R-equivalence and special unitary groups

V. Chernousov, A. Merkurjev *

The notion of R-equivalence, introduced by Manin in [6], is an important birational invariant of an algebraic variety defined over an arbitrary field. In the case of an algebraic group the set of R-equivalence classes has a natural group structure and it was studied by Colliot-Thélène and Sansuc in [2]. In particular, they computed this group for algebraic tori.

The group of R-equivalence classes for adjoint classical semisimple groups was computed in [9]. This computation enables to construct certain examples of non-rational adjoint groups. For simply connected semisimple groups the description of the group of R- equivalence classes is known only for inner forms of type A_n (Voskresenskii) and in some easy cases when this group is trivial (types C_n and G_2).

In the present paper we compute the group of R-equivalence classes for special unitary groups, i.e. for simply connected outer forms of type A_n. This computation comes as a result of the study of certain functorial properties of the group of R-equivalence classes for algebraic groups. To formulate these properties it is convenient to introduce the following category $\mathcal{C} = \mathcal{C}(F)$ for an arbitrary infinite field F. The objects of $\mathcal{C}$ are algebraic group homomorphisms $\alpha : G \to T$, where T is an algebraic torus over F and G is an algebraic group defined over F, such that $G(F)$ is dense in G and $G(F)/R = 1$. A morphism $(G \xrightarrow{\alpha} T) \to (G' \xrightarrow{\alpha'} T')$ in $\mathcal{C}$ is a group homomorphism $\beta : T \to T'$, such that for any field extension E/F the image of the composition $\beta_E \circ \alpha_E$ belongs to the image of $\alpha'_E : G'(E) \to T'(E)$.

It turns out (see section 2) that there is a natural functor from the category $\mathcal{C}$ to the category of abelian groups, taking $\alpha : G \to T$ to the group of R-equivalence classes $H(F)/R$, where $H = \ker(\alpha)$. As an application we show that this construction allows to define the *norm homomorphism* for groups of R-equivalence classes for certain algebraic groups. For example, if q is a non-degenerate quadratic form defined over F and E/F is a finite separable field extension, then there is a well-defined norm homomorphism for groups of R-equivalence classes of spinor groups

$$N_{E/F} : \mathrm{Spin}(q \otimes_F E)/R \longrightarrow \mathrm{Spin}(q)/R.$$

*The authors gratefully acknowledge the support of the Alexander von Humboldt-Stiftung and SFB 343 "Diskrete Strukturen in der Mathematik" and the hospitality of the University of Bielefeld.

Another application deals with the special unitary groups. Let L/F be a quadratic separable field extension, B be a central simple algebra over L with an involution τ of the second kind trivial on F. We prove (Theorem 4.3) that the group of R-equivalence classes of the special unitary group $\mathbf{SU}(B,\tau)$ is isomorphic to Σ/Σ', where Σ is the subgroup in $B^{\times}$ of all elements b, such that $\mathrm{Nrd}(b) \in F$ and Σ' is the subgroup in $B^{\times}$, generated by τ-symmetric elements. In particular, $SU(B,\tau)/R = 1$ if $\mathrm{ind}(B)$ is square-free.

The group Σ/Σ' does not depend on the involution τ. We explain this phenomenon in the second part of the paper. It turns out that the stable birational type of $\mathbf{SU}(B,\tau)$ depends only on the Brauer class of B (Corollary 5.7). We also prove (Theorem 5.4) that $\mathbf{SU}(B,\tau)$ is birationally isomorphic to $\mathbf{SU}(D,\sigma) \times \mathbf{A}^r_F$ for some r, where D is the central skewfield over L , Brauer equivalent to B, and σ is an involution of the second kind on D.

In particular, the group $\mathbf{SU}(B,\tau)$ is rational if $\mathrm{ind}(B) \leq 3$ (Corollary 5.5).

We use the following notation. For a vector space V over a field F and a field extension E/F we denote by V_E the E-vector space $V \otimes_F E$. For an algebraic variety X over F and for any F-scheme U the set $\mathrm{Mor}(U,X)$ of U-points of X is denoted by $X(U)$. If $U = \mathrm{Spec}(R)$, we simply write $X(R)$ for $X(U)$. For a morphism of varieties $\alpha : X \to Y$, the map of sets of points $X(R) \to Y(R)$ is denoted by α_R. $\mathbf{A}^n_F$ is the affine space over F. $\mathcal{AB}$ is the category of abelian groups.

1 The group of R-equivalence classes

Let F be an infinite field. Denote by $\mathcal{O}$ the semilocal subring in the field of rational functions $F(t)$ consisting of all functions defined at points $t = 0$ and $t = 1$. Evaluation at these points gives two F-algebra homomorphisms $e_i : \mathcal{O} \to F$, $i = 0, 1$.

Let X be an algebraic variety defined over a field F. For any element $x(t) \in X(\mathcal{O})$ and $i = 0, 1$ denote by $x(i)$ the *value of* $x(t)$ *at* i, i.e. the image of $x(t)$ under the map

$$X(e_i) : X(\mathcal{O}) \to X(F).$$

Lemma 1.1 *Let $\alpha : X \to Y$ be a morphism of algebraic varieties over a field F. Then*

$$\alpha_{\mathcal{O}}\big(x(t)\big)(i) = \alpha_F\big(x(i)\big)$$

for any $x(t) \in X(\mathcal{O})$ and $i = 0, 1$.

Proof. Follows from the commutativity of the diagram

$$\begin{array}{ccc} X(\mathcal{O}) & \xrightarrow{\alpha_{\mathcal{O}}} & Y(\mathcal{O}) \\ \big\downarrow X(e_i) & & \big\downarrow Y(e_i) \\ X(F) & \xrightarrow{\alpha_F} & Y(F). \end{array} \quad \square$$

Let G be an algebraic group defined over F. An element $g \in G(F)$ is called *R-trivial* if there is $g(t) \in G(\mathcal{O})$ such that $g(0) = 1$ and $g(1) = g$ (see [2]). The set of all R-trivial elements in $G(F)$ will be denoted by $RG(F)$.

Lemma 1.2 *Let H be a normal (closed) subgroup in G. Then*
1. $RH(F)$ is a normal subgroup in $G(F)$.
2. If G/H is a commutative group then so is $RG(F)/RH(F)$.

Proof. 1. Let $h_{1,2} \in RH(F)$. Choose $h_{1,2}(t) \in H(\mathcal{O})$ such that $h_{1,2}(0) = 1$, $h_{1,2}(1) = h_{1,2}$ and consider $h(t) = h_1(t) \cdot h_2(t)^{-1} \in H(\mathcal{O})$. Then $h(0) = 1$ and $h(1) = h_1 h_2^{-1}$, hence $h_1 h_2^{-1} \in RH(F)$, i.e. $RH(F)$ is a subgroup in $G(F)$. Let $g \in G(F)$ and $h \in RH(F)$. Choose $h(t) \in H(\mathcal{O})$ such that $h(0) = 1$ and $h(1) = h$ and consider $h'(t) = g \cdot h(t) \cdot g^{-1} \in H(\mathcal{O})$. Then $h'(0) = 1$ and $h'(1) = ghg^{-1}$, hence $ghg^{-1} \in RH(F)$ and $RH(F)$ is a normal subgroup in $G(F)$.

2. Let $g_{1,2} \in RG(F)$. Choose $g_1(t) \in G(\mathcal{O})$ such that $g_1(0) = 1$, $g_1(1) = g_1$ and consider the commutator $h(t) = [g_1(t), g_2] \in H(\mathcal{O})$. Then $h(0) = [1, g_2] = 1$ and $h(1) = [g_1, g_2]$, hence $[g_1, g_2] \in RH(F)$. $\square$

It follows from Lemma 1.2 that for an algebraic group G the group $RG(F)$ is a normal subgroup in $G(F)$. We simply denote the factor-group $G(F)/RG(F)$ by $G(F)/R$. If the variety of G is stably rational, then $G(F)/R = 1$ (see [2]).

Let $\alpha : G \to H$ be an algebraic group homomorphism over a field F. It induces homomorphisms

$$\alpha_F : G(F) \to H(F) \quad \text{and} \quad \alpha_{\mathcal{O}} : G(\mathcal{O}) \to H(\mathcal{O}).$$

Let $g \in RG(F)$, so that there exists $g(t) \in G(\mathcal{O})$ such that $g(0) = 1$ and $g(1) = g$. Denote by $h(t)$ the element $\alpha_{\mathcal{O}}\big(g(t)\big) \in H(\mathcal{O})$. By Lemma 1.1, $h(0) = \alpha_F\big(g(0)\big) = \alpha_F(1) = 1$ and $h(1) = \alpha_F\big(g(1)\big) = \alpha_F(g)$, hence, $\alpha_F(g) \in RH(F)$. In other words, f_F takes $RG(F)$ to $RH(F)$ and therefore induces a group homomorphism $G(F)/R \to H(F)/R$. Thus, the correspondence $G \mapsto G(F)/R$ induces a functor from the category of algebraic groups over F to the category of (abstract) groups.

We will use later the following technical

Proposition 1.3 *Let G be a connected algebraic group over a field F with the property that the group of rational points $G(F)$ is dense in G, $U \subset G$ be a non-empty open subset over F and $f : U(F) \to A$ be a map to some group A, such that $f(g_1 g_2) = f(g_1) f(g_2)$ for all $g_{1,2} \in U(F)$ with $g_1 \cdot g_2 \in U(F)$. Then f extends uniquely to a group homomorphism $\tilde{f} : G(F) \to A$.*

Proof. Let $g \in G(F)$. Since $gU^{-1} \cap U$ is not empty and F is infinite, there exist $g_{1,2} \in U(F)$ such that $g = g_1 g_2$. If an extension $\tilde{f}$ exists, then necessarily $\tilde{f}(g) = f(g_1) \cdot f(g_2)$, so that $\tilde{f}$ is unique. We define $\tilde{f}$ by this formula.

In order to prove that $\tilde{f}$ is well-defined, assume that $g = g_1 g_2 = g'_1 g'_2$ for some $g'_{1,2} \in U(F)$. Choose an F-point h in the intersection $U \cap g_1 U^{-1} \cap g'_1 U^{-1} \cap g U^{-1}$. We have $g_1 = hh_1$ and $g'_1 = hh'_1$ for some $h_1, h'_1 \in U(F)$. Moreover, $g = h \cdot h_1 g_2 = h \cdot h'_1 g'_2$, and $h_1 g_2 = h'_1 g'_2 \in U(F)$. Hence,

$$f(g_1) \cdot f(g_2) = f(hh_1) \cdot f(g_2) = f(h) \cdot f(h_1) \cdot f(g_2) = f(h) \cdot f(h_1 g_2) =$$

$$f(h) \cdot f(h'_1 g'_2) = f(h) \cdot f(h'_1) \cdot f(g'_2) = f(hh'_1) \cdot f(g'_2) = f(g'_1) \cdot f(g'_2).$$

Thus, the map $\tilde{f}$, being clearly the extension of f, is well-defined.

Now we prove that $\tilde{f}(g_1 g_2) = \tilde{f}(g_1) \cdot \tilde{f}(g_2)$ for any $g_{1,2} \in G(F)$. Assume first that $g_1 \in U(F)$. Choose an F-point h in $U \cap g_1^{-1} U \cap g_2 U^{-1}$. Then $g_2 = hg$ for some $g \in U(F)$ and $h, g_1 h \in U(F)$. We have

$$\tilde{f}(g_1 g_2) = \tilde{f}(g_1 h \cdot g) = f(g_1 h) \cdot f(g) = f(g_1) \cdot f(h) \cdot f(g) = \tilde{f}(g_1) \cdot \tilde{f}(g_2).$$

Consider now the general case. Let $g_1 = hg$ with $h, g \in U(F)$. Then

$$\tilde{f}(g_1 g_2) = \tilde{f}(h \cdot g g_2) = \tilde{f}(h) \cdot \tilde{f}(g g_2) = \tilde{f}(h) \cdot \tilde{f}(g) \cdot \tilde{f}(g_2) = \tilde{f}(g_1) \cdot \tilde{f}(g_2). \ \square$$

Remark 1.4 The condition for $G(F)$ to be dense in G holds if G is reductive or if F is a perfect field ([1, Cor.18.3]).

2 The category $\mathcal{C}(F)$

Assume we are given a diagram of connected algebraic groups and group homomorphisms

$$\begin{array}{ccc} G & & G' \\ \downarrow \alpha & & \downarrow \alpha' \\ T & \xrightarrow{\beta} & T', \end{array} \qquad (*)$$

such that for any field extension E/F the image of the composition $\beta_E \circ \alpha_E$ belongs to the image of $\alpha'_E : G'(E) \to T'(E)$.

Lemma 2.1 *There exists a rational morphism $\eta : G \to G'$ making the diagram commutative. If $G(F)$ is dense in G then η can be chosen in such a way that $\eta(1)$ is defined and equals 1.*

Proof. Let E be the function field $F(G)$ of the group G. By assumption, there is $\eta \in G'(E)$, such that

$$\alpha'_E(\eta) = \beta_E\big(\alpha_E(\xi)\big),$$

where $\xi \in G(E)$ is the generic element. Viewing η as a rational morphism from G to G', we have $\alpha' \circ \eta = \beta \circ \alpha$.

Assume that $G(F)$ is dense in G. Denote by $U \subset G$ the open subset of definition of η and choose $g \in U(F)^{-1}$. By assumption, there is $g' \in G'(F)$ such that $\alpha'_F(g') = \beta_F\big(\alpha_F(g)\big)$. Consider rational morphism $\eta' : gU \to G'$ given by the formula $\eta'(gu) = g' \cdot \eta(u)$. We have

$$\alpha'\big(\eta'(gu)\big) = \alpha'\big(g' \cdot \eta(u)\big) = \alpha'(g') \cdot \alpha'\big(\eta(u)\big) =$$

$$\beta\big(\alpha(g)\big) \cdot \beta\big(\alpha(u)\big) = \beta\big(\alpha(gu)\big),$$

i.e. $\alpha' \circ \eta' = \beta \circ \alpha$. Since $1 \in (gU)(F)$, we can replace η by η' and assume that $\eta(1)$ is defined. Finally, replacing η by $\eta(1)^{-1} \cdot \eta$, we get the desired equality $\eta(1) = 1$. $\Box$

Under the conditions of Lemma 2.1 denote by $U \subset G$ an open non-empty subset, which is contained in the set of definition of η, so that one can view η as a regular morphism $U \to G'$. In particular, we have a map

$$\eta_F : U(F) \longrightarrow G'(F).$$

Denote by H' the kernel of α' and by $\bar{\eta}$ the composition (see Lemma 1.2)

$$U(F) \xrightarrow{\eta_F} G'(F) \longrightarrow G'(F)/RH'(F).$$

Assume in addition that $G(F)/R = 1$. We claim that $\bar{\eta}$ satisfies the conditions of Proposition 1.3, i.e. $\bar{\eta}(g_1 g_2) = \bar{\eta}(g_1)\bar{\eta}(g_2)$ for all $g_{1,2} \in U(F)$ with the property $g_1 \cdot g_2 \in U(F)$. In order to prove the claim, choose $g_1(t) \in G(\mathcal{O})$ such that $g_1(0) = 1$, $g_1(1) = g_1$ and consider

$$h(t) = \eta_{\mathcal{O}}\big(g_1(t)\big) \cdot \eta_{\mathcal{O}}(g_2) \cdot \eta_{\mathcal{O}}\big(g_1(t)g_2\big)^{-1} \in G'(\mathcal{O}).$$

Since $\alpha' \circ \eta = \beta \circ \alpha$ is a homomorphism, it follows that $\alpha'_{\mathcal{O}}\big(h(t)\big) = 1$, i.e. $h(t) \in H'(\mathcal{O})$. By Lemma 1.1, we have

$$h(0) = \eta_F\big(g_1(0)\big) \cdot \eta_F(g_2) \cdot \eta_F\big(g_1(0)g_2\big)^{-1} = \eta_F(1) \cdot \eta_F(g_2) \cdot \eta_F(g_2)^{-1} = 1$$

and similarly

$$h(1) = \eta_F(g_1) \cdot \eta_F(g_2) \cdot \eta_F(g_1 g_2)^{-1},$$

i.e. the r.h.s. of the last equality belongs to $RH'(F)$.

By Proposition 1.3, the map $\bar{\eta}$ extends uniquely to a group homomorphism $G(F) \to G'(F)/RH'(F)$. Clearly, it does not depend on the choice of an open subset U in the set of definition of η. We prove that it does not depend also on the choice of η. Let $\eta' : G \to G'$ be another rational morphism, making the diagram $(*)$ commutative, such that $\eta'(1) = 1$. Choose a non-empty open subset $U \subset G$, which is contained in the sets of definition of η and η', so that $\eta, \eta' \in G'(U)$. We have $\eta' = \eta \cdot \nu$ with $\nu \in H'(U)$ and it suffices to show that $\nu_F(g) \in RH'(F)$ for any $g \in U(F)$. Choose $g(t) \in G(\mathcal{O})$ such that $g(0) = 1$ and $g(1) = g$. Consider the element $h(t) = \nu_{\mathcal{O}}\big(g(t)\big) \in H'(\mathcal{O})$. Clearly, by Lemma 1.1,

$$h(0) = \nu_F\big(g(0)\big) = \nu_F(1) = \eta'_F(1) \cdot \eta_F(1)^{-1} = 1$$

and $h(1) = \nu_F\big(g(1)\big) = \nu_F(g)$, i.e. $\nu_F(g) \in RH'(F)$.

Thus, the homomorphism $G(F) \to G'(F)/RH'(F)$ depends only on β (if we fix α and α') and we denote it by $\tilde{\beta}$.

The next step is to prove that $\tilde{\beta}\Big(RH(F)\Big) = 1$, where H is the kernel of α. Take any $h \in RH(F)$ and choose $h(t) \in H(\mathcal{O})$ such that $h(0) = 1$ and $h(1) = h$. As in the proof of Proposition 1.3 we can write $h = g_1 g_2$ with $g_{1,2} \in U(F)$. Since $G(F)/R = 1$, one can choose $g_1(t) \in G(\mathcal{O})$ such that $g_1(0) = 1$ and $g_1(1) = g_1$. Then we have $h(t) = g_1(t) \cdot g_2(t)$ for some $g_2(t) \in G(\mathcal{O})$, such that $g_2(0) = 1$ and $g_2(1) = g_2$. Denote $\eta_{\mathcal{O}}\big(g_{1,2}(t)\big)$ by $g'_{1,2}(t) \in G'(\mathcal{O})$. By Lemma 1.1, $g'_{1,2}(0) = \eta_F\big(g_{1,2}(0)\big) = \eta_F(1) = 1$ and $g'_{1,2}(1) = \eta_F\big(g_{1,2}(1)\big) = \eta_F(g_{1,2})$. Consider the product $h'(t) = g'_1(t) \cdot g'_2(t) \in G'(\mathcal{O})$. Equalities

$$\alpha'_{\mathcal{O}}\big(h'(t)\big) = \alpha'_{\mathcal{O}}\big(g'_1(t)\big) \cdot \alpha'_{\mathcal{O}}\big(g'_2(t)\big) = \alpha'_{\mathcal{O}}\Big(\eta_{\mathcal{O}}\big(g_1(t)\big)\Big) \cdot \alpha'_{\mathcal{O}}\Big(\eta_{\mathcal{O}}\big(g_2(t)\big)\Big) =$$

$$\beta_{\mathcal{O}}\Big(\alpha_{\mathcal{O}}\big(g_1(t)\big)\Big) \cdot \beta_{\mathcal{O}}\Big(\alpha_{\mathcal{O}}\big(g_2(t)\big)\Big) = \beta_{\mathcal{O}}\Big(\alpha_{\mathcal{O}}\big(h(t)\big)\Big) = 1$$

imply that $h'(t) \in H'(\mathcal{O})$. Since by Lemma 1.1, $h'(0) = g'_1(0) \cdot g'_2(0) = 1$ and

$$h'(1) = g'_1(1) \cdot g'_2(1) = \eta_F(g_1) \cdot \eta_F(g_2),$$

it follows that

$$\tilde{\beta}(h) = \eta_F(g_1) \cdot \eta_F(g_2) \cdot RH'(F) = 1 \cdot RH'(F) = 1 \in G'(F)/RH'(F).$$

We have proved that $\tilde{\beta}$ induces a homomorphism

$$G(F)/RH(F) \longrightarrow G'(F)/RH'(F)$$

which we also denote by $\tilde{\beta}$. The restriction of $\tilde{\beta}$ also gives a homomorphism $H(F)/R \to H'(F)/R$.

It is convenient to use the categorical language. We define a category $\mathcal{C} = \mathcal{C}(F)$ as follows. Objects are the algebraic group homomorphisms $\alpha : G \to T$ over F, such that T is an algebraic torus, $G(F)$ is dense in G and $G(F)/R = 1$. A morphism from $G \xrightarrow{\alpha} T$ to $G' \xrightarrow{\alpha'} T'$ is a group homomorphism $\beta : T \to T'$, such that for any field extension E/F the image of the composition $\beta_E \circ \alpha_E$ belongs to the image of α'_E. The composition in $\mathcal{C}$ is defined in an obvious way.

We also define a functor

$$\Phi : \mathcal{C} \longrightarrow \mathcal{AB},$$

taking an object $G \xrightarrow{\alpha} T$ to $G(F)/RH(F)$, where $H = \ker(\alpha)$ (see Lemma 1.2) and a morphism β - to the homomorphism $\tilde{\beta} : G(F)/RH(F) \to G'(F)/RH'(F)$ defined above. In order to prove that Φ takes compositions to compositions, consider two composable morphisms in $\mathcal{C}$:

$$(G \xrightarrow{\alpha} T) \xrightarrow{\beta} (G' \xrightarrow{\alpha'} T') \xrightarrow{\beta'} (G'' \xrightarrow{\alpha''} T'').$$

Denote by $\eta : G \to G'$ and $\eta' : G' \to G''$ two rational morphisms, such that $\alpha' \circ \eta = \beta \circ \alpha$, $\alpha'' \circ \eta' = \beta' \circ \alpha'$ and $\eta(1) = 1$, $\eta'(1) = 1$. Since the image of η contains 1, belonging to the set of definition of η', the composition $\eta'' = \eta' \circ \eta$ is well-defined and $\alpha'' \circ \eta'' = \beta'' \circ \alpha$, where $\beta'' = \beta' \circ \beta$.

Let $U' \subset G'$ be the subset of definition of η'. Choose a non-empty open subset $U \subset G$, such that η is defined on U and $\eta(U) \subset U'$. Then η'' is defined on U and η''_F coincides with the composition

$$U(F) \xrightarrow{\eta_F} U'(F) \xrightarrow{\eta'_F} G''(F).$$

Hence, by Proposition 1.3, $\tilde{\beta}'' = \tilde{\beta}' \circ \tilde{\beta}$, since the restriction of the both sides on $U(F)$ equals η''_F followed by the natural surjection $G''(F) \to G''(F)/RH''(F)$, where $H'' = \ker(\alpha'')$. Therefore, Φ is a well-defined functor.

Restricting Φ, we get a functor

$$\Psi : \mathcal{C} \longrightarrow \mathcal{AB},$$

taking an object $G \xrightarrow{\alpha} T$ to $H(F)/R$, where $H = \ker(\alpha)$.

3 Norm homomorphism

Let $\alpha : G \to T$ be an object in the category $\mathcal{C}(F)$, and $H = \ker(\alpha)$. For a finite separable field extension E/F we can consider the homomorphism $\alpha_E : G_E \to T_E$ as an object in $\mathcal{C}(E)$. Denote by $R_{E/F}$ the Weil transfer functor. Then

$$R_{E/F}(\alpha_E) : R_{E/F}(G_E) \to R_{E/F}(T_E)$$

is again an object in $\mathcal{C}(F)$.

Assume that F is a perfect field. By the norm principle [8, Th.3.9], the norm homomorphism for algebraic tori $N_{E/F} : R_{E/F}(T_E) \to T$ can be considered as a morphism from $R_{E/F}(\alpha_E)$ to α in the category $\mathcal{C}(F)$. Applying the functor Ψ to this morphism we get the *norm homomorphism*

$$N_{E/F} : H(E)/R \longrightarrow H(F)/R.$$

Proposition 3.1 *The composition*

$$H(F)/R \longrightarrow H(E)/R \xrightarrow{N_{E/F}} H(F)/R$$

coincides with the $(E : F)$*-power homomorphism.*

Proof. Follows from the fact that in the category $\mathcal{C}(F)$ the composition

$$T \longrightarrow R_{E/F}(T_E) \xrightarrow{N_{E/F}} T$$

is the $(E : F)$-power map. □

Example 3.2 Let A be a central simple algebra over F, $G = \mathbf{GL}_{1,A}$ the reductive group of invertible elements in A. Since G is clearly rational variety, it follows that $G(F)/R = 1$. Hence the reduced norm homomorphism $\mathrm{Nrd} : G \to \mathbf{G}_m$ is an object in $\mathcal{C}(F)$. The kernel H of the reduce norm homomorphism is the group $\mathbf{SL}_{1,A}$. It is known that $H(F)/R = \mathrm{SK}_1(A)$ ([14]). One can show that the norm homomorphism $H(E)/R \to H(F)/R$ for a finite separable field extension E/F is induced by the norm homomorphism $K_1(A_E) \to K_1(A)$ in the algebraic K-theory ([11]).

Example 3.3 Let q be a non-degenerate quadratic form of even dimension over a field F, G be the even Clifford group of q. This group is known to be reductive and rational, hence the spinor norm homomorphism $\mathrm{Sn} : G \to \mathbf{G}_m$ is an object in $\mathcal{C}(F)$. The kernel of the spinor norm homomorphism is the *spinor group* $\mathbf{Spin}(q)$. Thus, for a finite separable field extension E/F there is the norm homomorphism

$$\mathrm{Spin}(q_E)/R \longrightarrow \mathrm{Spin}(q)/R.$$

Note that we don't need to assume F to be perfect, since the norm principle for spinor norms, known as the Knebusch norm principle, holds for any field F [5, Th.VII.5.1].

4 Special unitary groups

In this section we compute the group of R-equivalence classes for a special unitary group.

Let L/F be a separable quadratic field extension, B be a central simple L-algebra with an involution τ of the second kind, trivial on F. We consider the *unitary group* $\mathbf{U}(B,\tau)$ with the group of F-points

$$U(B,\tau) = \{b \in B^\times \text{ such that } \tau(b)\cdot b = 1\}.$$

This is a reductive rational group (see [15, Sec.4]), hence $U(B,\tau)/R = 1$.

Denote by T the norm 1 torus of the extension L/F. More precicely,

$$T = \ker\Big(R_{L/F}(\mathbf{G}_{m,L}) \stackrel{N_{L/F}}{\longrightarrow} \mathbf{G}_{m,F}\Big).$$

The reduced norm gives the homomorphism

$$\mathrm{Nrd} : \mathbf{U}(B,\tau) \to T,$$

which can be considered as an object in the category $\mathcal{C}(F)$. The kernel $\mathbf{SU}(B,\tau)$ of Nrd is the *special unitary group*.

Consider also the group homomorphism

$$\alpha = \alpha_B : R_{L/F}(\mathbf{GL}_{1,B}) \to T,$$

defined by the formula $\alpha(b) = \mathrm{Nrd}(b)/\tau\big(\mathrm{Nrd}(b)\big)$. Clearly, α can be also considered as an object in $\mathcal{C}(F)$. The kernel of α we denote by $K = K_B$. The group of F-points of K, consisting of all $b \in B^\times$, such that $\mathrm{Nrd}(b) \in F^\times$, we denote by $\Sigma' = \Sigma'_B$. Any invertible τ-symmetric element $b \in B^\times$ belongs to $\Sigma = K(F)$. Moreover, if we consider $b(t) = 1 + t(b-1) \in K\big(F(t)\big)$, then $b(0) = 1$ and $b(1) = b$, i.e. b is an R-trivial element in $K(F)$. Denote by $\Sigma = \Sigma_B$ the subgroup in Σ', generated by invertible τ-symmetric elements in B, so that $\Sigma \subset RK(F)$. The group Σ does not depend on the choice of involution of the second kind on B.

Lemma 4.1 $\Sigma = RK(F)$.

Proof. Assume first that B is a skewfield. Take any $b \in RK(F)$ and choose $b(t) \in K(\mathcal{O})$, such that $b(0) = 1$ and $b(1) = b$. The group Σ'/Σ does not change under purely transcendental extensions by [17, Th. 2], hence $b = b_0 \cdot s_1 \cdot \ldots \cdot s_n$ where $b_0 \in K(F)$ and s_i are τ-symmetric elements in $D_{F(t)}$. Since D is a skewfield, one can modify s_i, multiplying by appropriate powers of t and $t-1$, so that s_i are defined at points $t = 0$ and $t = 1$. Then $b = b(1) \cdot b(0)^{-1}$, being the product of $s_i(1) \cdot s_i(0)^{-1}$, belongs to Σ.

Consider now the general case. Let D be a central skewfield over L, Brauer equivalent to B, so that B is isomorphic to a matrix algebra over D. We consider $\mathbf{GL}_{1,D}$ as a subgroup in $\mathbf{GL}_{1,B}$ by identifying an element d with the diagonal matrix $\mathrm{diag}(d, 1, \ldots, 1)$. By the first part of the proof the left vertical homomorphism in the commutative diagram

$$\begin{array}{ccc} \Sigma'_D/\Sigma_D & \longrightarrow & \Sigma'_B/\Sigma_B \\ \downarrow & & \downarrow \\ K_D(F)/R & \longrightarrow & K_B(F)/R \end{array}$$

is an isomorphism. The bottom arrow is an isomorphism by [16, Lem. 2,3]. The group homomorphisms

$$\alpha_D : R_{L/F}(\mathbf{GL}_{1,D}) \to T \text{ and } \alpha_B : R_{L/F}(\mathbf{GL}_{1,B}) \to T$$

have the same images on the groups of points over any field extension of F, hence the identity automorphism of T represents an isomorphism between α_D and α_B in the category $\mathcal{C}$. Since the reduced norm of an element $d \in D$ equals the reduced norm of the corresponding diagonal matrix in B, the application of the functor Ψ to this isomorphism shows that the top arrow in the diagram is an isomorphism. Hence, the right vertical homomorphism is also an isomorphism, i.e. $\Sigma_B = RK_B(F)$. □

We will use several times the following

Proposition 4.2 [8, Prop.6.1] *For any field extension E/F the images of* Nrd_E *and α_E coincide.* □

It follows from Proposition 4.2 that the identity automorphism of T can be considered as an isomorphism between Nrd and α in the category $\mathcal{C}(F)$. The application of the functor Ψ and Lemma 4.1 give then the following computation of the group of R-equivalence classes of the group $\mathbf{SU}(B,\tau)$.

Theorem 4.3 *There is a natural isomorphism* $\mathrm{SU}(B,\tau)/R \xrightarrow{\sim} \Sigma'/\Sigma$. □

Remark 4.4 In [12] this statement was proved under assumption that the involution τ is isotropic.

By [16], the group Σ'/Σ is trivial if $\mathrm{ind}(B)$ is square-free. Thus, we obtain

Corollary 4.5 *If* $\mathrm{ind}(B)$ *is square-free, then* $SU(B,\tau)/R = 1$. □

Consider the isomorphism in Theorem 4.3 in detail. Choose an element $x \in L$ with the (L/F)-trace 1 (one can take $x = 1/2$ if $\mathrm{char}(F) \neq 2$). Denote by $U \subset \mathbf{U}(B,\tau)$ the open subset of all b such that the element $xb + \tau(x)$ is invertible in B. Clearly, $1 \in U$. Consider the morphism

$$\eta : U \to R_{L/F}(\mathbf{G}_{m,L}),$$

given by the formula $\eta(b) = xb + \tau(x)$. We can consider η as a rational morphism $\mathbf{U}(B,\tau) \to R_{L/F}(\mathbf{G}_{m,L})$. Since for any b in $\mathbf{U}(B,\tau)$ one has $\tau(b)b = 1$, it follows that

$$\big(\tau(x)\tau(b) + x\big)b = xb + \tau(x).$$

Hence,

$$\alpha\big(\eta(b)\big) = \alpha\big(xb + \tau(x)\big) =$$
$$\mathrm{Nrd}\big(xb + \tau(x)\big) / \mathrm{Nrd}\big(\tau(x)\tau(b) + x\big) = \mathrm{Nrd}(b),$$

i.e. $\alpha \circ \eta = \mathrm{Nrd}$. Hence, for any $b \in U(F)$ the isomorphism in Theorem 4.3 takes b to $xb + \tau(x)$. Of course, the latter does not depend on the choice of x and for any invertible b one can find $x \in L$ with the trace 1, such that the element $xb + \tau(x)$ is invertible.

Remark 4.6 It follows from Theorem 4.3 that the group of R-equivalence classes $\mathrm{SU}(B,\tau)/R$ does not depend on the choice of involution τ. This is a reflection of the fact that the groups $\mathbf{SU}(B,\tau)$ and K are stably birationally isomorphic. Indeed, consider the following commutative diagram with the cartesian square

$$\begin{array}{ccccc}
 & & \mathbf{SU}(B,\tau) & = & \mathbf{SU}(B,\tau) \\
 & & \downarrow & & \downarrow \\
K & \longrightarrow & G & \longrightarrow & \mathbf{U}(B,\tau) \\
\| & & \downarrow & & \downarrow \mathrm{Nrd} \\
K & = & R_{L/F}(\mathbf{GL}_{1,B}) & \xrightarrow{\alpha} & T.
\end{array}$$

Using the fact that on the group of points the images of Nrd and α are equal, as in the proof of [9, Prop.3], one deduces that the middle row and column are birationally split, hence the groups $\mathbf{SU}(B,\tau)\times R_{L/F}(\mathbf{GL}_{1,B})$ and $K\times\mathbf{U}(B,\tau)$ are birationally isomorphic.

5 Birational properties of special unitary groups

Let L/F be a separable quadratic field extension and D be a central skewfield of finite dimension over L with an involution of the second kind on D being trivial on F. Consider a right vector space V over D of finite dimension and a non-degenerate hermitian form h on V. Then the *unitary group* $\mathbf{U}(V,h)$ of isometries of V with respect to h coincides with $\mathbf{U}(B,\tau)$, where $B=\mathrm{End}_D(V)$ and τ is the adjoint involution of h (see [13] and [4]). The *special unitary group* $\mathbf{SU}(V,h)$ equals $\mathbf{SU}(B,\tau)$.

Let $w\in V$ be anisotropic vector and $W=w^{\perp}$ be the orthogonal complement to w. A non-trivial isometry $\psi\in\mathbf{U}(V,h)$, which is identity on W, is called a *reflection* on V. The image of $\psi-\mathrm{id}$ equals wD. Hence the space W is determined by ψ as the orthogonal complement to $\psi(v)-v$ for any $v\in V$, such that $\psi(v)\neq v$ (see [13, p.268]).

Lemma 5.1 *Let v and v' be anisotropic vectors in V, such that $h(v,v)=h(v',v')$ and $v'-v$ is anisotropic vector. Then there is a unique reflection $\psi\in\mathbf{U}(V,h)$, such that $\psi(v)=v'$.*

Proof. Denote $v'-v$ by w and $w^{\perp}$ by W. Since

$$h(v,v)=h(v',v')=h(v,v)+h(v,w)+h(w,v)+h(w,w),$$

it follows that $v\notin W$. Hence the linear map ψ, defined by the conditions $\psi(v)=v'$ and $\psi|_W=\mathrm{id}\,|_W$, is clearly a reflection.

Conversely, let ψ be a reflection, such that $\psi(v)=v'$. Since $w=\psi(v)-v$ is non-trivial, ψ must be identical on W and hence is uniquely determined. □

Let $v\in V$ be anisotropic vector. Denote by V' the orthogonal complement of v in V and by h' the restriction of h on V'. Then h' is a non-degenerate hermitian form on V' and $\mathbf{U}(V',h')$ can be identified with the subgroup in $\mathbf{U}(V,h)$ consisting of all φ, such that $\varphi(v)=v$. Denote by X the factor-variety of right cosets $\mathbf{U}(V,h)/\,\mathbf{U}(V',h')$ and by $I\subset\mathbf{U}(V,h)$ the subvariety of reflections.

Lemma 5.2 *The restriction of the natural surjection $\pi:\mathbf{U}(V,h)\to X$ to I is a birational isomorphism.*

Proof. Denote by I' the open subvariety in I consisting of all reflections ψ, such that $\psi(v)\neq v$, i.e. $\psi(v)-v$ is anisotropic vector, and by U' the open subvariety in $\mathbf{U}(V,h)$ consisting of all φ, such that $\varphi(v)-v$ is anisotropic. By Lemma 5.1,

$$U' = I' \cdot \mathbf{U}(V',h') \simeq I' \times \mathbf{U}(V',h'),$$

i.e. the restriction of π on I' yields an isomorphism between I' and the image of U' in X. □

Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \to & \mathbf{SU}(V',h') & \longrightarrow & \mathbf{SU}(V,h) & \xrightarrow{\rho} & X & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \mathbf{U}(V',h') & \longrightarrow & \mathbf{U}(V,h) & \xrightarrow{\pi} & X & \to & 1.
\end{array}$$

By Lemma 5.2, the map π splits rationally. We would like to show that ρ is also rationally split. Let E be the function field of X. Since π splits, there is $\varphi \in U(V_E, h_E)$ such that $\pi_E(\varphi)$ is the generic point in $X(E)$. By Proposition 4.2, the image of the reduced norm homomorphism

$$\mathrm{Nrd} : U(V_E, h_E) \to (L \otimes_F E)^\times$$

equals $(1-\varepsilon)\,\mathrm{Nrd}(B_E) = (1-\varepsilon)\,\mathrm{Nrd}(D_E)$ (here ε is the generator of the Galois group of $L \otimes_F E/E$) and hence does not depend on the hermition form h. In particular, the reduced norm homomorphism

$$\mathrm{Nrd} : U(V'_E, h'_E) \to (L \otimes_F E)^\times$$

has the same image. Therefore, we can modify φ by an element from $U(V'_E, h'_E)$ such that $\mathrm{Nrd}(\varphi) = 1$, i.e. $\varphi \in SU(V_E, h_E)$ and hence ρ splits rationally. In particular, $\mathbf{SU}(V,h)$ is birationally isomorphic to $\mathbf{SU}(V',h') \times X$.

Lemma 5.3 *The variety $\mathbf{SU}(V,h)$ is birationally isomorphic to the product $\mathbf{SU}(V',h') \times \mathbf{A}_F^k$ for some k.*

Proof. By Lemma 5.2, it suffices to show that I is a rational variety. Denote by $\mathbf{P}_D(V)$ the subvariety of 1-dimensional D-subspaces in V. This is a rational projective homogeneous variety being a twisted form of a Grassmannian variety (see [10]). Consider the dominant morphism $\mu : I \to \mathbf{P}_D(V)$, taking a reflection ψ to the image of $\psi - \mathrm{id}$. A fiber over anisotropic 1-dimensional D-subspace $V_1 \subset V$ is isomorphic to the unitary group $\mathbf{U}(V_1, h_1)$, where h_1 is the restriction of h on V_1. Since the variety of the unitary group is rational, the generic fiber of μ and hence I is also rational. □

Theorem 5.4 *There is an involution σ of the second kind on D, such that $\mathbf{SU}(V,h)$ is birationally isomorphic to $\mathbf{SU}(D,\sigma) \times \mathbf{A}_F^r$ for some r.*

Proof. By Lemma 5.3 and induction, it suffices to consider the case $\dim(V) = 1$. But in this case $\mathbf{SU}(V,h) = \mathbf{SU}(D,\sigma)$ for a certain involution σ on D of the second kind. □

Corollary 5.5 *Let B be a central simple L-algebra with an involution τ of the second kind trivial on F. If* $\mathrm{ind}(B) \le 3$*, then the variety of the group* $\mathbf{SU}(B,\tau)$ *is rational.*

Proof. By the Wedderburn theorem, $B \simeq \mathrm{End}_D(V)$, where D is a central skewfield over L and V is a vector space over D. Then the involution τ is isomorphic to the adjoint involution with respect to a certain non-degenerate hermitian form h on V ([3], [4]). By Theorem 5.4, it suffices to show that the group $\mathbf{SU}(D,\sigma)$ is rational. But the latter is a semisimple group of rank at most 2, and hence is rational. □

Remark 5.6 Since $\mathbf{SU}(B,\tau)_L \simeq \mathbf{SL}_{1,B}$, it follows from [7] that the group $\mathbf{SU}(B,\tau)$ is not rational if $\mathrm{ind}(B)$ is divisible by 4.

Corollary 5.7 *Let (B,τ) and (B',τ') be central simple L-algebras with involutions of the second kind. If B and B' are Brauer equivalent algebras, then the groups* $\mathbf{SU}(B,\tau)$ *and* $\mathbf{SU}(B',\tau')$ *are stably birationally isomorphic.*

Proof. By Theorem 5.4, it suffices to show that the groups $\mathbf{SU}(D,\sigma)$ and $\mathbf{SU}(D,\sigma')$, where σ and σ' are involutions of the second kind on a skewfield D, are stably birationally isomorphic. We have $\sigma' = \mathrm{Int}(d) \circ \sigma$ for some σ-symmetric invertible element $d \in D$ ([3], [4]). Consider 2-dimensional D-space $V = D \oplus D$ and hermitian diagonal form $h = \langle 1, d \rangle$ on V. By Lemma 5.3, the groups $\mathbf{SU}(D,\sigma)$ and $\mathbf{SU}(D,\sigma')$, being subgroups in $\mathbf{SU}(V,h)$, are stably birationally isomorphic to this group. □

Remark 5.8 Corollary shows that the stable birational type of the group variety $\mathbf{SU}(B,\tau)$ depends only on the Brauer class of B and does not depend on the involution τ.

References

[1] A. Borel. Linear Algebraic Groups, Second Enlarged Edition. *Graduate Texts in Mathematics.* Vol. 126, Springer, Berlin, 1991.

[2] J.-L. Colliot-Thélène, J.-J. Sansuc. La R-équivalence sur les tores. *Ann. Scient. Éc. Norm. Sup., 4^e série.* **10**, (1977), 175–230.

[3] M.-A. Knus. *Quadratic and Hermitian Forms over Rings.* Grundlehren math. Wiss. 294, Springer-Verlag, Berlin, 1991.

[4] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol. *Book of involutions,* in preparation.

[5] T.-Y. Lam. *The Algebraic theory of Quadratic Forms.* Benjamin, Reading, Massachusetts, 1973.

[6] Yu.I. Manin. *Cubic forms.* North-Holland, Amsterdam, 1974.

[7] A.S. Merkurjev. Generic element in SK_1 for simple algebras. *K-theory.* **7** (1993), 1–3.

[8] A.S. Merkurjev. Norm principle for algebraic groups, *St. Petersburg Math.J.* **7** (1996), 243–264.

[9] A.S. Merkurjev. R-equivalence and rationality problem for semisimple adjoint classical algebraic groups, to appear in *Publ. Math. I.H.E.S.*

[10] A.S. Merkurjev, I.A. Panin, A. Wadsworth. Index reduction formulas for twisted flag varieties. Preprint, 1995.

[11] J. Milnor. *Introduction to algebraic K-theory.* Ann. Math. Stud., N 72, Princeton, 1971.

[12] A.P. Monastyrnyĭ, V.I. Yanchevskiĭ. Whitehead groups of spinor groups. *Math. USSR Izvestija* **36** (1991), 61–100.

[13] W. Scharlau. *Quadratic and Hermitian Forms.* Grundlehren math., Wiss. 270, Springer-Verlag, Berlin, 1985.

[14] V.E. Voskresenskiĭ. *Algebraic tori*, Nauka, Moscow, 1977 (Russian).

[15] A. Weil. Algebras with involutions and the classical groups. *J. Indian Math. Soc.* **24** (1960), 589–623.

[16] V.I. Yanchevskiĭ. Simple algebras with involution and unitary groups. *Math. Sbornik* **22** (1974), 372–385.

[17] V.I. Yanchevskiĭ. Reduced unitary K-theory. Applications to algebraic groups. *Math. Sbornik* **38** (1981), 533–561.