Dobrushin's uniqueness for quantum lattice systems with nonlocal interaction

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Abstract

Based on Dobrushin's fundamental criterion, we prove uniqueness of Euclidean Gibbs states for a certain class of quantum lattice systems with unbounded spins, nonharmonic pair potentials and infinite radius of interaction. The necessary estimates on Dobrushin's coefficients are obtained from the Log-Sobolev inequality which holds for the one-point conditional distributions on the infinite dimensional single spin (= loop) spaces.

Dedication

This paper is dedicated to the memory of Ronald Dobrushin, to whom we are forever grateful, for all he has given us, by his inspired work, his warm friendship, his being in many ways a true teacher for us.

1 Temperature loop space representation for quantum Gibbs states

Let $\mathbb{Z}^d$, $d \in \mathbb{N}$, be the integer lattice with the Euclidean distance $|k-j|$, $k, j \in \mathbb{Z}^d \subset \mathbb{R}^d$. By $S(\mathbb{Z}^d)$ and $S'(\mathbb{Z}^d)$ we denote the mutually dual spaces of fastly decreasing resp. slowly increasing sequences over $\mathbb{Z}^d$, i.e.,

$$S(\mathbb{Z}^d) := \left\{ x = (x_k)_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d} \mid \forall\, p \in \mathbb{Z}_+,\ \sum_{k\in\mathbb{Z}^d} (1+|x|)^p x_k^2 < \infty \right\},$$

$$S'(\mathbb{Z}^d) := \left\{ x = (x_k)_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d} \mid \exists\, p \in \mathbb{Z}_+,\ \sum_{k\in\mathbb{Z}^d} (1+|x|)^{-p} x_k^2 < \infty \right\}.$$

Note that $S'(\mathbb{Z}^d)$ obviously is a sub-algebra of $\mathbb{R}^{\mathbb{Z}^d}$. We consider a translation invariant system of interacting quantum anharmonic oscillators with the heuristic Hamiltonian

$$H := -\frac{1}{2m}\sum_{k\in\mathbb{Z}^d}\Delta_k + \frac{a^2}{2}\sum_{k\in\mathbb{Z}^d} x_k^2 + \sum_{\{k,j\}\subset\mathbb{Z}^d} J(|k-j|)U(x_k - x_j) + \sum_{k\in\mathbb{Z}^d} V(x_k), \tag{1.1}$$

where $m > 0$ (= physical mass). The two-particle interactions $U(x_k - x_j)$ have intensities

$$J : \mathbb{R}_+ \to \mathbb{R}_+,\ J(0) = 0,\ (J(|k|))_{k\in\mathbb{Z}^d} \in S(\mathbb{Z}^d),\ \widetilde{J} := \sum_{k\in\mathbb{Z}^d} J(|k|), \quad (1.2)$$

and $U \in C^2(\mathbb{R}^1)$ is a convex function such that $\exists b_-, b_+ > 0$: $\forall q \in \mathbb{R}^1$

$$b_-^2 \le U''(q) \le b_+^2,\ 0 = U(0) \le U(q) = U(-q) \le b_+^2 q^2/2\ . \quad (1.3)$$

The harmonic self-interaction is defined by $a^2 x_k^2/2$ with a constant $a^2 > 0$. The anharmonic self-interaction potential has the form

$$V(q) = V_0(q) + W(q)\ . \quad (1.4)$$

Here $V_0 \in C^2(\mathbb{R}^1)$ is a convex function with polynomially bounded derivatives, that is, $\exists\ b,\ C > 0,\ M \in \mathbb{N}$: $\forall q \in \mathbb{R}^1$

$$V_0''(q) \ge b^2\ ,\ |V_0^{(l)}(q)| \le C(1 + |q|)^M,\ l = 0, 1, 2. \quad (1.5)$$

The potential W describes the presence of (possible) wells in the self-interaction V and is given by a bounded function

$$W \in C_b(\mathbb{R}^1). \quad (1.6)$$

We set $\delta(W) := \sup_{\mathbb{R}^1} W - \inf_{\mathbb{R}^1} W$.

Models of the above type in the classical case where the single spin spaces are just $\mathbb{R}^1$ have been studied in [BH-K].

A fundamental problem of equilibrium quantum statistical physics is the study of temperature (i.e., Gibbs) states. We will take the Euclidean approach (see e.g. the early work [AH-K] and its further developments in [KL], [GK]) to give a rigorous meaning to the Gibbs state G_β of the lattice system (1.1) at inverse temperature $\beta > 0$, i.e., we construct (via a path space representation) the corresponding Euclidean Gibbs measure ν_β on the "loop lattice" Ω_β defined as follows:

Let S_β be the circle of length β. The spaces $C(S_\beta)$ and $\mathcal{H}_\beta := L_2(S_\beta)$ consist of all continuous resp. (relative to Lebesgue measure) square integrable functions $\omega : S_\beta \to \mathbb{R}^1$, equipped with the sup-norm $\|\cdot\|_{C(S_\beta)}$ resp. L_2-norm $\|\cdot\|_\beta = (\cdot,\cdot)_\beta^{1/2}$. For the corresponding Borel σ-algebras we have:

$$\mathcal{B}(C(S_\beta)) = \mathcal{B}(\mathcal{H}_\beta) \cap C(S_\beta).$$

As the configuration space we introduce the temperature loop space ("loop lattice")

$$\Omega_\beta := C(S_\beta)^{\mathbb{Z}^d} = \{\omega = (\omega_k)_{k\in\mathbb{Z}^d} | \omega : S_\beta \to \mathbb{R}^{\mathbb{Z}^d},\ \omega_k \in C(S_\beta)\}\ ,$$

endowed with the product topology and with the Borel σ-algebra $\mathcal{B}(\Omega_\beta)$ (= σ-algebra generated by the cylinder sets

$$\{\omega \in \Omega_\beta | (\omega_k)_{k\in\Lambda} \in B_\Lambda\} \in \mathcal{B}(C(S_\beta)^\Lambda),\ ,\ \Lambda \subset \mathbb{Z}^d \text{ with } |\Lambda| < \infty).$$

Because of the infinite radius of interaction, we also need the subspace of tempered configurations

$$\Omega_\beta^t := \{\omega \in \Omega_\beta \mid (\|\omega_k\|_\beta)_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d)\}$$

with the topology and Borel structure induced by Ω_β.

The Euclidean Gibbs measure μ_β is described by a corresponding family of local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$ (cf. [Do1], [Do2], [Pre], [Ge]). Let Δ_β be the Laplace-Beltrami operator on the circle S_β and $\gamma_\beta(d\omega_k)$ be the Gaussian measure on $(\mathcal{H}_\beta, \mathcal{B}(\mathcal{H}_\beta))$ with zero mean value and correlation operator $(-m\Delta_\beta + a^2 1\!\!1)^{-1}$. Actually, the set $C(S_\beta)$ of continuous loops has full measure, i.e., $\gamma_\beta(C(S_\beta)) = 1$, and the measure γ_β on the space $(C(S_\beta), \mathcal{B}(C(S_\beta)))$ can be viewed as the canonical realization of the well-known oscillator bridge process of length β (see e.g. [Si]).

We then define

$$d\sigma_\beta(\omega_k) := \frac{1}{Z_\beta} \exp\{-\int_{S_\beta} V(\omega_k(\tau))d\tau\} d\gamma_\beta(\omega_k) \ . \tag{1.7}$$

as a probability measure on $(C(S_\beta), \mathcal{B}(C(S_\beta)))$ and consider the reference measure $\times_{k\in\mathbb{Z}^d} d\sigma_\beta(\omega_k)$ on $(\Omega_\beta^t, \mathcal{B}(\Omega_\beta^t))$. Obviously, the definition (1.7) is meaningful for the potentials V dealt with in (1.4) - (1.6), and moreover, the measure σ_β has all the moments of the form

$$E_{\sigma_\beta}[\|\omega_\beta\|^M_{C(S_\beta)}] < \infty, \quad M \in \mathbb{N}, \tag{1.8}$$

due to the same property of the Gaussian measure γ_β.

The local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, are defined as stochastic kernels on $(\Omega_\beta^t, \mathcal{B}(\Omega_\beta^t))$ in the following way: $\forall B \in \mathcal{B}(\Omega_\beta^t)$, $\xi \in \Omega_\beta^t$

$$\pi_{\beta,\Lambda}(B|\xi) := \frac{1}{Z_{\beta,\Lambda}(\xi)} \int_{\Omega_{\beta,\Lambda}} \exp\{-I_{\beta,\Lambda}(\omega|\xi)\} 1\!\!1_B(\omega_\Lambda \times \xi_{\Lambda^c}) \times_{k\in\Lambda} d\sigma_\beta(\omega_k). \tag{1.9}$$

Here,

$$\Omega_{\beta,\Lambda} := C(S_\beta)^\Lambda, \quad \omega_\Lambda := (\omega_k)_{k\in\Lambda} \in \Omega_{\beta,\Lambda} \ ,$$

$Z_{\beta,\Lambda}(\xi)$ is a normalization constant, and

$$I_{\beta,\Lambda}(\omega|\xi) := \int_{S_\beta} [\sum_{\{k,j\}\subset\Lambda} J(|k-j|)U(\omega_k(\tau) - \omega_j(\tau)) + \sum_{\substack{\{k,j\}\subset\mathbb{Z}^d \\ k\in\Lambda, j\in\Lambda^c}} J(|k-j|)U(\omega_k(\tau) - \xi_j(\tau))]d\tau \tag{1.10}$$

is the pair interaction in the volume Λ under the external boundary condition $\xi_{\Lambda^c} := (\xi_j)_{j\in\Lambda^c}$, $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$. Due to assumption (1.2), $I_{\beta,\Lambda}(\omega|\xi)$ is well-defined for all $\xi \in \Omega_\beta^t$ and can be extended by the continuity to the class of boundary conditions

$$\xi \in \tilde{\Omega}_\beta^t := \left\{\omega \in L_2(S_\beta)^{\mathbb{Z}^d} \mid (\|\omega_k\|_\beta)_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d)\right\}. \tag{1.11}$$

An essential point is that for the stochastic kernels (1.9) the consistency condition holds [Do1, Do2], [Pre], [Ge]: $\forall\ \Lambda \subset \Lambda',\ |\Lambda'| < \infty,\ B \in \mathcal{B}(\Omega^t_\beta)$, $\xi \in \Omega^t_\beta$

$$\pi_{\beta,\Lambda'}(\pi_{\beta,\Lambda}(B|\cdot) \mid \xi) := \int_{\Omega_\beta} \pi_{\beta,\Lambda'}(d\omega|\xi)\pi_{\beta,\Lambda}(B|\omega) = \pi_{\beta,\Lambda'}(B|\xi)\ . \qquad (1.12)$$

Definition 1.1 *A probability measure μ_β on $(\Omega^t_\beta, \mathcal{B}(\Omega^t_\beta))$ is called (Euclidean) Gibbs state for the local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, if it satisfies the DLR (Dobrushin-Lanford-Ruelle) equations, i.e., for all $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$,*

$$\mu_\beta \pi_{\beta,\Lambda} = \mu_\beta\ . \qquad (1.13)$$

As usual, we restrict our considerations to the the subset $\mathcal{G}^t_\beta$ of tempered Gibbs measures, i.e., those μ_β from Definition 1.1 with

$$(< \|\omega_k\|_\beta >_{\mu_\beta})_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d)\ , \qquad (1.14)$$

where $< f >_{\mu_\beta} = \int_{\Omega_\beta} f(\omega) d\mu_\beta(\omega)$ for μ_β-integrable $f : \Omega_\beta \to \mathbb{R}$. Note that any probability measure μ on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$, which suits (1.14), is in fact supported on Ω^t_β, i.e., $\mu(\Omega^t_\beta) = 1$.

The initial step in the study of Gibbs measures is to ensure that the set $\mathcal{G}^t_\beta$ is nonempty, which is otherwise not evident for unbounded spin systems. The existence problem goes back to Dobrushin's papers [Do1, Do2], where general existence criteria for Gibbs distribution were first given. For further research on the existence problem for quantum lattice models, based on various techniques, see [AH-K], [BaK1, BaK2], [GK], [Pa], [PY1], [PY2]. Regarding the quantum lattice system with Hamiltonian (1.1), the simplest way to show that $\mathcal{G}^t_\beta \neq \emptyset$ is to take advantage of the fact that the interaction is superstable as soon as assumptions (1.2)-(1.4) hold. As was shown in [PY1], for a subclass of boundary conditions $\xi \in {\Omega_\beta}^{t,0} \subset \Omega^t_\beta$ (for instance, such that $\sup_k \|\xi_k\|_\beta < \infty$) the family of local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, has at least one accumulation point μ_β from the subset of so-called Ruelle type "superstable" Gibbs measures $G^{t,0}_\beta \subset G^t_\beta$. The aim of our paper is to prove a sufficient condition implying $|G^t_\beta| = 1$. All corresponding results below have been announced in [AKRT2].

2 Uniqueness of Gibbs states

We present the main result of this paper.

Theorem 2.1 *Suppose that the parameters of the system specified in* (1.1) *(resp. its rigorous implementation through Definition 1.1) satisfy the relation*

$$\frac{e^{\beta\delta(W)}\ b_+^2}{b_-^2 + \widetilde{J}^{-1}(a^2 + b^2)} < 1\ , \qquad (2.1)$$

then the set $\mathcal{G}^t_\beta$ of tempered Gibbs measures consists of exactly one point.

Proof The proof is based on Dobrushin's fundamental uniqueness criterion (see [Do2], [DSK], [Fö]). With full details the proof is contained in [AKRT1], but for a more particular class of quantum lattice models than (1.1). Below we outline the basic ideas and new feature of our more general case here.

In fact we shall prove more, namely that there exists at most one tempered Gibbs measure μ_β on the enlarged configuration space

$$\widetilde{\Omega}^t_\beta := \{\omega \in \mathcal{H}_\beta^{\mathbb{Z}^d} \,|\, (\|\omega_k\|_\beta)_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d)\}, \tag{2.2}$$

which satisfies the growth condition (1.14) and corresponds to the family of local specifications (1.9) with the definition of the interaction $I_{\beta,\Lambda}(\omega|\xi)$ extended to all $\xi \in \widetilde{\Omega}^t_\beta$. It is enough to consider the one-dimensional Gibbs distributions $\nu_k(d\omega|\xi)$ in volumes $\Lambda_k = \{k\}$, $k \in \mathbb{Z}^d$, with boundary conditions $\xi = (\xi_j)_{j\in\mathbb{Z}^d} \in \widetilde{\Omega}^t_\beta$, which are defined by

$$\nu_k(B|\xi) := \pi_{\{k\}}(\{\omega \in \widetilde{\Omega}^t_\beta \mid \omega_k \in B\}|\,\xi),\ \ B \in \mathcal{B}(\mathcal{H}_\beta)\ .$$

These are explicitly given by

$$\nu_k(d\omega_k|\xi) = \frac{1}{Z_{\beta,k}(\xi)} \exp\{-I_{\beta,k}(\omega_k|\xi)\} d\sigma_\beta(\omega_k), \tag{2.3}$$

where $Z_{\beta,k}(\xi)$ is a normalization constant and

$$I_{\beta,k}(\omega_k|\xi) = \int_{S_\beta} [\sum_{\substack{i\in\mathbb{Z}^d \\ i\neq k}} J(|k-i|) U(\omega_k(\tau) - \xi_i(\tau))] d\tau\ . \tag{2.4}$$

As is obvious from (1.8), for all $\xi \in \widetilde{\Omega}^t_\beta$

$$\int_{H_\beta} \|\omega_k\|^M_\beta\ \nu_k(d\omega_k|\xi) < \infty, \quad M \in \mathbb{N}. \tag{2.5}$$

The Dobrushin coefficients are defined by

$$C_{kj} := \sup_{\substack{\xi,\eta\in\widetilde{\Omega}^t_\beta \\ \forall i\neq j:\, \xi_i=\eta_i}} \frac{R(\nu_k(\cdot|\xi),\ \nu_k(\cdot|\eta))}{\|\xi_j - \eta_j\|_\beta}\ ,\ \ k \neq j, \tag{2.6}$$

where R denotes the Wasserstein distance between the distributions $\nu_k(\cdot|\xi)$ and $\nu(\cdot|\eta)$ on $\mathcal{H}_\beta$. That is (cf. [Do2], [Ra])

$$R(\nu_k(\cdot|\xi),\ \nu_k(\cdot|\eta)) := \sup_{f\in Lip_1(\mathcal{H}_\beta)} \left| \int_{\mathcal{H}_\beta} f\ d\nu_k(\cdot|\xi) - \int_{\mathcal{H}_\beta} f\ d\nu_k(\cdot|\eta) \right|, \tag{2.7}$$

where

$$Lip_1(\mathcal{H}_\beta) := \{f : \mathcal{H}_\beta \to \mathbb{R}^1 | [f]_{Lip} := \sup_{\substack{z,z'\in\mathcal{H}_\beta \\ z\neq z'}} \frac{|f(z) - f(z')|}{\|z - z'\|_{\mathcal{H}_\beta}} \leq 1\}.$$

Actually, in the definition (2.7) one can take the supremum only over the subclass $\mathcal{F}C^\infty(\mathcal{H}_\beta)$ of smooth cylinder functions from $Lip_1(\mathcal{H}_\beta)$ (it follows e.g. from the proof of Corollary 5.2 in [AKRT1]). Due to the Dobrushin uniqueness criterion, it sufficies to prove that

$$\sup_{k\in\mathbb{Z}^d} \sum_{\substack{j\in\mathbb{Z}^d\\ j\neq k}} C_{kj} < 1 \ . \tag{2.8}$$

For $k, j \in \mathbb{Z}^d$, $k \neq j$, $f \in Lip_1(\mathcal{H}_\beta)$, and $\xi = (\xi_i)_{i\in\mathbb{Z}^d} \in \widetilde{\Omega}^t_\beta$, let us consider the mapping

$$\mathcal{H}_\beta \ni \xi_j \mapsto < f >_{\nu_k(d\omega_k|\xi)} := \int_{\mathcal{H}_\beta} f(\omega_k)\nu_k(d\omega_k|\xi) \in \mathbb{R}^1 \ . \tag{2.9}$$

Because of (1.2) - (1.4) and (2.5) this mapping is Fréchet differentiable and the derivative in direction $\varphi \in \mathcal{H}_\beta$ is the following:

$$\begin{aligned} & \left(\nabla_{\xi_j}\langle f\rangle_{\nu_k(d\omega_k|\xi)},\, \varphi\right)_\beta \\ = \quad & J(|k-j|)[\langle f(\omega_k)\,(U'(\omega_k-\xi_j),\varphi)_\beta\rangle_{\nu_k(d\omega_k|\xi)} \\ & \qquad -\langle f\rangle_{\nu_k(d\omega_k|\xi)}\langle (U'(\omega_k-\xi_j),\varphi)_\beta\rangle_{\nu_k(d\omega_k|\xi)}] \\ = \quad & J(|k-j|)\mathrm{Cov}_{\nu_k(d\omega_k|\xi)}(f(\omega_k), (U'(\omega_k-\xi_j),\varphi)_\beta) \ . \end{aligned} \tag{2.10}$$

The latter can be estimated as

$$\begin{aligned} & |\left(\nabla_{\xi_j}\langle f\rangle_{\nu_k(d\omega_k|\xi)}, \varphi\right)_\beta| \leq \\ & J(|k-j|)\left(\mathrm{Var}_{\nu_k(d\omega_k|\xi)} f\right)^{\frac{1}{2}}\left(\mathrm{Var}_{\nu_k(d\omega_k|\xi)}(U'(\omega_k-\xi_j),\varphi)_\beta\right)^{\frac{1}{2}}, \end{aligned} \tag{2.11}$$

where as usual

$$\mathrm{Var}_{\nu_k(d\omega_k|\xi)} f := \int_{\mathcal{H}_\beta} (f- < f >_{\nu_k(d\omega_k|\xi)})^2 d\nu_k(d\omega_k|\xi)$$

denote the variance of the function f w.r.t. the measure $\nu_k(d\omega_k|\xi)$ and $\mathrm{Cov}_{\nu_k}(d\omega_k|\xi)$ is defined correspondingly.

Now we are at the crucial point in the proof. For the measures $\nu_k(d\omega_k|\xi)$ uniformly w.r.t. the boundary conditions ξ, the Log-Sobolev (LS) inequality (see Lemma 2.2 below) holds and as a consequence the variance estimate (2.19) (see Corollary 2.3). Therefore,

$$\mathrm{Var}_{\nu_k(d\omega_k|\xi)} f \leq C_{LS} := \frac{e^{\beta\delta(W)}}{\widetilde{J}b_-^2 + a^2 + b^2}$$

and by (1.3)

$$\mathrm{Var}_{\nu_k(d\omega_k|\xi)}\left(U'(\omega_k-\xi_j),\varphi\right)_\beta \leq C_{LS} b_+^4 \|\varphi\|_\beta^2 \ . \tag{2.12}$$

Hence from (2.11) and (2.12)

$$|(\nabla_{\xi_j}\langle f\rangle_{\nu_k(d\omega_k|\xi)},\varphi)_\beta| \leq J(|k-j|) C_{LS} b_+^2 \|\varphi\|_\beta \ ,$$

which by the mean-value theorem implies that for all $\xi, \eta \in \widetilde{\Omega}^t_\beta$ such that $\xi_i = \eta_i$, $\forall i \neq j$,

$$\left| \int_{\mathcal{H}_\beta} f \, d\nu_k(\cdot|\xi) - \int_{\mathcal{H}_\beta} f \, d\nu_k(\cdot|\eta) \right| \leq J(|k-j|) C_{LS} b_+^2 \|\xi_j - \eta_j\|_\beta \ ,$$

or, in terms of the Wasserstein distance,

$$R(\nu_k(\cdot|\xi), \, \nu_k(\cdot|\eta)) \leq J(|k-j|) C_{LS} \, b_+^2 \ . \tag{2.13}$$

By (2.6), (2.13), and the assumption (2.1), the estimate (2.8) follows, which in turn results

$$|\mathcal{G}^t_\beta| = 1 \ .$$

□

In order to formulate the LS-inequality and its consequences, first we introduce the spaces of smooth functions on $\mathcal{H}_\beta$. We define the space of k-times, $k \in \mathbb{N} \cup \{\infty\}$, continuously differentiable cylinder functions on $\mathcal{H}_\beta$

$$\begin{aligned} \mathcal{F}C^k(\mathcal{H}_\beta) := g((\cdot, l_1)_\beta, \ldots, (\cdot, l_n)_\beta) \mid n \in \mathbb{N}\,, \\ g \in C^k(\mathbb{R}^n), \ \ l_1, \ldots, l_n \in \mathcal{H}_\beta\} \end{aligned} \tag{2.14}$$

and its subspace

$$\begin{aligned} \mathcal{F}C^k_b(\mathcal{H}_\beta) := g((\cdot, l_1)_\beta, \ldots, (\cdot, l_n)) \mid n \in \mathbb{N}\,, \\ g \in C^k_b(\mathbb{R}^n), l_1, \ldots, l_n \in \mathcal{H}_\beta\}. \end{aligned} \tag{2.15}$$

We shall use the symbol $\nabla u(z) \in \mathcal{H}_\beta$, $z \in \mathcal{H}_\beta$ for the gradient realization of the derivative $u'(z) \in \mathcal{L}(\mathcal{H}_\beta, \mathbb{R}^1)$.

Having fixed a boundary condition $\xi \in \widetilde{\Omega}^t_\beta$, the subject of our study now will be the one-particle measure $\nu^\xi(\cdot) := \nu_k(\cdot|\xi)$ on the spin space $C(S_\beta) \subset L_2(S_\beta) := \mathcal{H}_\beta$ given by (2.3), (2.4).

On the domain $\mathcal{F}C^\infty_b(\mathcal{H}_\beta)$ we define the classical pre-Dirichlet form associated with the measure ν^ξ:

$$\mathcal{E}_{\nu^\xi}(u,v) := \int_{\mathcal{H}_\beta} (\nabla u(\omega), \nabla v(\omega))_\beta \, d\nu^\xi(\omega) \ . \tag{2.16}$$

Clearly, $(\mathcal{E}_{\nu^\xi}, \ \mathcal{F}C^\infty_\beta(\mathcal{H}_\beta))$ is closable on $L^2(\nu_\xi) := L^2(\mathcal{H}_\beta, \nu^\xi)$ [AR1]. We denote its closure by $(\mathcal{E}_{\nu^\xi}, D(\mathcal{E}_{\nu^\xi}))$. For a more detailed discussion of infinite–dimensional Dirichlet forms and additional references see [AR], [AKR].

Lemma 2.2 *(Log–Sobolev inequality).*

For all $u \in \mathcal{F}C^\infty_b(\mathcal{H}_\beta)$ the following inequality is true

$$\int_{\mathcal{H}_\beta} |u|^2 \log|u| \, d\nu^\xi \ \leq \ C_{LS} \int_{\mathcal{H}_\beta} ||\nabla u||^2_\beta \, d\nu^\xi + ||u||^2_{L^2(\nu^\xi)} \log ||u||_{L^2(\nu^\xi)} \tag{2.17}$$

with the Sobolev coefficient

$$C_{LS} := \frac{e^{\beta\delta(W)}}{\widetilde{J} b_-^2 + a^2 + b^2} \ . \tag{2.18}$$

Proof. Note first that obiously by approximation we can at once restrict our considerations to the case of finite radius of interaction (i.e., when $\exists\, 0 < \rho < \infty :\ J(|k-j|) = 0,\ |k-j| > \rho$). We follow completely the method of the proof of [AKRT1, Theorem 5.1]. Secondly, in the case $W = 0$, which corresponds to the quantum lattice system (1.1) with convex interactions, we use an approximation approach (similar to [AKR, Theorem 4]) to obtain an infinite-dimensional version of the well-known Bakry-Emery criterion in the case of the single spin space of loops $C(S_\beta)$. Then the statement of the lemma for general potentials $W \neq 0$ results from the simple perturbation theorem for the LS-constants, see [St]. □

Corollary 2.3 *(Variance estimate).*

$Lip(\mathcal{H}_\beta) \subset D(\mathcal{E}_{\nu^\xi})$, and for all $f \in Lip(\mathcal{H}_\beta)$ the following inequality holds:

$$Var_{\nu^\xi} f \ := \int (f - < f >_{\nu^\xi})^2 d\nu^\xi \leq C_{LS} [f]^2_{Lip} \,. \tag{2.19}$$

Proof. The proof is completely analogous to the proof of [AKRT1, Corollary 5.2]. □

3 Concluding remarks

(i) The assertion of Theorem 2.1 in particular implies that uniqueness of μ_β can be achieved by keeping fixed the other parameters of the system (1.1), but choosing a sufficiently high temperature, i.e., a small $\beta > 0$, or a small intensity of the pair interaction $\widetilde{J} > 0$, or a W with a small oscillation. These effects have an obvious physical interpretation. For an extended analysis of the connection between the shape of the one-particle potentials V and the uniqueness of the corresponding Gibbs states μ_β, see [AKRT1].

(ii) Without discussion of the existence problem, the fact that $|\mathcal{G}^t_\beta| \leq 1$ in Theorem 2.1 can also be proved for the more general quantum lattice systems with the heuristic Hamiltonian

$$H = -\frac{1}{2m} \sum_{k \in \mathbb{Z}^d} \Delta_k + \frac{1}{2} \sum_{k \in \mathbb{Z}^d} a_k^2 x_k^2 + \sum_{\{k,j\} \in \mathbb{Z}^d} U_{kj}(x_k - x_j) + \sum_{k \in \mathbb{Z}^d} V_k(x_k). \tag{3.1}$$

The pair interactions are given here by functions $U_{kj} \in C^2(\mathbb{R}^1)$ such that $\forall q \in \mathbb{R}^1$

$$0 < B^-_{kj} \leq U''_{kj}(q) \leq B^+_{kj} < \infty,\ 0 \leq U_{kj}(q) = U_{kj}(-q) \leq C_{kj}(1 + |q|)^2 \,.$$

The anharmonic self-interactions have the form $V_k(q) = V_{0,k}(q) + W_k(q)$, where $V_{0,k} \in C^2(\mathbb{R}^1)$, $W_k \in C_b(\mathbb{R}^1)$ and $\forall q \in \mathbb{R}^1$

$$0 < b_k^2 \leq V''_{0,k}(q),\ |V^{(l)}_{0,k}(q)| \leq C_k (1 + |q|)^M,\ l = 0, 1, 2,\ M \in \mathbb{N} \,.$$

The sequence $(C_k)_{k \in \mathbb{Z}^d}$ and the matrices $(B^-_{kj})_{k,j \in \mathbb{Z}^d}$, $(B^+_{kj})_{k,j \in \mathbb{Z}^d}$, $(C_{kj})_{k,j \in \mathbb{Z}^d}$ are supposed to be fastly decreasing. In this non-translation-invariant case the uniqueness condition then reads as

$$\sup_{k \in \mathbb{Z}^d} \frac{e^{\beta \delta(W_k)} \sum_{j \in \mathbb{Z}^d, j \neq k} B^+_{kj}}{\sum_{j \in \mathbb{Z}^d, j \neq k} B^-_{kj} + b_k^2 + a_k^2} < 1 \,. \tag{3.2}$$

(iii) A uniqueness result (via cluster expansion techniques) in the more restricted class $\mathcal{G}^t_{\beta,0}$ ($\subsetneq \mathcal{G}^t_\beta$) of quantum Gibbs states satisfying an a priori assumption on the support (the so-called "super-stable" Ruelle type Gibbs measures, cf. [COPP]) has been obtained in [PY2].

(iv) It is worth noting that our methods work both in quantum and classical cases. The analogous uniqueness result for classical lattice systems with harmonic nearest-neighbour and polynomial self-interactions has been proved in [Roy]. The Dobrushin matrix has been also estimated in [BP], using a technique different from our's for classical systems with potentials $V = V_0 + W$ such that $\inf_{\mathbb{R}^1} V_0'' \geq 0$ and, in addition, $\sup_{\mathbb{R}^1} |W'| < \infty$. Actually, the idea to get a bound on Dobrushin's coefficients through the spectral gap for the Dirichlet operators associated with one-particle conditional distributions goes back to [DS], [Wa], which concern compact manifolds as spin spaces for classical lattice models.

Acknowledgments Financial support by the DFG (through SFB 237 Bochum-Düsseldorf-Essen, SFB 343 Bielefeld as well as the Research Project AL 214/9-2), by the Alexander-von-Humboldt Foundation (fellowship for the fourth named author) and by the EC-Science Project SC1*CT92-0784 is gratefully acknowledged.

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