ON THE EQUIVARIANT HOMOTOPY OF FREE ABELIAN GROUPS ON G-SPACES AND G-SPECTRA

PAULO LIMA-FILHO

Contents

1. Introduction

The classical Dold-Thom theorem [DT56] asserts that the singular homology of a space X can be obtained as the homotopy groups $\pi_j(\mathrm{AG}\,(X))$ of the topological free abelian group $\mathrm{AG}\,(X)$ on X. One of the main purposes of this paper is to provide *unstable and stable equivariant* versions of this result.

We obtain results both in the unstable and in the stable equivariant worlds, for a fixed finite group G. In the unstable context, we start with the simple observation that for a G-space X, $\mathrm{AG}\,(X)$ gives rise to a Mackey functor F_X with values in the category of topological abelian groups. It follows that the composition of any homotopy functor with

Research Partially supported by NSF Grant # : DMS-9401533.

F_X yields ordinary Mackey functors. For example one could use any generalized homology or cohomology theory H_* and obtain Mackey functors $H_* \circ F_X$, etc.

Our first and preliminary result is the following Dold-Thom type theorem.

Theorem 4.5. *The homotopy groups of F_X compute the Bredon homology of X with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$.*

The main technical tool here is the fact, proven in Theorem 2.7, that *for a G-$\mathcal{CW}$-pair (X, A) the projection $AG(X) \to AG(X/A)$ is a locally trivial, numerable $(G, \alpha, AG(A))$-bundle.* It is worthwhile to mention that the restriction and transfer maps for Bredon homology have a very simple description in terms of the topological Mackey functor structure of $\mathrm{AG}(X)$ itself.

It is well-known that the Bredon homotopy groups of $\mathcal{Z}(X)$ do not completely determine its equivariant homotopy type, contrary to the non-equivariant situation. In fact, Nishida [Nis84] proves that the homotopy category of topological $k[G]$-modules, for a ring k, is equivalent to the derived category of complexes of $k[G]$-modules, and this result is also called an equivariant Dold-Thom theorem. See also [Tri83].

In the stable context our contructions become more subtle, although the results are equally natural. We assume very little knowledge from the reader about the equivariant stable category developed in [LMS85], and we provide essentially all basic definitions and detailed references to all key results used. We first apply the functor $\mathrm{AG}(-)$ spacewise to G-prespectra E and, after suitably defining the structural maps, one then obtains a G-prespectrum $\mathrm{AG}(E)$. This assignment is functorial and reasonably well-behaved with respect to basic operations on prespectra. One then uses the machinery of [LMS85] to pass from prespectra to spectra and denote by $\mathbb{Z}(E)$ the resulting G-spectrum associated to E.

The first stable result obtained identifies $\mathbb{Z}(\mathbb{S})$, where $\mathbb{S}$ is the sphere spectrum:

Theorem 6.4. *Given a finite group G, let A_G denote its Burnside ring Mackey functor. Then one has a natural equivalence of spectra $\mathbb{Z}(\mathbb{S}) \cong K(A_G, 0)$, where $K(A_G, 0)$ denotes an equivariant Eilenberg-MacLane spectrum.*

We then proceed to identify $\mathbb{Z}(E)$ in the stable homotopy category $hG\mathcal{S}$ of G-spectra for an arbitrary G-spectrum E. We construct a natural morphism of spectra $R_E : \mathbb{Z}(\mathbb{S}) \wedge E \to \mathbb{Z}(E)$ which satisfies the following property.

Theorem 6.7. *The natural maps $R_E : \mathbb{Z}(\mathbb{S}) \wedge E \to \mathbb{Z}(E)$ are weak homotopy equivalences.*

One immediately obtains several consequences.

Corollary 6.8 [Stable Dold-Thom theorem]. *For any G-spectrum E, the homotopy groups of $\mathbb{Z}(E)$ compute the ordinary $RO(G)$-graded homology of E with coefficients on the Burnside ring Mackey functor, as defined in* [LMM81]. *In other words, for a G-spectrum E and subgroup $H < G$, one has $\pi_*^H(\mathbb{Z}(E)) \cong H_*^G(E, \mathcal{A}_G)[G/H]$.*

Corollary 6.9. *Given G-spectra E and F one has weak equivalences $\mathbb{Z}(E)\wedge F \sim \mathbb{Z}(E\wedge F) \sim E\wedge \mathbb{Z}(F)$.*

There are several questions left aside in this paper which may deserve further consideration. For example, our constructions are natural enough to preserve various multiplicative structures, such as ring spectra, E_∞-ring spectra and modules over such objects. The reader should keep this in mind when searching for examples. In a sequel to this paper we intend to investigate the case of compact Lie groups in general, and extend the results to a suitable notion of *equivariant functors with smash products*, of which $\mathcal{Z}(-)$ is an example.

The paper is organized as follows. In §2 we provide the background results on the abelian group functor on G-spaces, discussing fibration properties and fixed point set behavior. We end the section presenting various natural maps which will play fundamental role in later constructions, and which exhibit $\mathcal{Z}(-)$ as an equivariant functor with smash products. In §3 we briefly recall the notion of Mackey and Hecke functors, and introduce the topological Mackey functors which are used throughout the paper, and in §4 we prove the unstable equivariant Dold-Thom result. Our main construction in the categories of G-spectra and G-prespectra are presented in §5, where we investigate functorial properties and preservation of subcategories, such as Σ-inclusion prespectra under various constructions. In the last section we present and prove our main result, which identifies the spectra $\mathbb{Z}(E)$ in the stable homotopy category, and establishes the stable equivariant Dold-Thom theorem as a corollary.

We must finally say that this work developed as a consequence of discussions with H. Blaine Lawson, Jr., and Marie-Louise Michelsohn, where in a joint work [LLFM] still under progress, we investigate the topological behavior of algebraic cycles on algebraic varieties under actions of finite groups.

Acknowledgements: The author would like to express his gratitude to H. Blaine Lawson, Jr., and Marie-Louise Michelsohn for inspiring discussions and encouragement; to Roland Schwänzl for helpful assistance with the subtleties of the stable category and for pointing out to Nishida's results, and to Rainer Vogt for showing me several mistakes and for useful suggestions. Finally the author wants to thank the support of the *Sonderforschungsbereich 343: Diskrete Strukturen in der Mathematik* and the warm hospitality of Osnabrück Univ. and SUNY Stony Brook during the elaboration of part of this work.

2. The free abelian group on a space

2.1. **Non-equivariant background.** Throughout this paper we work in the compactly generated category ($\mathfrak{CG}$) in the sense of [McC69, p. 276], i.e. weak Hausdorff k-spaces, with products, quotients and function spaces taken in that category, and we always use $F(X,Y)$ to denote the space of *based* maps between *based* spaces in $\mathfrak{CG}$.

Given X in $\mathfrak{CG}$, let $SP_d(X)$ denote its d-fold symmetric product ($d \geq 0$), that is, the quotient space $X^d/\mathfrak{S}_d$ of the d-fold product of X with itself modulo the action of the symmetric group $\mathfrak{S}_d$. We topologize the free abelian group $\mathcal{Z}(X)$ on X as the quotient of $\coprod_{d,e}(SP_d(X)\times SP_e(X))$ under the natural map

$$p:\coprod_{d,e}(SP_d(X)\times SP_e(X))\to \mathcal{Z}(X)$$

which sends (σ,τ) to $\sigma-\tau$. Note that if $\mathcal{Z}(X)^{\leq n}\overset{\text{def}}{=}p\left(\coprod_{d+e\leq n}SP_d(X)\times SP_e(X)\right)$, then $\mathcal{Z}(X)$ has the topology of the union of the expanding sequence of closed embeddings

$$\{0\}\subseteq\ldots\subseteq\mathcal{Z}(X)^{\leq n}\subseteq\mathcal{Z}(X)^{\leq n+1}\subseteq\ldots. \tag{2.0.1}$$

It is then easy to see, using the same arguments as in [McC69, 2.6, 6.5 and 6.6], that $\mathcal{Z}(X)$ is a topological group in the category $\mathfrak{CG}$.

Let us summarize a few well-known properties of $\mathcal{Z}(-)$:

Properties 2.1. **a.** The assignment $X\mapsto\mathcal{Z}(X)$ defines a covariant functor from $\mathfrak{CG}$ to the category of abelian topological groups in $\mathfrak{CG}$. See [McC69, 6.6].

b. The groups $\mathcal{Z}(X)$ come equipped with a continuous augmentation $\deg:\mathcal{Z}(X)\to\mathbb{Z}$ whose kernel we denote by $\mathrm{AG}\,(X)$. It is easy to see that when X is path-connected, then $\mathrm{AG}\,(X)$ is the connected component of $0\in\mathcal{Z}(X)$. One also verifies that $\mathrm{AG}\,(X)$ is topologically isomorphic to the group $\mathrm{AG}(X,0)$ defined in [DT56], once a base-point for X is specified. The latter group, is in turn topologically isomorphic to the group $B(\mathbb{Z},X)$ of [McC69].

c. If A is a closed subspace of X, then the natural homomorphism $\mathcal{Z}(A)\to\mathcal{Z}(X)$ is a closed inclusion. Furthermore, if (X,A) is a triangulable pair, then $\mathcal{Z}(X/A)$ has the structure of a $\mathcal{CW}$-complex and sequence

$$\mathrm{AG}\,(A)\to\mathrm{AG}\,(X)\to\mathrm{AG}\,(X/A)$$

becomes a numerable principal fibration; cf. [McC69, 8.8].

d. The natural map $j_X:X\to\mathrm{AG}\,(X)$ sending $x\in X$ to $x-x_o$, where x_o is the base-point of X, has the universal property that any map $X\to M$ from X to an abelian topological group M sending x_0 to $0\in M$ must factor through $\mathrm{AG}\,(X)$ via a topological group homomorphism $\mathrm{AG}\,(X)\to M$.

e. Any homotopy $H:I\times X\to Y$ between two maps $f_0,f_1:X\to Y$ induces a homotopy $H_*:I\times\mathcal{Z}(X)\to\mathcal{Z}(Y)$ between f_{0*} and f_{1*}, through homomorphisms H_{t*}.

Given a collection $\{X_\alpha\}_{\alpha\in\Lambda}$ of spaces with base points $x_\alpha^o\in X_\alpha$, we denote by $\oplus_{\alpha\in\Lambda}\mathcal{Z}(X_\alpha)$ the weak product of the corresponding topological abelian groups. One has a natural "multi-degree" homomorphism

$$\oplus\deg_\alpha:\bigoplus_{\alpha\in\Lambda}\mathcal{Z}(X_\alpha)\to\bigoplus_{\alpha\in\Lambda}\mathbb{Z},$$

whose composition with the homomorphism $t:\bigoplus_{\alpha\in\Lambda}\mathbb{Z}\to\mathbb{Z}$, sending $(n_\alpha)_{\alpha\in\Lambda}$ to $\sum_\alpha n_\alpha$, induces a "total degree" map $\deg\overset{\text{def}}{=}t\circ\oplus\deg_\alpha:\bigoplus_{\alpha\in\Lambda}\mathcal{Z}(X_\alpha)\to\mathbb{Z}$.

Lemma 2.2.

a. *The natural inclusions* $X_\alpha \overset{i_\alpha}{\hookrightarrow} \coprod_{\alpha\in\Lambda} X_\alpha$ *induce a topological isomorphism* $I : \oplus_\alpha \mathcal{Z}(X_\alpha) \to \mathcal{Z}(\coprod_\alpha X_\alpha)$ *which is compatible with degrees.*

b. *The projection* $\coprod_{\alpha\in\Lambda} X_\alpha \xrightarrow{\pi} \bigvee_{\alpha\in\Lambda} X_\alpha$ *induces a surjective homomorphism*

$$\mathcal{Z}(\coprod_\alpha X_\alpha) \xrightarrow{\pi_*} \mathcal{Z}(\bigvee_\alpha X_\alpha)$$

whose kernel $\mathcal{K}$ *consists of the intersection of the discrete subgroup generated by the base-points* $x^o_\alpha \in X_\alpha$ *with the kernel of the augmentation* deg. *In other words,*

$$\mathcal{K} = \{\sum_\alpha m_\alpha x^o_\alpha \mid m_\alpha \in \mathbb{Z} \text{ and } \sum_\alpha m_\alpha = 0\}.$$

In particular, $\pi_* \circ I$ *induces a natural topological isomorphism:*

$$\oplus_\alpha AG(X_\alpha) \;\overset{\cong}{\rightarrow}\; AG\left(\bigvee_\alpha X_\alpha\right).$$

Proof. See Appendix A. □

2.2. **Equivariant considerations.** Let G be a finite group, and let X be a G-space. Our first observation is that G acts on $\mathcal{Z}(X)$ via automorphisms, i.e., the action of G on X induces a continuous homomorphism

$$\alpha : G \to \operatorname{Aut}(\mathcal{Z}(X)). \tag{2.2.1}$$

It can be seen that Properties 2.1(a), (b), (d) and (e) now hold in the category of G-spaces. In particular all induced homomorphisms are G-maps and all induced homotopies are G-homotopies.

We refer the reader to [tD87, §II.1] for the basic definitions and facts about equivariant G-$\mathcal{CW}$-complexes, whenever needed. For the moment, we just recall that a G-$\mathcal{CW}$ decomposition of a G-space X is given by a "skeletal filtration" $X_0 \subset X_1 \subset \ldots$ of X by G-subspaces such that X is the topological colimit of the X_j's, and each X_j is obtained from X_{j-1} by adjoining j-cells of the form $(G/H) \times D^j$. Here H is a subgroup of G and D_j is the usual unit disk of dimension j with trivial G action. In particular, the quotient X_j/X_{j-1} can be written as $\bigvee_{\lambda\in\Lambda_j} (G/H_\lambda)_+ \wedge S^j$, where Λ_j is the indexing set of all j-cells of X.

We use $\mathrm{AG}\,(X)^H$ to denote the fixed point set of $\mathrm{AG}\,(X)$ under H, and point out that, even when X^H is connected, the space $\mathrm{AG}\,(X)^H$ is quite different from the connected component of the identity in $\mathcal{Z}(X)^H$.

Lemma 2.3.

a. *Given a pair of* G*-spaces* (X,Y)*, the projection* $\mathcal{Z}(X) \xrightarrow{p} \mathcal{Z}(X)/\mathcal{Z}(Y)$ *induces a natural topological isomorphism* $\mathcal{Z}(X)^H/\mathcal{Z}(Y)^H \overset{\cong}{\rightarrow} \{\mathcal{Z}(X)/\mathcal{Z}(Y)\}^H$.

b. *The projection* $q : X \to X/Y$ *induces a natural equivariant isomorphism*

$$\mathcal{Z}(X)/AG(Y) \cong \mathcal{Z}(X/Y).$$

In particular, it induces an equivariant isomorphism

$$AG(X)/AG(Y) \cong AG(X/Y).$$

Proof. See Appendix A. $\square$

Remark 2.4. It follows from the lemma that one has a natural topological isomorphism $\left\{\mathrm{AG}\,(X)^H/\mathrm{AG}\,(Y)^H\right\} \cong \mathrm{AG}\,(X/Y)^H$, for every pair (X,Y) of connected spaces and every subgroup $H \leq G$.

Proposition 2.5. *Let S be a finite G-set and let X be a G-space. Then there is a natural equivariant isomorphism between $AG(S_+ \wedge X)$ and $F(S_+, AG(X))$, where $F(S_+, AG(X))$ denotes the space of based maps from S_+ to $AG(X)$ with the natural G-action.*

Proof. This is simply a restatement, in the equivariant context, of the isomorphism in Lemma 2.2(b), once one observes that $S_+ \wedge X$ is G-isomorphic to $\bigvee_{s\in S} X_s$, where X_s denotes a copy of X indexed by the element $s \in S$, and G acts "diagonally" by permuting copies of X while acting on X itself at the same time. This corresponds to the natural G-action on $F(S_+, \mathrm{AG}\,(X))$ under the isomorphism of Lemma 2.2(b). The explicit isomorphism

$$\rho_S : F(S_+, \mathrm{AG}\,(X)) \to \mathrm{AG}\,(S_+ \wedge X) \tag{2.5.1}$$

sends $f \in F(S_+, \mathrm{AG}\,(X))$ to $\sum_{s\in S} L_{S_+,X}(s \wedge f(s))$, where $L_{S_+,X} : S_+ \wedge \mathrm{AG}\,(X) \to \mathrm{AG}\,(S_+ \wedge X)$ is the map introduced in 2.12.2, and $s \wedge \sigma$ denotes an element in $S_+ \wedge \mathrm{AG}\,(X)$. $\square$

As described in Properties 2.1(c), given a $\mathcal{CW}$ pair (X,A), the projection $\mathrm{AG}\,(X) \to \mathcal{Z}(X/A)$ is a $\mathcal{Z}(A)$-principal bundle, where $\mathcal{Z}(A)$ acts on the right on $\mathcal{Z}(X)$ via translations. We need to generalize this fact to the appropriate equivariant context, which is established by the following standard definitions; cf. [tD87, §I.(8.7)].

Definition 2.6. 1. Given two topological groups Π and K, together with a continuous homomorphism $\alpha : \Pi \to \mathrm{Aut}(K)$, one defines a (Π, α, K)-bundle to be a locally trivial principal K-bundle $p : E \to B$, where E and B are left Π-spaces satisfying:

a: p is Π-equivariant;

b: Given $\gamma \in \Pi$, $k \in K$ and $e \in E$, one has $\gamma(ek) = \gamma(e)\alpha_\gamma(k)$.

2. In this context, one calls *local objects* those (Π, α, K)-bundles over orbits Π/Λ which have (weak) Hausdorff total space.

3. A (Π, α, K)-bundle $p : E \to B$ is called **locally trivial** if B admits an open covering $\mathfrak{U} = (U_j | J \in J)$ by Π-sets U_j such that each restriction $p^{-1}(U_j) \to U_j$ admits a (Π, α, K)-bundle map into a local object.

It follows from the observations preceding this definition that the map $\mathcal{Z}(X) \to \mathcal{Z}(X)/\mathcal{Z}(A)$ is a $(G, \alpha, \mathcal{Z}(A))$-bundle, where $\alpha : G \to \mathrm{Aut}(\mathcal{Z}(A))$ is induced by the action of G on $\mathcal{Z}(A)$; cf. (2.2.1).

One can actually prove the following stronger result.

Theorem 2.7. *Given G-$\mathcal{CW}$-pair (X,A), the projection $AG(X) \to AG(X/A)$ is a locally trivial $(G,\alpha,AG(A))$-bundle. Furthermore, if (X,A) is a G-simplicial pair, then the bundle is numerable*[1].

The key step in the proof of this theorem is the following result.

Proposition 2.8. *Given a G-$\mathcal{CW}$-complex pair (X,A), there is a system of G-invariant neighborhoods $\{V_j\}$ of $0 \in \mathcal{Z}(X)/\mathcal{Z}(A)$ together with equivariant sections $\sigma_j : V_j \to \mathcal{Z}(X)$ which send 0 to 0. Furthermore, one can have $V_j \subseteq AG(X)/AG(A)$ and σ_j taking values in $AG(X)$.*

Proof. Let us first consider the case where

$$X = \coprod_{j\in J}(G/H_j)\times D^n \text{ and } A = \coprod_{j\in J}(G/H_j)\times S^{n-1},$$

with D^n an n-cell and S^{n-1} its boundary. Using the natural identification $\mathcal{Z}(X) \cong \mathrm{AG}\,(X_+)$, together with Lemmas 2.2(b) and 2.3(b), one obtains a commutative diagram, where all horizontal are equivariant topological isomorphisms

$$\begin{array}{ccc}
\bigoplus_{j\in J}\mathrm{AG}\left(G/H_{j+}\wedge S^{n-1}_+\right) & \xrightarrow{\cong} & \mathrm{AG}\,(A_+)\\
\downarrow & & \downarrow\\
\bigoplus_{j\in J}\mathrm{AG}\left(G/H_{j+}\wedge D^n_+\right) & \xrightarrow{\cong} & \mathrm{AG}\,(X_+)\\
\downarrow & & \downarrow\\
\bigoplus_{j\in J}\mathrm{AG}\left(G/H_{j+}\wedge S^n\right) & \xrightarrow{\cong} & \mathrm{AG}\,(X/A)\,,
\end{array}$$

and where we are identifying S^n with D^n/S^{n-1} together with its natural basepoint.

Notice that the action of G does not interchange the factors in the products appearing on the bottom row, and hence this allows one to assume that $J=\{*\}$. In this case we can use Lemma 2.5 to observe that there is an equivariant isomorphism $\mathrm{AG}\,((G/H)_+\wedge S^n) \cong F(G/H_+,\mathrm{AG}\,(S^n))$.

It follows from [LF93, Thm. 5.2] that one can find a system of neighborhoods $\{U_j\}$ of $0\in \mathrm{AG}\,(S^n)$ together with local sections $\sigma_j : U \to \mathrm{AG}\left(D^n_+\right)$ of $\mathrm{AG}\left(D^n_+\right) \to \mathrm{AG}\,(S^n)$. By applying the function space functor, one obtains a local cross-section

$$\sigma_{j*} : V_j \to F(G/H_+, \mathrm{AG}\left(D^n_+\right)) \cong \mathrm{AG}\,((G/H)_+\wedge (D^n)_+)$$

of the projection $\mathrm{AG}\,((G/H)_+\times (D^n)_+) \to \mathrm{AG}\,(G/H_+\wedge S^n)$ defined over the invariant neighborhood $V_j = F(G/H_+,U_j)$ of 0 in $\mathrm{AG}\,(G/H_+\wedge S^n)$. The proposition is then proven in this particular case.

[1] A similar result has been shown in [Koz86, Prop. 6].

Let us suppose that one has a push-out square in the category of G-spaces

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

with (X, A) as above, i.e., Y is an n-cellular extension of B. One now has a diagram with commuting squares

$$\begin{tikzcd} & \mathrm{AG}\,(A) \arrow[r, "f_1"] \arrow[d, "i"] & \mathrm{AG}\,(B) \arrow[d, "i'"] \\ & \mathrm{AG}\,(X) \arrow[r, "f_2"] \arrow[d, "p"] & \mathrm{AG}\,(Y) \arrow[d, "p'"] \\ V_j \subseteq \arrow[ur, bend left, "\sigma_j"] & \mathrm{AG}\,(X/A) \arrow[r, "f_3"] & \mathrm{AG}\,(Y/B) \end{tikzcd}$$

whose bottom arrow is a G-isomorphism, according to Lemma 2.3(b). Choose an equivariant section $\sigma_j : V_j \to \mathrm{AG}\,(X)$ of $\mathrm{AG}\,(X) \to \mathrm{AG}\,(X/A)$ defined on the G-invariant neighborhood $V_j \subseteq \mathrm{AG}\,(X/A)$, introduced above, and define $V_j' = f_3(V_j) \subset \mathrm{AG}\,(Y/B)$. The composition $\sigma_j' = f_2 \circ \sigma_j \circ f_3^{-1}$ defines a local equivariant section of p'.

Applying this process inductively over the successive skeleta of a G-$\mathcal{CW}$-pair (X, A) completes the proof. $\square$

We are now able to prove Theorem 2.7.

Proof. The main observation here is that given any point $c \in \mathrm{AG}\,(X/A)$, one can associate (in a canonical way) a point $\alpha_c \in \pi^{-1}(c)$ such that the isotropy group G_c of c coincides with the isotropy group G_{α_c} of α_c. This occurs because one can write $c = \sum_i n_i\, \pi(x_i) + m[x_o]$, where the x_i's lie in $X - A$, and $[x_o]$ is the basepoint of X/A. The element $\alpha_c = \sum_i n_i x_i + m x_o$ has the required properties, where $x_o \in A$ is the basepoint. Note that the assignment $c \mapsto \alpha_c$ is not a continuous section of π.

Given $c \in \mathrm{AG}\,(X/A)$, write $G/G_c = \{g_1 G_c, \ldots , g_r G_c\}$ and choose V_n as in (2.8) small enough so that $(g_i * c + V_n) \cap (g_j * c + V_n) = \emptyset$ whenever $i \neq j$. Now define $U_c = \coprod_{i=1}^{r} (g_i * c + V_n)$ and notice that U_c is an invariant neighborhood of c. Furthermore the map $\sigma_c(g_i * c + v) \stackrel{\text{def}}{=} g_i * \alpha_c + \sigma(v)$ defines a section of π on U_c.

We show the equivariance of σ_c as follows. For any $h \in G$ and $i \in \{1, \ldots , r\}$ let $j_{h,i} \in \{1, \ldots , r\}$ and $k_{h,i} \in G_c$ be such that $h g_i = g_{j_{h,i}} k_{h,i}$. One has

$$
\begin{aligned}
h * \sigma_c(g_i * c + v) &= h * (g_i * \alpha_c + \sigma(v)) = (g_{j_{h,i}} k_{h,i}) * \alpha_c + h\sigma(v) \\
&= g_{j_{h,i}} * \alpha_c + \sigma(hv) = \sigma_c(g_{j_{h,i}} * c + hv) \ = \sigma_c((hg_i) * c + hv) \\
&= \sigma_c(h(g_i * c + v)).
\end{aligned}
$$

This gives an equivariant section σ_c of π, defined on the open G-set U_c. This section now provides a G-isomosphism $\pi^{-1}(U_c) \xrightarrow{\cong} U_c \times \mathrm{AG}\,(A)$, which together with the "indexing" G-map $U_c \to G/G_c$ (from the construction of U_c), shows that $\mathrm{AG}\,(X) \to \mathrm{AG}\,(X/A)$ is a locally trivial $(G, \alpha, \mathrm{AG}\,(A))$-bundle; cf. Definition 2.6(3).

When the pair (X, A) is G-simplicial, it is shown in [McC69] that $\mathrm{AG}\,(X/A)$ is a (non-equivariant) $\mathcal{CW}$-complex, and hence it is paracompact. Since G is finite, one can easily obtain a locally finite equivariant partition of unity subordinated to the covering $\{U_c\}$ of $\mathrm{AG}\,(X/A)$, thereby showing numerability; cf. [tD87, p. 47]. ☐

Corollary 2.9. *If X is a G-simplicial complex then $AG\,(X)$ is a G-$\mathcal{CW}$-complex.*

Proof. We proceed inductively. If H is a subgroup of G, and $X = G/H_+$, then $\mathrm{AG}\,(X) = \mathbb{Z}[G/H]$ can be considered a discrete G-$\mathcal{CW}$-complex in the obvious way.

Let (X, A) be a G-simplicial pair G-$\mathcal{CW}$-pair, such that

$$
X/A = \bigvee_{\lambda \in \Lambda} (G/H_\lambda)_+ \wedge S^{p_\lambda},
$$

and assume that $\mathrm{AG}\,(A)$ is a G-$\mathcal{CW}$-complex. The theorem states that $\mathrm{AG}\,(A) \to \mathrm{AG}\,(X) \to \mathrm{AG}\,(X/A)$ is a numerable $(G, \alpha, \mathrm{AG}\,(A))$-bundle, and Lemmas 2.2(b) and (2.5) give the isomorphisms

$$
\begin{aligned}
\mathrm{AG}\,(X/A) = \mathrm{AG}\left(\bigvee_{\lambda \in \Lambda} (G/H_\lambda)_+ \wedge S^{p_\lambda} \right) &\cong \bigoplus_{\lambda \in \Lambda} \mathrm{AG}\,((G/H_\lambda)_+ \wedge S^{p_\lambda}) \\
&\cong \bigoplus_{\lambda \in \Lambda} F((G/H_\lambda)_+, \mathrm{AG}\,(S^{p_\lambda})).
\end{aligned}
$$

One can use the fact that G/H is finite, and that $\mathrm{AG}\,(S^n)$ is a (non-equivariant) $\mathcal{CW}$-complex, to conclude that $F((G/H_\lambda)_+, \mathrm{AG}\,(S^{p_\lambda}))$ is a G-$\mathcal{CW}$-complex. Finally observe that the total space of a numerable $(G, \alpha, \mathrm{AG}\,(A))$-bundle, whose fiber and base spaces are G-$\mathcal{CW}$-complexes, admits a structure of G-$\mathcal{CW}$-complex.

The proof then proceeds by induction over the skeleta of X. ☐

Remark 2.10. The above results then show that the functor $\mathrm{AG}\,(-)$ preserves spaces of the homotopy type of G-$\mathcal{CW}$-complexes.

Corollary 2.11. *If $C(f)$ denotes the cofiber of a G-map $f : X \to Y$ of G-$\mathcal{CW}$-complexes, then the sequence $AG\,(X) \to AG\,(Y) \to AG\,(C(f))$ is homotopy equivalent to a G-fibration. In other words, any* **homotopy lifting problem** *for the map $AG(Y) \to AG(C(f))$, in the category of G-spaces, has a solution Φ.*

Proof. It follows from [Wan80] that one can find a diagram of G-$\mathcal{CW}$-complexes

$$\begin{array}{ccccc} A & \longrightarrow & T & \longrightarrow & T/A \\ \uparrow & & \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y & \longrightarrow & C(f) \end{array}$$

where (T,A) is a G-simplicial pair, and all vertical arrows are equivariant homotopy equivalences. Applying the functor $\mathrm{AG}\,(-)$ gives a commutative diagram whose vertical arrows are again homotopy equivalences, and whose top row is a numerable, $(G,\alpha,\mathrm{AG}\,(A))$-bundle. □

Corollary 2.12. *Given X and Y as above, and any subgroup $H<G$, the sequence*

$$AG\,(Y)^H \to AG\,(X)^H \to \{AG\,(X)\,/AG\,(Y)\}^H$$

is a principal fibration.

Proof. Since $\mathrm{AG}\,(Y)^H$ is a subcomplex of $\mathrm{AG}\,(X)^H$, the equivariant local section constructed in the proposition restricts to a local section of the corresponding fixed point set sequence. The result now follows from standard arguments, e.g. [Ste51, §7.4]. □

2.3. **Equivariant FSP structure.** We now introduce a few natural maps which will play an importante role in later constructions. Given based G-spaces X and Y, let

$$\mathrm{AG}\,(X)\wedge \mathrm{AG}\,(Y) \to \mathrm{AG}\,(X\wedge Y) \tag{2.12.1}$$

be the natural pairing which is given as the bilinear extension of the natural inclusion $j_{X\wedge Y}: X\wedge Y \to \mathrm{AG}\,(X\wedge Y)$. The composition of this map with $j_X\wedge 1 : X\wedge \mathrm{AG}\,(Y) \to \mathrm{AG}\,(X)\wedge\mathrm{AG}\,(Y)$ or $1\wedge j_Y : \mathrm{AG}\,(X)\wedge Y \to \mathrm{AG}\,(X)\wedge\mathrm{AG}\,(Y)$ yields equivariant embeddings

$$L_{X,Y} : X\wedge \mathrm{AG}\,(Y) \to \mathrm{AG}\,(X\wedge Y)\,, \tag{2.12.2}$$

and

$$R_{X,Y} : \mathrm{AG}\,(X)\wedge Y \to \mathrm{AG}\,(X\wedge Y)\,, \tag{2.12.3}$$

respectively.

Remark 2.13. When G is the trivial group, the map (2.12.1) induces a pairing in homotopy groups $\pi_j(\mathrm{AG}\,(X))\otimes\pi_k(\mathrm{AG}\,(Y)) \to \pi_{j+k}(\mathrm{AG}\,(X\wedge Y))$, which corresponds, under the Dold-Thom isomorphism [DT56] to the usual exterior product in reduced singular homology: $\tilde{H}_j(X,\mathbb{Z})\otimes\tilde{H}_k(Y,\mathbb{Z}) \to \tilde{H}_{j+k}(X\wedge Y,\mathbb{Z})$.

Now, let $h_{X,Y} : \mathrm{AG}\,(X) \to F(Y,\mathrm{AG}\,(X\wedge Y))$ be the adjoint of $R_{X,Y}$, and note that this is a topological group homomorphism (in the category of G-spaces) when $F(Y,\mathrm{AG}\,(X\wedge Y))$ is given the group structure under pointwise addition.

In the particular case where $Y=S^V$, for a G-module V, we denote this map by

$$h_V : \mathrm{AG}\,(X) \to \Omega^V\mathrm{AG}\,\left(S^V X\right). \tag{2.13.1}$$

Remark 2.14. When G is the trivial group and $V = \mathbb{R}^n$, the map (2.13.1) induces a homomorphism $h_{n*} : \pi_j(\mathrm{AG}\,(X)) \to \pi_j(\Omega^n \mathrm{AG}\,(S^n X)) \cong \pi_{j+n}(\mathrm{AG}\,(S^n X))$, which under the Dold-Thom isomorphism [DT56] is nothing else than the classical suspension isomorphism $s_n : \tilde{H}_j(X;\mathbb{Z}) \xrightarrow{\cong} \tilde{H}_{j+n}(S^n X;\mathbb{Z})$.

The other map of interest is constructed as follows. The natural inclusion $X \to \mathrm{AG}\,(X)$ induces a G-map $F(Y,X) \to F(Y,\mathrm{AG}\,(X))$ which, by universality, induces a homomorphism $t_{X,Y} : \mathrm{AG}\,(F(Y,X)) \to F(Y,\mathrm{AG}\,(X))$.

In the particular case where $Y = S^V$, for a G-module V, one has a natural G-map

$$t_V : \mathrm{AG}\,\left(\Omega^V X\right) \to \Omega^V \mathrm{AG}\,(X)\,. \tag{2.14.1}$$

Remark 2.15. When G is trivial, and $V = \mathbb{R}^n$, the map t_n induces, once again under the Dold-Thom isomorphism, a homomorphism $t_{n*} : \tilde{H}_j(\Omega^n X,\mathbb{Z}) \to \tilde{H}_{j+n}(X,\mathbb{Z})$, which corresponds to the homology suspension homomorphism. See [Whi78, §VIII].

Consider a G-map $f : X \wedge Y \to Z$ and denote its adjoint by $\tilde{f} : X \to F(Y,Z)$. One also has associated induced maps $f_* : \mathrm{AG}\,(X \wedge Y) \to \mathrm{AG}\,(Z)$ and $F(1,f_*) : F(Y,\mathrm{AG}\,(X \wedge Y)) \to F(Y,\mathrm{AG}\,(Z))$.

Lemma 2.16. *The compositions* $AG(X) \xrightarrow{\tilde{f}_*} AG(F(Y,Z)) \xrightarrow{t_{Y,Z}} F(Y,AG(Z))$ *and* $AG(X) \wedge Y \xrightarrow{R_{X,Y}} AG(X \wedge Y) \xrightarrow{f_*} AG(Z)$ *are adjoint maps.*

Proof. Canonical adjunction arguments imply that $F(1,f_*) \circ h_{X,Y} = (f_* \circ R_{X,Y})\tilde{}$, where tilde mean adjoint maps, and one has a commutative diagram

$$\begin{array}{ccc} \mathrm{AG}\,(X) & \xrightarrow{h_{X,Y}} & F(Y,\mathrm{AG}\,(X \wedge Y)) \\ {\scriptstyle \tilde{f}_*}\big\downarrow & & \big\downarrow{\scriptstyle F(1,f_*)} \\ \mathrm{AG}\,(F(Y,Z)) & \xrightarrow{t_{Y,Z}} & F(Y,\mathrm{AG}\,(Z)). \end{array} \tag{2.16.1}$$

The two assertions together prove the lemma. □

2.4. **Behavior on G-LEC's and cofibrations.** Our last technical results analyze how the functor $\mathrm{AG}\,(-)$ and the various natural maps introduced above behave with respect to equivariant cofibrations.

Given a collection $\Gamma = \{\Gamma_1,\dots,\Gamma_r\}$ of subsets Γ_j of $[k] = \{1,\dots,k\}$, we denote by Δ_X^Γ the subset $\cup_{j=1}^r \Delta_X^{\Gamma_j}$ of X^k, where $\Delta_X^{\Gamma_j}$ is the "diagonal" in X^k consisting of those k-tuples of points in X^k whose coordinates indexed by Γ_j coincide. The *small* diagonal Δ_X^k is defined to be Δ_X^Γ, where $\Gamma = [k]$, in other words, all coordinates coincide.

Definition 2.17.

a. A G-map $i : A \to X$ is said to be a G-cofibration if it satisfies the *equivariant homotopy extension property* (G-HEP); see [BV73, App. 2].

b. A G-space X is said to be G-locally equiconnected (G-LEC) if the diagonal map $\Delta_X : X \to X \times X$ is a G-cofibration.

c. We say that a G-space X is a *strong* G-LEC if the inclusion of the small diagonal $\delta_X^k : X \to X^k$ is a $(G \times \mathfrak{S}_k)$-cofibration (see [LF]), for all $k \geq 1$, where $\mathfrak{S}_k$ is the symmetric group on k letters.

Remark 2.18. As in the non-equivariant case, one can prove that a G-$\mathcal{CW}$-complex is a strong G-LEC; cf. [LF].

Proposition 2.19. *Let X be a based strong G-LEC. Then:*

a. *For each $k \geq 1$ the k-fold symmetric product $SP^k(X)$ is a strong G-LEC.*

b. *Let $\{0\} \subseteq \ldots \subseteq AG(X)^{\leq n} \subseteq AG(X)^{\leq n+1} \subseteq \ldots$ be the filtration of $AG(X)$ given by $AG(X)^{\leq n} = \mathcal{Z}(X)^{\leq 2n} \cap AG(X)$, cf.(2.0.1). Then every $AG(X)^{\leq n}$ is a G-LEC and all inclusions $AG(X)^{\leq n} \hookrightarrow AG(X)^{\leq n+1}$ are G-cofibrations. In particular, $AG(X)$ is a G-LEC.*

c. *The natural map $j_X : X \to AG(X)$ is a G-cofibration.*

Proof. See Appendix A. □

Proposition 2.20. *Let X and Y be based strong G-LEC's. Then:*

a. *The smash product $X \wedge Y$ is a strong G-LEC.*

b. *If $f : X \to Y$ is a G-cofibration, then $f_* : AG(X) \to AG(Y)$ is a G-cofibration.*

c. *The natural map $R_{X,Y} : AG(X) \wedge Y \to AG(X \wedge Y)$ is a G-cofibration.*

Proof. See Appendix A. □

3. Mackey and Hecke Functors

Generalities

We first recall the notion of a Mackey functor for a finite group G, with values in the category of abelian groups $\mathcal{A}b$, introduced by Dress in [Dre72]. There are other equivalent formulations of this notion in the literature, e.g. [Lew], [LMS85, p. 288], which are more convenient for the study of equivariant stable homotopy theory, but the present approach suffices for our purposes.

Let $\mathcal{F}_G$ be the category of finite (left) G-sets and G-maps between them. A Mackey functor M with values in $\mathcal{A}b$ consists of a covariant and contravariant functor from $\mathcal{F}_G$ to $\mathcal{A}b$ which has a unique object function $S \mapsto M(S)$ converting disjoint unions to coproducts. For a G-map $\sigma : S \to T$ one has $M_*(\sigma) : M(S) \to M(T)$ and $M^*(\sigma) : M(T) \to M(S)$, the covariant and contravariant values of M on the morphism σ (usually called the transfer -

or induction - and restriction maps). Those maps must transform a pull-back diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

in $\mathcal{F}_G$ into a commutative diagram

$$\begin{tikzcd} M(A) \arrow[r, "M_*(f)"] & M(B) \\ M(C) \arrow[u, "M^*(g)"] \arrow[r, "M_*(k)"'] & M(D) \arrow[u, "M^*(h)"'] \end{tikzcd}$$

in $\mathcal{A}b$. Morphisms of Mackey functors are defined in an obvious way and the resulting category $\mathcal{M}_{\mathcal{A}b}$ of Mackey functors with values in $\mathcal{A}b$ is abelian.

Another related notion is that of a *Hecke functor*, defined as follows. Let $\mathbb{Z}G$ be the group-ring of G, and let $\mathbb{Z}\mathcal{G}$ denote the category with objects $\{G/H \mid H \in \mathcal{F}\}$ where $\mathcal{F}$ is a collection of subgroups of G containing exactly one subgroup in each conjugacy class of subgroups of G, and whose morphisms are the $\mathbb{Z}G$-module maps $\mathbb{Z}[G/H] \to \mathbb{Z}[G/K]$. Here $\mathbb{Z}[G/L]$ denotes the free $\mathbb{Z}G$-module generated by the G-set G/L. A *Hecke functor* T is defined to be a contravariant functor $T : \mathbb{Z}\mathcal{G} \to \mathcal{A}\lfloor$.

Lemma 3.1. *A Hecke functor T defines, in a natural way, a unique Mackey functor M_T with the property that $M_T(G/H) = T(G/H)$, for all $H \in \mathcal{F}$. This induces an embedding of the category of Hecke functors into $\mathcal{M}_{\mathcal{A}b}$.*

Proof. Extend M_T additively to all of $\mathcal{F}_G$, since any $S \in \mathcal{F}_G$ can be writen uniquely as $S = \coprod G/H,\ H \in \mathcal{F}$. One then just needs to define $M_T^*(f)$ and $M_{T*}(f)$ for G-maps $f : G/H \to G/K$, with $H, K \in \mathcal{F}$. Since such G-map induces a $\mathbb{Z}G$-homomorphism $f_* : \mathbb{Z}G/H \to \mathbb{Z}G/K$, one then defines $M_T^*(f) = T(f_*)$. On the other hand, f also induces a $\mathbb{Z}G$-map $f^* : \mathbb{Z}G/K \to G/H$ sending eK to $\sum_i k_i H$, where $\{k_i\}$ is a collection of representatives for K/H^a, and H^a is the subgroup of K conjugate to H which induces f. One now defines $M_{T*} = T(f^*)$. It is easy now to verify that (M_T, M_T^*, M_{T*}) is a Mackey functor. ☐

Given a $\mathbb{Z}G$-module M, let M^H denote the fixed point set of M under the action of a subgroup $H < G$, and let $F(S_+, M)^G$ denote the group of G-equivariant pointed maps from S_+ to M, where $0 \in M$ is the base point. Notice that, given a subgroup $H < G$, there is only one subgroup $H' \in \mathcal{F}$ such that $F(G/H_+, M)^G \cong Z^{H'}$. The following result is well-known, cf. [Nis84], as well as some variations of it, cf. [GM92, Defn. 1].

Proposition 3.2. *Given a $\mathbb{Z}G$-module M the assignment $\Phi_M(S) = F(S_+, M)^G$ can naturally be given the structure of a Mackey functor with values in $\mathcal{A}b$.*

Proof. For $H \in \mathcal{F}$ one has $F(S_+, M)^G \cong M^H \cong Mor_{\mathbb{Z}G}(\mathbb{Z}[G/H], M)$, and that the assignment $\underline{M} : G/H \mapsto Mor_{\mathbb{Z}G}(\mathbb{Z}[G/H], M)$ is obviously a Hecke functor. The functor Φ_M is precisely the Mackey functor associated to $\underline{M}$, as in the previous Lemma. $\square$

Corollary 3.3. *The assignment $M \mapsto \Phi_M$ defines a covariant functor from the category ${}_{\mathbb{Z}G}Mod$ of $\mathbb{Z}G$-modules to the category $\mathcal{M}_{\mathcal{A}b}$ of Mackey functors.*

Remark 3.4. 1. Recall that, given a Mackey functor $M \in \mathcal{M}_{\mathcal{A}b}$ the abelian group $M(G/e)$ has a natural action by G. This defines a covariant functor $F : \mathcal{M}_{\mathcal{A}b} \to {}_{\mathbb{Z}G}\mathrm{M}od$ from the category of Mackey functors to the category of G-modules. It is not hard to see that the functor $\Phi : {}_{\mathbb{Z}G}\mathrm{M}od \to \mathcal{M}_{\mathcal{A}b}$ is a left adjoint to F; cf. [GM92, Prop. 2].

Topological Mackey functors

The notion of Mackey functors that we presented can be extended in various ways, for example, by allowing the functor to take values in other categories than $\mathcal{A}b$. More generally, one can work with the notion of Mackey functors between suitable categories; cf. [Lin76]. Here we are concerned with an instance of such generalizations.

Definition 3.5. A *topological Mackey functor* is a Mackey functor with values in the category $\mathcal{ATOG}$ consisting of abelian topological groups and continuous homomorphisms.

We have seen in the previous section that one has a covariant functor $X \mapsto \mathrm{AG}\,(X)$ from the category G-$\mathfrak{CG}_*$ of based G-spaces to the category of topological G-modules. Composing this functor with Φ yields the following result.

Proposition 3.6. *The assignment of $F_X \stackrel{def}{=} \Phi_{AG(X)}$ to a based G-space X defines a covariant functor $F : G\text{-}\mathfrak{CG}_* \to \mathcal{M}_{\mathcal{ATOG}}$, from the category of based G-spaces to the category of topological Mackey functors.*

Corollary 3.7. *Composing F with any homotopy functor gives a functor from the category of based G-spaces to the category of Mackey functors.*

Example 3.8. There are various other instances where one has functors from suitable category of spaces to topological Mackey functors.

- **a:** Let M be a compact Riemannian G-manifold, on which G acts by isometries. The group $\mathcal{I}_k(M)$ of integral k-currents on M, endowed with the flat-norm topology, then becomes a topological abelian G-group. In particular, $\Phi_{\mathcal{I}_k(M)}$ becomes a topological Mackey functor, and the assignment $M \mapsto \Phi_{\mathcal{I}_k(M)}$ is also functorial.
- **b:** Given a complex algebraic variety X where G acts by algebraic automorphisms, then G also acts on the topological group $\mathcal{Z}_p(X)$ of algebraic p-cycles on X. For a detailed description of the topological group structure on $\mathcal{Z}_p(X)$ we refer the reader to [LF94]. Once again, $\Phi_{\mathcal{Z}_p(X)}$ becomes another topological Mackey functor, and this assignment is functorial in X.

4. Unstable equivariant Dold-Thom results

As an immediate consequence of Corollary 3.7 it follows that, given a G-$\mathcal{CW}$-pair (X, A), the assignment $G/H \mapsto \pi_j(F_{X/A}(G/H)) \cong \pi_j(\mathrm{AG}\,(X/A)^H)$ defines a Mackey functor $\underline{\pi}_j \circ F_{X/A}$, whose identification is the main purpose of this section.

Let us start with a G-$\mathcal{CW}$-pair (X, A) and recall that the projection $\mathrm{AG}\,(X) \to \mathrm{AG}\,(X/A)$ is homotopy equivalent to a locally trivial, numerable $(G, \alpha, \mathrm{AG}\,(A))$-bundle in the sense of Definition 2.6, as shown in Corollary 2.11. The standard homotopy lifting problem induced by the evaluation map $I \times \Omega\mathrm{AG}\,(X/A) \to \mathrm{AG}\,(X/A)$, produces a connecting G-map $\Omega\mathrm{AG}\,(X/A) \to \mathrm{AG}\,(A)$, unique up to G-homotopy which has the following property:

Proposition 4.1. *The connecting map $\Omega AG\,(X/A) \to AG\,(A)$ induces a exact sequence*

$$\cdots \to \Omega^n AG\,(X) \to \Omega^n AG\,(X/A) \to \Omega^{n-1} AG\,(A) \to \cdots$$

in the homotopy category of G-spaces. In other words, given a based G-space K, one has an exact sequence of abelian groups

$$\cdots \to [K, \Omega^n AG\,(X)]_G \to [K, \Omega^n AG\,(X/A)]_G \to [K, \Omega^{n-1} AG\,(A)]_G \to \cdots .$$

Given a G-module V, denote by $\underline{\pi}_{V+n} \circ F_X$ the Mackey functor which sends G/H to $\pi_j\left(F(S^V; \mathrm{AG}\,(X))^H\right)$.

Corollary 4.2. *For any G-module V one has a long exact sequence of Mackey functors*

$$\cdots \to \underline{\pi}_{V+n} \circ F_X \to \underline{\pi}_{V+n} \circ F_{X/A} \to \underline{\pi}_{V+n-1} \circ F_A \to \cdots ,$$

natural on G-$\mathcal{CW}$-pairs (X, A).

Corollary 4.3. *Given a finite G-$\mathcal{CW}$-complex X and a G-module V, there is a spectral sequence of Mackey functors converging to $\underline{\pi}_{V+*} \circ F_X$ whose E^1 term is given by*

$$E^1_{p,q}(-) = \oplus_{\lambda \in \Lambda_p} \pi_{V+p+q}\left(F(-,\ AG\,((G/H_\lambda)_+ \wedge S^p))\right),$$

where Λ_p is the indexing set of the p-cells $G/H_\lambda \times D^p$ of a G-$\mathcal{CW}$ decomposition of X.

Proof. This follows in the standard way from the previous corollary, the definition of G-$\mathcal{CW}$-complex and Lemma 2.2. □

Lemma 4.4. *For any G-module V, with $\dim V^G \geq 2$, one has $\underline{\pi}_{V+q} \circ F_{S^V} = 0$ as a Mackey functor, for $q > 0$.*

Proof. Let us denote by X_p the p-th skeleton of a G-$\mathcal{CW}$ decomposition of $X = S^V$ and let N be a subgroup of G. The Mackey funtor valued spectral sequence defined in the previous corollary, when applied to G/N, has the form

$$\begin{aligned} E^1_{p,q}(G/N) &= \oplus_\lambda [S^{p+q} \wedge S^V \wedge (G/N)_+\ ,\ \mathrm{AG}\,((G/H_\lambda)_+ \wedge S^p)]_G \\ &\cong [S^{p+q} \wedge S^V \wedge (G/N)_+\ ,\ F(G/H_\lambda, \mathrm{AG}\,(S^p))]_G \\ &\cong [S^{p+q} \wedge \left(S^V \wedge (G/N)_+ \wedge (G/H_\lambda)_+\right)\ ,\ \mathrm{AG}\,(S^p)]_G. \end{aligned}$$

However, since G acts trivially on both S^p and S^{p+q}, the latter group is isomorphic to $[S^{p+q} \wedge \left(S^V \wedge (G/N)_+ \wedge (G/H_\lambda)_+\right)/G\ ,\ \mathrm{AG}\,(S^p)]$. This is now clearly seen to be a trivial group whenever $p+q>0$, for $\mathrm{AG}\,(S^p)$ is an Eilenberg-MacLane space of type $K(\mathbb{Z},p)$. $\square$

We now proceed to collect a few properties of the Mackey functors $\underline{\pi}_j \circ F_{X/A}$, adopting the convention that the symbol $X/\emptyset$ means X_+.

1. Functoriality: The assignment

$$(X,A) \mapsto \mathfrak{h}_j(X,A) \stackrel{\text{def}}{=} \underline{\pi}_j \circ F_{X/A}$$

is covariantly functorial on pairs of G-$\mathcal{CW}$-complexes; cf. Corollary 3.7.

2. Long Exact Sequences: It follows from Proposition 4.1 that there is a natural transformation $\delta : \mathfrak{h}_j(X,A) \to \mathfrak{h}_{j-1}(A,\emptyset)$ which leads to an exact sequence:

$$\ldots \to \mathfrak{h}_j(A,\emptyset) \to \mathfrak{h}_j(X,\emptyset) \to \mathfrak{h}_j(X,A) \to \mathfrak{h}_{j-1}(A,\emptyset) \to \ldots$$

3. Homotopy Axiom: The fact that two G-homotopic maps $f_0, f_1 : (X,A) \to (Y,B)$ induce equivariant homomorphisms $f_{0*}, f_{1*} : \mathrm{AG}\,(X/A) \to \mathrm{AG}\,(Y/B)$ which are G-homotopic through homomorphisms, as explained in (2.1[e]), immediately shows that f_0 and f_1 induce the same map on $\mathfrak{h}_*(X,A)$.

4. Excision Axiom: If (X,A) and (Y,B) are relatively homeomorphic G-$\mathcal{CW}$-pairs, one immediately sees that $\mathrm{AG}\,(X/A)$ and $\mathrm{AG}\,(Y/B)$ are G-isomorphic topological groups. The excision axiom $\mathfrak{h}_j(X,A) \cong \mathfrak{h}_j(Y,B)$ follows immediately from this.

5: Dimension Axiom: Let H and K be subgroups of G. Since G/H is a finite G-set, the topological group $\mathrm{AG}\,(G/H_+)$ is discrete and naturally isomorphic to $\mathcal{Z}(G/H) = \mathbb{Z}[G/H]$, the free G-module generated by G/H. Therefore $\mathfrak{h}_j(G/H,\emptyset) = 0$, for $j>0$, and

$$\mathfrak{h}_0(G/H,\emptyset)[G/K] = \pi_0((\mathbb{Z}[G/H]^K)) \cong \mathbb{Z}[G/H]^K \cong \mathbb{Z}[K\backslash G/H].$$

In particular, $\mathfrak{h}_0(G/H,\emptyset)[G/G] \cong \mathbb{Z}$.

Gathering all those properties together, one obtains the following result:

Theorem 4.5. *The Mackey functors $\mathfrak{h}_j(X,A)$ give precisely the Bredon homology of the pair (X,A) with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$.*

Proof. Our functors satisfy all necessary axioms which uniquely characterize Bredon homology. See [Ill75] and [tD87, §II(6.7)]. $\square$

Remark 4.6. In order to compatibilize notation, let us denote $\mathfrak{h}_j(X,A)[G/G]$ by $H_j^G(X,A;\underline{\mathbb{Z}})$. This corresponds to the usual Bredon homology groups, which in the usual approach are extended to a Mackey functor by defining $H_j^G(X,A;\underline{\mathbb{Z}})[G/H] = H_j^G(G/H \times (X,A);\underline{\mathbb{Z}})$, see [tD87, p. 163]. Note however that here

$$\begin{aligned}\mathfrak{h}_j(X,A)[G/H] &= \pi_j(F(G/H_+, \mathrm{AG}\,(X/A))^G) \cong \pi_j(\mathrm{AG}\,(G/H_+ \wedge X/A)) \\ &= \pi_j(\mathrm{AG}\,((G/H \times X)/(G/H \times A))) = H_j^G(G/H \times (X,A);\underline{\mathbb{Z}}).\end{aligned}$$

It is a simple exercise to show that the transfer and restriction maps for both approaches coincide. However, it might be interesting to recall at this point how the transfer and restriction maps for Bredon homology are defined under our approach. Due to the additive character of Mackey functors, it suffices to understand the maps induced by a G-map $\phi : G/K \to G/H$.

For a given G-space X, the *transfer* map $\phi^* : H_j^G(X;\underline{\mathbb{Z}})[G/H] \to H_j^G(X;\underline{\mathbb{Z}})[G/K]$ is simply the map on homotopy groups induced by $\phi^* : F(G/H, \mathrm{AG}\,(X)) \to F(G/K, \mathrm{AG}\,(X))$. While the *restriction* map $\phi_* : H_j^G(X;\underline{\mathbb{Z}})[G/K] \to H_j^G(X;\underline{\mathbb{Z}})[G/K]$ uses $\phi \wedge 1 : G/K_+ \wedge X \to G/H_+ \wedge X$ to define ϕ_* as the map induced on homotopy groups by the composition of G-maps

$$F(G/K_+, \mathrm{AG}\,(X)) \cong \mathrm{AG}\,(G/K_+ \wedge X) \xrightarrow{(\phi\wedge 1)_*} \mathrm{AG}\,(G/H_+ \wedge X) \cong F(G/H_+, \mathrm{AG}\,(X)),$$

where the isomorphisms come from Proposition 2.5.

Notice that up to isomorphism one can assume that $K < H$, $F(G/H, \mathrm{AG}\,(X))^G \cong \mathrm{AG}\,(X)^H$ and $F(G/K, \mathrm{AG}\,(X))^G \cong \mathrm{AG}\,(X)^K$. In this case the restriction map is induced by the inclusion $\mathrm{AG}\,(X)^K \subset \mathrm{AG}\,(X)^H$, while the transfer is induced by the homomorphism $\mathrm{AG}\,(X)^H \to \mathrm{AG}\,(X)^K$ sending $\sigma \in \mathrm{AG}\,(X)^H$ to $\sum_i^r h_i * \sigma$, when one writes $H/K = \{h_1K, \dots, h_rK\}$.

5. Stable Constructions

In this section we provide the appropriate stable generalization of our previous results. We first recall a few definitions and facts which can be found in [LMS85, §1]. Throughout this whole discussion we fix a complete G-universe $\mathcal{U}$, i.e., a unitary G-module containing countably many copies of each irreducible representation of G. For example, take $\mathcal{U}$ to be a countable direct sum $V_\rho \oplus V_\rho \oplus \dots$ of the right regular representation V_ρ of G. In this context, G-modules are meant to be finite submodules of $\mathcal{U}$, the so-called *indexing spaces* in the terminology of [LMS85]. An *indexing set* $\mathcal{A}$ in $\mathcal{U}$ is a cofinal collection of indexing spaces containing $\{0\}$. The one-point compactification of a G-module V is denoted S^V and the V-th suspension of a G-space X is the G-space $\Sigma^V X \overset{\mathrm{def}}{=} X \wedge S^V$.

Definition 5.1. a: A G-**prespectrum** E indexed on an indexing set $\mathcal{A}$, is a collection $\{E(V) \mid V \in \mathcal{A}\}$ of G-spaces, together with G-maps $\sigma_V^W : \Sigma^{W-V}E(V) \to E(W)$, where V is a G-submodule of W and $W - V$ denotes the orthogonal complement of V in W. Those maps must satisfy appropriate compatibility conditions; cf. [LMS85, Defn. I.2.1]. One has the notion of a **morphism** of G-prespectra $f : E \to E'$, which is a compatible collection of G-maps $f_V : E(V) \to E'(V)$. The collection of all G-prespectra forms a category which we denote by $G\mathcal{P}$, where the set of morphims from E to E' is denoted by $G\mathcal{P}(E, E')$.

b: If all structural maps $\sigma_V^W : \Sigma^{W-V}E(V) \to E(W)$ are topological embeddings of spaces, we call E a *Σ-inclusion G-prespectrum*; if they are G-cofibrations we call E a *Σ-cofibrant* prespectrum.

c: Denoting the adjoint map to σ_V^W by $\tilde{\sigma}_V^W : E(V) \to \Omega^{W-V}E(W)$, one calls E an *inclusion* prespectrum if all $\tilde{\sigma}_V^W$'s are topological embeddings. Note that a Σ-inclusion prespectrum is necessarily an inclusion prespectrum. One denotes the full subcategory of all inclusion G-prespectra by $G\mathcal{I}$.

d: A **spectrum** E is a prespectrum where all structural maps $\tilde{\sigma}_V^W : E(V) \to \Omega^{W-V}E(W)$ are G-homeomorphisms. The collection of all spectra forms a full subcategory of $G\mathcal{P}$, which we denote by $G\mathcal{S}$, and we denote the set of morphisms of spectra between E and E' by $G\mathcal{S}(E, E')$.

e: Given a G-spectrum E and a G-space X, the collection $F(X, E(V))$ is again a G-spectrum under the evident natural maps. In particular, one can define for every G-spectrum E and G-module V, the V-th loop spectrum of E: $\Omega^V E \stackrel{\text{def}}{=} F(S^V, E)$.

Remark 5.2. Our notation agrees with that of [LMS85], except for the fact that we supress mention to the indexing set $\mathcal{A}$. Unless otherwise stated, we only consider the indexing set consisting of all submodules of $\mathcal{U}$.

It is clear from the definitions above that a G-spectrum is automatically a G-prespectrum and that this defines a "forgetful" functor $\ell : G\mathcal{S} \to G\mathcal{P}$. Unfortunately, the category of G-spectra is not closed under basic space-level constructions, such as taking smash products with a fixed G-space or taking space level colimits, which are operations which exist in the category of G-prespectra. To remedy this handicap one resorts to the existence of a left adjoint to ℓ, a functor L sending G-prespectra to G-spectra and having all the benefits of being a left adjoint functor. This is the content of the following fundamental result, found in [LMS85, Thm. I.2.2].

Theorem 5.3. *There is a left adjoint $L : G\mathcal{P} \to G\mathcal{S}$ to the forgetful inclusion functor $\ell : G\mathcal{S} \to G\mathcal{P}$. That is, $G\mathcal{P}(D, \ell E) \cong G\mathcal{S}(LD, E)$ for $D \in G\mathcal{P}$ and $E \in G\mathcal{S}$. Let $\eta : D \to \ell LD$ and $\epsilon : L\ell D \to E$ be the unit and counit of the adjunction. Then ϵ is an isomorphism for each G-spectrum E, hence η is an isomorphism if $D = \ell E$.*

Remark 5.4. When E is an inclusion prespectrum, LE is defined by

$$(LE)(V) = \varinjlim_{W \supset V} \Omega^{W-V}E(W),$$

where for $W \supset W' \supset V$ the maps in the colimit are given by the compositions:

$$\Omega^{W'-V}E(W') \xrightarrow{\Omega^{W'-V}(\sigma_{W'}^W)} \Omega^{W'-V}\left(\Omega^{W-W'}E(W)\right) \cong \Omega^{W-V}E(W).$$

This remark will be thoroughly used in our constructions.

Definition 5.5 ([LMS85, §I]).

a. For any G-space X and G-prespectrum E, one can define a G-prespectrum $E \wedge X$ by $E(V) \wedge X$ with the evident structural maps, and the associated spectrum is then defined as $L(E \wedge X)$. Both the spectrum and the prespectrum will be denoted by $E \wedge X$, leaving the context to appropriately determine which notion is being used. For example, given a

G-module V one defines the V-th suspension $\Sigma^V E$ of a G-spectrum E as $E \wedge S^V$; the **cone** $CE \stackrel{\text{def}}{=} E \wedge I$, where 1 is the basepoint of I, and the homotopical cofiber $Cf \stackrel{\text{def}}{=} E' \cup_f E$ of a map $f : E \to E'$ of G-spectra; one can also define the cylinder $E \wedge I_+$ and introduce the notions of *homotopies*, etc. With those notions available, we introduce the following notation:

1. $[X, X']_G$= equivariant homotopy classes of maps of G-spaces.
2. $\omega(X, X')_G$= stable equivariant homotopy classes of maps of G-spaces.
3. $\{E, E'\}_G$ = equivariant homotopy classes of maps of G-spectra. Note that $\omega(X, X')_G = \{\Sigma^\infty X, \Sigma^\infty X'\}_G$, where $\Sigma^\infty X$ is introduced in the following example.

b. A G-$\mathcal{CW}$-prespectrum D is a G-prespectrum D satisfying:

1. Every space $D(V)$ is a G-LEC of the homotopy type of a G-$\mathcal{CW}$-complex.
2. Every structural map $D(V) \wedge S^{W-V} \to D(W)$ is a G-cofibration.

A G-spectrum E is a G-$\mathcal{CW}$-spectrum if it has the form LD, where D is a G-$\mathcal{CW}$-prespectrum. This definition is equivalent to the usual one [LMS85], where one builds up the spectrum by attaching cells in the category of spectra; cf. [EK$^+$95].

c. For notational convenience we call D a *strong* G-$\mathcal{CW}$-prespectrum if in addition $D(V)$ is a *strong* G-LEC for every V; cf. Definition 2.17(c).

Example 5.6. One can associate to a G-space X a Σ-inclusion prespectrum $S^\infty X$ such that $S^\infty X(V) = \Sigma^V X$ and the structural maps σ_V^W are the identifications $\Sigma^{W-V}(\Sigma^V X) = \Sigma^W X$. This defines an embedding of the category of G-spaces into the category of G-prespectra. Define the suspension spectrum $\Sigma^\infty X$ as $L(S^\infty X)$, and when $X = S^0$ denote $S^\infty S^0$ by S^∞ and $\Sigma^\infty S^0$ by $\mathbb{S}$. We have chosen to differentiate the notations for the prespectrum $S^\infty X$ and the spectrum $\Sigma^\infty X$ in order to make later proofs more transparent. It is evident that $S^\infty X$ and $\Sigma^\infty X$ are G-$\mathcal{CW}$ objects in their respective categories.

Prespectrum level properties

The natural maps (2.12.3), (2.13.1) and (2.14.1) will be used to introduce the main construction of this section. Given a G-prespectrum E with structural maps $\sigma_V^W : \Sigma^{W-V} E(V) \to E(W)$, define $\mathrm{AG}\,(E)\,(V) \stackrel{\text{def}}{=} \mathrm{AG}\,(E(V))$, and given G-modules $V \subset W$, define $\underline{\sigma}_V^W : \Sigma^{W-V}\mathrm{AG}\,(E)\,(V) \to \mathrm{AG}\,(E)\,(W)$ as the composition

$$\mathrm{AG}\,(E(V)) \wedge S^{W-V} \xrightarrow{R_{E(V),S^{W-V}}} \mathrm{AG}\left(E(V) \wedge S^{W-V}\right) \xrightarrow{\sigma_{V*}^W} \mathrm{AG}\,(E(W)).$$

It follows from Lemma 2.16 that the adjoint map $\widetilde{\underline{\sigma}_V^W}$ to $\underline{\sigma}_V^W$ is given by the composition

$$\mathrm{AG}\,(E(V)) \xrightarrow{\tilde{\sigma}_{V*}^W} \mathrm{AG}\left(\Omega^{W-V} E(W)\right) \xrightarrow{t_{W-V}} \Omega^{W-V}\mathrm{AG}\,(E(W)).$$

The naturality of these contructions, together with the commutativity of the diagram in the proof of Lemma 2.16, implies the following result.

Proposition 5.7. *The collection $\{AG(E)(V),\ \underline{\sigma}_V^W\}$, defines a G-prespectrum indexed on $\mathcal{U}$.*

Definition 5.8. Given a G-prespectrum E, denote the G-prespectrum introduced in the previous proposition by $\mathrm{AG}\,(E)$, and denote by $\mathbb{Z}(E)$ the associated G-spectrum $L(\mathrm{AG}\,(E))$.

The various natural properties of the abelian group functor of G-spaces $\mathrm{AG}\,(-)$, discussed in §2, directly reflect upon the constructions just described.

Proposition 5.9.

1. *The assignment $E \mapsto AG(E)$ defines a covariant, homotopy preserving, functor from $G\mathcal{P}$ to $G\mathcal{P}$ which sends Σ-inclusion prespectra to Σ-inclusion prespectra.*

2. *For any prespectrum E and subgroup $H < G$, the prespectrum $AG(E \wedge G/H_+)$ is naturally isomorphic to $F(G/H_+, AG(E))$.*

3. *For prespectra E and F, one has a canonical isomorphism $AG(E \bigvee F) \cong AG(E) \times AG(F)$.*

4. *If Z is a G-module and $\Lambda_Z : G\mathcal{P} \to G\mathcal{P}$ is the shift suspension functor, defined in* [LMS85, Defn. I.7.1], *then $AG(-) \circ \Lambda_Z = \Lambda_Z \circ AG(-)$.*

5. *If D is a strong G-$\mathcal{CW}$-prespectrum, cf. Definition 5.5(c), then $AG(D)$ is a G-$\mathcal{CW}$-prespectrum.*

Proof. **1.** The covariance is immediate, from that of $\mathrm{AG}\,(-)$. We also know, from §2 and (2.12.2) that the natural map $R_{X,Y} : \mathrm{AG}\,(X) \wedge Y \to \mathrm{AG}\,(X \wedge Y)$ is an inclusion, for arbitrary spaces X and Y. Therefore, the structural maps σ_V^W are given as composition of inclusions

$$\mathrm{AG}\,(E(V)) \wedge S^{W-V} \hookrightarrow \mathrm{AG}\,\big(E(V) \wedge S^{W-V}\big) \hookrightarrow \mathrm{AG}\,(E(W))\,,$$

showing that $\mathrm{AG}\,(E)$ is a Σ-inclusion prespectrum.

2. This follows directly from Proposition 2.5 and canonical adjunction arguments.

3. The naturality of the isomorphism in Lemma 2.2 together with the distributivity of the smash product over a wedge product of spaces proves this last assertion.

4. For the sake of completeness we exhibit a proof of this assertion, which amounts essentially to exhibit the definition of Λ_Z. Write the universe $\mathcal{U}$ as $\mathcal{U}' + Z^\infty$, where Z^∞ is the sum of countably many copies of Z. By definition, any indexing space can be written as $V' + Z^n$, where $V' \subset \mathcal{U}'$, and $\Lambda_Z(D)(V' + Z^n) \overset{\mathrm{def}}{=} D(V' + Z^{n+1})$, for a prespectrum D. The structural maps coincide with the original $\tilde{\sigma}_{V+Z^n}^{W+Z^n}$ when $V + Z^n \subset W + Z^n$, and for $V + Z^n \subset V + Z^{n+1}$, the structural map is the composition of $\tilde{\sigma}_{V+Z^{n+1}}^{V+Z^{n+2}} : D(V + Z^{n+1}) \to \Omega^Z D(V+Z^{n+2})$ with the homeomorphism which regards the loop coordinate as the $(n+1)$-st rather than the $(n+2)$-nd copy of Z. Recall that the structural maps of $\mathrm{AG}\,(D)$ are simply compositions of the type $\mathrm{AG}\,(D(V)) \xrightarrow{\tilde{\sigma}} \mathrm{AG}\,\big(\Omega^{W-V}D(W)\big) \xrightarrow{t_{V,W}} \Omega^{W-V}\mathrm{AG}\,(D(W))$, and hence the application of Λ_Z commutes with $\mathrm{AG}\,(-)$.

5. Given a strong G-$\mathcal{CW}$-prespectrum D, it follows from Proposition 2.19(b) and Remark 2.10 that each $\mathrm{AG}\,(D)\,(V)$ is a G-LEC of the homotopy type of a G-$\mathcal{CW}$-complex. Applying Proposition 2.20(b) to the composition $\mathrm{AG}\,(D(V)) \wedge S^{W-V} \to \mathrm{AG}\left(D(V) \wedge S^{W-V}\right) \to \mathrm{AG}\,(D(W))$, concludes the proof that $\mathrm{AG}\,(D)$ is a G-$\mathcal{CW}$-prespectrum. ☐

The functor $\mathrm{AG}\,(-)$ also behaves quite well with respect to *twisted half smash products* in the prespectrum level. More precisely, let X be a compact G-space, and let $\mathcal{U}$, $\mathcal{U}'$ be universes, with indexing sets $\mathcal{A}$ and $\mathcal{A}'$, respectively. Given a G-map $\chi : X \to \mathcal{I}(\mathcal{U},\mathcal{U}')$ and a χ-*connection* $(\mu, \nu) : \mathcal{A} \to \mathcal{A}'$ in the sense of [LMS85, Defn. VI.2.7], one defines a functor $\chi \ltimes - : G\mathcal{P}(\mathcal{A}) \to G\mathcal{P}(\mathcal{A}')$.

Proposition 5.10. *Given $\chi : X \to \mathcal{I}(\mathcal{U},\mathcal{U}')$ as above, and $D \in G\mathcal{P}(\mathcal{A})$ there is a map $\Psi_\chi : \chi \ltimes AG\,(D) \to AG\,(\chi \ltimes D)$ which is natural on χ and on D.*

Proof. Recall that for any prespectrum $F \in G\mathcal{P}(\mathcal{A})$ and $V' \in \mathcal{A}'$ one defines $\chi \ltimes F(V') \stackrel{\text{def}}{=} T(\nu(V'), V') \wedge F(\nu(V'))$, for suitable Thom spaces $T(\nu(V'), V')$ defined in [LMS85, Defn. VI.2.1]. Therefore, $(\chi \ltimes \mathrm{AG}\,(E))(V') = T(\nu(V'), V') \wedge \mathrm{AG}\,(E(\nu(V')))$ and $\mathrm{AG}\,(\chi \ltimes E)\,(V') = \mathrm{AG}\,(T(\nu(V'), V') \wedge E(\nu(V')))$. One then defines $\Phi_\chi : (\chi \ltimes \mathrm{AG}\,(E))(V') \to \mathrm{AG}\,(\chi \ltimes E)\,(V')$ to be $R_{E(\nu(V')),T(\nu(V'),V')}$; cf. (2.12.3). This is clearly seen to be a map of prespectra, and the naturality conditions follow from the naturality of all functors involved in the definitions. ☐

In the particular case where X is a point, and hence $\chi(X)$ is a G-linear isometry $f \in \mathcal{I}(\mathcal{U},\mathcal{U}')$, we denote Ψ_χ by $\Psi_f : f_* \mathrm{AG}\,(D) \to \mathrm{AG}\,(f_* D)$.

Proposition 5.11. *If $f \in \mathcal{I}(\mathcal{U},\mathcal{U}')$ is a G-isometric isomorphism and $(\mu, \nu) : \mathcal{A} \to \mathcal{A}'$ is the standard connection* [LMS85, Exmp. IV.2.11], *then for all prespectra $D \in G\mathcal{P}(\mathcal{A})$ the map $\Psi_f : f_* AG\,(D) \to AG\,(f_* D)$ is an isomorphism. In particular, $\Psi_\imath$ is a homotopy equivalence for any G-linear isometry $\imath \in \mathcal{I}(\mathcal{U},\mathcal{U}')$, provided that $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic G-universes.*

Proof. Given $V' \in \mathcal{A}'$ one has equalities

$$f_* \mathrm{AG}\,(D)\,(V') \stackrel{\text{def}}{=} \mathrm{AG}\left(D(f^{-1}(V))\right) = \mathrm{AG}\left((f_* D)(V')\right) = \mathrm{AG}\,(f_* D)\,(V)$$

which suffice to show the isomorphism of prespectra. The standard argument involving a path $\chi : I \to \mathcal{I}(\mathcal{U},\mathcal{U}')$ of G-linear isometries from $f = \chi(0)$ to $\imath = \chi(1)$ shows that the diagram commutes:

$$\begin{array}{ccccc}
f_* \mathrm{AG}\,(E) = \chi(0) \ltimes \mathrm{AG}\,(E) & \xrightarrow{\simeq} & \chi \ltimes \mathrm{AG}\,(E) & \xleftarrow{\simeq} & \chi(1) \ltimes \mathrm{AG}\,(E) = \imath_* \mathrm{AG}\,(E) \\
\cong\downarrow & & \downarrow & & \downarrow \\
\mathrm{AG}\,(f_* E) = \mathrm{AG}\,(\chi(0) \ltimes E) & \xrightarrow{\simeq} & \mathrm{AG}\,(\chi \ltimes E) & \xleftarrow{\simeq} & \mathrm{AG}\,(\chi(1) \ltimes E) = \mathrm{AG}\,(\imath_* E),
\end{array}$$

and that all rows are inclusion maps of deformation retracts. ☐

Spectrum level properties

In the spectrum level one has the following couterpart of Proposition 5.9.

Proposition 5.12.

1. *The assignment $D \mapsto \mathbb{Z}D$ defines a homotopy preserving, covariant functor from the category of G-prespectra to the category of G-spectra. This functor sends a strong G-$\mathcal{CW}$-prespectrum to a G-$\mathcal{CW}$-spectrum.*

2. *If D is a Σ-inclusion prespectrum, then the unit $\eta : D \to \ell LD$ of the adjunction in Theorem 5.3 induces an isomorphism $\eta_* : \mathbb{Z}(D) \xrightarrow{\cong} \mathbb{Z}(\ell LD)$ of G-spectra.*

3. *If D is a Σ-inclusion G-prespectrum and Z is a G-module, then there are natural isomorphisms $\mathbb{Z}(\Lambda_Z D) \cong \Lambda_Z(\mathbb{Z}(D))$ and $\mathbb{Z}(\Lambda^Z D) \cong \Lambda^Z(\mathbb{Z}(D))$, where Λ^Z and Λ_Z are the shift functors of* [LMS85, Defn. I.7.1].

4. *Given a Σ-inclusion prespectrum D and a subgroup $H < G$, the G-spectrum $\mathbb{Z}(D \wedge G/H_+)$ is naturally isomorphic to $F(G/H_+, \mathbb{Z}(D))$.*

Proof. **1.** The covariance is evident, since $\mathbb{Z}(-)$ is a composition of covariant functors. Analogous argument, together with $R_{-,I} : \mathrm{AG}\,(E(V)) \wedge I \to \mathrm{AG}\,(E(V) \wedge I)$ applies to show that it also preserves homotopies. If D is a strong G-$\mathcal{CW}$-prespectrum, then Proposition 5.9(5) shows that $\mathrm{AG}\,(D)$ is a G-$\mathcal{CW}$-prespectrum and hence, $L(\mathrm{AG}\,(D))$ is a G-$\mathcal{CW}$-spectrum; cf. Definition 5.5(b).

2. We exhibit an inverse to the map $\eta_* : \mathbb{Z}(D) \to \mathbb{Z}(\ell LD)$, defined as follows. Given a G-module V, it follows from Proposition 5.9(1) and Remark 5.4 that $\mathbb{Z}(\ell LD)(V)$ is given as the double limit

$$\begin{aligned}
\mathbb{Z}(\ell LD)(V) &= L(\mathrm{AG}\,(\ell LD))(V) \\
&= \varinjlim{}_{W \supset V} \Omega^{W-V} \mathrm{AG}\,(\ell LD(W)) \\
&= \varinjlim{}_{W \supset V} \Omega^{W-V} \mathrm{AG}\,\left(\varinjlim{}_{T \supset W} \Omega^{T-W} D(T) \right) \\
&= \varinjlim{}_{W \supset V} \varinjlim{}_{T \supset W} \Omega^{W-V} \mathrm{AG}\,\left(\Omega^{T-W} D(T) \right).
\end{aligned}$$

Notice that the natural maps $t_{T-W} : \mathrm{AG}\,\left(\Omega^{T-W} D(T)\right) \to \Omega^{T-W} \mathrm{AG}\,(D(T))$ induce a map

$$\begin{aligned}
\mathbb{Z}(\ell LD)(V) &= \varinjlim{}_{W \supset V} \varinjlim{}_{T \supset W} \Omega^{W-V} \mathrm{AG}\,\left(\Omega^{T-W} D(T) \right) \\
&\to \varinjlim{}_{W \supset V} \varinjlim{}_{T \supset W} \Omega^{W-V} \Omega^{T-W} \mathrm{AG}\,(D(T)) \\
&= \varinjlim{}_{W \supset V} \varinjlim{}_{T \supset W} \Omega^{T-V} \mathrm{AG}\,(D(T)) \\
&= \varinjlim{}_{T \supset V} \Omega^{T-V} \mathrm{AG}\,(D(T)) = \mathbb{Z}(D)(V),
\end{aligned}$$

which is easily seen to be the desired inverse to η_*.

3. Recall that Λ^Z commutes with L, since their adjoints Λ_Z and ℓ obviously commute, and that $\Lambda_Z \circ \Lambda^Z = id$, while there is a natural transformation $id \to \Lambda^Z \circ \Lambda_Z$, which induces isomorphisms on spectra. It is now a simple observation that, for any Σ-inclusion prespectrum E, the unit of the adjunction $\eta : E \to \ell L(E)$ induces a map $\Lambda_Z E \to \Lambda_Z(\ell L E)$, and hence a map

$$L(\Lambda_Z E) \to L(\Lambda_Z(\ell L E)) = \Lambda_Z L E, \tag{5.12.1}$$

where the latter equality comes from the fact that Λ_Z preserves spectra. On the other hand, the natural map $E \to \Lambda^Z \Lambda_Z(E)$ induces $L(E) \to L(\Lambda^Z \Lambda_Z E) = \Lambda^Z L(\Lambda_Z E)$, and an application of Λ_Z produces a map

$$\Lambda_Z L(E) \to L(\Lambda_Z E). \tag{5.12.2}$$

It is easy to verify that when E is a Σ-inclusion prespectrum, the maps (5.12.1) and (5.12.2) are inverses to each other. One now applies this result to the case $E = \mathrm{AG}\,(D)$, where D is a Σ-inclusion prespectrum, and Proposition 5.9(1) and (4) finishes the proof for Λ_Z.

The adjoint of the isomorphism $\mathbb{Z}(D) = \mathbb{Z}(\Lambda_Z \Lambda^Z D) \xrightarrow{\cong} \Lambda_Z \mathbb{Z}(\Lambda^Z D)$ gives the desired natural isomorphism $\Lambda^Z \mathbb{Z}(D) \to \mathbb{Z}(\Lambda^Z D)$. We want to point out here that, contrary to Λ_Z, Λ^Z does not commute with $\mathrm{AG}\,(-)$ in the level of prespectra, otherwise this last result would be an equality instead of an isomorphism.

4. Since $F(G/H_+, \mathrm{AG}\,(D))$ is also a Σ-inclusion prespectrum, it follows from Remark 5.4 and from usual adjunctions that one has an isomorphism

$$F(G/H_+, \mathbb{Z}(D)) = F(G/H_+, L(\mathrm{AG}\,(D))) \cong L(F(G/H_+, \mathrm{AG}\,(D))),$$

which combined with Proposition 5.9(2) proves the assertion. □

In order to place our construction under the proper perspective, suppose that G is a trivial group, and let us compute the homotopy groups of the spectrum $\mathbb{Z}(\Sigma^\infty X)$. For $j \geq 0$ one has

$$\begin{aligned}\pi_j(\mathbb{Z}(\Sigma^\infty X)) &= \{\mathbb{S}^j, \mathbb{Z}(\Sigma^\infty X)\} = \varinjlim{}_n [S^{j+n}, \mathbb{Z}(\Sigma^\infty X)(S^n)] \\ &= \varinjlim{}_{n,m} [S^{j+n}, \Omega^m \mathrm{AG}\left(S^{n+m} X\right)] = \varinjlim{}_{n,m} [S^{j+n+m}, \mathrm{AG}\left(S^{n+m} X\right)].\end{aligned}$$

Now, notice that all maps in the composition $[S^j, \mathrm{AG}\,(X)] \to [S^{j+1}, \mathrm{AG}\,(SX)] \to \ldots \to [S^{j+k}, \mathrm{AG}\left(S^k X\right)]$ are simply suspension isomorphisms under the Dold-Thom theorem, as pointed out in Remark 2.14. In other words, the homotopy group of $\mathbb{Z}(\Sigma^\infty X)$ precisely compute the (reduced) singular homology of X.

Our goal is to generalize the latter considerations to the equivariant context, where this phenomenon will be properly understood.

6. Stable Dold-Thom results

6.1. **Identification of $\mathbb{Z}(\mathbb{S})$.** We first need to perform some space level computations. Let us start with a fixed G-module V which will be assumed to contain at least two copies of the trivial G-module, i.e. $\dim V^G \geq 2$. We denote by $n(K)$ the dimension function associated with V, which sends a subgroup $K \leq G$ to $n(K) \overset{\text{def}}{=} \dim V^K$.

Lemma 6.1. *Let V be a G-module as above.*

1. *For every $K \leq G$, $AG\left(S^V\right)^K$ is $(n(K)-1)$-connected.*
2. $\pi_{n(K)}(AG\left(S^V\right)^K) \cong \mathbb{Z}$.
3. *The natural inclusion ${S^V}^K \xrightarrow{j_V^K} AG\left(S^V\right)^K$ induces an isomorphism:*

$$j_{V*}^K : \pi_{n(K)}({S^V}^K) \xrightarrow{\cong} \pi_{n(K)}(AG\left(S^V\right)^K).$$

Proof. Recall the isomorphism $\pi_j(\mathrm{AG}\left(S^V\right)^K) \cong \tilde{H}_j^K(S^V, \underline{\mathbb{Z}})$, where $\tilde{H}_j^K(-,\underline{\mathbb{Z}})$ denotes (reduced) Bredon homology with coefficients in the Mackey functor $\underline{\mathbb{Z}}$; cf. Theorem 4.5. Writing $V = \mathbb{R}^{n(K)} \oplus \overline{V}$, where $\overline{V}$ has no copies of the trivial K-module one has

$$\tilde{H}_j^K(S^V, \underline{\mathbb{Z}}) \cong \tilde{H}_j^K(S^{\overline{V}} \wedge S^{n(K)}, \underline{\mathbb{Z}}) \cong \tilde{H}_{j-n(K)}^K(S^{\overline{V}}, \underline{\mathbb{Z}}) = \begin{cases} 0, & \text{if } j < n(K) \\ \mathbb{Z}, & \text{if } j = n(K). \end{cases}$$

This shows 1 and 2.

The proof of 3 is now straightforward. □

Let $\mathcal{C}(G)$ denote the set of all integral valued functions on the set of conjugacy classes of subgroups of G. The previous lemma together with [tDP82, Prop. (3.1)] allows one to associate a *degree* $d(f) \in \mathcal{C}(G)$ for every G-map $f : S^V \to \mathrm{AG}\left(S^V\right)$. This *degree* is defined on the conjugacy class of K to be the degree of the homomorphism $f_*^K : \pi_{n(K)}({S^V}^K) \to \pi_{n(K)}(\mathrm{AG}\left(S^V\right)^K)$.

The degree map stabilizes to define a homomorphism $\deg' : \omega(S^V, \mathrm{AG}\left(S^V\right))_G \to \mathcal{C}(G)$, between stable equivariant homotopy classes of maps; cf. [tDP82, Defn. (3.3)]. Using the same argument one also obtains a degree homomorphism $\deg : \omega(S^V, S^V)_G \to \mathcal{C}(G)$. Recall that $\omega(S^V, S^V)_G \cong \{\mathbb{S}, \mathbb{S}\}_G \cong A(G)$, where $A(G)$ is the Burnside ring of G; cf. [LMS85, Thm. V.2.11].

Proposition 6.2. **a.** *Both* $\deg$ *and* $\deg'$ *are injective homomorphisms.*

b. *If $j_V : S^V \to AG\left(S^V\right)$ is the natural map, the induced homomorphism $j_{V*} : \omega(S^V, S^V)_G \to \omega(S^V, AG\left(S^V\right))_G$ satisfies* $\deg' \circ j_{V*} = \deg$.

Proof. The first assertion is proven in [tDP82, Prop. 3.4].

In order to prove the second assertion, observe that, given a representative $f : S^{V+W} \to S^{V+W}$ of an element in $\omega(S^V, S^V)_G$ and a conjugacy class $K \leq G$, the commutative diagram of spaces

$$\begin{array}{ccc} S^{(V+W)^K} & \xrightarrow{f_K} & S^{(V+W)^K} \\ {\scriptstyle j_{V+W}}\big\downarrow & & \big\downarrow{\scriptstyle j_{V+W}} \\ \mathrm{AG}\left(S^{(V+W)}\right)^K & \xrightarrow{f_{K*}} & \mathrm{AG}\left(S^{(V+W)}\right)^K \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} \cong \pi^G_{n(K)}(S^{(V+W)^K}) & \xrightarrow{f_K} & \pi^G_{n(K)}(S^{(V+W)^K}) \cong \mathbb{Z} \\
{\scriptstyle j_{V+W}}\big\downarrow & & \big\downarrow{\scriptstyle j_{V+W}} \\
\mathbb{Z} \cong \pi^G_{n(K)}\left(\mathrm{AG}\left(S^{(V+W)}\right)^K\right) & \xrightarrow{f_{K*}} & \pi^G_{n(K)}\left(\mathrm{AG}\left(S^{(V+W)^K}\right)^K\right) \cong \mathbb{Z}
\end{array}$$

whose vertical arrows are isomorphisms, according to Lemma 6.1. (Notice that here $n(-)$ denotes the dimension function associated to $V+W$.) Therefore the degrees of the top and bottom horizontal arrows coincide. $\square$

We must recall that since $A(G)$ is finitely generated and free as an abelian group, then the isomorphism $\{\mathbb{S},\mathbb{S}\}_G \cong A(G)$ can be realized at a finite level, in order words, there is a sufficiently large G-module V such that $[S^V, S^V]_G \cong A(G)$.

Corollary 6.3. *For V as above, $j_{V*}: A(G) \to \omega(S^V, AG\left(S^V\right))_G$ is injective.*

Combining the results just developed, we are able to identify the spectrum $\mathbb{Z}(\mathbb{S})$. Recall that, given a Mackey functor M, an Eilenberg-MacLane G-spectrum $K(M,0)$ is a spectrum satisfying $\underline{\pi}^G_k(K(M,0)) = \begin{cases} 0 & ;\ \text{if } k \neq 0 \\ M & ;\ \text{if } k = 0, \end{cases}$ and that this condition uniquely characterizes $K(M,0)$, up to homotopy.

Theorem 6.4. *Given a finite group G, let A_G denote its Burnside Mackey functor. Then one has an equivalence of spectra*

$$\mathbb{Z}(\mathbb{S}) \cong K(A_G, 0).$$

Proof. We first observe that

$$\pi^H_k(\mathbb{Z}(\mathbb{S})) \equiv \underline{\pi}^G_k(\mathbb{Z}(\mathbb{S}))[G/H] = \varinjlim{}_V [S^{V+k} \wedge G/H_+, \mathrm{AG}\left(S^V\right)]_G = 0,$$

for $k > 0$, according to Lemma 4.4. On the other hand, $\mathbb{Z}(\mathbb{S})$ has no cells of negative dimension, and hence the negative homotopy groups all vanish. We proceed to compute $\underline{\pi}^G_0(\mathbb{Z}(\mathbb{S}))$.

Lemma 6.1 together with the equivariant Hurewicz theorem of [Lew92] imply that

$$[X, S^V]_G \to\ [X, \mathrm{AG}\left(S^V\right)]_G$$

is surjective for any $n(-)$-connected G-space X, where $n(-)$ denotes the dimension function associated to a sufficiently large G-module V. In particular, this holds for $X = (G/N)_+ \wedge S^V$, where N is an arbitrary subgroup of G, and this shows that the morphism of Mackey functors $A_G(N) = A(N) = \{\mathbb{S} \wedge G/N_+,\ \mathbb{S}\}_G \to \underline{\pi}^G_0(\mathbb{Z}(\mathbb{S}))[G/N]$ is surjective, and hence an isomorphism;cf. Corollary 6.3. $\square$

6.2. **The general case.** In this subsection we identify the functor $\mathbb{Z}(-)$ in the stable homotopy category of G-spectra $\overline{h}G\mathcal{S}$, where all weak equivalences are inverted, following the treatment of [LMS85]. We use the symbols $\cong$, $\simeq$ and $\sim$ to denote isomorphisms in the categories $G\mathcal{S}$, $hG\mathcal{S}$ and $\overline{h}G\mathcal{S}$, respectively.

Remark 6.5. We observe that in the passage to the stable category, one defines $\mathbb{Z}(E)$ to be $\mathbb{Z}(E')$, where E' is a G-$\mathcal{CW}$-approximation of E; cf. [EK$^+$95, p. 11]. Furthermore, we may assume that E' has the form $E' = LD$, where D is a strong G-$\mathcal{CW}$-complex, as shown in [LF, Thm. 4.6].

Let $\mathcal{A}$ denote the *standard indexing set* for $\mathcal{U}$, cf. Remark 5.2, and note that in our previous notation $G\mathcal{S}$ actually meant $G\mathcal{S}(\mathcal{A})$ in the language of [LMS85]. We start with the "external" smash product $\wedge : G\mathcal{S}(\mathcal{A}) \times G\mathcal{S}(\mathcal{A}) \to G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$, which sends prespectra A and B indexed on $\mathcal{A}$ to the prespectrum $A \wedge B$, indexed on $\mathcal{A} \oplus \mathcal{A}$, via the assignment $(V, W) \mapsto A(V) \wedge B(W)$, together with the evident structural maps; cf. [LMS85, Defn. II.3.1]. For A and B in $G\mathcal{S}(\mathcal{A})$, one then defines $A \wedge B$ in $G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$ as $L(\ell A \wedge \ell B)$.

Given G-spectra E, F and a G-$\mathcal{CW}$-complex K we introduce natural maps

$$r_K : \mathbb{Z}(E) \wedge K \to \mathbb{Z}(E \wedge K) \tag{6.5.1}$$

and

$$R_E : \mathbb{Z}(F) \wedge E \to \mathbb{Z}(F \wedge E) \tag{6.5.2}$$

in $G\mathcal{S}(\mathcal{A})$ and $G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$, respectively.

Given $V \in \mathcal{A}$ define $\tilde{r}_K : \mathrm{AG}\,(E(V)) \wedge K \to \mathrm{AG}\,(E(V) \wedge K)$ to be $R_{E(V),K}$; cf. (2.12.3). This gives a map of prespectra $\tilde{r}_K : \mathrm{AG}\,(E) \wedge K \to \mathrm{AG}\,(E \wedge K)$, which induces the desired map

$$\begin{aligned} r_K :& \mathbb{Z}(E) \wedge K \overset{\text{def}}{=} L(\ell L(\mathrm{AG}\,(\ell E)) \wedge K) \\ \cong& L(\mathrm{AG}\,(\ell E) \wedge K) \xrightarrow{L(\tilde{r}_K)} L(\mathrm{AG}\,(\ell E \wedge K)) = \mathbb{Z}(E \wedge K) \end{aligned}$$

in $G\mathcal{S}(\mathcal{A})$.

The definition of R_E is slightly more elaborate. Start, once again, in the prespectra level with the map

$$\rho_E : \mathrm{AG}\,(F) \wedge \ell E \to \mathrm{AG}\,(F \wedge \ell E) \tag{6.5.3}$$

defined as $R_{F(V),E(W)} : \mathrm{AG}\,(F(V)) \wedge E(W) \to \mathrm{AG}\,(F(V) \wedge E(W))$ on $(V \oplus W) \in \mathcal{A} \oplus \mathcal{A}$; cf. (2.12.3). It is evident that this is a natural map of prespectra. In order to pass to the spectrum level we use the natural isomorphism given by the composition

$$\Phi : \mathbb{Z}(F) \wedge E \overset{\text{def}}{=} L\,(\ell L(\mathrm{AG}\,(\ell F)) \wedge \ell E) \xrightarrow{\cong} L(\mathrm{AG}\,(\ell F) \wedge \ell E) \tag{6.5.4}$$

where the last step is the canonical isomorphism $L(\ell L(A) \wedge \ell B) \cong L(A \wedge \ell B)$ which holds for arbitrary prespectrum A and spectrum B.

We use the prespectrum map $\mu_* : \mathrm{AG}\,(F \wedge \ell E) \to \mathrm{AG}\,(\ell L(F \wedge E))$, induced by the unit of the adjunction in Theorem 5.3, to define R_E as the composition of maps in $G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$

$$R_E : \mathbb{Z}(\mathbb{S}) \wedge E \underset{\cong}{\xrightarrow{\Phi}} L(\mathrm{AG}\,(\ell F) \wedge \ell E) \xrightarrow{L(\rho_E)} L(\mathrm{AG}\,(\ell F \wedge \ell E))$$

$$\xrightarrow{L(\mu_*)} L(\mathrm{AG}\,(\ell L(\ell F \wedge \ell E))) = \mathbb{Z}(\mathbb{S} \wedge E). \tag{6.5.5}$$

Defining the linear isometry $\imath : \mathcal{U} \to \mathcal{U} \oplus \mathcal{U}$, as the inclusion in the first coordinate, one obtains a change of universe functor $\imath_* : G\mathcal{S}(\mathcal{A}) \to G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$; cf. [LMS85, Defn. II.1.1].

Lemma 6.6. *Given $F \in G\mathcal{S}(\mathcal{A})$ and a finite G-$\mathcal{CW}$-complex K, the maps $R_{\Sigma^\infty K}$ and r_K introduced above are related by the following commutative diagram in $G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$.*

$$\begin{tikzcd}
\mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty K = \imath_*(\mathbb{Z}(\mathbb{S}) \wedge K) \arrow[rr, "\imath_*(r_K)"] \arrow[dr, "R_{\Sigma^\infty K}"'] & & \imath_*(\mathbb{Z}(F \wedge K)) \arrow[dl, "\Psi_\imath"] \\
& \mathbb{Z}(F \wedge \Sigma^\infty K) = \mathbb{Z}(\imath_*(F \wedge K)) &
\end{tikzcd}$$

In particular $R_{\Sigma^\infty K}$ is homotopy equivalence in $G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$ whenever so is r_K in $G\mathcal{S}(\mathcal{A})$.

Proof. Clear. ☐

We devote the rest of this section to the proof of this identification of $\mathbb{Z}(E)$ in $\overline{h}G\mathcal{S}(\mathcal{A})$:

Theorem 6.7. *The natural map $R_E : \mathbb{Z}(\mathbb{S}) \wedge E \to \mathbb{Z}(\mathbb{S} \wedge E)$ is an equivalence in the stable category $\overline{h}G\mathcal{S}(\mathcal{A} \oplus \mathcal{A})$. In particular, if one uses $f \in \mathcal{I}(\mathcal{U} \oplus \mathcal{U}, \mathcal{U})$ to induce the internal smash product, this becomes an equivalence in $\overline{h}G\mathcal{S}(\mathcal{A})$.*

Corollary 6.8 (Equivariant Stable Dold-Thom Theorem)**.** *For any G-spectrum E, the homotopy groups of $\mathbb{Z}(E)$ compute the ordinary $RO(G)$-graded homology of E with coefficients on the Burnside ring Mackey functor, as defined in* [LMM81]. *In other words, for a G-spectrum E and subgroup $H < G$, one has $\pi_*^H(\mathbb{Z}(E)) \cong H_*^G(E, \mathcal{A}_G)[G/H]$.*

Corollary 6.9. *Given G-spectra E and F in $G\mathcal{S}(\mathcal{A})$, one has weak equivalences*

$$\mathbb{Z}(E) \wedge F \sim \mathbb{Z}(E \wedge F) \sim E \wedge \mathbb{Z}(F)$$

in $G\mathcal{S}(\mathcal{A})$.

Proof. Just observe that $\mathbb{Z}(E \wedge F) \sim \mathbb{Z}(\mathbb{S}) \wedge (E \wedge F)$ and use the associativity and commutativity of $\wedge$ in $\overline{h}G\mathcal{S}$. ☐

The first major simplification towards the proof of the main result is given by this lemma.

Lemma 6.10. *If the theorem holds for spectra of type $\Sigma^\infty X$, where X is a finite G-$\mathcal{CW}$-complex, then it holds for arbitrary spectra.*

Proof. Assuming that E is a G-$\mathcal{CW}$-spectrum, notice that the cells of both $\mathbb{Z}(\mathbb{S}) \wedge E$ and $\mathbb{Z}(\mathbb{S} \wedge E)$ are obtained from products of cells of $\mathbb{S}$ and E and, in particular, any compact subspectrum of $\mathbb{Z}(\mathbb{S}) \wedge E$ is contained in a subspectrum of type $\mathbb{Z}(\mathbb{S}) \wedge \Lambda^Z(\Sigma^\infty K)$, where K is a finite G-$\mathcal{CW}$-complex. Similarly, any compact subspectrum of $\mathbb{Z}(\mathbb{S} \wedge E)$ is contained in a subspectrum of type $\mathbb{Z}(\mathbb{S} \wedge \Lambda^Z(\Sigma^\infty K))$. Therefore, given homotopies $\mathbf{g} : I_+ \wedge \mathbb{S}^n_H \to \mathbb{Z}(\mathbb{S}) \wedge E$ and $\mathbf{h} : I_+ \wedge \mathbb{S}^n_H \to \mathbb{Z}(\mathbb{S} \wedge E)$, one can find a finite G-$\mathcal{CW}$-complex K such that $\mathbf{g}$ and $\mathbf{h}$ factor through g' and h' in the following commutative diagram

$$\begin{tikzcd}
 & \mathbb{Z}(\mathbb{S}) \wedge E \arrow[r, "R_E"] & \mathbb{Z}(\mathbb{S} \wedge E) & \\
I_+ \wedge \mathbb{S}^n_H \arrow[ur, "\mathbf{g}"] \arrow[r, "g'"'] & \mathbb{Z}(\mathbb{S}) \wedge \Lambda^Z(\Sigma^\infty K) \arrow[u] \arrow[r, "R_{\Lambda^Z(\Sigma^\infty K)}"] & \mathbb{Z}(\mathbb{S} \wedge \Lambda^Z(\Sigma^\infty K)) \arrow[u] & I_+ \wedge \mathbb{S}^n_H . \arrow[ul, "\mathbf{h}"'] \arrow[l, "h'"]
\end{tikzcd}$$

Recall that $\mathbb{Z}(\mathbb{S}) \wedge \Lambda^Z(\Sigma^\infty K) \cong \Lambda^Z(\mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty K)$ and that $\mathbb{Z}(\mathbb{S} \wedge \Lambda^Z(\Sigma^\infty K)) \cong \mathbb{Z}(\Lambda^Z(\mathbb{S} \wedge \Sigma^\infty K)) \cong \Lambda^Z \mathbb{Z}(\mathbb{S} \wedge \Sigma^\infty K)$, where the last isomorphism comes from Proposition 5.12(3). It is easy to verify that under these isomorphisms one has $R_{\Lambda^Z(\Sigma^\infty K)} = \Lambda^Z \circ R_{\Sigma^\infty K}$, since the isomorphisms involved all come from units of adjunctions. This suffices to prove the lemma. $\square$

Remark 6.11. Observe that one can apply the same argument at the end of the proof above and Proposition 5.12(3) to conclude that $R_{\Lambda_Z(\Sigma^\infty K)}$ is a weak equivalence if and only if $R_{\Sigma^\infty K}$ is a weak equivalence.

Our next technical result provides the necessary tool to perform induction over skeleta of G-$\mathcal{CW}$-complexes.

Proposition 6.12. *Let X and Y be finite G-$\mathcal{CW}$-complexes, and let $A \subset X$ be a subcomplex. Then the sequence*

$$\{\Sigma^\infty Y, \mathbb{Z}(\Sigma^\infty A)\}_G \xrightarrow{j_*} \{\Sigma^\infty Y, \mathbb{Z}(\Sigma^\infty X)\}_G \xrightarrow{\pi_*} \{\Sigma^\infty Y, \mathbb{Z}(\Sigma^\infty (X/A))\}_G$$

is exact at the middle group.

Proof. Let $\mathbf{f} \in G\mathcal{S}(\Sigma^\infty Y,\ \mathbb{Z}(\Sigma^\infty X))$ represent an element in $\ker(\pi_*)$, and let $\mathbf{h} \in G\mathcal{S}(I_+ \wedge \Sigma^\infty Y,\ \mathbb{Z}(\Sigma^\infty (X/A)))$ be a homotopy from $\mathbf{f} \circ \pi$ to zero. It follows from the canonical adunctions $G\mathcal{S}(\Sigma^\infty Y,\ \mathbb{Z}(\Sigma^\infty X)) \cong \text{G-}\mathfrak{CG}_*(Y,\ \mathbb{Z}(\Sigma^\infty X)(0))$ and $G\mathcal{S}(I_+ \wedge \Sigma^\infty Y,\ \mathbb{Z}(\Sigma^\infty (X/A))) \cong \text{G-}\mathfrak{CG}_*(I_+ \wedge Y,\ \mathbb{Z}(\Sigma^\infty (X/A))(0))$ that $\mathbf{f}$ and $\mathbf{h}$ are induced by space level maps $\mathbf{f}_0 : Y \to \mathbb{Z}(S^\infty X)(0)$ and $\mathbf{h}_0 : I_+ \wedge Y \to \mathbb{Z}(S^\infty (X/A))(0)$. Furthermore, since Y is compact, one can find a sufficiently large indexing space W such that $\mathbf{f}_0$ and $\mathbf{h}_0$ factor through maps f and h

which fit into a diagram of G-maps

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "j_0"'] & \Omega^W \mathrm{AG}\left(S^W X\right) \arrow[d, "\pi_*"'] \\
I_+ \wedge Y \arrow[r, "h"'] & \Omega^W \mathrm{AG}\left(S^W (X/A)\right).
\end{tikzcd}$$

Since the sequence $\mathrm{AG}\left(S^W A\right) \to \mathrm{AG}\left(S^W X\right) \to \mathrm{AG}\left(S^W (X/A)\right)$ is left exact in the homotopy category of G-spaces, according to Proposition 4.1, one concludes that the sequence $[S^W Y, \mathrm{AG}\left(S^W A\right)]_G \to [S^W Y, \mathrm{AG}\left(S^W X\right)]_G \to [S^W Y, \mathrm{AG}\left(S^W (X/A)\right)]_G$ is exact, thereby producing an element $g : Y \to \Omega^W \mathrm{AG}\left(S^W A\right)$, whose class in $[Y, \Omega^W \mathrm{AG}\left(S^W A\right)]$ maps to the class of f under the first map of the sequence. This gives a map $\mathbf{g}_0 : Y \to \mathbb{Z}(\Sigma^\infty A)(0)$ which is sent to the class of $\mathbf{f}_0$. The adjoint map $\mathbf{g} : \Sigma^\infty Y \to \mathbb{Z}(\Sigma^\infty A)$ is mapped into the class of $\mathbf{f}$ under j_*, and this proves the lemma. $\square$

Now we can prove Theorem 6.7.

Proof. We just need to prove the theorem for spectra of type $E = \Sigma^\infty X$, where X is a finite G-$\mathcal{CW}$-complex. We use induction, adding one cell at a time. Notice that when X is obtained from A by adjoining a cell of type $G/H \times D^n$ one has a cofiber sequence of spectra $\mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty A \to \mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty X \to \mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty (X/A)$. Applying homotopy groups one obtains a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi^H_{j+1}\left(\mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty (G/H_+ \wedge S^n)\right) \arrow[r] \arrow[d] & \pi^H_j\left(\mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty A\right) \arrow[r] \arrow[d] & \pi^H_j\left(\mathbb{Z}(\mathbb{S}) \wedge \Sigma^\infty X\right) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi^H_{j+1}\left(\mathbb{Z}(\mathbb{S} \wedge \Sigma^\infty (G/H_+ \wedge S^n))\right) \arrow[r] & \pi^H_j\left(\mathbb{Z}(\mathbb{S} \wedge \Sigma^\infty A)\right) \arrow[r] & \pi^H_j\left(\mathbb{Z}(\mathbb{S} \wedge \Sigma^\infty X)\right) \arrow[r] & \cdots
\end{tikzcd}$$

where the top row is exact by the cofiber condition and the vertical arrows are induced by the various natural maps R_-. A successive application of Proposition 6.12 together with space-level cofibers and Corollary 2.11 shows that the bottom row is also exact. One then concludes that if both $R_{\Sigma^\infty A}$ and $R_{\Sigma^\infty (G/H_+ \wedge S^n)}$ are weak equivalences, then so is $R_{\Sigma^\infty X}$.

It follows from Lemma 6.6 that we only need to show that $r_X : \mathbb{Z}(\mathbb{S}) \wedge X \to \mathbb{Z}(\mathbb{S} \wedge X)$ is homotopy equivalence in $G\mathcal{S}(\mathcal{A})$. In this case, the observations above, together with Remark 6.11 reduce us to the case where $X = G/H_+$, for $\Sigma^\infty(G/H_+ \wedge S^n) \cong \Lambda_n \Sigma^\infty (G/H_+)$. Recall the isomorphism $\rho : \mathbb{Z}(\mathbb{S} \wedge G/H_+) \cong F(G/H_+, \mathbb{Z}(\mathbb{S}))$ given in Proposition 5.12(4).

The composition $\mathbb{Z}(\mathbb{S}) \wedge G/H_+ \xrightarrow{R_{G/H_+}} \mathbb{Z}(\mathbb{S} \wedge G/H_+) \xrightarrow{\rho} F(G/H_+, \mathbb{Z}(\mathbb{S}))$ is precisely the map $\Psi : \mathbb{Z}(\mathbb{S}) \wedge G/H_+ \to F(G/H_+, \mathbb{Z}(\mathbb{S}))$ such that $\epsilon \circ (\Psi \wedge 1) = \mu$, where $\epsilon : F(G/H_+, \mathbb{Z}(\mathbb{S})) \wedge G/H_+ \to \mathbb{Z}(\mathbb{S})$ is the evaluation map, while $\mu : \mathbb{Z}(\mathbb{S}) \wedge G/H_+ \to \mathbb{Z}(\mathbb{S})$ is the projection which sends $\mathbb{Z}(\mathbb{S}) \wedge (eH)$ isomorphically onto $\mathbb{Z}(\mathbb{S})$ and everything else to the base point. One sees that Ψ coincides with the map $\Psi : G \ltimes_H \mathbb{Z}(\mathbb{S}) \to F_H[G, \mathbb{Z}(\mathbb{S}))$ defined in [LMS85, p. 90], under the natural isomorphisms $G \ltimes_H \mathbb{Z}(\mathbb{S}) \cong \mathbb{Z}(\mathbb{S}) \wedge G/H_+$ and $F_H[G, \mathbb{Z}(\mathbb{S})) \cong F(G/H_+, \mathbb{Z}(\mathbb{S}))$ which come from the fact that $\mathbb{Z}(\mathbb{S})$ is a G-spectrum and not only an H-spectrum. The homotopy inverse to Ψ is provided by Wirthmüller's isomorphism, see e.g. [LMS85, Thm. II.6.2], using a simple Pontrjagin-Thom construction, which we briefly recall.

Take a G-module V_0 in which G/H embeds equivariantly and choose disjoint little disks around the points of G/H in V_0 which are permuted by the action of G. One then gets a G-map $t : S^{V_0} \to G/H_+ \wedge S^{V_0}$ by collapsing the complement of the disks to a base point. The homotopy inverse to Ψ is given by the composition

$$\begin{aligned}\omega : F(G/H_+, \mathbb{Z}(\mathbb{S})) &\cong \Lambda^{V_0}\left(F(G/H_+, \mathbb{Z}(\mathbb{S}))\right) \wedge S^{V_0} \\ &\xrightarrow{1\wedge t} \Lambda^{V_0}\left(F(G/H_+, \mathbb{Z}(\mathbb{S}))\right) \wedge (S^{V_0} \wedge G/H_+) \cong \left(\Lambda^{V_0}\left(F(G/H_+, \mathbb{Z}(\mathbb{S}))\right) \wedge S^{V_0}\right) \wedge G/H_+ \\ &\cong \mathbb{Z}(\mathbb{S}) \wedge G/H_+;\end{aligned}$$

cf. [LMS85, Defn. II.6.1]. It is an easy task now to verify that Ψ and ω are homotopy inverses, and an explicit proof of this can be found in [LMS85, Thm. II.6.2]. We warn the reader that the tangent space L to G/H at eH which appears in [LMS85], in the context of arbitrary compact Lie groups, obviously does not play any role here. $\square$

Appendix A. Proofs of technical lemmas

Proof. [of Lemma 2.2] Define a homomorphism I by $I((x_\alpha)_{\alpha\in\Lambda}) = \sum_\alpha i_{\alpha *}(x_\alpha)$ and observe that, since $\mathcal{Z}(-)$ is an abelian-group-valued functor, it follows that I is a topological homomorphism. Furthermore, since the spaces X_α are disjoint and $\mathcal{Z}(-)$ is a free abelian group functor, standard arguments show that I is bijective.

We have thus shown that I is a continuous isomorphism. However, given a compact subset $S \subset \mathcal{Z}(\coprod_{\alpha\in\Lambda} X_\alpha)$, one knows that $S \subset \mathcal{Z}_{0,\leq D}(K)$ for some compact subset $K \subset \coprod_{\alpha\in\Lambda} X_\alpha$ and some $D \geq 0$. Furthermore, since K is compact, one knows that there is a finite collection $\alpha_1, \ldots, \alpha_n$ of α's such that $K \cap X_\alpha = \emptyset$ whenever $\alpha \neq \alpha_j$ for $j = 1, \ldots, n$.

Observe that the assertion in the proposition about degrees is evident, and one obtains $I^{-1}(\mathcal{Z}_{0,\leq D}(K)) \subset \bigcup_{r_1+\ldots+r_n\leq D} [\mathcal{Z}_{0,\leq r_1}(X_{\alpha_1}) \oplus \ldots \oplus \mathcal{Z}_{0,\leq r_n}(X_{\alpha_n})]$. Since the right hand side is compact and $I^{-1}(K)$ is closed, one concludes that $I^{-1}(K)$ is compact. This fact easily implies that I is a closed map, for the spaces in questions are compactly generated, and the lemma follows. $\square$

Proof. [of Lemma 2.2] The surjectivity of π_* follows immediately from that of π. Given $c \in \ker(\pi_*)$, write $c = \sum_{i=1}^n i_{\alpha *}(\sigma_i)$, notation as in the previous lemma. Therefore one has $\pi_*(c) = \sum_{i=1}^n \pi_* \circ i_{\alpha_i *}(\sigma_i) = \sum_{i=1}^n \imath_{\alpha_i *}(\sigma_i)$, where $\imath_\alpha$ denote the inclusion of X_α in $\bigvee_\alpha X_\alpha$.

This shows that $\imath_{\alpha_i *}(\sigma_i) = -\left(\sum_{j\neq i} \imath_{\alpha_j *}(\alpha_j)\right)$ and hence $\imath_{\alpha_i *}(\sigma_i)$ is supported in $\imath_{\alpha_i}(X_{\alpha_i}) \cap \left(\bigcap_{j\neq i} X_{\alpha_j}\right) = \{z_0\}$, where $z_0 \in \bigvee_\alpha X_\alpha$ is the basepoint. It follows that $\sigma_i = m_i x^o_{\alpha_i}$, for some $m_i \in \mathbb{Z}$ and for all $i = 1, \ldots, n$. The fact that $\pi_*(\sum_i i_{\alpha_i}(\sigma_i)) = \sum_i m_i z_0 = 0$ computes the kernel of π_*, as desired. $\square$

Proof. [Lemma 2.3] **a.** Let $\sigma \in \mathcal{Z}(X)$ be a representative for an H-invariant element of $\mathcal{Z}(X)/\mathcal{Z}(Y)$ such that $\mathrm{supp}(\sigma)$ contains no points of Y. In particular, $\sigma - h * \sigma \in \mathcal{Z}(Y)$ for all $h \in H$. Since Y is G-invariant, one concludes that $\mathrm{supp}(h * \sigma)$ also does not contain points of Y, for all $h \in H$. This proves that $\sigma = h * \sigma$, for all $h \in H$. In other words, $\mathcal{Z}(X)^H \to \{\mathcal{Z}(X)/\mathcal{Z}(Y)\}^H$ is surjective. Now, the elements of $\mathcal{Z}(X)^H$ sent to zero under this map are precisely the elements of $\mathcal{Z}(X)^H \cap \mathcal{Z}(Y) = \mathcal{Z}(Y)^H$.

b. Observe that the surjection $X \to X/Y$ induces an equivariant proclusion $p : \mathcal{Z}(X) \to \mathcal{Z}(X/Y)$; cf. [McC69, 6.7]. Now, suppose that $\sigma \in \ker(p)$ and write $\sigma = \sum_i n_i x_i + \sum_j m_j y_j$, with $x_i \in X - Y$ and $y_j \in Y$, for all i, j. Since $p_*(\sigma) = \sum_i n_i p(x_i) + (\sum_j m_j) z_0 = 0$, where z_0 is the natural basepoint of X/Y, and all $p(x_i)$'s are different from z_0, one concludes that $n_i = 0$ for all i, and that $\sum_j m_j = 0$. In other words, $\sigma \in \mathrm{AG}\,(Y)$ and the rest of the proof follows the usual pattern. $\square$

We will use the following well-known result several times.

Proposition A.1 ([Lew82]). *Assume that in the commutative diagram*

$$\begin{tikzcd}
A \arrow[rr, "\alpha"] \arrow[dr, "\imath"] \arrow[dd, "f"'] & & A' \arrow[dr, "\imath'"] \arrow[dd, "f'" near start] & \\
& X \arrow[rr, "\beta" near end] \arrow[dd, "p" near start] & & X' \arrow[dd, "p'"] \\
Y \arrow[rr, "\gamma" near start] \arrow[dr, "q"'] & & Y' \arrow[dr, "q'"'] & \\
& P \arrow[rr, "\epsilon"] & & P'
\end{tikzcd}$$

of G-spaces and G-maps, the maps $\imath$, $\imath'$, α, β and γ are G-cofibrations. If both the left and the right faces are equivariant push-out squares and the top face is an equivariant pull-back square. Then ϵ is a G-cofibration.

In order to proceed, we first describe some auxiliary spaces which are relevant to our constructions. Given positive integers n, r and k, let $S^n_{k,r}$ denote the collection of all k-tuples $\underline{I} = (I_1, \dots, I_k)$ such that $I_j = (i_{1j}, \dots, i_{rj}) \in [n]^r$ for all $j = 1, \dots, k$, where $[n] = \{1, \dots, n\}$. For each $\underline{I} \in S^n_{k,r}$ define $\Delta_{\underline{I}}(X) \subset \underbrace{X^n \times \cdots \times X^n}_{k\text{-times}}$ to be the "multidiagonal" consisting of those points $(\underline{a}_1, \dots \underline{a}_k)$, with $\underline{a}_j = (a_{1j}, \dots, a_{nj}) \in X^n$, which satisfy $a_{1i_{1j}} = \dots = a_{ki_{kj}}$, for all $j = 1, \dots, r$. Finally define $\Delta^n_{k,r}(X) = \cup_{\underline{I}} \Delta_{\underline{I}}$, where the union is taken over all $\underline{I} \in S^n_{k,r}$.

Lemma A.2.

a. *If X is a strong G-LEC , then the inclusion $\Delta^n_{k,r}(X) \to X^n \times \cdots \times X^n$ is a $(G \times (\mathfrak{S}_n \wr \mathfrak{S}_k))$-cofibration, for all n, k and r. In particular, if one defines $Q^n_{k,r}(X)$ as the quotient of $\Delta^k_n(X)$ by the normal subgroup $\mathfrak{S}_n \times \mathfrak{S}_n$ then the inclusion $Q^n_{k,r}(X) \to SP^n(X) \times \cdots \times SP^n(X)$ is a $G \times \mathfrak{S}_k$-cofibration.*

b. *The following commutative diagram is a pullback and a pushout square in the category of G-spaces*

$$\begin{tikzcd}
Q^n_{2,1}(X) \arrow[r] \arrow[d] & SP^n(X) \times SP^n(X) \arrow[d] \\
AG\,(X)^{\leq n-1} \arrow[r] & AG\,(X)^{\leq n},
\end{tikzcd}$$

and the map of pairs $(SP^n(X) \times SP^n(X), Q^n_{2,1}(X)) \to (AG\,(X)^{\leq n}, AG\,(X)^{\leq n-1})$ *is a relative homeomorphism*

Proof. **a.** The first assertion is shown in [LF, Proposition 2.5] while the second follows from the general fact that whenever $j : A \to B$ is a $\tilde{G}$-cofibration, for any topological group $\tilde{G}$, and $\tilde{H} \trianglelefteq \tilde{G}$ is a normal subgroup, then the induced map $j/\tilde{H} : A/\tilde{H} \to B/\tilde{H}$ is a $G/\tilde{H}$-cofibration.

b. This follows by simple inspection. □

Proof. [Proposition 2.19]

a. Note that the $G \times \mathfrak{S}_k$-cofibration $Q^n_{k,n} \to SP^n(X) \times \cdots \times SP^n(X)$ described in the previous lemma is precisely the inclusion of the small diagonal.

b. First observe that $\mathrm{AG}\,(X)^{\leq 1} = (X \times X)/\Delta_X$ and that the inclusion $\mathrm{AG}\,(X)^{\leq 0} = \{0\} \to \mathrm{AG}\,(X)^{\leq 1}$ is just the inclusion of the base point, which is a G-cofibration, since the diagonal is cofibered. Using the pushout diagram in Lemma A.2(b), and assuming that $\mathrm{AG}\,(X)^{\leq n-1}$ is a G-LEC one concludes that $\mathrm{AG}\,(X)^{\leq n}$ is a G-LEC ; cf. [LF, Theorem 3.1]. Furthermore, the relative homeomorphism of Lemma A.2(b) implies that each inclusion $\mathrm{AG}\,(X)^{\leq n-1} \subset \mathrm{AG}\,(X)^{\leq n}$ is a G-cofibration and hence the colimit $\mathrm{AG}\,(X) = \varinjlim \mathrm{AG}\,(X)^{\leq n}$ is a G-LEC ; cf. [LF, Corollary 3.2].

c. The inclusion $X \to \mathcal{Z}(X)$ which sends $x \in X$ to $x - x_o$, factors through a G-map $\epsilon : X \to \mathcal{Z}(X)^{\leq 1} = (X \times X)/\Delta_X$, and one has a commutative diagram of G-spaces

$$\begin{tikzcd}
\{x_o\} \arrow[rr] \arrow[dr, "\imath"] \arrow[dd] & & \Delta_X \arrow[dr, "\delta_X"] \arrow[dd] & \\
& X \arrow[rr, "\epsilon_{x_o}"] \arrow[dd, "1_X"] & & X \times X \arrow[dd, "\pi"] \\
\{x_o\} \arrow[rr] \arrow[dr, "\imath"] & & \{*\} \arrow[dr] & \\
& X \arrow[rr, "\epsilon"] & & (X \times X)/\Delta_X
\end{tikzcd}$$

where $\epsilon_{x_o} : X \to X \times X$ sends $x \in X$ to (x, x_o). It now follows from Proposition A.1 that ϵ is a G-cofibration and from Proposition 2.19 that $X \to \mathrm{AG}\,(X)$ is a G-cofibration, since composition of cofibrations is a cofibration. □

Proof. [Proposition 2.20]

a. This is proven in [LF, Corollary 3.4].

b. Given a G-cofibration $f : X \to Y$ between strong G-LEC's, we show that the induced map $f_n : \mathrm{AG}\,(X)^{\leq n} \to \mathrm{AG}\,(Y)^{\leq n}$ is a cofibration for all $n \geq 1$, and this suffices to prove the assertion.

These maps fit into a commutative diagram

$$\begin{tikzcd}
Q^n_{2,1}(X) \arrow[rr, "f^Q_{n+1}"] \arrow[dr, "\imath_X"] \arrow[dd, "\pi_X"'] & & Q^n_{2,1}(Y) \arrow[dr, "\imath_Y"] \arrow[dd, "\pi_Y"'] & \\
& SP^{n+1}(X) \times SP^{n+1}(X) \arrow[rr, "f^{n+1} \times f^{n+1}"] \arrow[dd, "p"] & & SP^{n+1}(Y) \times SP^{n+1}(Y) \arrow[dd, "p'"] \\
\mathrm{AG}\,(X)^n \arrow[rr, "f_n"] \arrow[dr, "i_X"'] & & \mathrm{AG}\,(Y)^n \arrow[dr, "i_Y"'] & \\
& \mathrm{AG}\,(X)^{n+1} \arrow[rr, "f_{n+1}"'] & & \mathrm{AG}\,(Y)^{n+1}
\end{tikzcd}$$

where the left and right faces are pushout squares, according to Lemma A.2(b), and the top face is a pullback where $f^{n+1} \times f^{n+1}$, $\imath_X$ and $\imath_Y$ arrows are all cofibrations. Using induction we may assume that f_n is a G-cofibration and hence f_{n+1} will be shown to be a G-cofibration once one shows that f^Q_{n+1} is a G-cofibration; cf. Proposition A.1. The latter assertion follows from the definitions and a direct application of [BV73, two props].

c. Given based strong G-LEC 's X and Y, we assume that $R^n : \mathrm{AG}\,(X)^{\leq n} \wedge Y \to \mathrm{AG}\,(X \wedge Y)^{\leq n}$ is a cofibration, for a fixed $n \geq 0$, and construct a commutative diagram

$$\begin{tikzcd}
T_{n+1}(X,Y) \arrow[rr] \arrow[dr] \arrow[dd] & & Q^{n+1}_{2,1}(X \wedge Y) \arrow[dr] \arrow[dd] & \\
& P_{n+1}(X,Y) \arrow[rr] \arrow[dd, "p"] & & SP^{n+1}(X \wedge Y) \times SP^{n+1}(X \wedge Y) \arrow[dd, "p'"] \\
\mathrm{AG}\,(X)^n \wedge Y \arrow[rr, "R^n"] \arrow[dr, "i_X \wedge 1_Y"'] & & \mathrm{AG}\,(X \wedge Y)^n \arrow[dr, "i_{X \wedge Y}"'] & \\
& \mathrm{AG}\,(X)^{n+1} \wedge Y \arrow[rr, "R^{n+1}"'] & & \mathrm{AG}\,(X \wedge Y)^{n+1}
\end{tikzcd}$$

satisfying the conditions of Proposition A.1. Notice that the right face is a pushout square, according to Lemma A.2.

Now define $S_{n+1}(X,Y) \subset (X \wedge Y)^{n+1} \times (X \wedge Y)^{n+1}$ to be the set of all points

$$(\alpha_1, \dots, \alpha_{n+1}; \beta_1, \dots, \beta_{n+1})$$

which can be represented as $\alpha_j = a_j \wedge y$, $\beta_j = b_j \wedge y$ for $a_j, b_j \in X$ and $y \in Y$, for all $j = 1, \ldots, n$. It is easy to see that the inclusion $S_{n+1}(X,Y) \to (X \wedge Y)^{n+1} \times (X \wedge Y)^{n+1}$ is a $(G \times \mathfrak{S}_n \times \mathfrak{S}_n)$-cofibration, and hence $P_{n+1}(X,Y) \stackrel{\text{def}}{=} S_{n+1}(X,Y)/(\mathfrak{S}_n \times \mathfrak{S}_n) \subset SP^{n+1}(X \wedge Y) \times SP^{n+1}(X \wedge Y)$ is a G-cofibration. Finally, define $T_{n+1}(X,Y) \stackrel{\text{def}}{=} P_{n+1}(X,Y) \cap Q^n_{2,1}(X \wedge Y)$. A careful inspection of the definitions shows that these spaces satisfy the required conditions. $\square$

References

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, LNM, vol. 347, Springer-Verlag, New York, 1973.

[Dre72] A. Dress, *Contributions to the theory of induced representations*, Algebraic K-Theory, Proc. Conf., Seattle (New York), Lecture Notes in Mathematics, vol. 342, Springer-Verlag, 1972, pp. 182–240.

[DT56] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische produkte*, Ann. of Math. (2) **67** (1956), 230–281.

[EK+95] A. D. Elmendorf, I. Kriz, , M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Preprint, 1995.

[GM92] J. P. C. Greenlees and J. P. May, *Some remarks on the structure of Mackey functors*, Proc. Amer. Math. Soc. **115** (1992), 237–243.

[Ill75] S. Illman, *Equivariant singular homology and cohomology - I*, Mem. Amer. Math. Soc., vol. 156, AMS, Providence, Rhode Island, 1975.

[Koz86] A. Kozlowski, *Equivariant bundles and cohomology*, Trans. Amer. Math. Soc. **296** (1986), 181–190.

[Lew] L. G. Lewis, Jr., *The theory of Green functors*, Mimeographed notes.

[Lew82] L. G. Lewis, Jr., *When is the natural map $X \to \Omega\Sigma X$ a cofibration*, Trans. Amer. Math. Soc. **273** (1982), no. 1, 147–155.

[Lew92] L. G. Lewis, Jr., *The equivariant Hurewicz map*, Trans. Amer. Math. Soc. **129** (1992), 433–472.

[LF] P. Lima-Filho, *A note on strong equivariant locally equiconnected spaces*, Preprint, August 1995.

[LF93] P. Lima-Filho, *Completions and fibrations for topological monoids*, Trans. Amer. Math. Soc. **340** (1993), 127–147.

[LF94] P. Lima-Filho, *The topological group structure of algebraic cycles*, Duke Math. J.. **75** (1994), no. 2, 467–491.

[Lin76] H. Lindner, *A remark on Mackey functors*, Manuscripta Math. **18** (1976), 273–278.

[LLFM] H. B. Lawson, Jr., P. Lima-Filho, and M.-L. Michelsohn, *Algebraic cycles and equivariant cohomology theories*, SUNY Preprint, April 1995. Submitted for publication.

[LMM81] L. G. Lewis, Jr., J. P. May, and J. E. McClure, *Ordinary $RO(G)$-graded cohomology*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 208–212.

[LMS85] L. G. Lewis, Jr., J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, LNM, vol. 1213, Springer-Verlag, New York, 1985.

[McC69] M. C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. **146** (1969), 273–298.

[Nis84] Goro Nishida, *Hecke functors and the equivariant Dold-Thom theorem*, Publ. RIMS, Kyoto Univ. **20** (1984), 65–77.

[Ste51] N. Steenrod, *The topology of fiber bundles*, Princeton Univ. Press, Princeton, NJ, 1951.

[tD87] T. tom Dieck, *Transformation groups*, de Gruyter Studies in Mathematics, vol. 8, Walter de Gruyter, New York, NY, 1987.

[tDP82] T. tom Dieck and T. Petrie, *Homotopy representations of finite groups*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 129–169.

[Tri83] G. Triantafillou, *Rationalization for Hopf G-spaces*, Math. Z. **182** (1983), 485–500.
[Wan80] S. Waner, *Equivariant homotopy theory and milnor's theorem*, Trans. Amer. Math. Soc. **258** (1980), 351–368.
[Whi78] G. W. Whitehead, *Elements of homotopy theory*, 1st ed., Grad. Texts in Math., vol. 61, Springer-Verlag, New York, NY, 1978.

Texas A&M University
E-mail address: plfilho@math.tamu.edu