Uniqueness of Gibbs states on loop lattices

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Abstract — We prove uniqueness of Gibbs states for certain quantum lattice systems with unbounded spins, based on Dobrushin's fundamental uniqueness criterion. The necessary estimates for the Vasershtein distance between one-point conditional distributions, with boundary conditions differing only at one site, are obtained from the Log-Sobolev inequality which holds on the infinite dimensional single spin (= loop) spaces. Detailed proofs are contained in [2] resp. [3]. The latter deals with the more general case of nonharmonic pair potentials with possibly infinite radius of interaction.

Unicité des états de Gibbs sur des réseaux de lacets

Résumé — Une preuve de l'unicité des états de Gibbs pour certains réseaux quantiques de spins non bornés est donnée. La preuve utilise le critère fondamental de Dobrushin, les estimations nécessaires de la distance de Vasershtein des distributions conditionées à un point, avec des conditions aux bords qui diffèrent en un seul point, étant obtenues en utilisant l'inégalité de Sobolev logarithmique dans l'espaces des lacets. Les preuves détaillées sont contenues dans [2] et [3] (le cas plus général de potentiels à deux corps non harmoniques avec portée infinie est aussi étudié).

Version française abrégée —

1 Représentation des états de Gibbs quantiques

On considére un système d'oscillateurs anharmoniques quantiques avec Hamiltonien heuristique donné par

$$H := -\frac{1}{2m}\sum_{k\in\mathbb{Z}^d}\Delta_k + \frac{a^2}{2}\sum_{k\in\mathbb{Z}^d}x_k^2 + \sum_{\{k,j\}\subset\mathbb{Z}^d}J(k-j)U(x_k-x_j) + \sum_{k\in\mathbb{Z}^d}V(x_k).$$

$m>0$ est la masse, $U\in C^2(\mathbb{R}^1)$ l'interaction à deux corps convexe, satisfaisant à (1.3), J une fonction d'intensité, positive, nulle en zéro. De plus, $a^2>0$ et $V=V_0+W$, où $V_0\in C^2(\mathbb{R}^1)$ satisfait à (1.4) et $W\in C_b(\mathbb{R}^1)$, avec $\delta(W):=\sup_{\mathbb{R}^1}W-\inf_{\mathbb{R}^1}W$.

Nous étudions les états de Gibbs quantiques (à température $1/\beta$, $\beta>0$) au moyen de la mesure euclidienne de Gibbs ν_β sur un réseau de lacets Ω_β ν_β est définie comme mesure de probabilité sur Ω_β satisfaisant aux équations DLR formulées au moyen de la spécification locale $\pi_{\beta,\Lambda}$, $\Lambda\subset\mathbb{Z}^d$, $|\Lambda|<\infty$, associée à l'Hamiltonien H et décrite par (1.5)–(1.7). On appelle ν_β tempérée si $(\int\|\omega_k\|_\beta d\nu_\beta(\omega))_{k\in\mathbb{Z}^d}\in S'(\mathbb{Z}^d)$, $\omega=(\omega_k)_{k\in\mathbb{Z}^d}\in\Omega_\beta$.

2 Unicité des états de Gibbs

Le résultat principal est le suivant:
Théorème 2.1 *Soit*

$$\frac{e^{\beta\delta(W)}b_+^2}{b_-^2 + \widetilde{J}^{-1}(a^2+b^2)} < 1,$$

$b_\pm$ resp. b étant les constantes apparaissant en (1.3) resp. (1.4) et où $\widetilde{J} := \sum_{k\in\mathbb{Z}^d} J(k)$. Alors l'ensemble des mesures de Gibbs tempérées est constitué par un seul point.

La preuve est basée sur le critère de Dobrushin [5], [6] et une évaluation de la distance de Vasershtein des probabilités conditionnelles "en un point", données par (2.2), (2.3). Cette estimation à son tour est une conséquence de l'inégalité de Sobolev logarithmique démontrée en [2].

Le résultat d'unicité s'étend au cas d'interactions à deux corps sans invariance de translation et à des anharmonicités plus générales, la condition suffisante pour l'unicité étant alors (3.2).

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1 Temperature loop space representation for quantum Gibbs states

Let $\mathbb{Z}^d$, $d\in\mathbb{N}$, be the integer lattice with the Euclidean distance $|k-j|$, $k,j\in\mathbb{Z}^d\subset\mathbb{R}^d$. By $S(\mathbb{Z}^d)$ and $S'(\mathbb{Z}^d)$ we denote the mutually dual spaces of fastly decreasing resp. slowly increasing sequences over $\mathbb{Z}^d$.

We consider a translation invariant system of interacting quantum anharmonic oscillators with the heuristic Hamiltonian

$$H := -\frac{1}{2m}\sum_{k\in\mathbb{Z}^d}\Delta_k + \frac{a^2}{2}\sum_{k\in\mathbb{Z}^d}x_k^2 + \sum_{\{k,j\}\subset\mathbb{Z}^d} J(k-j)U(x_k-x_j) + \sum_{k\in\mathbb{Z}^d}V(x_k), \tag{1.1}$$

where $m>0$ (= physical mass). The two-particle interactions $U(x_k-x_j)$ have intensities

$$J(k-j) = J(|k-j|)\geq 0,\ J(0)=0,\ (J(k))_{k\in\mathbb{Z}^d}\in S(\mathbb{Z}^d),\ \widetilde{J} := \sum_{k\in\mathbb{Z}^d}J(k), \tag{1.2}$$

and $U\in C^2(\mathbb{R}^1)$ is a convex function such that $\exists b_-,b_+>0$: $\forall q\in\mathbb{R}^1$

$$b_-^2 \leq U''(q)\leq b_+^2,\ 0=U(0)\leq U(q)=U(-q)\leq b_+^2q^2/2\ . \tag{1.3}$$

The harmonic self-interaction is defined by $a^2x_k^2/2$ with a constant $a^2>0$. The anharmonic self-interaction potential has the form $V(q)=V_0(q)+W(q)$. Here $V_0\in C^2(\mathbb{R}^1)$ is such that $\exists b,\ C>0,\ M\in\mathbb{N}$: $\forall q\in\mathbb{R}^1$

$$V_0''(q)\geq b^2\ ,\ |V_0^{(l)}(q)|\leq C(1+|q|)^M,\ l=0,1,2, \tag{1.4}$$

and $W\in C_b(\mathbb{R}^1)$. Set $\delta(W) := \sup_{\mathbb{R}^1}W - \inf_{\mathbb{R}^1}W$.

A fundamental problem of equilibrium quantum statistical physics is the study of temperature (i.e., Gibbs) states. We will take the Euclidean approach (see e.g. [1]) to give a rigorous meaning to the Gibbs state G_β of the lattice system (1.1) at the inverse temperature $\beta > 0$, i.e., we construct (a path space representation) via the corresponding Euclidean Gibbs measure ν_β on the "loop lattice" Ω_β defined as follows:

Let S_β be the circle of length β. The spaces $C(S_\beta)$ and $\mathcal{H}_\beta := L_2(S_\beta)$ consist of continuous resp. (relative to Lebesgue measure) square integrable functions $\omega : S_\beta \to \mathbb{R}^1$, equipped with the sup-norm $\|\cdot\|_{C(S_\beta)}$ resp. L_2-norm $\|\cdot\|_\beta = (\cdot,\cdot)_\beta^{1/2}$. We define as the configuration space the temperature loop space ("loop lattice")

$$\Omega_\beta := C(S_\beta)^{\mathbb{Z}^d} = \{\omega = (\omega_k)_{k\in\mathbb{Z}^d} | \omega : S_\beta \to \mathbb{R}^{\mathbb{Z}^d},\ \omega_k \in C(S_\beta)\} ,$$

endowed with the product topology and with the Borel σ-algebra $\mathcal{B}(\Omega_\beta)$. Because of the infinite radius of interaction, we also need the subspace of tempered configurations $\Omega_\beta^t := \{\omega \in \Omega_\beta | (\|\omega_k\|_\beta)_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d)\}$.

The Euclidean Gibbs measure ν_β is described by a corresponding family of local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$ (cf. [7]). Let Δ_β be the Laplace-Beltrami operator on the circle S_β and $\gamma_\beta(d\omega_k)$ be the Gaussian measure on $(\mathcal{H}_\beta, \mathcal{B}(\mathcal{H}_\beta))$ with zero mean value and correlation operator $(-m\Delta_\beta + a^2 1\!\!1)^{-1}$. Consider the reference measure $\times_{k\in\mathbb{Z}^d} d\sigma_\beta(\omega_k)$, where

$$d\sigma_\beta(\omega_k) := \frac{1}{Z_{\beta,\Lambda}} \exp\{-\int_{S_\beta} V(\omega_k(\tau))d\tau\} d\gamma_\beta(\omega_k) \ . \tag{1.5}$$

Define the local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$: for all $B \in \mathcal{B}(\Omega_\beta)$, $\xi \in \Omega_\beta$

$$\pi_{\beta,\Lambda}(B|\xi) := \begin{cases} \dfrac{1}{Z_{\beta,\Lambda}(\xi)} \displaystyle\int_{\Omega_{\beta,\Lambda}} \exp\{-I_{\beta,\Lambda}(\omega_\Lambda|\xi)\} 1\!\!1_B(\omega_\Lambda \times \xi_{\Lambda^c}) \\ \qquad\qquad \times_{k\in\Lambda} d\sigma_\beta(\omega_k), & \xi \in \Omega_\beta^t, \\ 0, & \xi \in \Omega_\beta \setminus \Omega_\beta^t \ . \end{cases} \tag{1.6}$$

Here, $Z_{\beta,\Lambda}(\xi)$ is a normalization constant, $\Omega_{\beta,\Lambda} := C(S_\beta)^\Lambda$, $\omega_\Lambda := (\omega_k)_{k\in\Lambda} \in \Omega_{\beta,\Lambda}$, $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$, and

$$I_{\beta,\Lambda}(\omega|\xi) := \tag{1.7}$$
$$\int_{S_\beta} [\sum_{\{k,j\}\subset\Lambda} J(k-j)U(\omega_k(\tau) - \omega_j(\tau)) + \sum_{\substack{\{k,j\}\subset\mathbb{Z}^d \\ k\in\Lambda, j\in\Lambda^c}} J(k-j)U(\omega_k(\tau) - \xi_j(\tau))]d\tau \, .$$

Definition 1.1 *A probability measure ν_β on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ is called (Euclidean) Gibbs state for the local specifications $\pi_{\beta,\Lambda}$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, if it satisfies the DLR (Dobrushin-Lanford-Ruelle) equations, i.e., for all $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$,*

$$\nu_\beta \pi_{\beta,\Lambda} = \nu_\beta \ . \tag{1.8}$$

As usual we restrict our considerations to the the subset $\mathcal{G}^t_\beta$ of tempered Gibbs measures, i.e., those ν_β from Definition 1.1 with

$$(< \|\omega_k\|_\beta >_{\nu_\beta})_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d) \ , \tag{1.9}$$

where $< f >_{\nu_\beta} = \int_{\Omega_\beta} f(\omega) d\nu_\beta(\omega)$. It is well-known and can be proved by the standard "super-stability" techniques due to Ruelle that $\mathcal{G}^t_\beta \neq \emptyset$ (cf. e.g. [8]).

2 Uniqueness of Gibbs states

We present the main result of this paper.

Theorem 2.1 *Suppose that the parameters of the system* (1.1) *(resp. its rigorous implementation through* (1.6) *and Definition 1.1) satisfy the relation*

$$\frac{e^{\beta\delta(W)} b_+^2}{b_-^2 + \widetilde{J}^{-1}(a^2+b^2)} < 1 \ , \tag{2.1}$$

then the set $\mathcal{G}^t_\beta$ of tempered Gibbs measures consists of exactly one point.

Proof The proof is based on Dobrushin's fundamental uniqueness criterion (see [5], [6]). The details are contained in [3] resp. [2] (the latter, however, is concerned with a more particular class of quantum lattice models than (1.1)). Below we outline the basic ideas and new features. We shall prove even more, namely that there exists at most one tempered Gibbs measure ν_β on the configuration space $\widetilde{\Omega}_\beta := \mathcal{H}_\beta^{\mathbb{Z}^d} \supset \Omega_\beta$ with the local specifications (1.6) extended by continuity (due to (1.2) and (1.3)) to all $\xi \in \widetilde{\Omega}^t_\beta := \{\omega \in \widetilde{\Omega}_\beta | (\|\omega_k\|_\beta)_{k\in\mathbb{Z}^d} \in S'(\mathbb{Z}^d)\}$.

The corresponding one-dimensional Gibbs distribution $\nu_k(d\omega_k|\xi)$ in volumes $\Lambda_k = \{k\}$, $k \in \mathbb{Z}^d$, with boundary conditions $\xi = (\xi_j)_{j\in\mathbb{Z}^d} \in \widetilde{\Omega}_\beta$, are given by

$$\nu_k(d\omega_k|\xi) = \begin{cases} \dfrac{1}{Z_{\beta,k}(\xi)} \exp\{-I_{\beta,k}(\omega_k|\xi)\} d\sigma_\beta(\omega_k), & \xi \in \widetilde{\Omega}^t_\beta, \\ 0, & \xi \in \widetilde{\Omega}_\beta \setminus \widetilde{\Omega}^t_\beta, \end{cases} \tag{2.2}$$

where $Z_{\beta,k}(\xi)$ is a normalization constant and

$$I_{\beta,k}(\omega_k|\xi) = \int_{S_\beta} [\sum_{\substack{j\in\mathbb{Z}^d \\ j\neq k}} J(k-j) U(\omega_k(\tau) - \xi_j(\tau))] d\tau \ . \tag{2.3}$$

The Dobrushin coefficients are defined by

$$C_{kj} := \sup_{\substack{\xi,\eta\in\widetilde{\Omega}_\beta \\ \forall i\neq j: \xi_i = \eta_i}} \frac{R(\nu_k(\cdot|\xi),\ \nu_k(\cdot|\eta))}{\|\xi_j - \eta_j\|_\beta} \ , \ k \neq j, \tag{2.4}$$

where R denotes the Vasershtein distance, that is,

$$R(\nu_k(\cdot|\xi),\ \nu_k(\cdot|\eta)) := \sup_{f\in Lip_1(\mathcal{H}_\beta)} \left| \int_{\mathcal{H}_\beta} f\ d\nu_k(\cdot|\xi) - \int_{\mathcal{H}_\beta} f\ d\nu_k(\cdot|\eta) \right| \tag{2.5}$$

(here $Lip_1(\mathcal{H}_\beta)$ is the unit in the space of real-valued Lipschitz functions on $\mathcal{H}_\beta$). Due to the Dobrushin uniqueness criterion, it suffices to prove that

$$\sup_{k\in\mathbb{Z}^d} \sum_{\substack{j\in\mathbb{Z}^d \\ j\neq k}} C_{kj} < 1 \ . \tag{2.6}$$

For $k, j \in \mathbb{Z}^d$, $k \neq j$, $f \in Lip_1(\mathcal{H}_\beta)$, and $\xi = (\xi_i)_{i\in\mathbb{Z}^d} \in \widetilde{\Omega}^t_\beta$, the map

$$\mathcal{H}_\beta \ni \xi_j \mapsto < f >_{\nu_k(d\omega_k|\xi)} := \int_{\mathcal{H}_\beta} f(\omega_k)\nu_k(d\omega_k|\xi) \in \mathbb{R}^1 \tag{2.7}$$

is Fréchet differentialble with the derivative in direction $\varphi \in \mathcal{H}_\beta$

$$\left(\nabla_{\xi_j}\langle f\rangle_{\nu_k(d\omega_k|\xi)}, \varphi\right)_\beta = J(k-j)\mathrm{Cov}_{\nu_k(d\omega_k|\xi)}(f, (U'(\omega_k - \xi_j), \varphi)_\beta),$$

which can be estimated as

$$|\left(\nabla_{\xi_j}\langle f\rangle_{\nu_k(d\omega_k|\xi)}, \varphi\right)_\beta| \leq J(k-j)\left(\mathrm{Var}_{\nu_k(d\omega_k|\xi)} f\right)^{1/2}\left(\mathrm{Var}_{\nu_k(d\omega_k|\xi)}(U'(\omega_k - \xi_j), \varphi)_\beta\right)^{1/2} . \tag{2.8}$$

Here as usual Cov_ν, Var_ν denote the covariance resp. variance relative to a probability measure ν. Now we are at the crucial point in the proof. Following the method of the proof of Theorem 5.1 in [2], we can prove the Log-Sobolev (LS) inequality and, as a sequel, the variance estimates for $\nu_k(d\omega_k|\xi)$ (uniformly w.r.t. the boundary conditions ξ). In particular,

$$\mathrm{Var}_{\nu_k(d\omega_k|\xi)} f \leq C_{LS} := \frac{e^{\beta\delta(W)}}{\widetilde{J}b_-^2 + a^2 + b^2}$$

and

$$\mathrm{Var}_{\nu_k(d\omega_k|\xi)}\left(U'(\omega_k - \xi_j), \varphi\right)_\beta \leq C_{LS} b_+^4 \|\varphi\|_\beta^2 \ . \tag{2.9}$$

Hence from (2.8) and (2.9)

$$|(\nabla_{\xi_j}\langle f\rangle_{\nu_k(d\omega_k|\xi)}, \varphi)_\beta| \leq J(k-j)C_{LS}b_+^2\|\varphi\|_\beta \ ,$$

which by the mean-value theorem implies that for all $\xi, \eta \in \widetilde{\Omega}^t_\beta$ such that $\xi_i = \eta_i$, $\forall i \neq j$,

$$\left|\int_{\mathcal{H}_\beta} f\, d\nu(\cdot|\xi) - \int_{\mathcal{H}_\beta} d\nu_k(\cdot|\eta)\right| \leq J(k-j)C_{LS}b_+^2\|\xi_j - \eta_j\|_\beta \ . \tag{2.10}$$

By (2.4), (2.5), (2.10) and the assumption (2.1), estimate (2.6) follows.

□

3 Concluding remarks

(i) The fact that $|\mathcal{G}^t_\beta| \leq 1$ in Theorem 2.1 also can be proved for the more general quantum lattice systems with the heuristic Hamiltonian

$$H = \frac{1}{2m}\sum_{k\in\mathbb{Z}^d}\Delta_k + \frac{1}{2}\sum_{k\in\mathbb{Z}^d} a_k^2 x_k^2 + \sum_{\{k,j\}\in\mathbb{Z}^d} U_{kj}(x_k - x_j) + \sum_{k\in\mathbb{Z}^d} V_k(x_k). \tag{3.1}$$

The pair interactions are given by functions $U_{kj} \in C^2(\mathbb{R}^1)$ such that $\forall q \in \mathbb{R}^1$

$$0 < B_{kj}^- \leq U_{kj}''(q) \leq B_{kj}^+ < \infty,\ 0 \leq U_{kj}(q) = U_{kj}(-q) \leq C_{kj}(1+|q|)^2 \ .$$

The anharmonic self-interaction have the form $V_k(q) = V_{0,k}(q) + W_k(q)$, where $V_{0,k} \in C^2(\mathbb{R}^1)$, $W_k \in C_b(\mathbb{R}^1)$ and $\forall q \in \mathbb{R}^1$

$$V_{0,k}''(q) \geq b_k^2 > 0,\ |V_{0,k}^{(l)}(q)| \leq C_k(1+|q|)^M,\ l = 0,1,2,\ M \in \mathbb{N} \ .$$

The sequence $(C_k)_{k\in\mathbb{Z}^d}$ and the matrices $(B_{kj}^-)_{k,j\in\mathbb{Z}^d}$, $(B_{kj}^+)_{k,j\in\mathbb{Z}^d}$, $(C_{kj})_{k,j\in\mathbb{Z}^d}$ are supposed to be fastly decreasing. In the non-translation-invariant case the uniqueness condition can be written as

$$\sup_{k\in\mathbb{Z}^d} \frac{e^{\beta\delta(W_k)} \sum_{j\in\mathbb{Z}^d, j\neq k} B_{kj}^+}{\sum_{j\in\mathbb{Z}^d, j\neq k} B_{kj}^- + b_k^2 + a_k^2} < 1 \ . \tag{3.2}$$

(ii) A uniqueness result (via cluster expansion techniques) in the more restricted class $\mathcal{G}_{\beta,0}^t$ ($\subsetneq \mathcal{G}_\beta^t$) of Gibbs states satisfying an a priori assumption on the support (the so-called "super-stable" Ruelle type Gibbs measures, cf. [4]) has been obtained in [9].
(iii) The results of this paper have been presented in various talks by the authors in Bielefeld, Bochum, and Bonn. Detailed proofs, more references, and applications to physics (cf. also [2, Sects. 6, 7]) will be contained in the forthcoming paper [3].

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References

[1] S. Albeverio, R. Høegh-Krohn, Homogeneous random fields and quantum statistical mechanics, J.Func.Anal., 19, 1975, pp. 242-272.

[2] S. Albeverio, Yu.G. Kondratiev, M. Röckner, T.V. Tsikalenko, Uniqueness of Gibbs states for quantum lattice systems, Preprint, Universität Bielefeld, 1996.

[3] S. Albeverio, Yu.G. Kondratiev, M. Röckner, T.V. Tsikalenko, Uniqueness of Gibbs states on loop lattices for a general class of interactions, in preparation.

[4] M. Cassandro, E. Olivieri, A. Pellegrinotti, E. Presutti, Existence and uniqueness of DLR measures for unbounded spin systems, Z. Wahrsch. verw. Gebiete, 41, 1978, pp. 313-334.

[5] R.L. Dobrushin, Prescribing a system of random variables by conditional distributions, Theory Prob. Appl., 15, 1970, pp. 458-489.

[6] H. Föllmer, A covariance estimate for Gibbs measures, J. Funct. Anal. 46, 1982, pp. 387-395.

[7] H.O. Georgii, Gibbs measures and phase transitions, Studies in Mathematics, 9, Walter de Gruyter, Berlin-New York, 1988.

[8] Y.M. Park, H.H. Yoo, A characterization of Gibbs states of lattice boson systems, J. Stat. Phys., 75, 1994, pp. 215-239.

[9] Y.M. Park, H.H. Yoo, Uniqueness and clustering properties of Gibbs states for classical and quantum unbounded spin systems, J. Stat. Phys., 80, 1995, pp. 223-271.

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