Tilting decompositions of semisimple complex Lie algebras *

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Abstract. Let B be a tilted algebra of Dynkin type Δ. Let $\mathbf{g}$ be the semisimple complex Lie algebra of type Δ. It is proved in this note that the Lie subalgebra of the degenerate Hall algebra $\mathcal{H}_1(B)$ determined by all indecomposable B-modules, can be embedded in the Lie algebra $\mathbf{g}$ in a natural way. This leads to a decomposition of $\mathbf{g}$ analogous to the triangular decomposition of $\mathbf{g}$.

Key Words: tilted algebras, Hall algebras, derived categories, Lie algebras.

0. Introduction

Let A be an associative hereditary algebra over some finite field k of Dynkin type Δ. Let $\mathbf{g}$ be the semisimple complex Lie algebra of type Δ with triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$. C.M.Ringel in [1] showed that A, as a directed algebra, has Hall polynomials. So he could make the free abelian group $\oplus_{[M]}\mathbf{Z}\mathbf{u}_{[\mathbf{M}]}$, indexed by the set of isomorphism classes $[M]$ of all finitely generated A-modules M, into an associative ring by defining:

$$u_{[M]}u_{[N]} = \sum_{[L]} \phi^L_{MN}(1)u_{[L]},$$

for any $M, N \in \mathrm{mod}A$, where ϕ^L_{MN} are the Hall polynomials corresponding to the filtration sets $\{X \subset L : X \simeq N \text{ and } L/X \simeq M\}$; this ring is called the degenerate Hall algebra of A and denoted by $\mathcal{H}_1(A)$. Let $K(\mathrm{mod}A)$ be the Grothendieck group of $\mathrm{mod}A$, the category of all finitely generated (right) A-modules, modulo split exact sequences. Then $K(\mathrm{mod}A)$ can be identified with the subgroup $\oplus_{[X]}\mathbf{Z}u_{[X]}$ of $\mathcal{H}_1(A)$, indexed by the set of isomorphism classes $[X]$ of all indecomposable modules in $\mathrm{mod}A$. He showed that $K(\mathrm{mod}A)$ is a Lie subalgebra of $\mathcal{H}_1(A)$, and furthermore he has proved in [2] that $K(\mathrm{mod}A) \otimes_{\mathbf{Z}} \mathbf{C}$ is isomorphic to $\mathbf{n}_+$ as Lie algebras. So we can think that the triangular decomposition of $\mathbf{g}$ is one corresponding to a hereditary partition.

Let B be a tilted algebra of type Δ. Similarly, we have the degenerate Hall algebra $\mathcal{H}_1(B)$ of B and $K(\mathrm{mod}B)$ is a Lie subalgebra of $\mathcal{H}_1(B)$. In this note, we will point out that $\mathbf{g}$ has another decomposition $\mathbf{g} = \mathbf{a}_- \oplus \mathbf{b}_- \oplus \mathbf{h} \oplus \mathbf{b}_+ \oplus \mathbf{a}_+$ with $\mathbf{a}_+ \simeq K(\mathrm{mod}B) \otimes_{\mathbf{Z}} \mathbf{C}$ as Lie algebras, where $\mathbf{h}$ is the Cartan subalgebra. We call it as the tilting decomposition of $\mathbf{g}$.

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1. A Realization of $\mathbf{g}$

As mentioned, Ringel has given a realization of the positive part of a semisimple complex Lie algebra $\mathbf{g}$ via the module category of a hereditary algebra. This stimulated the authors to consider how to realize the whole Lie algebra $\mathbf{g}$ using Hall algebra methods. In [4], we have done this successfully via the derived category of a hereditary algebra. Now in this section we recall this in some details and fix notations, which will be used in the next section to establish and prove the tilting decomposition of $\mathbf{g}$.

As before, let A be a hereditary algebra over a finite field k of Dynkin type Δ, and let $\mathbf{g}$ be the semisimple complex Lie algebra of type Δ.

Let $D^b(A)$ be the derived category of mod$A$, which is a triangulated category with the translation functor T. Denote by $\mathcal{R}$ the orbit category $D^b(A)/T^2$, called the root category of mod$A$(or, Simply, of A), and let $F : D^b(A) \longrightarrow \mathcal{R}$ be the canonical covering functor. The translation functor T under F induces a translation functor of $\mathcal{R}$, still denoted by T, and so $T^2 = 1$ in $\mathcal{R}$. The root category $\mathcal{R}$ with T still is a triangulated category (see Appendix of [4]) and F is exact.

For $X, Y, L \in \mathcal{R}$, let

$$W(X, Y; L) = \{(f, g, h) : X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TX \quad \text{is a triangle}\};$$

for $(f, g, h) \in W(X, Y; L)$, denote

$$(f, g, h)^{\wedge} = \{(\alpha f, g\beta^{-1}, \beta h(T\alpha)^{-1}) : \alpha \in \mathrm{AutX}, \beta \in \mathrm{AutY}\},$$

clearly $(f, g, h)^{\wedge} \subset W(X, Y; L)$; let

$$V(X, Y; L) = \{(f, g, h)^{\wedge} : (f, g, h) \in W(X, Y; L)\}.$$

Let E be a field extension of k. For any k-space V, denote $V^E = V \otimes_k E$. So A^E naturally becomes an E-algebra and M^E an A^E-module for any A-module M. Hence for any object $N \in D^b(A)$, a bounded complex of A-modules, N^E makes sense in a natural way, and so does X^E for any $X \in \mathcal{R}$. We call E conservative (for $\mathcal{R}$) if $(\mathrm{End}X/\mathrm{rad}(\mathrm{End}X))^E$ is a field for any indecomposable object $X \in \mathcal{R}$. Note that there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{R}$. Then it is easy to see that there are infinitely many isomorphism classes of conservative finite field extensions of k.

For $M, N, L \in \mathcal{R}$, an integral polynomial $\phi^L_{NM}(q) \in \mathbf{Z}[q]$ is called a Hall polynomial of $\mathcal{R}$ if

$$\phi^L_{NM}(|E|) = |V(M^E, N^E; L^E)|$$

for any conservative finite field extension E of k.

Proposition 1.1. $\mathcal{R}$ has the Hall polynomials ϕ^L_{MN} and ϕ^L_{NM} for any $M, N, L \in \mathcal{R}$ with N indecomposable.

Now denote by $\mathcal{H}$ the abelian group with generators $\{H_{[M]}\}_{[M]}$, indexed by the set of isomorphism classes $[M]$ of all objects M of $\mathcal{R}$, and with relations

$$H_{[M\oplus N]} - H_{[M]} - H_{[N]}$$

and

$$\sum_{i=1}^{l} d(X_i)H_{[X_i]} - \sum_{i=1}^{m} d(Y_i)H_{[Y_i]} + \sum_{i=1}^{n} d(Z_i)H_{[Z_i]}$$

provided there exists a triangle of form in $\mathcal{R}$

$$\oplus_{i=1}^{l} X_i \to \oplus_{i=1}^{m} Y_i \to \oplus_{i=1}^{n} Z_i \to T(\oplus_{i=1}^{l} X_i)$$

with all X_i, Y_i, Z_i indecomposable, here $d(X) = \dim_k(\mathrm{End}X/\mathrm{rad}(\mathrm{End}X))$ for any indecomposable object $X \in \mathcal{R}$. We call $\mathcal{H}$ the weighted Grothendieck group of $\mathcal{R}$.

We also consider the Grothendieck group of $\mathcal{R}$ as usual, denoted by $G(\mathcal{R})$, it is an abelian group with generators $\{[M] : M \in \mathcal{R}\}$ modulo relations $[X] - [Y] + [Z]$ provided there exists a triangle of form $X \to Y \to Z \to TX$ in $\mathcal{R}$.

We have a bilinear function $\langle - | - \rangle$ on $\mathcal{H} \times \mathcal{G}(\mathcal{R})$ by defining:

$$\langle H_{[X]} \mid [Y] \rangle = (1/d(X))(\dim_k \mathrm{Hom}(X,Y) - \dim_k \mathrm{Hom}(X,TY) + \dim_k \mathrm{Hom}(Y,X) - \dim_k \mathrm{Hom}(Y,TX))$$

for any $X, Y \in \mathcal{R}$ with X indecomposable, and then generate them to the whole $\mathcal{H}$ linearly, here $d(X) = \dim_k(\mathrm{End}X/\mathrm{rad}(\mathrm{End}X))$ as before. Note that by definitions $\mathcal{H}$, $G(\mathcal{R})$ and $\langle - | - \rangle$ are independent on the conservative field extension of k.

Let $\mathcal{N}$ be a free abelian group with a basis $\{u_{[X]}\}_{[X]}$, indexed by the set of isomorphism classes $[X]$ of all indecomposable objedts X in $\mathcal{R}$. Let

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{N},$$

a direct sum of the abelian groups. We now define a bilinear operator $[-,-]$ on $\mathcal{G}$ by

(a) for any two indecomposable objects $X, Y \in \mathcal{R}$,

$$[u_{[X]}, u_{[Y]}] = \begin{cases} \sum_{[L], L \in \mathrm{ind}\mathcal{R}} \gamma_{XY}^{L} u_{[L]}, & \text{if } \mathrm{Y} \not\simeq \mathrm{TX}, \\ H_{[X]}, & \text{if } \mathrm{Y} \simeq \mathrm{TX}. \end{cases}$$

where $\gamma_{XY}^{L} = \phi_{XY}^{L}(1) - \phi_{YX}^{L}(1)$;

(b) for any objects $X, Y \in \mathcal{R}$ with Y indecomposable,

$$[H_{[X]}, u_{[Y]}] = -\langle H_{[X]} | [Y] \rangle \, u_{[Y]} \quad \text{and} \quad [\mathrm{u}_{[\mathrm{Y}]}, \mathrm{H}_{[\mathrm{X}]}] = -[\mathrm{H}_{[\mathrm{X}]}, \mathrm{u}_{[\mathrm{Y}]}]$$

(c) $[\mathcal{H}, \mathcal{H}] = 0$.

Proposition 1.2. $\mathcal{G}$ with the operation $[-,-]$ is a Lie algebra. Moreover $\mathcal{G} \otimes_{\mathbf{Z}} \mathbf{C} \simeq \mathbf{g}$ as Lie algebras, and under this isomorphism, we have $\mathcal{H} \otimes_{\mathbf{Z}} \mathbf{C} \simeq \mathbf{h}$, the Cartan subalgebra of $\mathbf{g}$.

2. Tilting Decomposition of g

The assumptions and the notations are the same as in Section 1.

We know that mod$A$, as the stalk complex category of some fixed stalk place, is embedded in $D^b(A)$. This induces a full embedding mod$A \to \mathcal{R}$ under the covering functor F. In this way, we identify mod$A$ with a full subcategory of $\mathcal{R}$.

Let S be a tilting A-module and $B = \mathrm{End}(S_A)$ the tilted algebra. Denote

$$\mathcal{T} = \{M \in \mathrm{mod}A : \mathrm{Ext}^1_A(S, M) = 0\}$$

and

$$\mathcal{F} = \{N \in \mathrm{mod}A : \mathrm{Hom}_A(S, N) = 0\},$$

then $(\mathcal{F}, \mathcal{T})$ is a torsion theory of $\mathrm{mod}A$ with $\mathcal{T}$ torsion and $\mathcal{F}$ torsionfree. It is not difficult to see that there exists a subcategory $\mathcal{C}$ of $\mathcal{R}$ (usually, not full) such that $\mathcal{C} \simeq \mathrm{mod}\mathcal{B}$, and $\mathrm{mod}A \cap \mathcal{C} = \mathcal{T}$ and $T(\mathrm{mod}A) \cap \mathcal{C} = T\mathcal{F}$. We identify $\mathrm{mod}B$ with $\mathcal{C}$.

Denote by $\mathrm{ind}\mathcal{R}$ a set of representatives of isomorphism classes of all indecomposable objects of $\mathcal{R}$ which is closed under T. Similarly write $\mathrm{ind}A$, $\mathrm{ind}B$, $\mathrm{ind}\mathcal{T}$ and $\mathrm{ind}\mathcal{F}$ as the subsets of $\mathrm{ind}\mathcal{R}$. Therefore we have $\mathrm{ind}\mathcal{R} = (\mathrm{indA}) \cup \mathrm{T}(\mathrm{indA})$, $\mathrm{indB} = (\mathrm{ind}\mathcal{T}) \cup \mathrm{T}(\mathrm{ind}\mathcal{F})$.

Now we denote some subgroups of $\mathcal{G}$ determined by the tilted algebra B and the tilting module S as follows:

$$\mathbf{a}_+ = \oplus_{X \in \mathrm{indB}} \mathbf{Z}u_{[X]}, \quad \mathbf{a}_- = \oplus_{TX \in \mathrm{indB}} \mathbf{Z}u_{[X]},$$

and

$$\mathbf{b}_+ = \oplus_{X \in (\mathrm{indA})\backslash(\mathcal{T}\cup\mathcal{F})} \mathbf{Z}u_{[X]}, \qquad \mathbf{b}_- = \oplus_{TX \in (\mathrm{indA})\backslash(\mathcal{T}\cup\mathcal{F})} \mathbf{Z}u_{[X]}$$

With these notations, we have the following main result.

Theorem 2.1. We have a direct sum of abelian subgroups $\mathcal{G} = \mathbf{a}_- \oplus \mathbf{b}_- \oplus \mathcal{H} \oplus \mathbf{a}_+ \oplus \mathbf{b}_+$. And $\mathbf{a}_+$, $\mathbf{a}_-$, $\mathbf{b}_+$, $\mathbf{b}_-$, $\mathbf{a}_+ \oplus \mathbf{b}_+$, $\mathbf{a}_+ \oplus \mathbf{b}_-$, $\mathbf{a}_- \oplus \mathbf{b}_+$, $\mathbf{a}_- \oplus \mathbf{b}_-$ all are Lie subalgebras of $\mathcal{G}$.

By Proposition 1.2 we can think the decomposition in the above theorem as the tilting decomposition for $\mathbf{g}$ with respect to the tilting A-module S.

Proof. By definitions it is easy to see that

$$\mathcal{G} = \mathbf{a}_- \oplus \mathbf{b}_- \oplus \mathcal{H} \oplus \mathbf{a}_+ \oplus \mathbf{b}_+$$

as abelian groups.

Now let $X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} TX$ be a triangle in $\mathcal{R}$ with X, Y, Z indecomposable. Then clearly $f \neq 0$, $g \neq 0$ and $h \neq 0$.

Assume that $X, Y \in \mathrm{indB}$. We claim that $Z \in \mathrm{indB}$ and hence $\mathbf{a}_+$ is a Lie subalgebra of $\mathcal{G}$. In fact, if $X, Y, \in \mathcal{T}$, then clearly $Z \in \mathcal{T} \subseteq \mathrm{indB}$. If $X, Y \in T\mathcal{F}$, then also clearly $Z \in T\mathcal{F} \subseteq \mathrm{mod}B$. Suppose that $X \in T\mathcal{F}$ and $Y \in \mathcal{T}$. Because $\mathrm{Hom}_{D^b(A)}(\mathcal{T}, T^{2m}\mathcal{F}) = 0$ for any $m \in \mathbf{Z}$, then $h : Y \to TX$ is zero, a contradiction. So finally we consider the case $X \in \mathcal{T}$ and $Y \in T\mathcal{F}$. If $Z \in \mathrm{mod}A$, then

$$0 \longrightarrow T^- Y \stackrel{-(T^- h)}{\longrightarrow} X \stackrel{f}{\longrightarrow} Z \longrightarrow 0$$

is exact in $\mathrm{mod}A$ and so $Z \in \mathcal{T} \subseteq \mathrm{mod}B$. If $Z \in T(\mathrm{mod}A)$, then

$$0 \longrightarrow T^- Z \stackrel{-(T^- g)}{\longrightarrow} T^- Y \stackrel{-(T^- h)}{\longrightarrow} X \longrightarrow 0$$

is exact in $\mathrm{mod}A$ and hence $Z \in T\mathcal{F} \subseteq \mathrm{mod}B$. So we have proved our claim.

Assume that $X, Y \in (\text{indA})\backslash(\mathcal{T} \cup \mathcal{F})$. We want to show that $Z \in (\text{indA})\backslash(\mathcal{T} \cup \mathcal{F})$ and so $\mathbf{b}_+$ is a Lie subalgebra of $\mathcal{G}$. In fact, we have that the sequence

$$0 \longrightarrow X \xrightarrow{f} Z \xrightarrow{g} Y \longrightarrow 0$$

is exact in mod$A$. It is easy to see that $Z \in (\text{indA})\backslash(\mathcal{T} \cup \mathcal{F})$.

Assume that $X, Y \in (\text{indB}) \cup ((\text{indA})\backslash(\mathcal{T} \cup \mathcal{F}))$. We claim that $Z \in (\text{indB}) \cup ((\text{indA})\backslash(\mathcal{T} \cup \mathcal{F}))$ and so $\mathbf{a}_+ \oplus \mathbf{b}_+$ is a Lie subalgebra of $\mathcal{G}$. In fact, by the above proofs, it is enough to consider two cases:

(1) $X \in (\text{indB})\backslash(\text{indA})$, i.e. $X \in T\mathcal{F}$ and $Y \in (\text{indA})\backslash(\mathcal{T} \cup \mathcal{F})$;

(2) $X \in (\text{indA})\backslash(\mathcal{T} \cup \mathcal{F})$ and $Y \in T\mathcal{F}$.

In the first case, suppose that $Z \notin (\text{indB}) \cup ((\text{indA})\backslash(\mathcal{T} \cup \mathcal{F}))$. If $Z \in \text{mod}A$, then $Z \in \mathcal{F}$ and the sequence

$$0 \longrightarrow Z \xrightarrow{g} Y \xrightarrow{h} TX \longrightarrow 0$$

is exact in mod$A$. So we have $Y \in \mathcal{F}$ since $TX = T^{-}X \in \mathcal{F}$, a contradiction. If $Z \in T(\text{mod}A)$, then

$$0 \longrightarrow Y \xrightarrow{h} TX \xrightarrow{-(Tf)} TZ \longrightarrow 0$$

is exact in mod$A$ and so $Y \in \mathcal{F}$, also a contradiction. Therefore we always have $Z \in (\text{indB}) \cup ((\text{indA})\backslash(\mathcal{T} \cup \mathcal{F}))$. In the second case, if $Z \in \text{mod}A$, then

$$0 \longrightarrow T^{-}Y \xrightarrow{-(T^{-}h)} X \xrightarrow{f} Z \longrightarrow 0$$

is exact in mod$A$. So we have $Z \notin \mathcal{F}$, i.e. $Z \in (\text{indA})\backslash\mathcal{F} \subseteq (\text{indB}) \cup ((\text{indA})\backslash(\mathcal{T} \cup \mathcal{F}))$. If $Z \in T(\text{mod}A)$, then

$$0 \longrightarrow T^{-}Z \xrightarrow{-(T^{-}g)} T^{-}Y \xrightarrow{-(T^{-}h)} X \longrightarrow 0$$

is exact in mod$A$ and so $T^{-}Z \in \mathcal{F}$ and hence $Z \in T(\text{ind}\mathcal{F}) \subseteq \text{indB}$. Therefore we also always have $Z \in (\text{indB}) \cup ((\text{indA})\backslash(\mathcal{T} \cup \mathcal{F}))$.

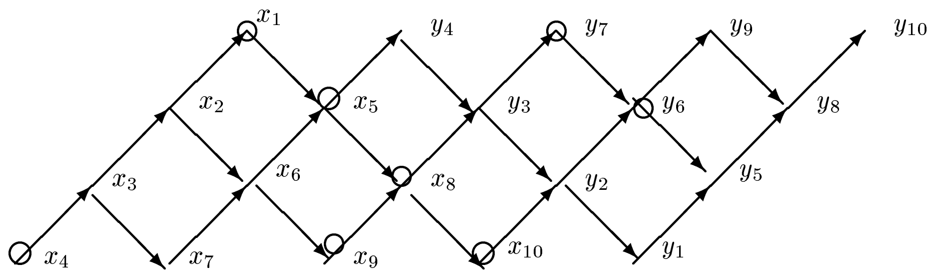
Similarly, we can prove that $\mathbf{a}_-$, $\mathbf{b}_-$, $\mathbf{a}_+ \oplus \mathbf{b}_-$, $\mathbf{a}_- \oplus \mathbf{b}_-$, $\mathbf{a}_- \oplus \mathbf{b}_+$ are Lie subalgebras of $\mathcal{G}$.

This finishes the proof of the theorem.

Remark Following the above proof we see that the Lie subalgebra $\mathbf{a}_+$ of $\mathcal{G}$ is canonically isomorphic to the Lie algebra corresponding to mod$B$ in the sence of Ringel[5].

Let us see an example as follows.

Let A be a hereditary k–algebra determined by the Dynkin quiver $1 \to 2 \to 3 \to 4$. Then the Auslander–Reiten quiver Γ_A of $\mathcal{R}$ is as follows.



with $Tx_i = y_i,\ i = 1, 2, \cdots, 10$.

We can think the full subquiver determined by $\{x_i | i = 1, 2, \cdots, 10\}$ as the Auslander–Reiten quiver Γ_A of mod$A$. Let S be the tilting A–module determined by $\{x_1, x_4, x_8, x_9\}$. Then for the tilted algebra $B = \mathrm{End(S_A)}$, indB is equivalent to a subcategory of $\mathcal{R}$ whose objects are all vertices signed by $\circ$. We see that $\mathcal{F} = \{x_6, x_7\}$, $(\mathrm{indA})\backslash(\mathcal{T} \cup \mathcal{F}) = \{\mathrm{x_2, x_3}\}$. We should point out that usually $\mathbf{a}_- \oplus \mathcal{H} \oplus \mathbf{a}_+$ and $\mathbf{b}_- \oplus \mathcal{H} \oplus \mathbf{b}_+$ are not Lie subalgebras of $\mathcal{G}$. In fact, in the about example there are triangles $x_4 \to x_3 \to x_7 \to y_4$ and $x_2 \to x_9 \to y_3 \to y_2$. We have that $x_4 \in \mathbf{a}_+,\ x_7 = Ty_7 \in \mathbf{a}_-$ but $x_3 \notin \mathbf{a}_- \oplus \mathcal{H} \oplus \mathbf{a}_+$ and that $x_2 \in \mathbf{b}_+,\ y_3 \in \mathbf{b}_-$ but $x_9 \notin \mathbf{b}_- \oplus \mathcal{H} \oplus \mathbf{b}_+$.

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