Canonical Dirichlet operator and distorted Brownian motion on Poisson spaces

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Abstract

We derive the precise relation of the *internal* differential geometry of Poisson spaces developed in [1] to the corresponding *external* geometry given by the associated Fock structure. In particular, we identify the (internal) Laplace-Beltrami operator $H^{\Gamma}_{\pi_\sigma}$ on Poisson space as the second quantization of the Laplacian on $\mathbb{R}^d$. Furthermore, the diffusion process generated by $H^{\Gamma}_{\pi_\sigma}$, which is a kind of distorted Brownian motion on the configuration space, is described explicitly. All this generalizes to the case where the underlying Euclidean space $\mathbb{R}^d$ is replaced by an (infinite-dimensional) manifold.

L'opérateur de Dirichlet canonique et le mouvement Brownien tordu sur les espaces de Poisson

Résumé — Une relation précise est établie entre la "géométrie différentielle intrinsèque" des espaces de Poisson développée dans la note précédante [1] et la "géométrie extérieure" associée à la structure de Fock correspondante. En particulier, l'opérateur de Laplace-Beltrami $H^{\Gamma}_{\pi_\sigma}$ (intrinsèque) sur l'espace de Poisson est identifié avec la deuxième quantisation du Laplacien sur $\mathbb{R}^d$. De plus, le processus de diffusion engendré par $H^{\Gamma}_{\pi_\sigma}$ est identifié. Tous les résultats ont une généralisation au cas où $\mathbb{R}^d$ est remplacé par une variété (de dimension finie ou infinie).

1 La structure de Fock des espaces de Poisson

Soit Γ l'ensemble de toutes les configurations localement finies dans $\mathbb{R}^d$. Γ est vu comme un sous-espace fermé de l'espace $\mathcal{M}(\mathbb{R}^d)$ des mesures de Radon sur $\mathbb{R}^d$ avec la topologie de la convergence vague. Soit $\sigma \in \mathcal{M}(\mathbb{R}^d)$ de la forme $d\sigma(x) = \rho(x)\, dm(x)$, m étant la mesure de Lebesgue sur $\mathbb{R}^d$ et $\rho = \phi^2$, $\phi \in H^{1,2}_{loc}(\mathbb{R}^d)$, $\phi(x) > 0\ \forall x \in \mathbb{R}^d$. Soit π_σ la mesure de Poisson d'intensité σ sur Γ. Soit $\mathcal{F}C_b^\infty$ l'espace des fonctions cylindriques lisses et bornées. Nous considérons $L^2(\sigma)$ comme espace tangent à Γ dans un point $\gamma \in \Gamma$ et définissons le gradient de Fock ∇^F par

$$(\nabla^F f)(\gamma, x) = f(\gamma + \varepsilon_x) - f(\gamma), \quad x \in \mathbb{R}^d,\ f \in \mathcal{F}C_b^\infty$$

comme élément de $L^2(\pi_\sigma) \otimes L^2(\sigma)$. La dérivée dans la direction φ est définie par

$$(\nabla^F_\varphi f)(\gamma) \equiv (\nabla^F f(\gamma,\cdot), \varphi(\cdot))_{L^2(\sigma)}$$

pour tout $\varphi \in \mathcal{D}$ (espace de Schwartz).

Il y a une correspondance unitaire entre $L^2(\pi_\sigma)$ et l'espace de Fock symétrique Exp $L^2(\sigma)$ donnée par $f(\gamma) = \sum_{n=0}^{\infty} Q_n^{\pi_\sigma}(f^{(n)};\gamma)$, $Q_n^{\pi_\sigma}(f^{(n)};\gamma)$ étant le n-ple intégrale stochastique de $f^{(n)}$ par rapport au processus de Poisson engendré par la mesure de Poisson π_σ, voir [10].

2 Deuxième quantisation, opérateur de Dirichlet canonique et mouvement Brownien tordu

Théorème: Soit A un opérateur positif autoadjoint dans $L^2(\sigma)$ et soit D un domaine d'essentielle auto-adjonction pour A. Alors l'image $H_{\pi_\sigma,A}$ de la deuxième quantisation de A dans l'espace $L^2(\pi_\sigma)$ est essentiellement autoadjointe et positive sur $\mathcal{F}C_b^\infty$ et la forme bilinéaire correspondante est

$$(H_{\pi_\sigma,A} f, g)_{L^2(\pi_\sigma)} = \int_\Gamma (\nabla^F f, A\nabla^F g)_{L^2(\sigma)} \, d\pi_\sigma \ .$$

Soit H_σ l'opérateur donné sur D par

$$(H_\sigma \ \varphi) = -\Delta\varphi(x) - < \nabla \log \rho(x), \nabla\varphi(x) > \ .$$

Si H_σ est essentiellement autoadjoint sur D, alors $H_{\pi_\sigma}^\Gamma = H_{\pi_\sigma,H_\sigma}$, où $H_{\pi_\sigma}^\Gamma$ est l'opérateur de Dirichlet correspondant à la forme de Dirichlet canonique $\mathcal{E}_{\pi_\sigma}^\Gamma$ sur l'espace de Poisson.

Nous associons à $H_{\pi_\sigma}^\Gamma$ un processus de Markov X ayant Γ comme espace des états ("mouvement brownien tordu " sur Γ). X est en fait le processus de diffusions associé (au sens de [8]) avec la forme de Dirichlet canonique $\mathcal{E}_{\pi_\sigma}^\Gamma$ sur l'espace de Poisson.

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1 Fock structure of Poisson spaces

Let $\Gamma \equiv \Gamma_{\mathbb{R}^d}$ be the set of all locally finite configurations in $\mathbb{R}^d$:

$$\Gamma = \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}.$$

We will identify γ with the (integer-valued) measure $\sum_{x\in\gamma} \varepsilon_x$. Then for any $\varphi \in C_0(\mathbb{R}^d)$ we have a functional $\Gamma \ni \gamma \mapsto < \varphi, \gamma > = \sum_{x\in\gamma} \varphi(x) \in \mathbb{R}$. The space Γ is endowed with the vague topology.

Let $d\sigma(x) := \rho(x) dm(x)$, where m is the Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$, and $\rho = \phi^2$ with $\phi \in H_{loc}^{1,2}(\mathbb{R}^d)$, $\phi > 0$. We recall that the Poisson measure π_σ (with intensity measure σ) on $(\Gamma, \mathcal{B}(\Gamma))$ is defined via its Laplace transform by

$$\int_\Gamma e^{<\varphi,\gamma>} d\pi_\sigma(\gamma) = \exp[\int_{\mathbb{R}^d} (e^{\varphi(x)} - 1) d\sigma(x)], \ \varphi \in C_0(\mathbb{R}^d) \ . \tag{1.1}$$

In the following we will use a well-known isomorphism between the Poisson space $L^2(\pi_\sigma) := L^2(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$ and a Fock space in an essential way.

Our approach is based on [6], but we should also mention [4], [7], [9], [10] for related considerations and references therein.

Let $\mathcal{D} \equiv C_0^\infty(I\!R^d)$ We introduce a set of smooth cylinder functions $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ (dense in $L^2(\pi_\sigma)$) which consists of all functions of the form

$$f(\gamma) = F(<\varphi_1,\gamma>,\ldots,<\varphi_N,\gamma>),\ \gamma\in\Gamma, \tag{1.2}$$

where $\varphi_1,\ldots,\varphi_N \in \mathcal{D}$ and $F\in C_b^\infty(I\!R^N)$. Let us define the *Fock gradient* ∇^F of a function $f:\Gamma\to I\!R$. To this end we will consider as the tangent space to Γ at any point $\gamma\in\Gamma$ the same space $L^2(I\!R^d,\mathcal{B}(I\!R^d),\sigma) =: L^2(\sigma)$ and define a mapping

$$\nabla^F : \mathcal{F}C_b^\infty(\mathcal{D},\Gamma) \ni f \mapsto \nabla^F f \in L^2(\pi_\sigma)\otimes L^2(\sigma)$$

by

$$(\nabla^F f)(\gamma,x) := f(\gamma+\varepsilon_x) - f(\gamma),\ \ \gamma\in\Gamma,\ x\in I\!R^d. \tag{1.3}$$

We stress that the transformation $\Gamma\ni\gamma\mapsto\gamma+\varepsilon_x\in\Gamma$ is well-defined because $\pi_\sigma(\{\gamma\in\Gamma|x\in\gamma\}) = 0$ for any $x\in I\!R^d$. The directional derivative is then defined as

$$(\nabla_\varphi^F f)(\gamma) = (\nabla^F f(\gamma,\cdot),\varphi(\cdot))_{L^2(\sigma)} = \int_{I\!R^d}[f(\gamma+\varepsilon_x)-f(\gamma)]\varphi(x)d\sigma(x)$$

for any $\varphi\in\mathcal{D}$. The Fock gradient ∇^F yields (via a corresponding "integration by parts" formula) an orthogonal system of Charlier polynomials on the Poisson space $(\Gamma,\mathcal{B}(\Gamma),\pi_\sigma)$.

For any $n\in I\!N$ and all $\varphi\in\mathcal{D}$ we introduce a function in $L^2(\pi_\sigma)$ by

$$Q_n^{\pi_\sigma}(\varphi^{\otimes n};\gamma) := ((\nabla_\varphi^F)^{*n}1)(\gamma). \tag{1.4}$$

and define $Q_0^{\pi_\sigma} := 1$. These functions have the representation $Q_n^{\pi_\sigma}(\varphi^{\otimes n};\gamma) = <\varphi^{\otimes n}, Q_n^{\pi_\sigma}(\gamma)>$ with generalized symmetric kernels $Q_n^{\pi_\sigma}(\gamma):\Gamma\mapsto \mathrm{Exp}_n\mathcal{D}'$. Here and below by $\mathrm{Exp}_n E$ we denote the n-th symmetric tensor power of a linear topological space E (see e.g. [3]). Then for any smooth symmetric kernel $\varphi^{(n)}$ we introduce the function $Q_n^{\pi_\sigma}(\varphi^{(n)};\gamma) := <\varphi^{(n)}, Q_n^{\pi_\sigma}(\gamma)>$ such that for all $\varphi^{(n)}\in\mathrm{Exp}_n\mathcal{D},\ \psi\in\mathrm{Exp}_m\mathcal{D}$

$$\int_\Gamma Q_n^{\pi_\sigma}(\varphi^{(n)};\gamma)Q_m^{\pi_\sigma}(\psi^{(m)};\gamma)d\pi_\sigma(\gamma) = \delta_{nm}n!(\varphi^{(n)},\psi^{(n)})_{L^2(\sigma^{\otimes n})}. \tag{1.5}$$

Hence (1.4) extends to the case of kernels from the so-called n-particle Fock space $\mathrm{Exp}_n L^2(\sigma)$, $n\in I\!N$, and we set $\mathrm{Exp}_0 L^{(2)}(\sigma) = I\!R$. As usuall we introduce the *symmetric Fock space* (or symmetric tensor algebra) over the Hilbert space $L^2(\sigma)$ as $\mathrm{Exp}L^2(\sigma) := \oplus_{n=0}^\infty \mathrm{Exp}_n L^2(\sigma)$ [3]. The square of the norm of a vector $(f^{(n)})_{n=0}^\infty \in \mathrm{Exp}L^2(\sigma)$ is given by $\sum_n n!\|f^{(n)}\|^2_{L^2(\sigma^{\otimes n})}$. For any $f\in L^2(\pi_\sigma)$ there exists such a Fock vector, so that we have the following chaos decomposition:

$$f(\gamma) = \sum_{n=0}^\infty Q_n^{\pi_\sigma}(f^{(n)};\gamma), \tag{1.6}$$

and moreover $\|f\|^2_{L^2(\pi_\sigma)} = \sum_n n!\|f^{(n)}\|^2_{L^2(\sigma^{\otimes n})}$. And vice versa, by formula (1.6) any Fock vector generates a function from $L^2(\pi_\sigma)$. In stochastic analysis the function $Q_n^{\pi_\sigma}(f^{(n)};\gamma)$ is called *n-tuple stochastic integral* of the function $f^{(n)}$ with respect to a Poisson process generated by the Poisson white noise measure π_σ, see e.g. [9].

2 Second quantization and canonical Dirichlet operators

For any contraction B in $L^2(\sigma)$ it is possible to define an operator $\mathrm{Exp}B$ as a contraction in $\mathrm{Exp}L^2(\sigma)$ which in any subspace $\mathrm{Exp}_n L^2(\sigma)$ is given by $B\otimes\cdots\otimes B$ (n-times). For a positive self-adjoint operator A in $L^2(\sigma)$ we introduce the second quantization operator $d\mathrm{Exp}A$ as the generator of the semigroup $\mathrm{Exp}(e^{-tA}), t\geq 0$.

Let $H_{\pi_\sigma,A}$ denote the image of the operator $d\mathrm{Exp}A$ in the space $L^2(\pi_\sigma)$ under the decribed isomorphism.

Theorem 2.1 *Suppose $\mathcal{D}\subset D(A)$ is a core for the operator A. Then $H_{\pi_\sigma,A}$ is a positive essentially self-adjoint operator on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ and for all $f,g\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ the bilinear form corresponding to the operator $H_{\pi_\sigma,A}$ has the following representation*

$$(H_{\pi_\sigma,A}f,g)_{L^2(\pi_\sigma)} = \int_\Gamma (\nabla^F f, A\nabla^F g)_{L^2(\sigma)}d\pi_\sigma. \tag{2.7}$$

Let us consider another gradient on the Poisson space (Γ,π_σ) the so-called *intrinsic gradient* ∇^Γ introduced in [1]. We do not recall the coordinate free definition of ∇^Γ here, but only recall that for any f of the form (1.2)

$$(\nabla^\Gamma f)(\gamma,x) = \sum_{k=1}^N \frac{\partial F}{\partial s_k}(<\varphi_1,\gamma>,\ldots,<\varphi_N,\gamma>)\,\nabla\varphi_k(x),\ \gamma\in\Gamma, x\in I\!R^d,$$

where ∇ denotes the usual gradient on $I\!R^d$. For any $\gamma\in\Gamma$ the value $\nabla^\Gamma f(\gamma,\cdot)$ is considered as an element of the *tangent space* $T_\gamma(\Gamma)$ to Γ at the point γ, that is the space of vector fields $L^2(I\!R^d\to I\!R^d,\gamma)$ square summable with respect to the measure γ. This gradient gives the *canonical Dirichlet form* $(\mathcal{E}_{\pi_\sigma}, D(\mathcal{E}_{\pi_\sigma}))$ on the Poisson space, i.e., the closure on $L^2(\pi_\sigma)$ of

$$\mathcal{E}^\Gamma_{\pi_\sigma}(f,g) = \int_\Gamma (\nabla^\Gamma f,\nabla^\Gamma g)_{T_\gamma(\Gamma)}d\pi_\sigma,\ f,g\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma),$$

and the *canonical Dirichlet operator* $H^\Gamma_{\pi_\sigma}$ which is the generator of $\mathcal{E}^\Gamma_{\pi_\sigma}$, see [1]. For a description of the canonical Dirichlet operator in the Fock space we need to introduce the Dirichlet operator H_σ associated with the classical Dirichlet form $(\mathcal{E}_\sigma, D(\mathcal{E}_\sigma))$ of the measure σ on $I\!R^d$, i.e., the closure on $L^2(\sigma)$ of $\mathcal{E}_\sigma(u,v) = \int <\nabla u,\nabla v>_{I\!R^d} d\sigma,\ u,v\in\mathcal{D}$. Then $\mathcal{D}\subset D(H_\sigma)$ and for all $\varphi\in\mathcal{D},\ (H_\sigma\varphi)(x) = -\triangle\varphi(x) - <\nabla\log\rho(x),\nabla\varphi(x)>$, see e.g. [3].

Theorem 2.2 *Suppose $\mathcal{D}$ is a core for H_σ, then*

$$H^\Gamma_{\pi_\sigma} = H_{\pi_\sigma,H_\sigma},$$

i.e.,$H^\Gamma_{\pi_\sigma}$ coincides with the second quantization of the Dirichlet operator H_σ.

Remark 2.1 *The condition that $\mathcal{D}$ is a core for H_σ is, in fact, superfluous. However, to see this, H_{π_σ,H_σ} has to be replaced resp. identified as the Friedrichs' extension of $dExpH_\sigma$ restricted to $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$.*

Theorem 2.2 gives full information about the spectrum of the canonical Dirichlet operator H_{π_σ} on the Poisson space in terms of the underlying Dirichlet operator H_σ coming from the intensity measure σ.

3 Distorted Brownian motion on Γ

The canonical Dirichlet operator can be considered as a Laplace-Beltrami operator on Γ [1]. Now we will describe the Markov process (in fact a diffusion) with state space Γ generated by $H^\Gamma_{\pi_\sigma}$. Such a process (whose construction is indeed straight-forward and basically known) has then a natural interpretation as a *distorted Brownian motion* on Γ. (From this point of view the *Brownian motion* on Γ corresponds to the case σ=Lebesgue measure.) Let $(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in\mathbb{R}^d})$ be the diffusion process on $\mathbb{R}^d$ associated with $(\mathcal{E}_\sigma, D(\mathcal{E}_\sigma))$ (cf. e.g. [8]). For $\gamma\in\Gamma$ define $\mathbb{P}_\gamma$ to be the distribution of the map

$$(\omega_x)_{x\in\mathbb{R}^d} \mapsto \sum_{x\in\gamma} \varepsilon_{X_\cdot(\omega_x)}$$

from $\Omega^{\mathbb{R}^d}$ to $\mathbf{\Omega} := C([0,\infty)\to\Gamma)$ under $\otimes_{x\in\mathbb{R}^d} P_x$. Let $\mathbb{X}_t : \mathbf{\Omega}\to\Gamma, t\geq 0$, be the evaluation map at t and $\mathbb{F} := \sigma\{\mathbb{X}_t | t\geq 0\}$.

Theorem 3.1 $(\mathbf{\Omega}, \mathbb{F}, (\mathbb{X}_t)_{t\geq 0}, (\mathbb{P}_\gamma)_{\gamma\in\Gamma})$ *(i.e., the corresponding* independent particle process*) is the diffusion process associated (in the sense of [8]) with the canonical Dirichlet form $(\mathcal{E}^\Gamma_{\pi_\sigma}, D(\mathcal{E}^\Gamma_{\pi_\sigma}))$ on the Poisson space, that is, for all $f\in L^2(\pi_\sigma)$, $t\geq 0$, $\gamma\mapsto\int_{\mathbf{\Omega}} f(\mathbb{X}_t)d\mathbb{P}_\gamma$ is a π_σ-version of $T_t^\Gamma f$.*

As usual we set

$$P_\sigma := \int_{\mathbb{R}^d} P_x d\sigma(x), \quad \mathbb{P}_{\pi_\sigma} := \int_\Gamma \mathbb{P}_\gamma d\pi_\sigma(\gamma).$$

$\mathbb{P}_{\pi_\sigma}$ is nothing but the so-called *equilibrium process* given by H_σ (cf. e.g. [5],[11]). Let π_{P_σ} be the Poisson measure on the configuration space $\Gamma_{\mathcal{C}}$ over $\mathcal{C} := C([0,\infty)\to\mathbb{R}^d)$ with intensity measure P_σ. Note that $\Gamma_{\mathcal{C}}$ and $\mathbf{\Omega}$ can be identified canonically.

Proposition 3.1 $\pi_{P_\sigma} = \mathbb{P}_{\pi_\sigma}$.

Hence the measure $\mathbb{P}_{\pi_\sigma}$ can be considered as the analogue of an "Euclidean measure" corresponding to the canonical Dirichlet form $\mathcal{E}^\Gamma_{\pi_\sigma}$.

Remark 3.1 *As in [1] all the above generalizes to the case where the underlying Euclidean space $\mathbb{R}^d$ is replaced by a possibly infinite dimensional manifold (such as e.g. loop spaces, cf. [1], Sect. 3, Remark (iv)).*

Detailed proofs for all results of this paper are contained in [2].

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