Differential geometry of Poisson spaces

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Abstract

We identify the differential geometry of the Poisson measure over $I\!R^d$. We prove an integration by parts formula and determine explicitly the corresponding gradient, divergence, Dirichlet form, generalized Laplacian and heat semigroup. We also briefly describe immediate generalizations, for example to Poisson measures over (infinite dimensional) manifolds or other random fields (resp. Gibbs measures).

Géometrie différentielle des espaces de Poisson

Résumé — La géométrie différentielle associée à une mesure de Poisson sur $\mathbb{R}^d$ est décrite. Une formule d'intégration par parties est donnée, ainsi qu'une description des opérateurs gradient et divergence, la forme de Dirichlet, l'opérateur de Laplace généralisé et le semigroupe de la chaleur correspondants. Une brève discussion de généralisations au cas des mesures de Poisson des variétés infini-dimensionelles et à d'autres champs aléatoires (ou mesures de Gibbs) est donnée.

Version française abrégée —

1 Dérivées, intégration par partie

Soit Γ l'ensemble de toutes les configurations localement finies dans $\mathbb{R}^d$. Nous identifions $\gamma \in \Gamma$ avec la mesure $\sum_{x\in\gamma} \varepsilon_x$ et considérons Γ comme étant topologiquement un sous-ensemble fermé de l'espace $\mathcal{M}(\mathbb{R}^d)$ des mesures de Radon sur $\mathbb{R}^d$ avec la topologie de la convergence vague. Soit $\sigma \in \mathcal{M}(\mathbb{R}^d)$ de la forme $d\sigma(x) = \rho(x)dm(x)$ (m = mesure de Lebesgue sur $\mathbb{R}^d$), $\rho > 0$ m-p.p., $\sqrt{\rho} \in H^{1,2}_{loc}(\mathbb{R}^d)$. Soit π_σ la mesure de Poisson avec mesure d'intensité σ sur Γ. Une application $\psi : \mathbb{R}^d \to \mathbb{R}^d$ définit une transformation de Γ par $\gamma \mapsto \psi\ \gamma = \sum_{x\in\gamma} \varepsilon_{\psi x}$. Soit $\psi_{t,v}$ le groupe à un paramètre de difféomorphismes de $\mathbb{R}^d$ engendré par le champ de vecteurs $v \in C^1_0(\mathbb{R}^d \to \mathbb{R}^d)$. La dérivée directionelle $\nabla^\Gamma_v f$ de $f : \Gamma \to \mathbb{R}$ le long de v est définie par $(\nabla^\Gamma_v f)(\gamma) \equiv \frac{d}{dt} f(\psi_{t,v} v)|_{t=0}$, par exemple pour les f dans l'ensemble dense $\mathcal{F}C^\infty_b$ de $L^2(\Gamma; \pi_\sigma)$ (fonctions cylindriques lisses).

Nous définissons la dérivée logarithmique $\beta^{\pi_\sigma}_v$ de π_σ le long de v par

$$\beta^{\pi_\sigma}_v(\gamma) = \int_{\mathbb{R}^d} [< \beta^\sigma(x), v(x) >_{\mathbb{R}^d} + \mathrm{div}(v(x))]\, d\gamma(x)\ ,$$

avec $\beta^\sigma \equiv \frac{\nabla\rho}{\rho}$ ($\beta^\sigma \equiv 0$ sur $\{\rho = 0\}$). On a alors (intégration par parties), sur $\mathcal{F}C^\infty_b$:

$$\nabla^{\Gamma^*}_v = \nabla^\Gamma_v - \beta^{\pi_\sigma}_v\ .$$

2 Le gradient et la forme de Dirichlet canonique

Nous définissons l'espace tangent $T_\gamma(\Gamma)$ au point $\gamma \in \Gamma$ comme l'espace de Hilbert $L^2(\mathbb{R}^d \to \mathbb{R}^d;\gamma)$. Le gradient intrinsèque ∇^Γ d'une fonction $f \in \mathcal{F}C_b^\infty$ est, par définition, tel que $(\nabla_v^\Gamma f)(\gamma) = \langle \nabla^\Gamma f(\gamma), v\rangle_{T_\gamma(\Gamma)}$ pour tous les $v \in C_0^1(\mathbb{R}^d \to \mathbb{R}^d)$.

Nous définissons par dualité div^Γ sur les champs de vecteurs. On a alors pour tout champ de vecteurs de la forme $V(\gamma,x) = g(\gamma)v(x)$, $g \in \mathcal{F}C_b^\infty$, $v \in C_0^1(\mathbb{R}^d \to \mathbb{R}^d)$:

$$\begin{aligned}(\mathrm{div}^\Gamma V)(\gamma) &= < \nabla^\Gamma g(\gamma), v >_{T_\gamma(\Gamma)} \\ &\quad + < \mathrm{div} v, \gamma > g(\gamma) + < \beta^\sigma, v >_{T_\gamma(\Gamma)} \ g(\gamma) \ .\end{aligned}$$

La forme de Dirichlet engendrée par le gradient intrinsèque ∇^Γ est définie pour $f, g \in \mathcal{F}C_b^\infty$ par

$$\mathcal{E}_{\pi_\sigma}^\Gamma(f,g) := \int_\Gamma < \nabla^\Gamma f(\gamma), \nabla^\Gamma g(\gamma) >_{T_\gamma(\Gamma)} \ d\pi_\sigma(x) \ .$$

On montre que $\mathcal{E}_{\pi_\sigma}^\Gamma$ est fermable et sa fermeture est une forme de Dirichlet sur $L^2(\Gamma;\pi_\sigma)$. Le générateur correspondant $H_{\pi_\sigma}^\Gamma$ coïncide avec l'extension de Friedrichs de $\Delta^\Gamma f(\gamma) + < \mathrm{div}(\nabla^\Gamma f(\gamma,\cdot)), \gamma > + \langle < \beta^\sigma, \nabla^\Gamma f(\gamma) >_{\mathbb{R}^d}, \gamma\rangle$, Δ^Γ étant définie sur $\mathcal{F}C_b^\infty$ par $\mathrm{Tr}_{T_\sigma(\Gamma)\otimes T_\gamma(\Gamma)}(\nabla^\Gamma\nabla^\Gamma)$. T_t^Γ, le semigroupe de la chaleur correspondant, peut être calculé sur un ensemble dense comme suit:

$$T_t^\Gamma \exp[\langle \log(1+\varphi), \gamma\rangle] = \exp[< \log(1 + e^{-tH_\sigma}\varphi, \gamma >] \ ,$$

pour π_σ-p.t. $\gamma \in \Gamma$, H_σ étant l'extension de Friedrichs de la restriction de $(-\Delta - \langle \beta^\sigma, \nabla\rangle)$ à $C_0^\infty(\mathbb{R}^d)$.

L'enveloppe linéaire des fonctions $\exp\langle \ \langle \log(1+\varphi), \cdot\rangle$, $\varphi \in H_0^{1,2}(\mathbb{R}^d;\sigma)$, $\varphi > -1$, est un domaine d' autoadjonction essentielle pour $H_{\pi_\sigma}^\sigma$.

Nous avons donc trouvé un analogue naturel pour l'espace de Poisson des objects géometriques du calcul de Malliavin. Deux applications de ces idées en analyse stochastique, et en théorie des champs, sont en préparation.

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1 Derivatives and integration by parts

Let $\Gamma \equiv \Gamma_{I\!R^d}$ be the set of all locally finite configurations in $I\!R^d$:

$$\Gamma := \{\gamma \subset I\!R^d \mid |\gamma \cap K| < \infty \ \text{ for any compact } K \subset I\!R^d\}.$$

We will indentify γ with the (positive integer-valued) measure $\sum_{x\in\gamma}\varepsilon_x$. Then for any $\varphi \in C_0(I\!R^d)$ we have a functional $\Gamma \ni \gamma \mapsto < \varphi, \gamma > = \sum_{x\in\gamma}\varphi(x) \in I\!R$. The space Γ is endowed with the vague topology.

Let $d\sigma(x) := \rho(x)dm(x)$, where m is Lebesgue measure on $\mathcal{B}(I\!R^d)$, and $\rho > 0$ m-a.e. such that $\rho^{\frac{1}{2}} \in H_{loc}^{1,2}(I\!R^d)$. We recall that the Poisson measure π_σ (with intensity measure σ) on $(\Gamma, \mathcal{B}(\Gamma))$ is defined via its Laplace transform by

$$\int_\Gamma e^{<\varphi,\gamma>} d\pi_\sigma(\gamma) = \exp[\int_{I\!R^d}(e^{\varphi(x)} - 1)d\sigma(x)], \ \varphi \in C_0(I\!R^d) \ . \tag{1.1}$$

A map $\psi : I\!\!R^d \to I\!\!R^d$ defines a transformation of Γ by

$$\Gamma \ni \gamma \mapsto \psi\gamma = \{\psi x | x \in \gamma\} = \sum_{x\in\gamma} \varepsilon_{\psi x} \in \Gamma.$$

Any vector field $v \in C_0^1(I\!\!R^d \to I\!\!R^d)$ defines a one-parameter group $\psi_{t,v},\ t \in I\!\!R$, of diffeomophisms of $I\!\!R^d$.

Definition 1.1 *For $f : \Gamma \to I\!\!R$ we define the directional derivative along the vector field v as (provided the rigth hand side exists)*

$$(\nabla_v^\Gamma f)(\gamma) = \frac{d}{dt} f(\psi_{t,v}\gamma)|_{t=0}\ .$$

This definition applies to f in the following (dense in $L^2(\Gamma, \pi_\sigma)$) class of so-called smooth cylinder functions. We define $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ as the set of all functions on Γ of the form $f(\gamma) = F(<\varphi_1, \gamma>, \ldots, <\varphi_N, \gamma>)$, where $\varphi_1, \ldots, \varphi_N \in \mathcal{D} := C_0^\infty(I\!\!R^d)$ and $F \in C_b^\infty(I\!\!R^N)$. For any $f \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we have

$$(\nabla^\Gamma_v f)(\gamma) = \sum_{j=1}^N \frac{\partial F}{\partial s_j}(<\varphi_1, \gamma>, \ldots, <\varphi_N, \gamma>) <\nabla_v \varphi_j, \gamma>, \tag{1.2}$$

where $(\nabla_v \varphi)(x) = <\nabla\varphi(x), v(x)>$ is the usual directional derivative on $I\!\!R^d$ along the vector field v.

The logarithmic derivative of the measure σ is given by the vector field $\beta^\sigma = \nabla \log \rho = \frac{\nabla\rho}{\rho}$. Then the logarithmic derivative of σ along v is the function $x \mapsto \beta_v^\sigma(x) = <\beta^\sigma(x), v(x)>_{I\!\!R^d} + div\ v(x), x \in I\!\!R^d$, see e.g. [3].

Definition 1.2 *For any $v \in C_0^1(I\!\!R^d \to I\!\!R^d)$ we define the logarithmic derivative of π_σ along v as the following function on Γ:*

$$\Gamma \ni \gamma \mapsto \beta_v^{\pi_\sigma}(\gamma) := <\beta_v^\sigma, \gamma> = \tag{1.3}$$

$$\int_{I\!\!R^d} [<\beta^\sigma(x), v(x)>_{I\!\!R^d} + div\ v(x)] d\gamma(x).$$

Theorem 1.1 *For all $f, g \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and any $v \in C_0^1(I\!\!R^d \to I\!\!R^d)$ the following integration by parts formula holds:*

$$\int_\Gamma \nabla_v^\Gamma f\ g\ d\pi_\sigma = -\int_\Gamma f\ \nabla_v^\Gamma g\ d\pi_\sigma - \int_\Gamma f\ g\ \beta_v^{\pi_\sigma}\ d\pi_\sigma, \tag{1.4}$$

or

$$\nabla_v^{\Gamma\ *} = -\ \nabla_v^\Gamma\ -\ \beta_v^{\pi_\sigma},$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ in $L^2(\Gamma, \pi_\sigma)$.

2 Gradient and canonical Dirichlet form

By Riesz's representation theorem we can define a corresponding intrinsic gradient as follows:

Definition 2.1 *We define the tangent space $T_\gamma(\Gamma)$ at the point $\gamma \in \Gamma$ as the Hilbert space $L^2(\mathbb{R}^d \to \mathbb{R}^d, \gamma)$ of vector fields $u : \mathbb{R}^d \to \mathbb{R}^d$ with scalar product $< u, u >_{T_\gamma(\Gamma)} := \int_{\mathbb{R}^d} < u(x), u(x) >_{\mathbb{R}^d} d\gamma(x)$. The correspondig tangent bundle is denoted by $T(\Gamma)$. The intrinsic gradient of a function $f \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ is a mapping $\Gamma \ni \gamma \mapsto \nabla^\Gamma f(\gamma) \in T_\gamma(\Gamma)$ such that $(\nabla_v^\Gamma f)(\gamma) = < \nabla^\Gamma f(\gamma), v >_{T_\gamma(\Gamma)}$ for all $v \in C_0^1(\mathbb{R}^d \to \mathbb{R}^d)$.*

Definition 2.2 *For a vector field $V : \Gamma \to T(\Gamma)$ the divergence $div^\Gamma V \equiv div_{\pi_\sigma}^\Gamma V$ is defined via the duality relation*

$$\int_\Gamma < V(\gamma), \nabla^\Gamma f(\gamma) >_{T_\gamma(\Gamma)} d\pi_\sigma(\gamma) = - \int_\Gamma (div^\Gamma V)(\gamma) f(\gamma) d\pi_\sigma(\gamma) \qquad (2.5)$$

for all $f \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$.

Proposition 2.1 *For any vector field $V(\gamma, x) = g(\gamma)v(x)$ with $g \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, $v \in C_0^1(\mathbb{R}^d \to \mathbb{R}^d)$, we have*

$$(div^\Gamma V)(\gamma) = < \nabla^\Gamma g(\gamma), v >_{T_\gamma(\Gamma)} + < div\ v, \gamma > g(\gamma) + < \beta^\sigma, v >_{T_\gamma(\Gamma)} g(\gamma). \qquad (2.6)$$

For $f, g \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we introduce the *pre-Dirichlet form* which is generated by the *intrinsic gradient* ∇^Γ as

$$\mathcal{E}_{\pi_\sigma}^\Gamma(f, g) = \int_\Gamma < \nabla^\Gamma f(\gamma), \nabla^\Gamma g(\gamma) >_{T_\gamma(\Gamma)} d\pi_\sigma(\gamma).$$

We will also need the classical (pre-)Dirichlet form for the intensity measure σ which is given as $\mathcal{E}_\sigma(\varphi_1, \varphi_2) = \int_{\mathbb{R}^d} < \nabla\varphi_1, \nabla\varphi_2 >_{\mathbb{R}^d} d\sigma$ for $\varphi_1, \varphi_2 \in \mathcal{D}$. The closure of this form generates a Dirichlet operator H_σ which is given as the Friedrichs extension on $L^2(\mathbb{R}^d, \sigma)$ of

$$(H_\sigma\varphi)(x) = -\triangle\varphi(x) - < \beta^\sigma(x), \nabla\varphi(x) >, \ \varphi \in \mathcal{D},$$

see e.g. [3], [8].

For any $f \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ $(\nabla^\Gamma\nabla^\Gamma f)(\gamma, x, y) \in T_\gamma(\Gamma) \otimes T_\gamma(\Gamma)$ and we can define the Γ-*Laplacian* $(\triangle^\Gamma f)(\gamma) := Tr_{T_\gamma(\Gamma)\otimes T_\gamma(\Gamma)}(\nabla^\Gamma\nabla^\Gamma f)(\gamma) \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. We introduce a differential operator in $L^2(\Gamma, \pi_\sigma)$ on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ by the formula

$$(H_{\pi_\sigma}^\Gamma f)(\gamma) := -\triangle^\Gamma f(\gamma) - < div(\nabla^\Gamma f(\gamma, \cdot)), \gamma > - << \beta^\sigma, \nabla^\Gamma f(\gamma) >_{\mathbb{R}^d}, \gamma > . \qquad (2.7)$$

Theorem 2.1 *$H_{\pi_\sigma}^\Gamma = -div^\Gamma\nabla^\Gamma$ on $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, i.e.,*

$$\mathcal{E}_{\pi_\sigma}^\Gamma(f, g) = (H_{\pi_\sigma}^\Gamma f, g)_{L^2(\pi_\sigma)}, \ \ f, g \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma).$$

In particular, $(\mathcal{E}_{\pi_\sigma}^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma))$ is closable on $L^2(\Gamma, \pi_\sigma)$ and its closure $(\mathcal{E}_{\pi_\sigma}^\Gamma, D(\mathcal{E}_{\pi_\sigma}^\Gamma))$ is a Dirichlet form on $L^2(\Gamma, \pi_\sigma)$. The corresponding generator (again denoted by $H_{\pi_\sigma}^\Gamma$) is the Friedrichs extension of (2.3).

Denote by $T_t^\Gamma, t \geq 0$, the semigroup in $L^2(\Gamma, \pi_\sigma)$ with generator $-H_{\pi_\sigma}^\Gamma$, i.e., $T_t^\Gamma = e^{-tH_{\pi_\sigma}^\Gamma}, t \geq 0$

Proposition 2.2 *For any $\varphi \in L^2(I\!\!R^d, \sigma)$, $\varphi > -1, t \geq 0$, we have*

$$T_t^\Gamma \ \exp[< \log(1+\varphi), \gamma >] \ = \ \exp[< \log(1+e^{-tH_\sigma}\varphi), \gamma >], \ \textit{for } \pi_\sigma - a.e. \gamma \in \Gamma. \tag{2.8}$$

Corollary 2.1 *Let $H_0^{1,2}(I\!\!R^d, \sigma)$ be the closure of $\mathcal{D}$ with respect to the norm $(\mathcal{E}_\sigma + (\cdot, \cdot)_{L^2(I\!\!R^d, \sigma)})^{\frac{1}{2}}$. Then the linear hull $\mathcal{K}$ of*

$$\{\exp[< \log(1+\varphi), \cdot >] \mid \varphi \in H_0^{1,2}(I\!\!R^d, \sigma), \ \varphi > -1\}$$

is contained in the domain of $H_{\pi_\sigma}^\Gamma$ and $(H_{\pi_\sigma}^\Gamma, \mathcal{K})$ is essentially self-adjoint on $L^2(\Gamma, \pi_\sigma)$.

3 Concluding remarks

(i) The analogy of the above gradient, tangent bundle, etc. with the corresponding objects in the Malliavin calculus [9] is apparent. However, in our case the tangent space really varies from point to point.

(ii) In the announcement [1] we identify the corresponding diffusion process (a kind of *distorted Brownian motion* on Γ) and the associated *external geometry*, i.e., the geometry based on the corresponding Fock space.

(iii) We can extend the differential geometry of the Poisson space to cases where the Poisson measure π_σ is replaced by other measures on Γ obtained by local perturbations as e.g. Gibbs measures in classical statistical physics, see e.g. [7].

(iv) We emphasize that everything above (including the results in [1]) generalizes to the case where $I\!\!R^d$ is replaced by a Riemannian manifold M. In fact, the only ingredient needed is a Dirichlet form of gradient type over a manifold-like state space (as described in [11]) for which one has integration by parts. In particular, one can e.g. start with the Dirichlet form on the loop space of [6] (and obtain a "Poisson cloud" of loop-valued diffusions).

(v) Many people have worked on the Poisson space (see e.g. [4] and the references therein). We would only like to mention here that e.g. the integration by parts formula for directional derivatives in [5, Theorem 1.4] (see also the corresponding result in [10]) can be interpreted as a special case of (1.5) with our manifold M (see (iv)) being just $[0, 1]$ (resp. $[0, \infty)$ in [10]). The tangent bundle and hence the gradient in [5], [10] are, however, *totally different* from ours.

Detailed discussions about connections with previous related works on this subject, as well as the proofs of all results of this paper, of [1] and of the statements in (i)-(v) will be contained in the forthcoming paper [2] which will, however, be probably published in several parts.

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