ML}_{1,1}$ the set of once punctured elliptic curves with Lagrangian sublattices;

$$\mathcal{ML}_{1,1} = \{\widetilde{E}(a, b, x)\}.$$

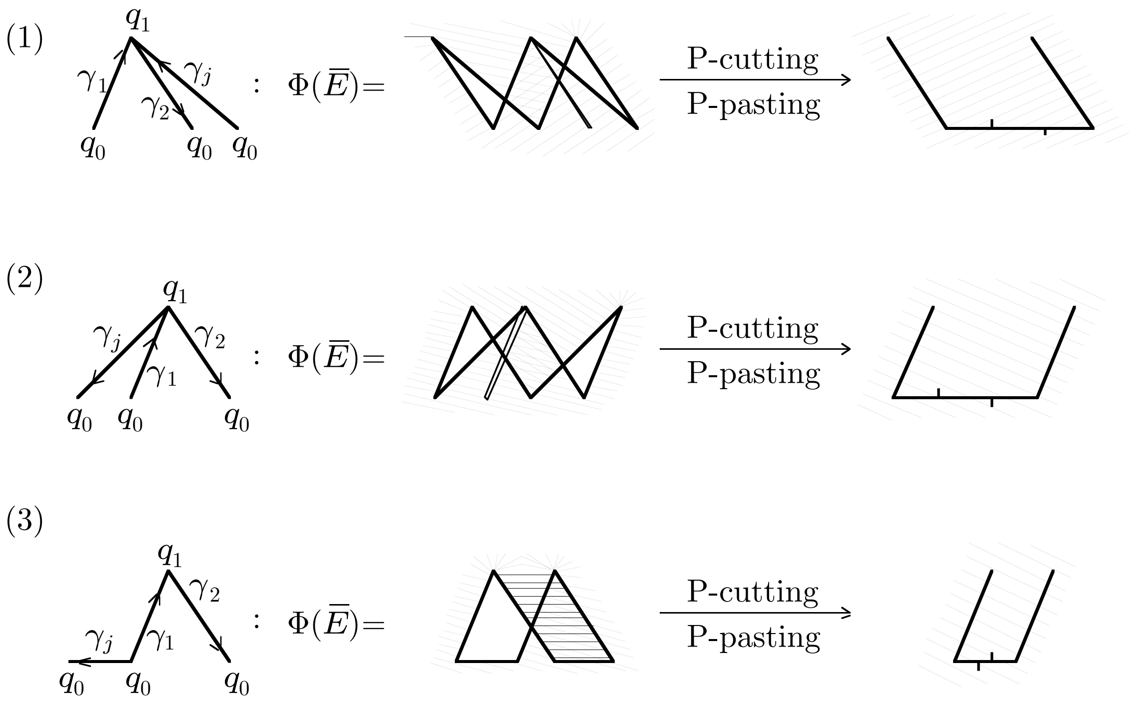


Figure 8

Furthermore we obtain the following inequality, and this is a contradiction.

$$\begin{aligned}\int_{\gamma_\mu+\gamma_\nu}\omega_\Lambda &= \int_{\gamma_\mu+\gamma_2}\omega_\Lambda \ - \int_{\gamma_2+\gamma_1}\omega_\Lambda \ + \int_{\gamma_1+\gamma_\nu}\gamma_\Lambda \\ &\geq \int_{\gamma_\mu+\gamma_2}\omega_\Lambda \\ &\geq 0.\end{aligned}$$

Therefore the following three cases occur:
(1) there is γ_j from q_0 to q_1 such that

$$\int_{\gamma_2+\gamma_j}\omega_\Lambda < 0.$$

(2) there is γ_j from q_1 to q_0 such that

$$\int_{\gamma_1+\gamma_j}\omega_\Lambda < 0.$$

(3) there is γ_j from q_0 to q_0 such that

$$\int_{\gamma_j}\omega_\Lambda < 0.$$

The figures on the left-hand side of Figure 8 indicate the images of γ_1, γ_2, and γ_j by the integral of ω_Λ in each case above.

Perform the P-cutting with E along γ_1 and γ_i, denote by $\bar{E}$ the resulting Riemann polygon, and set the signature of ω_Λ such that

$$\mathrm{Im}(\int_{\gamma_1} \omega_\Lambda) > 0 \quad \text{or} \quad \int_{\gamma_1} \omega_\Lambda > 0.$$

By choosing a point $\tilde{q}_0$ in $\partial \bar{E}$ suitably, we obtain a holomorphic map Φ from $\bar{E}$ to $\mathbb{C}P_1$:

$$\Phi(q) = \int_{\tilde{q}_0}^{q} \omega_\Lambda.$$

If we set

$$x = \int_{\gamma_1} \omega_\Lambda,$$

then Φ gives rise to an isomorphism between (E, p, Λ) and $\widetilde{E}(0, 1, x)$. (Igeta-construction)

(ii) Suppose

$$\int_{\gamma_i} \omega_\Lambda \neq - \int_{\gamma_1} \omega_\Lambda$$

for any γ_i, Λ-edge from q_1 to q_0 in $\gamma_{(0)}$.

(ii-a) When the integral of ω_Λ along γ_1 is not a real number, we choose the sign of ω_Λ such that

$$\mathrm{Im}(\int_{\gamma_1} \omega_\Lambda) > 0.$$

We may assume without loss of generality that there exists a Λ-edge γ_i from q_1 to q_0 in $\gamma_{(0)}$ such that the integral of ω_Λ along $\gamma_1 + \gamma_i$ is a positive integer. We may further assume by reordering that the integral of ω_Λ along $\gamma_1 + \gamma_2$ is minimal among such γ_i's.

$$\int_{\gamma_1+\gamma_i} \omega_\Lambda \geq \int_{\gamma_1+\gamma_2} \omega_\Lambda \geq 0$$

Since the integral of ω_Λ along $\gamma_{(0)}$ vanishes, we see from the inequality above that there occur the following two cases:

1: There exists in $\gamma_{(0)}$ a Λ-edge γ_j from q_0 to q_0 such that

$$\int_{\gamma_j} \omega_\Lambda < 0.$$

2: There exist in $\gamma_{(0)}$ Λ-edges γ_μ and γ_ν (γ_μ is from q_0 to q_1, γ_ν from q_1 to q_0) such that

$$\int_{\gamma_\mu+\gamma_\nu} \omega_\Lambda < 0.$$

Suppose the following two inequalities hold for any γ_μ from q_0 to q_1 and for any γ_ν from q_1 to q_0.

$$\int_{\gamma_2+\gamma_\mu} \omega_\Lambda \geq 0, \quad \int_{\gamma_1+\gamma_\nu} \omega_\Lambda \geq 0$$

Then we obtain the following inequality from the second inequality above and the choice of γ_2.

$$\int_{\gamma_1+\gamma_\nu} \omega_\Lambda \geq \int_{\gamma_1+\gamma_2} \omega_\Lambda$$

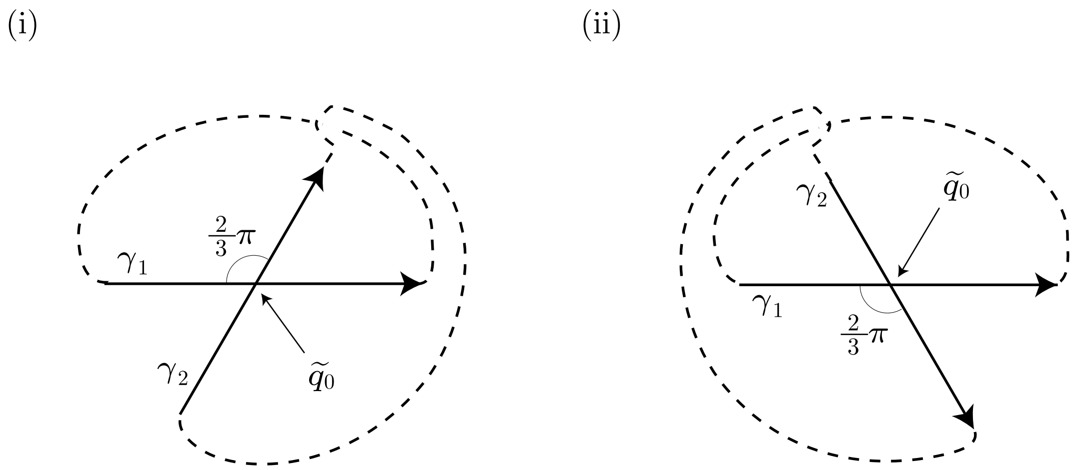


FIGURE 7

In case (i), set

$$a = \int_{\gamma_1} \omega_\Lambda,$$
$$b = -\int_{\gamma_2} \omega_\Lambda.$$

In case (ii), set

$$a = -\int_{\gamma_2} \omega_\Lambda,$$
$$b = \int_{\gamma_1} \omega_\Lambda.$$

Then the integers a and b are relatively prime because the pair $([\gamma_1], [\gamma_2])$ is a basis of $H_1(E;\mathbb{Z})$, and then Φ gives rise to an isomorphism between (E, p, Λ) and $\widetilde{E}(a, b, 0)$.

We next consider the other case: $\omega_\Lambda^{-1}(0)$ consists of two distinct points q_0 and q_1. In this case, from Lemma 2 there exists at least one Λ-edge from q_0 to q_1 and at least one Λ-edge from q_1 to q_0 in $\gamma_{(0)}$. We may assume that γ_1 is a Λ-edge from q_0 to q_1. We may further assume from Lemma 3 that $\gamma_{(0)}$ does not contain Λ-edges from q_1 to q_1 itself. Since γ_1 is a Λ-edge from q_0 to q_1, the Λ-edge γ_2 is from q_1 to q_0.

(i) Suppose there exists a Λ-edge γ_i from q_1 to q_0 in $\gamma_{(0)}$ such that

$$\int_{\gamma_i} \omega_\Lambda = -\int_{\gamma_1} \omega_\Lambda.$$

Because $\gamma_{(0)}$ has the smallest length, the simple loop (γ_1, γ_i) is non-trivial, and hence primitive (Fact (1)). Therefore we obtain $\gamma_{(0)} = (\gamma_1, \gamma_i)$. (Note that a loop γ representing a generator of Λ is characterized by two properties; one is that γ represents a primitive homology class, and the other is that the integral of ω_Λ along γ vanishes.)

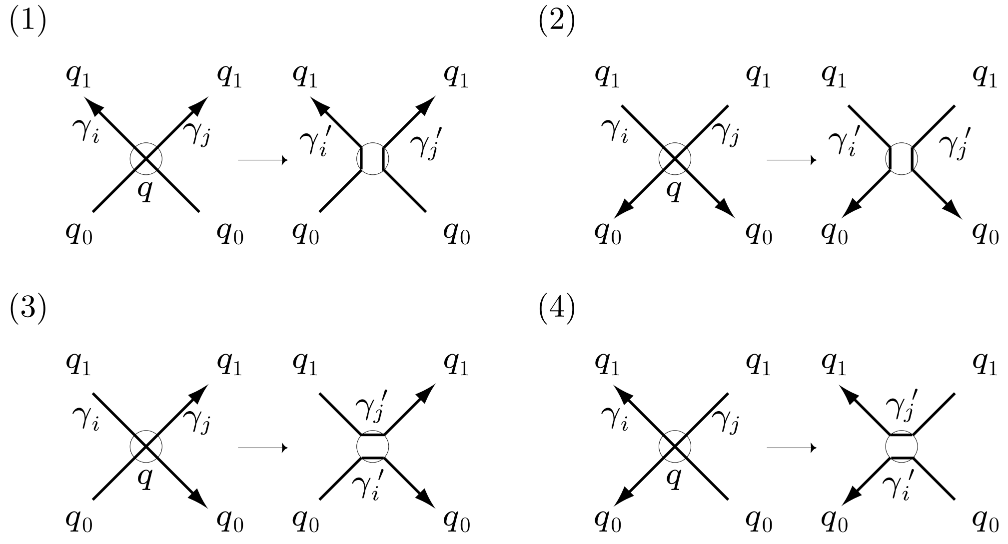


FIGURE 6

Λ-edges;

$$\gamma_{(0)} = (\gamma_1, \dots, \gamma_l).$$

We first consider the case where $\omega_\Lambda^{-1}(0)$ consists of only one point q_0. In this case, each element γ_k of $\gamma_{(0)}$ is an oriented simple loop on E, and the integral of ω_Λ along γ_k is a non-zero integer. Hence it follows from Fact (2) that γ_k represents a primitive class in $H_1(E;\mathbb{Z})$. We shall choose the sign of ω_Λ such that

$$\int_{\gamma_1} \omega_\Lambda > 0.$$

Since the integral of ω_Λ along $\gamma_{(0)}$ is equal to zero, we may further assume by reordering

$$\int_{\gamma_2} \omega_\Lambda < 0.$$

The loops γ_1 and $-\gamma_2$ represent different homology classes, because $\gamma_{(0)}$ has the smallest length. On the other hand, they intersect each other only at q_0. Hence from Facts (1) and (2) it follows that their intersection number at q_0 is equal to 1 or -1 and that the pair $([\gamma_1], [\gamma_2])$ is a basis of $H_1(E;\mathbb{Z})$.

Therefore we can describe the direction as in (i) or (ii) in Figure 7.

Perform the P-cuttings with E along γ_1 and γ_2, and denote by $\bar{E}$ the resulting Riemann polygon. In each case in Figure 7, by choosing a point $\tilde{q}_0$ in $\partial\bar{E}$ suitably (see Figure 7), we obtain a holomorphic map Φ from $\bar{E}$ to $\mathbb{C}P_1$:

$$\Phi(q) = \int_{\tilde{q}_0}^{q} \omega_\Lambda.$$

Fact (1): The homology class represented by a simple loop is trivial or primitive.

Fact (2): If γ and γ' are simple loops such that their homology classes $[\gamma]$, $[\gamma']$ are primitive and their intersection number $[\gamma]\cdot[\gamma']$ is equal to zero, then $[\gamma]=\pm[\gamma']$.

Fact (3): If γ and γ' are simple loops such that their intersection number is equal to ± 1, then the pair $([\gamma],[\gamma'])$ is a basis of $H_1(E;\mathbb{Z})$.

Since the integral value of ω_Λ along any Λ-edge does not vanish, the following lemma follows from Fact (2) and the definition of Λ-edge immediately, and we omit its proof. (Note that two distinct parallel Λ-edges have no common points except the points in $\omega_\Lambda^{-1}(0)$.)

Lemma 3. *Suppose $\omega_\Lambda^{-1}(0)$ consists of two distinct points q_0, q_1. If γ_1 (resp. γ_2) is a Λ-edge from q_0 to q_0 itself (resp. q_1 to q_1 itself), then $[\gamma_1]=\pm[\gamma_2]\in H_1(E;\mathbb{Z})$.*

Let $\gamma_{(0)}$ be an oriented Λ-edge-loop representing a generator of Λ and having the smallest length. When we consider $\gamma_{(0)}$ as an ordered set of oriented Λ-edges, we may obtain another loop $\gamma'_{(0)}$ representing the same homology class by reordering the oriented Λ-edges suitably. (We assume that each Λ-edge inherits its orientation from $\gamma_{(0)}$.) We obtain the following lemma about $\gamma_{(0)}$.

Lemma 4. *The image of $\gamma_{(0)}$ consists of Λ-edges which have no intersection with one another except $\omega^{-1}(0)$.*

Proof. When ω_Λ has a zero of order 2, the lemma is immediate because all the Λ-edges are parallel. We thus consider the case where ω_Λ has two points q_0, q_1 as zeros.

We consider $\gamma_{(0)}$ as an ordered set $(\gamma_1,\gamma_2,\dots,\gamma_l)$ of oriented Λ-edges. Suppose Λ-edges γ_i and γ_j $(1\le i\le j\le l)$ have an intersection at a point $q\notin\omega_\Lambda^{-1}(0)$. Then both γ_i and γ_j are Λ-edges between q_0 and q_1, and they intersect each other transversely.

We prepare paths γ'_i and γ'_j modifying γ_i and γ_j around q as in Figure 6.

We replace γ_i, γ_j with γ'_i, γ'_j respectively, and set

$$\gamma'_{(0)}=(\gamma_1,\dots,\gamma_i,\dots,\gamma_j,\dots,\gamma_l).$$

Then the cycle $\gamma'_{(0)}$ represents the homology class $[\gamma_{(0)}]$.

In the case (1) or (2), we can consider $\gamma'_{(0)}$ as a loop, and the length $\|\gamma'_{(0)}\|$ is smaller than $\|\gamma_{(0)}\|$. This contradicts the definition of $\gamma_{(0)}$.

In the case (3) or (4), it is necessary to reorder the elements of $\gamma'_{(0)}$. If there still exist Λ-edges between q_0 and q_1 in $\gamma'_{(0)}$, then it is easy to obtain a loop by reordering the paths and loops $\{\gamma_1,\dots,\gamma'_i,\dots,\gamma'_j,\dots,\gamma_k\}$ whose length is smaller than $\|\gamma_{(0)}\|$. This also contradicts the definition of $\gamma_{(0)}$. If there exist no more Λ-edges between q_0 and q_1 in $\gamma'_{(0)}$, we further replace the loop γ'_i by Λ-edges from q_0 to q_0 itself, the loop γ'_j by Λ-edges from q_1 to q_1 itself preserving the homology class. Moreover, by using Lemma 3, we can replace all the Λ-edges from q_1 to q_1 itself by Λ-edges from q_0 to q_0 preserving the homology class. Then we obtain a cycle representing the cohomology class $[\gamma_{(0)}]$, and consisting of only Λ-edges from q_0 to q_0 itself. This cycle can be considered as a loop, and this contradicts Lemma 2. ☐

Proof of Theorem 3. Let $\gamma_{(0)}$ be an oriented Λ-edge-loop representing a generator of Λ and having the smallest length as before. We describe $\gamma_{(0)}$ as an ordered set of oriented

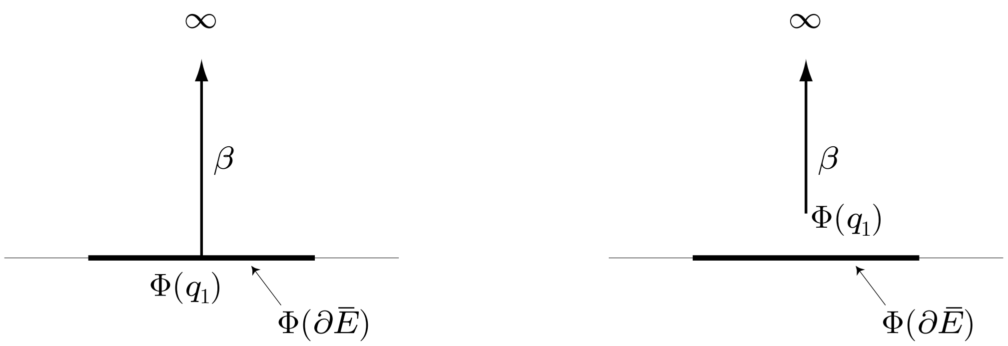


FIGURE 5

We perform the P-cutting along γ with E and denote by $\bar{E}$ the resulting Riemann polygon. Let $\tilde{q}_0$ be a point in $\bar{E}$ corresponding to q_0. The holomorphic map

$$\Phi(q) = \int_{q_0}^{q} \omega_\Lambda$$

from $E-(\gamma(S^1)\cup\{p\})$ to $\mathbb{C}$ extends to one from $\bar{E}$ to $\mathbb{C}P_1 (= \mathbb{C}\cup\{p_\infty\})$, and Φ maps a local neighborhood of any point which is not in $\omega_\Lambda^{-1}(0)$ to a subset of $\mathbb{C}P_1$ biholomorphically. Moreover, Φ maps the boundary $\partial\bar{E}$ of $\bar{E}$ into the real axis in $\mathbb{C}P_1$, because ω_Λ only has real periods.

Then we shall find two paths from q_1 to p

$$\alpha_1, \alpha_2 : \ [0,1] \longrightarrow \bar{E}$$

such that

$$\alpha_1([0,1]) \cap \alpha_2([0,1]) = \{q_1, p\},$$
$$(\Phi \circ \alpha_1)([0,1]) = (\Phi \circ \alpha_2)([0,1]).$$

We fix a path β on $\mathbb{C}P_1$ from $\Phi(q_1)$ to p_∞ such that $\beta([0,1]) \cap \Phi(\partial\bar{E}) \subseteq \{\Phi(q_1)\}$. (See Figure 5.) Around q_1 we can take two distinct paths

$$\alpha_1', \alpha_2' : \ [0,\epsilon] \longrightarrow \bar{E}$$

such that

$$\alpha_1'([0,1]) \cap \alpha_2'([0,1]) = \{q_1\},$$
$$(\Phi \circ \alpha_1')(t) = (\Phi \circ \alpha_2')(t) = \beta(t) \quad (t \in [0,\epsilon]).$$

Since Φ is a local biholomorphic map except q_0, q_1, we obtain the desired paths α_1, α_2 as the pull-back of β by Φ which coincide near q_1 with α_1', α_2' respectively. Meanwhile, the map Φ is biholomorphic around p. This is a contradiction. □

We recall three well-known facts about simple loops on elliptic curves.

When (E, p_∞, Λ_0) is of type III, we choose the sign of ω_{Λ_0} such that $\int_{\alpha_0} \omega_{\Lambda_0} > 0$. Then we obtain the following: (See Figure 3)

$$a = \int_{\alpha_0 - \alpha_2} \omega_{\Lambda_0},$$
$$b = \int_{\alpha_1 - \alpha_2} \omega_{\Lambda_0},$$
$$x = \int_{\alpha_0 + \alpha_1 - \alpha_2} \omega_{\Lambda_0}.$$

When (E, p_∞, Λ_0) is of type IV, we obtain the following: (See Figure 3)

$$a = |\int_{\alpha_0} \omega_{\Lambda_0}|,$$
$$b = |\int_{\alpha_1} \omega_{\Lambda_0}|.$$

We thereby recover the data (a, b, x) from (E, p_∞, Λ_0) in any case. Hence this corollary follows. ∎

Corollary 3 implies the uniqueness in Theorem 3. In order to finish our proof of Theorem 3, we next show the completeness; for a given once punctured elliptic curve with a Lagrangian sublattice (E, p, Λ), there exists an $\widetilde{E}(a, b, x)$ which is isomorphic to (E, p, Λ).

Let γ be an oriented loop or map from S^1 to E representing a generator of Λ. The Abelian differential ω_Λ is exact on $E - \gamma(S^1)$ because the integral of ω_Λ along a loop α equals zero if and only if α represents a homology class whose intersection number with the homology class $[\gamma]$ equals zero. When we denote by $\|\gamma\|$ the length of γ with respect to the metric g_Λ, it is verified from the following two facts (1), (2) that there exists a loop representing a generator of Λ which has the smallest length among such loops, and that the loop consists of Λ-edges. We shall call a loop consisting of Λ-edges a Λ-*edge-loop*.

(1): For any loop α on E, there exists a Λ-edge-loop α' consisting of Λ-edges such that α' is homotopic to α and $\|\alpha'\| \leq \|\alpha\|$.

(2): For any positive number r, there exist at most finitely many Λ-edge-loops with length smaller than r.

We can show that fact (1) follows this way: we first approximate α by a polygonal loop, and then we reduce the number of vertices which do not lie on $\omega_\Lambda(0)$. On the other hand, for a positive number r, it is obvious that there exist at most finitely many Λ-edges with length smaller than r, and then the fact (2) follows.

Lemma 2. *The image of an oriented* Λ*-edge-loop* γ *representing a generator of* Λ *contains* $\omega_\Lambda^{-1}(0)$.

Proof. The Abelian differential ω_Λ has one point as zero of order 2, or has two points as zeros of order 1. We show the lemma in the second case. (In the other case the statement is trivial.) We set $\omega_\Lambda^{-1}(0) = \{q_0, q_1\}$.

Suppose the image of γ contains q_0 and does not contain q_1. (The image of γ consists of only Λ-edges from q_0 to q_0 itself.)

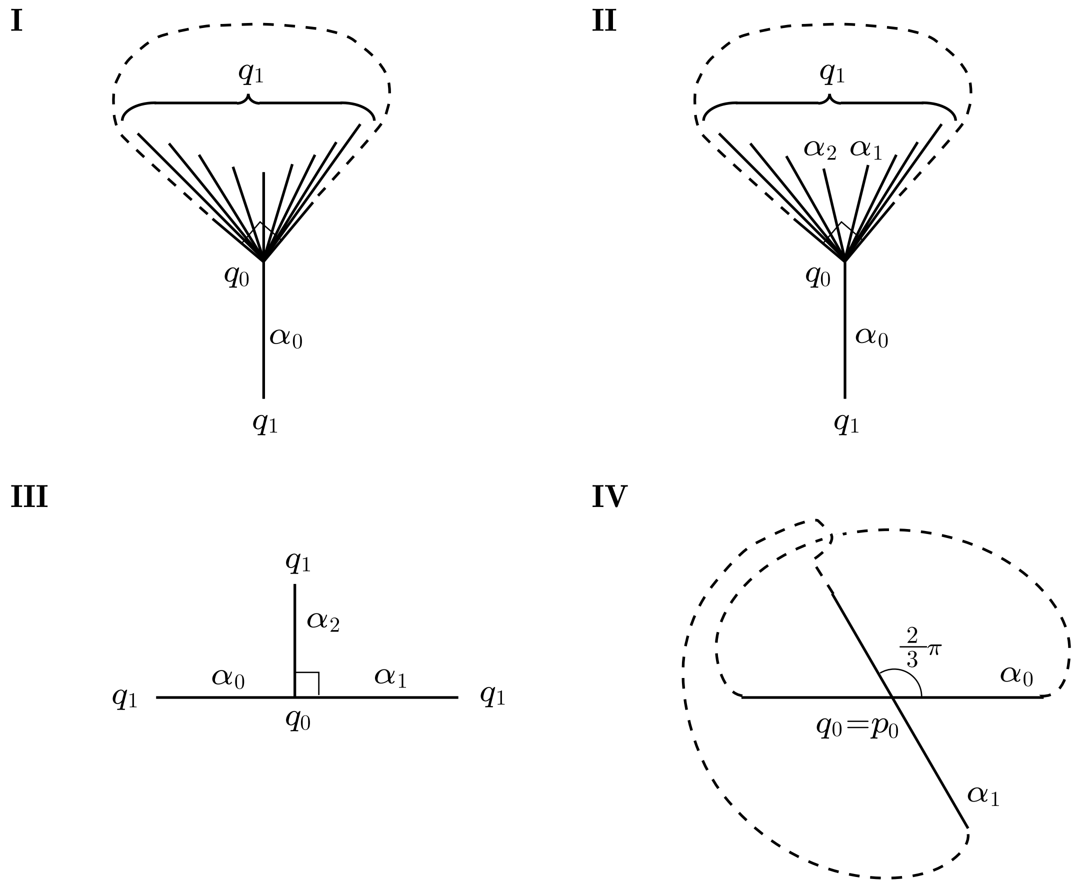


FIGURE 4

however in the case where (E, p_∞, Λ_0) is of type II there exists no Λ-edge of this kind. We thus specify the type of (E, p_∞, Λ). We shall orient the Λ_0-edges α_0, α_1 and α_2 in Figure 4, which are also in Figure 3, from q_0 to q_1 in the case of type I, II or III.

When (E, p_∞, Λ_0) is of type I or II, we choose the sign of ω_{Λ_0} such that $\operatorname{Im}\int_{\alpha_0}\omega_{\Lambda_0} > 0$. Then we obtain

$$x = \int_{\alpha_0} \omega_{\Lambda_0}.$$

In case (E, p_∞, Λ_0) is of type II, we also obtain the following: (See Figure 3)

$$a = \int_{\alpha_0 - \alpha_2} \omega_{\Lambda_0},$$
$$b = \int_{\alpha_1 - \alpha_0} \omega_{\Lambda_0}.$$

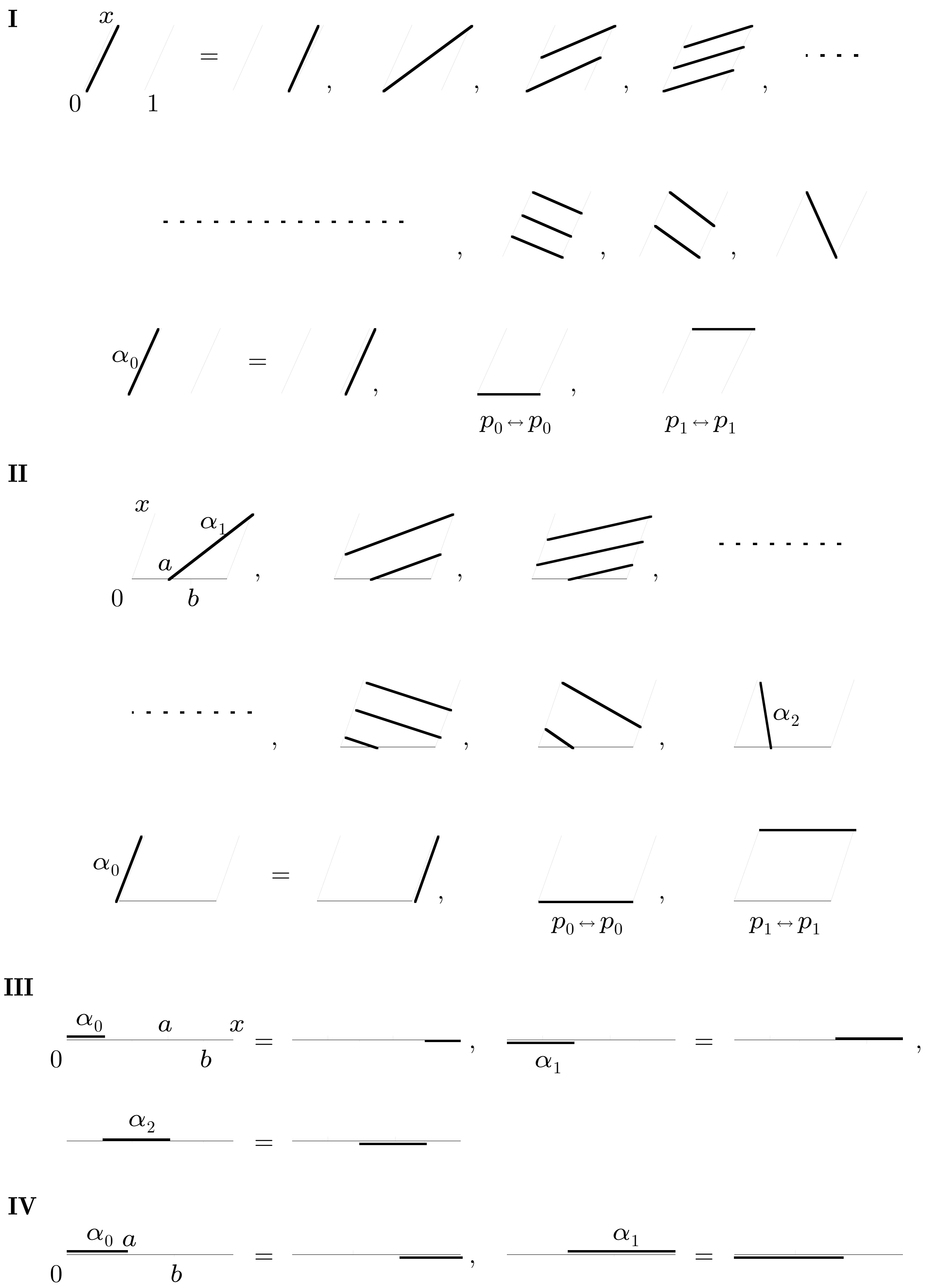


FIGURE 3. Λ_0-edges

IV: (The case where a and b are relatively prime positive integers, and $x = 0$.) Cut the complex plane along $[0, a+b]$ and paste the upper side of $[0, a]$ (resp. $[a, a+b]$) and the lower side of $[b, a+b]$ (resp. $[0, b]$) by a parallel transformation.

Note that each elliptic curve constructed in this way has a natural Abelian differential ω_0 induced by $d\zeta$ and that the Lagrangian sublattice Λ_0 of $\widetilde{E}(a,b,x)$ is characterized as the kernel of the period map of ω_0 from $H_1(E(a,b,x);\mathbb{Z})$ to $\mathbb{C}$. Further, the primitive period of ω_0 is equal to 1 because a and b are relatively prime. When $\widetilde{E}(a,b,x)$ is of type I, II, or III, the set $\omega_0^{-1}(0)$ consists of the two points p_0 and p_1 coming from the origin and the point x of $\mathbb{C}$, and when $\widetilde{E}(a,b,x)$ is of type IV, the set $\omega_0^{-1}(0)$ consists of the one point p_0 coming from the origin in $\mathbb{C}$. (See Figure 2.)

Our goal in this section is to prove

Theorem 3 . *For any once punctured elliptic curve with a Lagrangian sublattice (E, p, Λ), there exists one and only one $\widetilde{E}(a,b,x)$ isomorphic to (E, p, Λ).*

We call two once punctured Riemann surfaces with Lagrangian sublattices (R, p, Λ) and (R', p', Λ') *isomorphic*, if there exists a biholomorphic map from R to R' transforming p to p' and Λ into Λ'.

In order to prove Theorem 3 we investigate the geometry of geodesics on once punctured elliptic curves having metrics with conical singularities induced by Abelian differentials.

For a once punctured elliptic curve with a Lagrangian sublattice (E, p, Λ), there exists an Abelian differential ω_Λ, unique up to sign, in the kernel Z_Λ (see §3) such that the primitive period of ω_Λ is equal to 1, because Z_Λ is one-dimensional and any non-trivial element in Z_Λ has non-trivial periods. Then $\mathrm{PD}[\omega_\Lambda]$ is an integral homology class. Since the metric g_{ω_Λ} on $E - p$ depends only on Λ in this case, we shall shortly denote this metric by g_Λ. For the same reason we shall also use the term Λ-*edge* instead of ω_Λ-edge.

We first show that $\widetilde{E}(a,b,x)$'s are not isomorphic to one another by studying Λ_0-edges.

For each $\widetilde{E}(a,b,x)$, the Abelian differential ω_{Λ_0} coincides with ω_0 up to sign. So Λ_0-edges are line segments in $\mathbb{C}$ between the points of $\omega_0^{-1}(0)$ in each case of Figure 2. Accordingly we can easily obtain the following proposition.

Proposition 1 . *All the Λ_0-edges can be described as in Figure 3 for each $\widetilde{E}(a,b,x)$.*

Proposition 1 implies that all the closed geodesics containing some zeros of ω_0, which consist of Λ_0-edges, can be described visually for each $\widetilde{E}(a,b,x)$. This proposition yields the following corollary.

Corollary 3 . *If $(a,b,x) \neq (a',b',x')$, then $\widetilde{E}(a,b,x)$ is not isomorphic to $\widetilde{E}(a',b',x')$.*

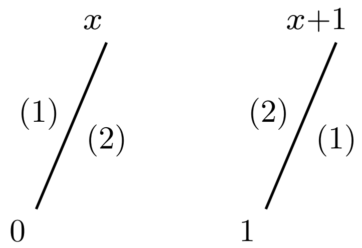
Proof. Suppose (E, p_∞, Λ_0) is isomorphic to $\widetilde{E}(a,b,x)$. We shall recover the data (a,b,x) from (E, p_∞, Λ_0).

We distinguish the type IV from others by the degeneracy of zeros of ω_{Λ_0}, and distinguish the types I, II and III from each other by the properties of Λ_0-edges on E. Let $\{q_0, q_1\}$ be the set of singular points of g_{Λ_0}. (When (E, p_∞, Λ_0) is of type IV, let q_0 be the unique singular point of g_{Λ_0}.)

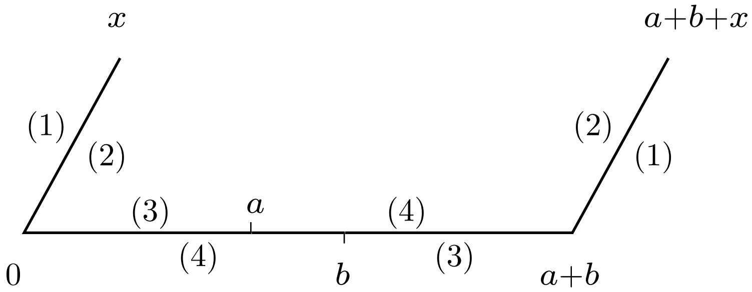
When we describe all the directions of Λ_0-edges around q_0, we obtain Figure 4 from Proposition 1. We can specify the Λ_0-edges α_0, α_1 and α_2 on E in Figure 4.

Note that these figures do not depend on the choice of q_0, and that there exists a Λ_0-edge which has the opposite direction of α_0 in the case where (E, p_∞, Λ_0) is of type I,

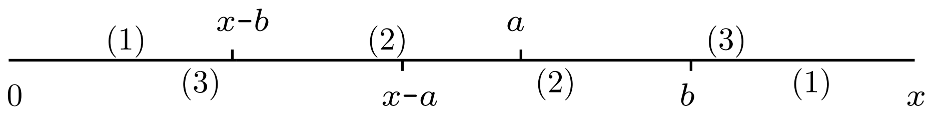
I $(E(0,1,x), p_\infty)$ where $x \in \mathbf{H} \cup (0,1)$.



II $(E(a,b,x), p_\infty)$ where $a,b \in \mathbb{Z}_+$, $(a,b)=1$, $x \in \mathbf{H}$.



III $(E(a,b,x), p_\infty)$ where $a,b \in \mathbb{Z}_+$, $a \neq b$, $(a,b)=1$, $x \in (\max(a,b), a+b)$.



IV $(\mathrm{E}(a,b,0), p_\infty)$ where $a,b \in \mathbb{Z}_+$, $(a,b)=1$.

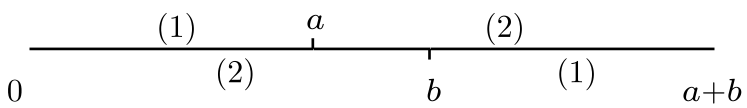


FIGURE 2. The list of $\widetilde{E}(a,b,x)$'s

Lagrangian sublattice can be obtained from $\mathbb{C}$ by cutting along line segments and pasting by parallel transformations. In this section, we consider a particular case: we give a description of all the once punctured elliptic curves with Lagrangian sublattices by using a natural extension of Igeta-construction.

We prepare some more notations and terms about the metrics induced by Abelian differentials. For an Abelian differential ω on a closed Riemann surface R, we denote by $\omega^{-1}(0)$ (resp. $\omega^{-1}(\infty)$) the set of zeros (resp. poles) of ω. We call a ω-line-segment $\gamma : [0,1] \to R - \omega^{-1}(\infty)$ *ω-edge* if $\gamma([0,1]) \cap \omega^{-1}(0) = \{\gamma(0), \gamma(1)\}$. We further denote by g_ω the flat metric on $R - \omega^{-1}(\infty)$ induced by the 2-form $\frac{i}{2}\omega \wedge \bar{\omega}$, which has conical singularities at $\omega^{-1}(0)$. Closed geodesics with respect to g_ω which contain some zeros of ω consist of ω-edges because the integral of ω gives rise to a local isometry between $R - (\omega^{-1}(\infty) \cup \omega^{-1}(0))$ and $\mathbb{C}$. An Abelian differential ω of the second kind on a closed Riemann surface R induces an element of $H^1(R;\mathbb{C})$, and further an element denoted by $\mathrm{PD}[\omega]$ of $H_1(R;\mathbb{C})$ via the Poincaré duality. It holds that

$$\int_\alpha \omega = \alpha \cdot \mathrm{PD}[\omega] \quad \text{for any } \alpha \in H_1(R;\mathbb{Z}).$$

We look for a singular 1-cycle σ representing $\mathrm{PD}[\omega]$ such that

$$\sigma = \sum_{k=1}^{N} c_k \gamma_k, \quad c_k \in \mathbb{C}$$

where $\gamma_1, \ldots, \gamma_N$ are ω-edges. (We can find it if $\omega^{-1}(0)$ is nonempty.)

From now on we consider the case of genus one.

Figure 2 is a list of some once punctured elliptic curves with Lagrangian sublattices denoted by $\widetilde{E}(a,b,x)$ (or $(E(a,b,x), p_\infty, \Lambda_0)$) constructed from $\mathbb{C}$ by cutting along the line segments and pasting by parallel transformations. (The numbers (1), (2), ... in each figure indicate pasting data or where to paste.) They split into the following four types, and each once punctured elliptic curve $(E(a,b,x), p_\infty)$ is constructed as follows:

I: (The case where $a = 0$, $b = 1$, and x is a complex number in the upper half plane $\mathbf{H}$ or a real number in the interval $(0,1)$.) Cut the complex plane along

$$x([0,1]) \cup (x([0,1]) + 1),$$

and paste each side of $x([0,1])$ and the opposite side of $(x([0,1]) + 1)$ by a parallel transformation. (Igeta-construction)

II: (The case where a and b are relatively prime positive integers, and x is a complex number in the upper half plane $\mathbf{H}$.) Cut the complex plane along

$$x([0,1]) \cup [0, a+b] \cup (x([0,1]) + a + b),$$

and paste each side of $x([0,1])$ and the opposite side of $(x([0,1]) + a + b)$ by a parallel transformation and paste the upper side of $[0,a]$ (resp. $[a, a+b]$) and the lower side of $[b, a+b]$ (resp. $[0,b]$) by a parallel transformation.

III: (The case where a and b are distinct relatively prime positive integers, and x is a real number such that $x \in (\max(a,b), a+b)$.) Cut the complex plane along $[0,x]$, and paste the lower side of $[0, x-a]$ (resp. $[x-a, b]$, $[b, x]$) and the upper side of $[a, x]$ (resp. $[x-b, a]$, $[0, x-b]$) by a parallel transformation.

on R, and that they have no intersection with C_j's. Let $(\tilde{R}_0, \omega)$ be a Riemann polygon obtained from (R_0, ω) by cutting along $a_1, b_1, \ldots, a_g, b_g$. We do the same with (R_0', ω') and denote by $(\tilde{R_0'}, \omega')$ the resulting Riemann polygon. It is sufficient for proving Theorem 2 to show that $(\tilde{R}_0, \omega)$ and $(\tilde{R_0'}, \omega')$ are piecewise parallel.

Now we can define a holomorphic function h on $\tilde{R}_0$ such that $dh = \omega$ because ω has no periods on $\tilde{R}_0$. Let g_ω be the metric on $\tilde{R}_0$ induced by the 2-form $\frac{i}{2}\omega \wedge \bar{\omega}$. The map h between $(\tilde{R}_0, g_\omega)$ and $(\mathbb{C}, g)$ is a local isometry at any point except zeros of ω, where g is the standard metric on $\mathbb{C}$. We deduce from Stokes formula

$$\int_{\tilde{R}_0} \frac{i}{2}\omega \wedge \bar{\omega} = S(\omega) + \int_{\sum_j C_j} \frac{i}{2} h\bar{\omega}$$

where C_j's are oriented counterclockwise.

The left-hand side of the equation above is the area of $\tilde{R}_0$ with respect to g_ω and the second term of the right hand side depends only on the behavior of the differential around its poles. So $S(\omega)$ is independent of the choice of the symplectic basis. We can decompose $\tilde{R}_0$ into small pieces each of which can be mapped to some polygon in $\mathbb{C}$ isometrically. Hence the conclusion follows from the theorem of Hadwiger-Glur. □

Now we return to the case of once punctured Riemann surfaces with Lagrangian sublattices. For the standard global coordinate ζ of $\mathbb{C}$, the differential $d\zeta$ uniquely extends to the complex projective line $\mathbb{C}P_1$ as an Abelian differential, which has a pole of order 2 at ∞. We also denote it by $d\zeta$.

Corollary 2 . *Let (R, p) be a once punctured Riemann surface and Λ be a Lagrangian sublattice of $H_1(R; \mathbb{Z})$. For a nonzero element $\omega \in Z_\Lambda$ the Riemann polygon (R, ω) is piecewise parallel to $(\mathbb{C}P_1, d\zeta)$.*

Proof. The assumption $\omega \in Z_\Lambda$ implies that $S(\omega) = 0$ and $PT(\omega) = PT(d\zeta)$. □

Corollary 2 indicates that any once punctured Riemann surface can be obtained from $\mathbb{C}$ by cutting along line segments and pasting by parallel transformations; the triple (R, p, Λ) is represented by a set of line segments on the complex plane plus pasting-data.

Remark 1. When we consider Riemann surfaces together with Abelian differentials of the first kind or holomorphic 1-forms, a result similar to Corollary 2 holds; any closed Riemann surface can be obtained from a fixed elliptic curve by cutting along line segments and pasting by parallel transformations. We first fix an elliptic curve and an Abelian differential on it. For instance, let E be the quotient $\mathbb{C}/L$ where L is the lattice generated by 1 and i, and we consider the standard Abelian differential $d\zeta$ of the first kind on E where ζ is the coordinate of $\mathbb{C}$. We next choose an Abelian differential ω of the first kind on any closed Riemann surface R so that $S(\omega) = S(d\zeta)(= 1)$. (It is easy to find such Abelian differentials.) We can show in the same way that (R, ω) and $(E, d\zeta)$ are piecewise parallel.

4. Elliptic curves with Lagrangian sublattices

We constructed in §2 a family of once punctured Riemann surfaces with Lagrangian sublattices via Igeta-construction for each genus g, which is locally complete and effectively parametrized. We showed in §3 that any once punctured Riemann surface with a

For Riemann polygons in $\mathbb{C}$, P-cutting indicates cutting along line segments, and P-pasting indicates pasting by parallel transformations. Igeta-construction is a special way of P-cutting and P-pasting.

It is obvious that P-pasting is the inverse operation of P-cutting. So these operators give rise to an equivalence relation between Riemann polygons: We call Riemann polygons (F,ω) and (F',ω') *piecewise parallel*, if (F,ω) is obtained from (F',ω') by finitely many P-cuttings and P-pastings.

We shall consider the special case where F is a closed Riemann surface of genus g. Let R be a closed Riemann surface of genus g. For an Abelian differential ω of the second kind on R, we define a sequence $PT(\omega)$ and a real number $S(\omega)$ as follows:

Definition 3.3.

$$PT(\omega) := \{n_i\}_{i\in\mathbb{Z}_+} \quad (n_i \text{ is the number of poles of order } i),$$

$$S(\omega) := \mathrm{Im}(\sum_{j=1}^{g}\int_{\alpha_j}\bar{\omega}\int_{\beta_j}\omega), \qquad ((\alpha_1,\beta_1,\dots,\alpha_g,\beta_g) \text{ is a symplectic basis of } H_1(R,\mathbb{Z})).$$

The value of S will be shown not to depend on the choice of the basis $(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g)$ in the proof of Theorem 2 below.

Theorem 2 . *Let ω and ω' be Abelian differentials of the second kind on closed Riemann surfaces R and R' such that $PT(\omega) = PT(\omega')$. Then the Riemann polygons (R,ω) and (R',ω') are piecewise parallel if and only if $S(\omega) = S(\omega')$.*

Before proving Theorem 2 we recall a result of Hadwiger-Glur about polygons in $\mathbb{R}^2$.

Let M be a finite set of polygons in $\mathbb{R}^2$, and let v be a unit vector in $\mathbb{R}^2$. (By our convention, M is a Riemann polygon.) Assume that the boundary of each polygon in M is oriented counterclockwise; we consider each boundary segment as a vector, and call them "boundary vectors". We denote by $A(M)$ the sum of the area of polygons in M, and define $J_v(M)$ to be the algebraic sum of the boundary vectors of M which are parallel to v.

Hadwiger-Glur's Theorem [Mo], [S] **.** *Let M and M' be finite sets of polygons in $\mathbb{R}^2$ such that $A(M) = A(M')$. M and M' are piecewise parallel if and only if $J_v(M) = J_v(M')$ for any unit vector v.*

Obviously, the invariant $J_v(M)$ can be extended as an invariant $J_v(R,\omega)$ of any Riemann polygon (R,ω) with respect to P-cuttings and P-pastings.

Proof of Theorem 2. It is obvious that $J_v(R,\omega) = J_v(R',\omega') = 0$ for any unit vector v because both R and R' are closed Riemann surfaces.

Fix a one-to-one correspondence between the poles of ω and the ones of ω' such that the orders of corresponding poles are equal, and fix for each pole p_j a local biholomorphism h_j which maps a neighborhood of p_j onto a neighborhood of the corresponding pole p'_j transforming ω into ω'. Let C_j be a small simple loop consisting of ω-line-segments around p_j. (We shall call such a loop simple polygonal loop.) Then $h_j(C_j)$ is a polygonal loop around p'_j. Cut off from R those components of $R - \sqcup_j C_j$ (resp. $R' - \sqcup_j h_j(C_j)$) which contain the poles. We then obtain a Riemann polygon (R_0,ω) (resp. (R'_0,ω')) with no poles. Furthermore, fix a symplectic basis $(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g)$ and $2g$ simple polygonal loops $(a_1,b_1,\dots,a_g,b_g)$ representing them such that their intersection is only one point

Problem . *Is it possible to construct any (R, p, Λ) by cutting and pasting the Gauss plane using line segments and parallel transformations as in the Igeta-construction?*

We next introduce a concept "Riemann polygon" so that we can consider cutting and pasting of Riemann surfaces with Abelian differentials using "line segments" and "parallel transformations".

Let R be a Riemann surface and ω an Abelian differential on it. We call a simple path or simple loop $\gamma\ : [0,1] \to R$ *ω-line-segment* if its image contains no poles of ω and the integral

$$\int_{\gamma(0)}^{\gamma(t)} \omega$$

depends on $t \in [0,1]$ linearly. We also call its image *ω-line-segment*. The 2-form $\frac{i}{2}\omega \wedge \bar{\omega}$ induces a metric g_ω on R which has conical singularities at the zeros of ω. ω-line-segments are geodesics for this metric g_ω. Let us cut R along ω-line-segments and separate into finitely many pieces. We call a collection of such pieces *Riemann polygon*:

Definition 3.2. A *Riemann polygon* (F, ω) is defined to be a pair consisting of a compact, not necessarily connected Riemann surface F and an Abelian differential ω on F such that the boundary of F, if not empty, consists of ω-line-segments.

A Riemann surface with boundary in our understanding is a 2-dimensional topological manifold F with boundary which is embedded in a Riemann surface R and whose interior inherits its complex structure from R; in addition an Abelian differential on F is a restriction of some Abelian differential on R. For example, any polygon P in the real plane $\mathbb{R}^2$ is considered as a Riemann polygon $(P, d\zeta|_P)$ when $\mathbb{R}^2$ is identified with $\mathbb{C}$. If we compactify $E^0_\Gamma \cup U_\infty$ in §2 by attaching line segments to both sides of $\sigma_j^\pm$, the pair consisting of the compactification $\overline{E^0_\Gamma}$ and the 1-form $d\zeta$ is a Riemann polygon. We can also consider a closed Riemann surface with an Abelian differential as a Riemann polygon with empty boundary.

We can easily generalize the concept of "the translation scissors congruence" (see [Mo], [S]) to the case of Riemann polygons. In order to do so, we introduce two operations "P-cutting" and "P-pasting" for getting a Riemann polygon from another Riemann polygon.

P-cutting: Let (F, ω) be a Riemann polygon, and let γ be an ω-line-segment on it such that the image $\gamma((0,1))$ is in the interior of F. We first remove the ω-line-segment $\gamma([0,1])$ from F, and then compactify by attaching one copy of $\gamma([0,1])$ to each side of $\gamma([0,1])$ obtaining a new Riemann polygon (F', ω'), where ω' is induced by ω naturally. (If $\gamma(0)$ (resp. $\gamma(1)$) is in the interior of F and is different from $\gamma(1)$ (resp. $\gamma(0)$), $\gamma(0)$ (resp. $\gamma(1)$) in one of the copies of $\gamma([0,1])$ should be identified with $\gamma(0)$ (resp. $\gamma(1)$) in the other.)

P-pasting: Let (F, ω) be a Riemann polygon. If there are ω-line-segments γ and γ' on the boundary ∂F such that

$$\int_{\gamma(0)}^{\gamma(t)} \omega = \int_{\gamma'(0)}^{\gamma'(t)} \omega$$

for any $t \in [0,1]$ and the interior of F sits on opposite sides of the paths, then we can paste $\gamma([0,1])$ and $\gamma'([0,1])$ by identifying $\gamma(t)$ with $\gamma'(t)$ and obtain a new Riemann polygon (F', ω'), where ω' is induced by ω naturally.

(by the residue theorem). Finally

$$\int_{C_j^+} \chi = 0, \quad j = 1, \dots, g$$

yields $\chi = 0$ by the bilinear relations of Riemann. □

The group $\mathrm{Aut}(\mathbb{C})$ of automorphisms of $\mathbb{C}$ acts on $I_g\eta$ as

$$(a_j, b_j^{\pm}) \mapsto (aa_j, b_j^{\pm} + b) \quad a \in \mathbb{C}^{\times}, b \in \mathbb{C}$$

preserving the complex structure of any once punctured Riemann surface $(R(\Gamma), p_\infty)$. Hence we obtain Corollary 1 below, which implies that the family of once punctured Riemann surfaces of genus g by Igeta-construction is complete and effectively parametrized at any point for each g.

Corollary 1 . *The Kodaira-Spencer map*

$$\rho_{\Gamma,0} \ : T(I_g\eta_0)_\Gamma \longrightarrow H^1(R(\Gamma); \Theta(-p_\infty))$$

is an isomorphism where

$$I_g\eta_0 = \{\Gamma \in I_g\eta \ ; \Gamma \ \textit{has}\ [0,1] \ \textit{as one of its}\ 2g\ \textit{line segments.}\}.$$

Note that the submanifold $I_g\eta_0$ gives a local manifold cover of the V-manifold $I_g\eta/\mathrm{Aut}(\mathbb{C})$ at any point.

3. Cutting and Pasting of Riemann Polygons

In the previous section, given a once punctured Riemann surface $(R(\Gamma), p_\infty)$ constructed from Igeta Γ, we used the Abelian differential ω_Γ of the second kind on $R(\Gamma)$ in order to prove Theorem 1. Let Λ_Γ be the subgroup of $H_1(R(\Gamma); \mathbb{Z})$ generated by C_j^+ $(j = 1, \dots, g)$. The integral of ω_Γ on any element λ of Λ_Γ vanishes, and the orthogonal complement of Λ_Γ with respect to the intersection form coincides with Λ_Γ itself.

We first give a definition of *Lagrangian sublattice*, which is deduced from the properties of Λ_Γ, for any closed Riemann surface as follows:

Definition 3.1. Let R be a closed Riemann surface. A *Lagrangian sublattice* is defined to be a subgroup Λ of $H_1(R;\mathbb{Z})$ satisfying $\Lambda = \Lambda^\perp$, where $\Lambda^\perp$ denotes the orthogonal complement with respect to the intersection form on $H_1(R;\mathbb{Z})$.

Let (R, p) be a once punctured Riemann surface of genus g and Λ a Lagrangian sublattice of $H_1(R;\mathbb{Z})$. The kernel Z_Λ of the homomorphism given by Abelian integrals

$$H^0(R; \Omega^1(2p)) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C}) \ (\cong \mathbb{C}^g)$$

is always one-dimensional because it holds that

$$\dim H^0(R; \Omega^1(2p)) = g + 1$$

from the Riemann-Roch formula and the surjectivity is implied by the bilinear relations of Riemann. Accordingly, a Lagrangian sublattice and a point on the surface determine an Abelian differential up to scalars. (Note that $\Lambda \cong \mathbb{Z}^g$.)

Each Igeta Γ is associated to a once punctured Riemann surface $(R(\Gamma), p_\infty)$ together with the Lagrangian sublattice Λ_Γ. In the case of $(R, p, \Lambda) = (R(\Gamma), p_\infty, \Lambda_\Gamma)$, the kernel Z_{Λ_Γ} is generated by ω_Γ. Now the problem is the following:

Then the (1,1)-form $\bar\partial(\psi\alpha)$ extends by 0 outside $U_1 \cap U_2$ and the evaluation of $[\alpha]$ is given by

$$\int_R \bar\partial(\psi\alpha) = \int_R d(\psi\alpha) = \int_{C_1+C_2} \psi\alpha = \int_{C_1} \alpha$$

up to sign. $\square$

Now we differentiate the coordinate transformations of $R(\Gamma)$. The map

$$U_\infty - \{0\} \ni w \longmapsto \zeta = w^{-1} \in E^0_\Gamma$$

is independent of the parameters of $I_g\eta$, and

$$V_j^\pm \ni z \longmapsto \zeta = \phi_j^\pm \circ J(z) \in E^0_\Gamma$$

depends only on $(a_j, b_j^\pm)$ and

$$\begin{aligned}\frac{\partial}{\partial a_j}(\phi_j^\pm \circ J(z)) &= \frac{1}{2}(z + z^{-1}),\\ \frac{\partial}{\partial b_j^\pm}(\phi_j^\pm \circ J(z)) &= 1.\end{aligned}$$

Thus

$$\begin{aligned}\langle \chi\omega,\ \rho(\frac{\partial}{\partial b_j^\pm})\rangle &= \langle \chi\omega,\ [\frac{\partial}{\partial\zeta} \ \text{ on } \ \phi_j^\pm \circ J(V_j^\pm)]\rangle\\ &= \langle \chi \text{ on } \phi_j^\pm \circ J(V_j^\pm)\rangle\\ &= \int_{C_j^\pm} \chi\end{aligned}$$

where $C_j^\pm = \{z \in V_j \ ; \ |z| = (1+\frac{\epsilon}{2})^{\pm 1}\}$ oriented appropriately. Since the left-hand side vanishes by the assumption,

$$\int_{C_j^\pm} \chi = 0.$$

Hence $\mathrm{Res}_{q_j^+}\chi + \mathrm{Res}_{q_j^-}\chi = \int_{C_j^+ + C_j^-} \chi = 0$. Further,

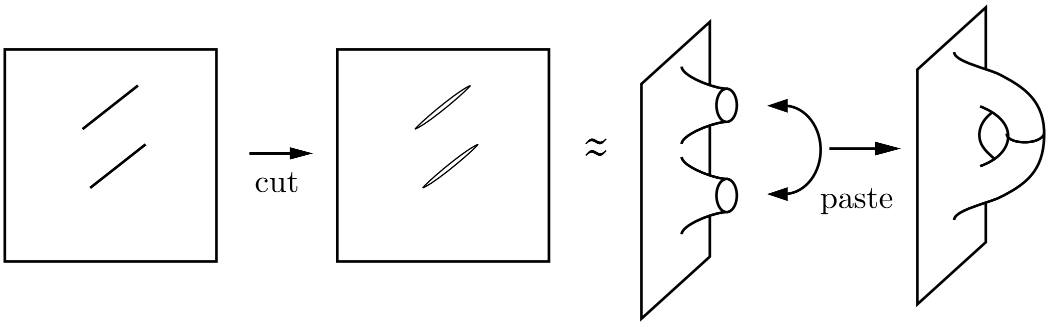
$$\begin{aligned}\langle \chi\omega,\ \rho(\frac{\partial}{\partial a_j})\rangle &= \langle \chi\omega,\ [\frac{1}{2}(z+z^{-1})\frac{\partial}{\partial\zeta} \ \text{ on } \ \phi_j^+ \circ J(V_j^+) \cup \phi_j^- \circ J(V_j^-)]\rangle\\ &= \langle \frac{1}{2}(z+z^{-1})\chi \ \text{ on } \ \phi_j^+ \circ J(V_j^+) \cup \phi_j^- \circ J(V_j^-)\rangle\\ &= \int_{C_j^+ + C_j^-} \frac{1}{2}(z+z^{-1})\chi\\ &= \mathrm{Res}_{q_j^+}\chi - \mathrm{Res}_{q_j^-}\chi\end{aligned}$$

and it vanishes, so we get

$$\mathrm{Res}_{q_j^\pm}\chi = 0$$

and

$$\chi \in H^0(\Omega^1(p_\infty)) = H^0(\Omega^1)$$



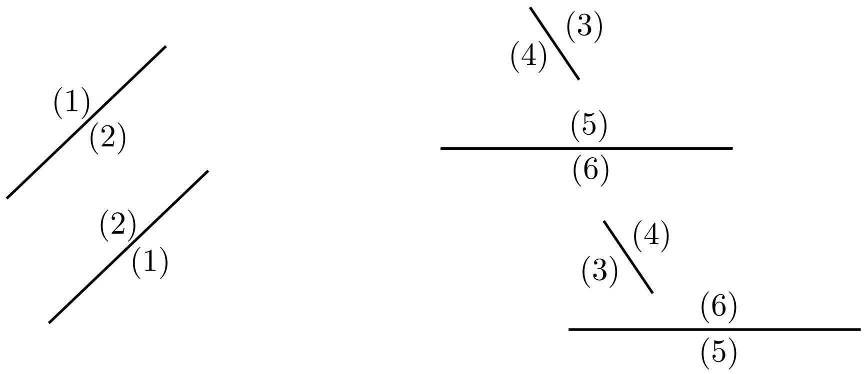


FIGURE 1. Igeta-construction

is nondegenerate because of the Serre duality. Note that the 1-form $d\zeta$ on E^0_Γ extends to a meromorphic 1-form $\omega = \omega_\Gamma$ on $R(\Gamma)$, which has a 2-pole at p_∞ and $2g$ zeros at $q_j^\pm$ $(j = 1, \ldots, g)$ corresponding to $\pm 1 \in V_j$. The multiplication by ω induces the isomorphism

$$N := H^0(\Omega^1(p_\infty + \sum_{j=1}^{g}(q_j^+ + q_j^-))) \to H^0(\Omega^1 \otimes \Omega^1(3p_\infty)).$$

Hence it is sufficient to show the following :

$$\chi \in N \text{ vanishes if } \langle \chi\omega,\ \rho(v)\rangle = 0 \text{ for any } v \in T(I_g\eta)_\Gamma.$$

The calculation of the pairing is based on the following lemma :

Lemma 1. *Let $R = \bigcup_{U \in S} U$ be a Riemann surface with an open covering. A holomorphic 1-form α on $U_1 \cap U_2$ for $U_1, U_2 \in S$ induces an element $[\alpha]$ of $H^1(R, \Omega^1)$ (the cocycle vanishes on the other intersections of the coordinate neighborhoods). If $U_1 \cap U_2$ is an annulus bounded by two circles $C_1 \subset U_2$, $C_2 \subset U_1$, then the evaluation of $[\alpha]$ is given by $\langle \alpha \rangle = \pm \int_{C_1} \alpha$.*

Proof. We introduce a cut-off function ψ on $U_1 \cap U_2$ such that

$$\mathrm{supp}\ \psi \subset U_2, \quad \mathrm{supp}\ (1 - \psi) \subset U_1.$$

maps the unit circle in $\mathbb{C}$ onto the interval $[-1,1]$. For sufficiently small $\epsilon > 0$, we paste the patches

$$E_B^0, \quad B \times U_\infty, \quad B \times V_j \ (j = 1, \ldots, g)$$

by the attaching maps

$$B \times (U_\infty - \{0\}) \ni (\Gamma, w) \longmapsto (\Gamma, w^{-1}) \in E_B^0,$$
$$B \times V_j^{\pm} \ni (\Gamma, z) \longmapsto (\Gamma, \phi_j^{\pm}[\Gamma] \circ J(z)) \in E_B^0$$

and obtain a complex manifold E_B, which is the total space of a holomorphic family of once punctured Riemann surfaces of genus g over B. As $I_g\eta$ is locally compact, we can construct the holomorphic family $\pi : E \to I_g\eta$ such that $\pi^{-1}(B) = E_B$ for any open and relatively compact subset B of $I_g\eta$. For a point Γ of $I_g\eta$ the Riemann surface $R(\Gamma) = \pi^{-1}(\Gamma)$ is constructed by pasting the patches

$$E_\Gamma^0, \quad U_\infty, \quad V_j \ (j = 1, \ldots, g)$$

through the attaching map

$$U_\infty \ni w \longmapsto \zeta = w^{-1} \in E_\Gamma^0,$$
$$V_j^{\pm} \ni z \longmapsto \zeta = \phi_j^{\pm} \circ J(z) \in E_\Gamma^0.$$

Denote by p_∞ the puncture on $R(\Gamma)$ corresponding to $0 \in U_\infty$.

We call such a pair (σ^+, σ^-) *Igeta* and the construction mentioned above *Igeta-construction.* Roughly speaking, the Igeta-construction is to cut the Gauss plane along the g pairs of line segments $(\sigma_j^{\pm} \ ; \ j = 1, \ldots, g)$ and to paste each side of σ_j^+ and the opposite side of σ_j^- by a parallel transformation. (Figure 1. The numbers (1), ... ,(6) in Figure 1 indicate where to paste.)

We investigate the infinitesimal deformation for this family $\pi : E \to I_g\eta$. We differentiate at a point Γ of $I_g\eta$ the coordinate transformation for the Riemann surface $R(\Gamma)$ by the parameters of $I_g\eta$, and obtain the *Kodaira-Spencer map* ([K]). We consider the Kodaira-Spencer map with respect to the deformation fixing p_∞ to order n:

$$\rho_\Gamma[-n] : T(I_g\eta)_\Gamma \longrightarrow H^1(R(\Gamma), \Theta(-np_\infty))$$

where Θ denotes the sheaf of holomorphic vector fields on $R(\Gamma)$. Now we state the following:

Theorem 1 . *The Kodaira-Spencer map*

$$\rho_\Gamma[-3] : \ T(I_g\eta)_\Gamma \longrightarrow H^1(R(\Gamma), \Theta(-3p_\infty))$$

is an isomorphism.

We first calculate the dimension of $H^1(\Theta(-3p_\infty))$. (Note that we omit $R(\Gamma)$ in $H^*(R(\Gamma), *)$.) By the Riemann-Roch formula,

$$\dim H^0(\Theta(-3p_\infty)) - \dim H^1(\Theta(-3p_\infty)) = 1 - g + c_1(\Theta(-3p_\infty)) = -3g.$$

As $c_1(\Theta(-3p_\infty)) = -(2g+1)$ is negative, $H^0(\Theta(-3p_\infty)) = 0$. So $\dim H^1(\Theta(-3p_\infty)) = 3g$, which coincides with the dimension of $I_g\eta$. Thus we show the surjectivity of $\rho = \rho_\Gamma[-3]$.

The pairing

$$\langle \ , \ \rangle \ : \ H^0(\Omega^1 \otimes \Omega^1(3p_\infty)) \times H^1(\Theta(-3p_\infty)) \to H^1(\Omega^1) \cong \mathbb{C}$$

differentials of the second kind, and show that any once punctured Riemann surface can be obtained by cutting the complex plane along line segments and pasting by parallel transformations. In §4 we give a description of all the once punctured elliptic curves with Lagrangian sublattices. In §5 we consider the complex structure of the moduli space of once punctured elliptic curves with Lagrangian sublattices by using the description given in §4.

This paper was prepared when the second named author stayed at the Max-Planck-Institut and the University of Bielefeld. The authors finished the manuscript in Mathematische Forschungsinstitut Oberwolfach supported by the Volkswagen-Stiftung (RiP-program at Oberwolfach). The authors would like to thank the many people who have contributed ideas and suggestions for this manuscript, among them C. F. Bödigheimer, V. Chueshev, K. Fukaya, M. Furuta, A. Hattori, S. Morita and K. Ono. The authors would like to express their gratitude to H. Helling for useful suggestions on how to improve the early drafts.

2. Igeta-construction and the Kodaira-Spencer maps

The Gauss plane is the complex affine line $\mathbb{A}^1$ with a fixed global coordinate $\zeta : \mathbb{A}^1 \to \mathbb{C}$. Consider the set η consisting of unordered pairs (σ^+, σ^-) of disjoint line segments in the Gauss plane $(\mathbb{A}^1, \zeta)$ such that σ^+ and σ^- are parallel and equilateral. We denote by $I_g\eta$ the collection of unordered sets Γ of g elements of η where

$$\Gamma = ((\sigma_j^+, \sigma_j^-) \in \eta \ ; \ j = 1, \dots, g)$$

such that $\sigma_j^\pm$ are pairwise disjoint. Let $\phi_j^\pm = \phi_j^\pm[\Gamma]$ be the affine map from the Gauss plane $(\mathbb{A}^1, \xi)$ to $(\mathbb{A}^1, \zeta)$ given by

$$\zeta = a_j \xi + b_j^\pm, \quad a_j \in \mathbb{C}^\times, \ b_j^\pm \in \mathbb{C}$$

such that $\sigma_j^\pm = \phi_j^\pm([-1, 1])$. The space $I_g\eta$ is a $3g$-dimensional open complex manifold with local coordinates $(a_j, b_j^\pm \ ; \ j = 1, \dots, g)$ for a fixed order of line segments.

We construct a holomorphic family of once punctured Riemann surfaces of genus g over $I_g\eta$ as follows. Let B be an open and relatively compact subset of $I_g\eta$. Set

$$E_\Gamma^0 = \mathbb{A}^1 - \bigcup_{j=1}^g (\sigma_j^+ \cup \sigma_j^-)$$

for $\Gamma = (\sigma_j^\pm) \in I_g\eta$ and set

$$E^0 = \bigsqcup_{\Gamma \in I_g\eta} E_\Gamma^0 \subset I_g\eta \times \mathbb{A}^1,$$
$$E_B^0 = E^0 \cap (B \times \mathbb{A}^1).$$

Let U_∞ be the disk $\{w \in \mathbb{C} \ ; \ |w| < \epsilon\}$ and V_j $(j = 1, \dots, g)$ copies of the annulus

$$\{z \in \mathbb{C} \ ; \ (1+\epsilon)^{-1} < |z| < 1 + \epsilon\}$$

for $\epsilon > 0$ and let

$$V_j^+ = \{z \in V_j \ ; \ |z| > 1\}, \quad V_j^- = \{z \in V_j \ ; \ |z| < 1\}.$$

Note that the Joukowski transform

$$J(z) = \frac{1}{2}(z + z^{-1})$$

is an isomorphism for any $\Gamma \in I_g\eta$*, where* $T(I_g\eta)_\Gamma$ *is the holomorphic tangent space of* $I_g\eta$ *at* Γ *and* $\Theta(-3p_\infty)$ *is the sheaf of germs of holomorphic vector fields on* $R(\Gamma)$ *having zero at* p_∞ *of order at least* 3*.*

Corollary 1 . *The Kodaira-Spencer map*

$$\rho_{\Gamma,0} \ : T(I_g\eta_0)_\Gamma \longrightarrow H^1(R(\Gamma); \Theta(-p_\infty))$$

is an isomorphism for any $\Gamma \in I_g\eta_0$*.*

For a closed Riemann surface R of genus g we define a *Lagrangian sublattice* Λ of R to be a subgroup of $H_1(R;\mathbb{Z})$ which equals its orthogonal complement with respect to the intersection form on $H_1(R;\mathbb{Z})$, i.e. a subgroup isomorphic to $\mathbb{Z}^g$ such that the quotient $H_1(R;\mathbb{Z})/\Lambda$ is also isomorphic to $\mathbb{Z}^g$ and the intersection number of any two elements in Λ equals zero. Moreover, for any once punctured Riemann surface (R,p) of genus g, a Lagrangian sublattice Λ of $H_1(R;\mathbb{Z})$ and the puncture p determine a certain Abelian differential ω_Λ of the second kind on the surface unique up to scalars. When we construct a once punctured Riemann surface $(R(\Gamma), p_\infty)$ from Γ, $R(\Gamma)$ has a natural Lagrangian sublattice Λ_Γ. On the other hand if we denote by ζ the standard coordinate of $\mathbb{C}$, $R(\Gamma)$ has a natural Abelian differential ω_Γ of the second kind induced by $d\zeta$. It turns out that ω_Γ is equal to ω_{Λ_Γ} up to scalars. We use ω_Γ to prove Theorem 1.

Furthermore, using ω_Λ of (R,p,Λ) we obtain the following result:

Corollary 2 . *For an arbitrary once punctured Riemann surface with a Lagrangian sublattice* (R,p,Λ)*,* (R,ω_Λ) *and* $(\mathbb{C}P_1, d\zeta)$ *are piecewise parallel.*

We call two Riemann surfaces (R,ω) and (R',ω') with Abelian differentials of the second kind *piecewise parallel* if after decomposing (R,ω) into small pieces having line-segment-boundaries we can obtain (R',ω') by pasting them together using parallel transformations in another way. This operation turns out to be reversible. (See §3.)

Corollary 2 indicates that any once punctured Riemann surface can be obtained from $\mathbb{C}$ by cutting along line segments and pasting by parallel transformations. We have to remark here that this corollary does not imply that any Riemann surface can be obtained by Igeta-construction. Nevertheless from this result we expect that any once punctured Riemann surface with a Lagrangian sublattice would appear in some natural extension of our family.

Igeta-construction leads us to consider the moduli space of once punctured Riemann surfaces with Lagrangian sublattices. In this paper we consider the case of genus 1, and describe the moduli space using a natural extension of Igeta-construction, that is, we make a complete list of once puctured elliptic curves with Lagrangian sublattices (see §4):

Theorem 3 . *For any once punctured elliptic curve with a Lagrangian sublattice* (E,p,Λ)*, there exists one and only one*
$(E(a,b,x), p_\infty, \Lambda_0)$ *isomorphic to* (E,p,Λ)*.*

To prove Theorem 3 we study the geometry of geodesics on once punctured elliptic curves having metrics with conical singularities induced by the Abelian differentials ω_Λ. For each $(E(a,b,x), p_\infty, \Lambda_0)$, all the closed geodesics can be described visually by using our construction.

This paper is organized as follows. In §2 we give an accurate definition of Igeta-construction, and prove Theorem 1. In §3 we consider Riemann surfaces with Abelian

CUTTING AND PASTING OF RIEMANN SURFACES WITH ABELIAN DIFFERENTIALS

YOSHITAKE HASHIMOTO AND KIYOSHI OHBA

Abstract. We introduce a new method of constructing once punctured Riemann surfaces by cutting the complex plane along "line segments" and pasting by "parallel transformations". The advantage of this construction is to give a good visualization of the deformation of complex structures of Riemann surfaces. In fact, given a positive integer g, there appears a family of once punctured Riemann surfaces of genus g which is complete and effectively parametrized at any point. Our construction naturally gives each of the resulting surfaces what we call a Lagrangian sublattice Λ, a certain subgroup of the first homology. Furthermore Λ and the puncture determine an Abelian differential ω_Λ of the second kind on the Riemann surface. Using Λ and ω_Λ we consider the Kodaira-Spencer maps and some extension of the family to obtain any once punctured Riemann surface with a Lagrangian sublattice. In particular we describe the moduli space of once punctured elliptic curves with Lagrangian sublattices.

1. Introduction

In this paper we introduce a new method of constructing once punctured Riemann surfaces. In our construction we use line segments in the complex plane $\mathbb{C}$ and parallel transformations: When there exists a pair of disjoint parallel line segments with the same length in $\mathbb{C}$, we first cut $\mathbb{C}$ along the segments and paste each side of one segment and the opposite side of the other segment by a parallel transformation obtaining a once punctured elliptic curve. The puncture is at infinity. (See §2, Figure 1.) We shall call such a pair an *Igeta*. (Igeta is a Japanese word coming from a technical term "Igeta-kuzushi" used in a Japanese martial art.) Suppose there exist g disjoint pieces of Igeta in $\mathbb{C}$, we obtain a once punctured Riemann surface of genus g in the same way. We denote a set of g disjoint Igeta by Γ and the resulting once punctured Riemann surface by $(R(\Gamma), p_\infty)$. Moreover when we move the position of g Igeta, there appears a family of once punctured Riemann surfaces of genus g. All the possible configurations of g disjoint Igeta up to the affine automorphisms of $\mathbb{C}$ form a $3g-2$-dimensional complex V-manifold and this dimension is the same as the dimension of the moduli space $\mathcal{M}_{g,1}$ of once punctured Riemann surfaces of genus g. We thus expect to have a visual image of the moduli space by using this construction.

We first consider the Kodaira-Spencer maps of the family. Let $I_g\eta$ be the collection of Γ's, and let $I_g\eta_0$ be the subset of $I_g\eta$ consisting of those Γ having $[0, 1]$ as one of its $2g$ line segments. $I_g\eta$ turns out to be a $3g$-dimensional complex manifold and $I_g\eta_0$ a $3g-2$-dimensional complex manifold. Our first main result is as follows:

Theorem 1 . *The Kodaira-Spencer map*

$$\rho_\Gamma[-3] \ : T(I_g\eta)_\Gamma \longrightarrow H^1(R(\Gamma); \Theta(-3p_\infty))$$