A NOTE ON INVARIANT MEASURES OF INFINITE DIMENSIONAL DIFFUSIONS

Nikolai A. Tolmachev

Department of Mechanics and Mathematics,
Moscow State University
Moscow 119899, RUSSIA
vbogach@mech.math.msu.su

Abstract

We construct two examples of a nondegenerate diffusion on a Hilbert space X governed by the SDE

$$d\xi_t = \sigma(\xi_t)dW_t + B(\xi_t)dt,$$

with $\sigma(x) = I + \sigma_0(x)$ and $B(x) = \Lambda x + v(x)$, where $\Lambda : X \to X$ is a bounded linear operator, σ_0 and v are smooth (Fréchet differentiable on X) mappings with values in the space of nuclear operators on X and X, respectively, such that:

(i) in the first example $\Lambda x = -\frac{1}{2}x$ and the diffusion ξ_t has the transition probabilities and an invariant measure (which is unique) without directions of differentiability (or even continuity);

(ii) in the second example, ξ_t has two different invariant probabilities ν_1 and ν_2 such that ν_1 is equivalent to some Gaussian measure and is differentiable, while ν_2 has no directions of differentiability (or even continuity).

In addition, given $\varepsilon > 0$, both σ_0 and v can be chosen vanishing outside the ε-ball around zero and with the norms (respectively, in the space of trace operators on X and X) bounded by ε. Some natural modification of our construction makes the appropriate measures to be exceptionally supported (in the sense of Aronszajn) and ultimately singular.

1. Introduction

Analytic properties of the transition probabilities and invariant measures of infinite dimensional diffusions have been actively studied over the last three decades. Under various restrictions on the coefficients, some positive results have been obtained in [16], [18], [19], [21], [1], [17], [24], [2], [13], [14], [15], [24], [10], [11], [8], [9] (see also references in [7]).

It has been shown in [10], [11] that if μ is an invariant probability for a diffusion ξ on R^n with constant diffusion operator and drift $B \in L^2(\mu, R^n)$, then μ admits

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a density p from the Sobolev class $H^{1,1}(R^n)$ and, moreover, it has the logarithmic gradient $\beta^\mu := \nabla p/p$ belonging to $L^2(\mu, R^n)$. In [9] this result has been extended to diffusions with non-constant uniformly Lipschitzian and uniformly nondegenerate diffusion operators σ. In fact, the corresponding results have been proved even in a more general setting of weak elliptic equations

$$L^*\mu = 0, \tag{$*$}$$

where

$$Lf = trace(\sigma f'') + (B, f'),$$

and the equation $(*)$ is understood in the weak sense

$$\int Lf(x)\,\mu(dx) = 0 \qquad \forall f \in C_0^\infty(R^n).$$

In [9] and [11] also some infinite dimensional analogues of these results have been obtained. More specifically, let (X, H, γ) be an abstract Wiener space (with a centered Gaussian measure γ), $B(x) = -x + v(x)$, where $v : X \to H$ is a bounded Borel mapping, and $\sigma(x) = I + \sigma_0(x)$, where $\sigma_0(\,\cdot\,)$ is a measurable map with values in the space of nuclear operators on H satisfying certain additional technical assumptions. Suppose that a probability measure μ on X satisfies the equation $(*)$ with

$$Lf = trace_H(\sigma f'') + \langle f', B\rangle$$

with $C_0^\infty(R^n)$ replaced by the class $\mathcal{FC}^\infty$ of all functions of the form

$$f(x) = \varphi(l_1(x), \dots, l_n(x)), \quad \varphi \in C_b^\infty(R^n),\ l_i \in X^*.$$

Then μ is absolutely continuous with respect to γ, admits a representation $\mu = \varrho^2\gamma$ with ϱ from the Sobolev class $H^{2,1}(\gamma)$, and μ has the vector logarithmic derivative β^μ (in particular, μ is differentiable along all directions in H). Note that this result gave a positive solution of the well-known Shigekawa problem [24].

Certainly, a question arises whether the finite dimensional results can be extended to infinite dimensions in the same generality. The aim of this note is construct two examples showing that in infinite dimensions certain extra assumptions (of the type considered in [9]) are necessary. We consider Hilbert spaces X.

In the first example we construct, for arbitrary $\varepsilon > 0$, a smooth map σ_0 with values in the space of nuclear operators on X and a smooth map $v : X \to X$ with $\|\sigma_0(\,\cdot\,)\|_{L^{(1)}(X)} \le \varepsilon$ and $\|v(\,\cdot\,)\| \le \varepsilon$ such that there is a unique probability measure μ invariant for the diffusion ξ generated by the stochastic differential equation

$$d\xi_t = \sigma(\xi_t)dW_t + B(\xi_t)dt$$

with $\sigma(x) = I + \sigma_0(x)$, $B(x) = -\frac{1}{2}x + v(x)$, but μ has no nonzero directions of differentiability (and even continuity).

In the second example we construct σ of the same type and B of the type $\Lambda x+v(x)$, but now there are two different invariant probabilities ν_1 and ν_2 for diffusion ξ such that ν_1 is equivalent to a Gaussian measure and possesses the vector logarithmic derivative β^{ν_1} while ν_2 has no directions of differentiability.

The problem remains open for $\sigma = I$.

2. Some preliminary results.

Let X be an infinite dimensional separable Hilbert space. By $L^{(1)}(X)$ denote the space of nuclear operator on X with its natural trace-norm.

We say that H is a Hilbert space densely embedded into X by a Hilbert–Schmidt operator if $H = T(X)$ for some injective nonnegative Hilbert–Schmidt operator on X and H is equipped with the inner product $(a,b)_H = (T^{-1}a, T^{-1}b)_X$.

With such a space H, one can always associate the Wiener process W_t defined by $W_t(\omega) = \sum_{n=1}^{\infty} w_t^n(\omega)e_n$, where $\{w_t^n\}$ is a sequence of independent standard Wiener processes on some probability space and $\{e_n\}$ is an orthogonal basis in H. Note that one can put $e_n = t_n\varphi_n$, where $\{\varphi_n\}$ is the orthonormal basis in X generated by the eigenvectors of T with the eigenvalues t_n. The pair (H,X) is called sometimes a rigged Hilbert space.

For any vector e in X, denote by $\mathcal{A}_e^{\mathcal{B}}$ the class of all Borel sets A such that $\mathrm{mes}(t : x+te \in A) = 0$ for every $x \in X$, where "mes" stands for the Lebesgue measure on R. For any sequence $\{e_n\}$ in X let $\mathcal{A}^{\mathcal{B}}\{e_n\}$ be the class of all sets of the form $A = \bigcup\limits_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{A}_{e_n}^{\mathcal{B}}$ for all n.

A set A is said to be *exceptional* (in the sense of Aronszajn) ($A \in \mathcal{A}^{\mathcal{B}}$) if it belongs to the class $\mathcal{A}^{\mathcal{B}}\{e_n\}$ for each sequence $\{e_n\}$ with dense linear span in X.

A Borel set A is called a *Gaussian null set* ($A \in \mathcal{G}^{\mathcal{B}}$) if it is a zero set for each nondegenerate (that is of the full topological support) Gaussian measure on X.

A Borel set A is called *negligible* ($A \in \mathcal{P}^{\mathcal{B}}$) if it is a zero set for each measure which is differentiable along the directions from a dense set.

All these classes coincide with Lebesgue zero Borel sets in finite dimensional spaces. It is known [3], [7] that $\mathcal{A}^{\mathcal{B}} \subset \mathcal{G}^{\mathcal{B}} = \mathcal{P}^{\mathcal{B}}$, but it remains open whether the inclusion is strict.

A measure μ on X is said to be differentiable along a vector h if for each Borel set B the function

$$t \mapsto \mu(B+th) \tag{1}$$

is differentiable at every point. If for any $B \in \mathcal{B}(X)$ the function in (1) is continuous, then μ is said to be continuous along the vector h.

Note that the differentiability above is known to be equivalent to the existence of a μ-integrable function β_h^μ (called the directional logarithmic derivative of μ along h) such that

$$\int \partial_h f(x)\,\mu(dx) = -\int f(x)\beta_h^\mu(x)\,\mu(dx)$$

for every smooth cylindrical function f (i.e., $f(x) = \varphi(l_1(x), \dots, l_n(x))$, where $\varphi \in C_b^\infty(R^n)$, $l_i \in X^*$).

Let H be embedded into X as above; we say that μ has the vector logarithmic derivative β_H^μ associated with H if $\beta_H^\mu : X \to X$ is a Borel map and for every $k \in X$ the function $(k, \beta_H^\mu)_X$ is the directional logarithmic derivative of μ along $T^2 k$.

For example, if μ is the centered Gaussian measure on X with the Cameron–Martin space H, then $\beta_H^\mu(x) = -x$. For more details on vector logarithmic derivatives see [7], [11].

As an auxiliary result we give an example of a quadratic transformation F on X which maps a Borel set of positive measure for some nondegenerate Gaussian measure onto an exceptional set. We use here an idea from [3].

For any selfadjoint nuclear operator A on X, denote by α_n the eigenvalues of A and by e_n its eigenvectors. Let $\Phi_2(x,y) = \sum\limits_{n=1}^{\infty} (x, e_n)(y, e_n) e_n$ and $\Phi(x) = \Phi_2(x, x)$. Clearly, Φ_2 is a continuous bilinear form on X and Φ is a quadratic form. Now, we can set $F(x) = x + \sum\limits_{n=1}^{\infty} \alpha_n (x, e_n)^2 e_n$ (i.e., $F = I + A\Phi$ is a quadratic perturbation of the identity mapping).

Proposition 1. *The transformation F above is a diffeomorphism of some neighborhoods of the origin U and V that takes a non negligible Borel subset of U onto an exceptional Borel subset of V.*

Proof. Let us take a self-adjoint injective compact operator K on X with dense range such that $A(X) \cap K(X) = 0$. One can do this according to the first Corollary of Theorem 1 of [23]. F is infinitely Fréchet differentiable with $F(0) = 0$ and $F'(0)$ being the identity mapping. According to the inverse function theorem, F is a diffeomorphism of some neighborhoods of zero U and V. Let us take any nondegenerate centered Gaussian measure γ such that $\gamma(K(X)) = 1$ (e.g., the measure γ with the Fourier transform $\widetilde{\gamma}(x) = \exp\{-(Qx, x)\}$, where Q is the nuclear operator with the same eigenvectors as K and the eigenvalues $q_n = n^{-2} k_n^2$, where the k_n's are the corresponding eigenvalues of K).

Thus, on the one hand, $\gamma(U \cap K(X)) = \gamma(U) > 0$, hence $U \cap K(X)$ is not negligible. On the other hand, we shall now prove that $F(K(X))$ is exceptional and so is $F(U \cap K(X))$. The fact that our sets are Borel follows from the Kuratowski theorem.

It suffices to show that every straight line crosses $F(K(X))$ at no more than two points. Let $F(y) - F(x) = t\ell$, $t \in R$, $\ell \in X$, where $x,\ y \in K(X)$. For $F(K(X))$ is a subset of $K(X) + A(X)$ we can write $\ell = a + Ab$, where $a \in K(X)$, $b \in X$. Recall that $K(X) \cap A(X) = 0$ and A is an injective operator, therefore the equality $y - x + A\Phi(y) - A\Phi(x) = ta + tAb$ yields

$$y = x + ta \quad \text{and} \quad \Phi(y) - \Phi(x) = tb. \tag{2}$$

So, we have $\Phi(x + ta) - \Phi(x) = tb$ i.e. $t\Phi(a) + 2\Phi_2(x, a) = b$, provided $t \neq 0$. If there exist at least two other different values of t such that the latter equation holds true, then $\Phi(a)$ must be zero which yields $a = 0$ and $\Phi_2(x, a) = 0$, hence $b = 0$ and $\ell = 0$. This ends the proof.

Let $\varphi : R_+ \to [0, 1]$ be a smooth function ($\varphi \in C_0^\infty(R_+)$) such that $\varphi(t) = 1$ if $0 \le t \le \delta/2$, $\varphi(t) = 0$ if $t \ge \delta$ and $\varphi'(t) < 0$ if $\delta/2 < t < \delta$. Put $V_\delta = \{x : \|x\| \le \delta\}$, $\delta > 0$. Set

$$F(x) \;=\; x + G(x) \equiv x + \varepsilon \cdot \varphi(\|x\|) \cdot A\Phi(x), \tag{3}$$

Proposition 2. *Let F be defined by (3), where A and Φ are from Proposition 1 and ε is sufficiently small. Then*

(i) *The transformations $F : X \to X$ as well as $F : V_\delta \to V_\delta$ are bijective and $F(x)\big|_{V_\delta^c} = x$.*
(ii) *F is infinitely Fréchet differentiable on X and locally diffeomorphic.*
(iii) *F has the same properties as F in Proposition 1, i.e. F is a diffeomorphism of some neighborhoods of the origin U and V that takes a non negligible borelian subset of U onto an exceptional borelian subset of V.*

Proof. The assertion (i) is proved using the contraction mapping principle. To (ii) and (iii) the same reasoning as in Proposition 1 applies.

Remark 1. It is important for the constructions described below that, given a Gaussian measure γ on X, one can always find a compact self-adjoint operator K on X with $\gamma(K(X)) = 1$. Then by [23], there is another injective compact operator K_0 with dense range such that $K_0(X) \cap K(X) = 0$, and, consequently, there is an injective nonnegative nuclear operator A with dense range such that $A(X) \cap K(X) = 0$. Hence, for every nondegenerate Gaussian measure γ, the nuclear operator A with the properties described in Proposition 1 and Proposition 2 can be found in such a way that F transforms a set of positive γ-measure into an exceptional set. Conversely, given an injective symmetric nuclear operator A, we can always find an operator K with the properties above.

3. Main results

Now we proceed to the main result of this article.

Theorem 1. *Given ε and $\delta > 0$, there exist a Hilbert space H densely embedded into X by a Hilbert–Schmidt operator, a smooth map σ_0 with values in the space of nuclear operators on X and a smooth map $v : X \to X$ with $\|\sigma_0(\,\cdot\,)\|_1 \le \varepsilon$, $\|v(\,\cdot\,)\| \le \varepsilon$ and both σ_0 and v vanishing outside the δ-ball around zero, such that there is a unique probability measure μ invariant for the diffusion ξ generated by the stochastic differential equation*

$$d\xi_t = \sigma(\xi_t)dW_t + B(\xi_t)dt$$

with $\sigma(x) = I + \sigma_0(x)$, $B(x) = -\frac{1}{2}x + v(x)$ and Wiener process W_t associated with H, but μ has no nonzero directions of differentiability (even continuity). In addition, *the transition probabilities $P_t(x,\cdot)$ of ξ have no directions of differentiability and continuity as well.*

Proof. Let A and K be the same as in the previous section. Let W_t be the Wiener process in $K(X)$ associated with $H := K^2(X)$ and let X_t be the corresponding Ornstein–Uhlenbeck process in $K(X)$ defined by

$$dX_t = dW_t - \frac{1}{2}X_t dt.$$

Then, as it is well known, there is a unique invariant measure γ for X_t and γ is the centered Gaussian measure with the reproducing kernel Hilbert space H (namely the law of W_1).

According to Ito's formula (see, e.g., Theorem 5.4 of [19]), one has for F from Proposition 2 that

$$F(X_t) = F(X_0) + \int_0^t [F'(X_s)\circ I]dW_s - \frac{1}{2}\int_0^t [F'(X_s)(X_s) + \zeta]ds,$$

where $\zeta = TR\{F''(X_s)\circ[I\times I]\}$ is the trace of the nuclear bilinear form on H with values in X (since it is the composition of the continuous bilinear form on $K(X)$ with the Hilbert–Schmidt natural embedding operator $I: H \to K(X)$).

Thus, the process $\xi_t := F(X_t)$ satisfies the stochastic differential equation

$$\xi_t = \xi_0 + \int_0^t \sigma(\xi_s)dW_s + \int_0^t B(\xi_s)ds,$$

where

$$\sigma(x) = F'(F^{-1}(x)), \quad B(x) = -\frac{1}{2}F'(F^{-1}(x))F^{-1}(x) - \frac{1}{2}\zeta(F^{-1}(x)).$$

Clearly, since $F(x) = x$ outside some ball U, $F' = I$ and $F'' = 0$ outside this ball. Therefore, $\sigma(x)$ and $B(x)$ can be written as $I + \sigma_0(x)$ and $-\frac{1}{2}x + v(x)$, respectively, with $\sup_x \|\sigma_0(x)\|_{L^{(1)}(X)}$ and $\sup_x \|v(x)\|_X$ arbitrary small if the operator A above and the function φ in (3) are chosen appropriately.

Hence $\gamma(K(X)) = 1$ and if $\gamma = \gamma_1 + \gamma_2$ is the decomposition of γ to $\gamma_1(\cdot) = \gamma(\cdot \cap V_\delta)$ and $\gamma_2(\cdot) = \gamma(\cdot \cap V_\delta^c)$ then the invariant measure $\mu = \gamma\circ F^{-1}$ for $F(X_t)$ may be written as $\mu_1 + \mu_2$, where μ_1 has the support on the exceptional set in V_δ and $\mu_2 = \gamma_2$ being absolutely continuous w.r.t. γ is continuous along all directions of H. As shown in [3], a Radon measure μ on X is continuous along $\{e_n\}$ if and only if it vanishes on all sets in $\mathcal{A}^{\mathcal{B}}\{e_n\}$. Hence μ_1 has no directions of continuity and so is μ.

Applying the same arguments to the transition probabilities $Q_t(x,dy)$ of the Ornstein-Uhlenbeck process and to $P_t(x,dy)$ of $F(X_t)$ respectively and noting that

$$P_t(x,B) = P\{F(X_t)\in B|F(X_0) = x\} = \\ P\{X_t \in F^{-1}B|X_0 = F^{-1}(x)\} = Q_t(F^{-1}(x), F^{-1}B),$$

one gets the corresponding properties of $P_t(x,\cdot)$.

Theorem 2. *Given ε and $\delta > 0$, there exist a Hilbert space H densely embedded into X by a Hilbert–Schmidt operator, a bounded linear operator $\Lambda : X \to X$, a smooth map σ_0 with values in the space of nuclear operators on X and a smooth map $v : X \to X$ with $\|\sigma_0(\,\cdot\,)\|_1 \le \varepsilon$, $\|v(\,\cdot\,)\| \le \varepsilon$ and their supports lying inside the δ-ball around zero, such that there are two different probability measures ν_1 and ν_2 invariant for the diffusion ξ generated by the stochastic differential equation*

$$d\xi_t = \sigma(\xi_t)dW_t + B(\xi_t)dt$$

with $\sigma(x) = I + \sigma_0(x)$, $B(x) = \Lambda x + v(x)$ and Wiener process W_t associated with H, such that ν_1 is equivalent to a Gaussian measure and possesses vector logarithmic derivative $\beta_H^{\nu_1}$, while ν_2 has no nonzero directions of differentiability (or even continuity).

Proof. We shall develop a construction from [11], where it was shown that there exist a Hilbert spaces H densely embedded into X by a Hilbert–Schmidt operator, and two different centered Gaussian measures λ_1, λ_2 on X such that there exists a bounded linear operator L on X which is a joint Borel version for $\beta_H^{\lambda_1}$ and $\beta_H^{\lambda_2}$.

Let $X = L^2[0,1]$, $H = W_0^{4,2}[0,1]$, i.e., the Sobolev space of functions with third absolutely continuous derivatives such that

$$f^{(4)} \in L^2[0,1],\ f^{(j)}(0) = f^{(j)}(1) = 0\ ,\ 0 \le j \le 3\ .$$

Define operators $(A_i, D(A_i))$, $i = 1, 2$, on X by

$$A_1 := -\Delta\ , \quad D(A_1) := \{u \in W^{2,2}[0,1] \,|\, u(0) = u(1) = 0\},$$

$$A_2 := -\Delta\ , \quad D(A_2) := \{u \in W^{2,2}[0,1] \,|\, u(1) - u'(1) = 0 = u(0) + u'(0)\}.$$

Then, as is well-known, $(A_i, D(A_i))$, $i = 1, 2$, are injective non-negative definite self-adjoint operators on X such that $T_i := A_i^{-1}$, $i = 1, 2$, are injective non-negative definite self-adjoint Hilbert–Schmidt operators on X. To get the one-to-one correspondence note that for any $u \in D(A_1)$

$$-\int_0^1 u''(t)u(t)dt = \int_0^1 u'(t)^2dt + u'u|_0^1 = \int_0^1 u'(t)^2dt \ge \int_0^1 u(t)^2dt,$$

and for any $u \in D(A_2)$

$$-\int_0^1 u''(t)u(t)dt = \int_0^1 u'(t)^2dt + u'u|_0^1 = \int_0^1 u'(t)^2dt + u(1)^2 + u(0)^2 \ge \frac{1}{2}\int_0^1 u(t)^2dt.$$

Since $H \subset X$ is Hilbert–Schmidt, there exists an injective non-negative definite self-adjoint Hilbert–Schmidt operator Q on X such that $Q(X) = H$ and $|\cdot|_H = |Q^{-1}\cdot|_X$. Clearly,

$$Q(X) = H \subset T_1^2(X) \cap T_2^2(X) \quad \text{and} \quad T_1^{-2}h = T_2^{-2}h \quad \forall\, h \in H\ .$$

Let X_n be the sequence of copies of the space X as above with $A_i^{(n)} := n^2 A_i$ on domain $D(A_i)$ and $T_i^{(n)} := (A_i^{(n)})^{-1} = n^{-2}T_i$. Let us take a new Hilbert space Y which is the Hilbert direct sum of X_n, i.e., Y consists of all the sequences (x_n), $x_n \in X$ with $\|x\|_0^2 = \sum_{n=1}^{\infty} \|x_n\|^2 < \infty$; define the operators S_i, $i = 1,\ 2$, on Y as the direct sums of $T_i^{(n)}$, i.e. $S_i x = (T_i^{(1)}x_1, T_i^{(2)}x_2, \dots, T_i^{(n)}x_n, \dots)$. Since the series of the Hilbert–Schmidt norms of the $T_i^{(n)}$'s converges, S_i are Hilbert–Schmidt operators. Let $\widetilde{Q}$ be the Hilbert–Schmidt operator on Y given by $\widetilde{Q}(x) = (Qx_1, \dots, n^{-2}Qx_n, \dots)$. Note that the space $H_0 = \widetilde{Q}(Y) = \{(h_n) \in Y : h_n \in H\}$ is dense in Y and in accordance with the construction above $S_1^{-1}h = S_2^{-1}h$ for all $h \in H_0$.

Since $Q^2(X) \subset Q(X)$ and $T_i^2(X) \subset T_i(X)$, $i = 1, 2$, we have $\widetilde{Q}^2(Y) \subset \widetilde{Q}(Y)$ and $S_i^2(Y) \subset S_i(Y)$.

Let μ_i be the centered Gaussian measure on Y with the Cameron–Martin space $S_i(Y)$, $i = 1, 2$. Taking into account the Example 2.5 in [11], the bounded operator $\Lambda = R^*$ with $R = S_1^{-2}\widetilde{Q}^2 = S_2^{-2}\widetilde{Q}^2$ is a joint Borel version for $\beta_{H_0}^{\mu_1}$ and $\beta_{H_0}^{\mu_2}$. In addition, μ_1 and μ_2 are invariant measures for the diffusion η_t, governed by the SDE

$$d\eta_t = dW_t + \Lambda\eta_t\, dt, \quad \eta_0 = y \in Y,$$

where W_t is the Wiener process with values in Y generated by the space H_0 (i.e., H_0 is the Cameron–Martin space of the law of W_1; clearly, W_t can be written as the countable collection of independent Wiener processes in the spaces X_n).

The linear space $E = S_1(Y) \cap S_2(Y)$ has infinite co-dimension in $S_1(Y)$. Indeed, it contains nontrivial vectors $v_n = (0, \dots, 0, \varphi_n, 0, \dots)$, where $\varphi_n(t) = n^{-2}t(1-t) \in D(A_1)\backslash D(A_2)$. Moreover, there is a compact operator K on Y with the following properties:

1) $\mu_2(K(Y)) = 1$,

2) $K(Y) \cap S_1(Y)$ has infinite co-dimension in $S_1(Y)$.

Indeed, the measure μ_2 is the direct product of the measures γ_n on X_n with the Cameron–Martin spaces $T_2^{(n)}(X_n)$. For every fixed n, there is a nontrivial vector $u_n \in T_1^{(n)}(X_n)\backslash T_2^{(n)}(X_n)$. It is known and can be checked directly (see, e.g., [6]) that then there is a compact injective operator K_n on X_n such that $u_n \notin K_n(X_n)$ and $\gamma_n(K_n(X_n)) = 1$. Let K_0 be the direct sum of the operators K_n. Then $\mu_2(K_0(Y)) = 1$ and $K_0(Y) \cap S_1(Y)$ has infinite co-dimension in $S_1(Y)$. Finally, there is a yet smaller subspace of full measure in $K_0(Y)$ having the form $K(Y)$, where K is a compact operator on Y (note that $K_0(Y)$ is a Hilbert space with the norm $\|K_0^{-1} \cdot\|_Y$, hence, every measure on it is concentrated on the image of certain compact operator).

Let us consider the space $Y_1 := S_1(Y)$ equipped with the Hilbert norm $\|S_1^{-1} \cdot\|_Y$. By virtue of [23], we can find an injective symmetric nuclear operator $A_0 : Y_1 \to Y_1$ such that $A_0(Y_1) \cap K_0(Y) = 0$. Consequently, there is an injective nonnegative nuclear operator A on Y such that $A(Y) \subset S_1(Y)$ and $A(Y) \cap K(Y) = 0$ (namely, there is an injective nonnegative operator A on Y such that $A(Y) = A_0(Y_1)$; then A is automatically nuclear, since so is A_0 and the natural embedding $Y_1 \to Y$ is continuous).

We shall take Y for the space X in the formulation of the theorem. Let us consider the process $\xi_t = F(\eta_t)$ on Y, where F is given by (3) with A chosen above. Now the reasoning used for γ in Proposition 1 applies to μ_2. Therefore $\nu_2 := \mu_2 \circ F^{-1}$ has no directions of continuity. The situation is different for μ_1. Indeed, the smooth mapping $F - I$ takes values in the Cameron–Martin space of μ_1 and satisfies all the conditions of the Ramer theorem (see [19]; certainly, it also satisfies the conditions of the Kusuoka theorem [20]). Hence, $\nu_1 := \mu_1 \circ F^{-1}$ is equivalent to μ_1; moreover, the explicit expression for the Radon–Nikodym density of $\mu_1 \circ F^{-1}$ with respect to μ_1 enables one to check directly the differentiability of this measure. Now the mappings σ and B with the announced properties are obtained by means of Ito's formula in the same way as in Theorem 1.

Remark 2. In the settings of the Theorems 1 and 2 all the invariant measures are obliged to have smooth components. Nevertheless, if we allow F not to be the identity mapping outside some ball and thus the coefficients of SDE not to coincide with those of the Ornstein–Uhlenbeck process outside some ball, we can construct diffusions satisfying the conclusions of the both theorems (without demands on supports of σ_0 and v) with the appropriate invariant measures being ultimately singular, i.e. singular w.r.t. any measure having at least one direction of continuity.

Remark 3. This article deals with the singularity of measures on the state-spaces of diffusion processes. But if one considers the measures induced by the diffusion processes in the path spaces, then even the distributions of the one-dimension diffusions with smooth coefficients have no directions of differentiability provided the diffusion coefficient is non-constant. This result proved in [4] (see also [5], [12]) settled the conjecture stated by Pitcher in [22]. Moreover, it is shown in [26] that in the case of the non-constant diffusion coefficient the distribution of the process is concentrated on the exceptional set and is ultimately singular even if the diffusion coefficient is a bounded analytical function without any restriction on the drift.

Remark 4. If we do not require the state-space to be a Hilbert space, one can construct the diffusion satisfying the hypothesis of Theorem 1 in a more easy way by taking $X = C[0,1]$ and the diffeomorphism $F[x](t) := f(x(t))$, where f is a mapping obtained by perturbing smoothly the identity map on $[-\delta, \delta]$.

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