ON A PROBLEM OF H. BASS FOR CHEVALLEY GROUPS OF TYPE E_7

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Abstract. This paper is devoted to the study of the surjective stability of the K_1–functor for Chevalley groups of type E_7. This is a particular case of the stability problem for Chevalley groups, which was posed by H. Bass. In the paper, surjective stability of the K_1–functor is proved under natural conditions on the dimension of the maximal spectrum of a ring and, independently, under a special condition which is meaningful from the point of view of equations defining a Chevalley group.

Introduction

The problem of K_1–functor stability for semisimple algebraic groups was mentioned by H. Bass in the book, "Algebraic K–theory"[Ba2, p.278]. He proposed to formulate stability hypotheses in terms of relations between the dimension of the maximal spectrum of a ring and the rank of a maximal split torus of a group.

In the framework of Chevalley groups, this problem was studied by M. Stein [S1],[S2] who developed an approach that considers, in the same way, both classical and exceptional cases. Despite this, however, the problem is still open. It consists of two, in fact, independent tasks. These are surjective and injective stability questions.

Traditionally, injective stability was considered as a more difficult question. However, recently it has been completely investigated. N.A. Vavilov has shown that *injective stability of K_1–functor for all regular embeddings of root systems holds under natural (depending on embedding) conditions* on **the stable rank of a ring**. This result is based on the Suslin–Tulenbaev proof [ST] of the Dennis–Vasershtein decomposition [D1],[D2],[V2].

For surjective stability the situation is as follows. It is known [Ba1],[V1],[BMS] that for the embeddings $A_{n-1} \to A_n$, $C_{n-1} \to C_n$, surjective stability holds under stable rank conditions. However, the cases of the orthogonal groups $B_{n-1} \to B_n$, $D_{n-1} \to D_n$ require stronger conditions on a ring [S2],[V2]. These are either the absolute stable rank condition ASR_n [S2], or some special condition V_n, introduced by L.Vaserstein [V2] for orthogonal and unitary groups, or just the condition on the maximal spectrum of a ring $\mathrm{dimMax}(R)$. Absolute stable rank condition turns out to be sufficient also for some other cases of Chevalley groups embeddings ($A_1 \to G_2$, $B_3, C_3 \to F_4$, $D_5 \to E_6$, [S1],[S2], [P1],[P2]).

The aim of this paper is to prove surjective stability of K_1–functor for the embedding of Chevalley groups of the type $E_6 \to E_7$ under some natural conditions on the ring. Then, for the embeddings of root systems of the same type, the only case that remains to be studied is $E_7 \to E_8$.

§ 1. Preliminaries

Let Φ be a reduced irreducible root system of rank l, let $G(\Phi,\ )$ be a simply connected Chevalley-Demazure group scheme over $\mathbb{Z}$ of type Φ (see [Ch],[DG]); and let $T(\Phi,\ )$ be a split maximal torus in it. If R is a commutative ring with 1, the value of the functor $G(\Phi,)$ on R is called the simply connected Chevalley group of type Φ over R and is denoted by $G(\Phi, R)$. To each root $\alpha \in \Phi$ there correspond elementary (with respect to T) root unipotent elements $x_\alpha(\xi)$, $\xi \in R$. All the elementary unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, generate a group $E(\Phi, R) = \langle x_\alpha(\xi), \alpha \in \Phi, \xi \in R\rangle$, which is called the elementary subgroup of $G(\Phi, R)$.

It is well-known that if Φ is an irreducible root system of rank $l \geq 2$, then $E(\Phi, R)$ is always *normal* in $G(\Phi, R)$ (see [Su], [SK], [Ko] for the classical groups and [Ta1], [Ta2],[A2],[V3],[Va1] for the Chevalley groups).

Thus, the K_1–functor of Chevalley group $G(\Phi, R)$ is naturally defined as the quotient group

$$\mathrm{K}_1(\Phi, R) = G(\Phi, R)/E(\Phi, R),$$

(see [S1] – [S3], [A1]).

Any inclusion of root systems $\Delta \subset \Phi$ induces the homomorphisms of groups $G(\Delta, R) \to G(\Phi, R)$, $E(\Delta, R) \to E(\Phi, R)$ taking roots into roots, and the homomorphism of the corresponding K_1–functors $\nu\colon K_1(\Delta, R) \to K_1(\Phi, R)$.

The surjective stability question is *to find conditions on the ring R, depending on $\Delta \subset \Phi$, which provide surjectivity of the homomorphism ν.*

Since we consider surjective stability for not necessary infinite series of Chevalley groups, it is better to reformulate this question as a question of the conditions on R which provide the existence of the decomposition $G(\Phi, R) = E(\Phi, R)G(\Delta, R)$.

Denote by $\langle X\rangle$ the ideal generated by X if $X \subset R$; the subgroup generated by X if X is a subset of a group; the minimal closed subsystem of roots containing X if $X \subset \Phi$.

We say that an element $e \in \mathrm{E}(\Phi, R)$ preserves the ideal $\mathfrak{a} = \langle a_1, \dots , a_i\rangle$ if $\langle (ae)_1, \dots , (ae)_i\rangle = \mathfrak{a}$, where $a = (a_1, \dots , a_i)$ and $i \leq n$, n is the dimension of a representation π of the group $E(\Phi, R)$.

Recall that a commutative ring R satisfies the absolute stable rank condition ASR_n if for any row $(r_1, \dots , r_n)$ with coordinates in R, there exist elements $t_1, \dots , t_{n-1} \in R$, such that every maximal ideal of R containing the ideal $\langle r_1 + t_1r_n, \dots , r_{n-1} + t_{n-1}r_n\rangle$ contains also the ideal $\langle r_1, \dots , r_n\rangle$. This notion was introduced in [EO] and used in [S1],[S2] and then in [P1],[P2] for stability problems. The description of various properties of ASR_n, as well as numerous applications (for not necessary commutative rings), can be found in [MVkV].

It was mentioned by A. Bak that the absolute stable rank condition ASR_n is equivalent to the following one. Let $(r_1, \dots, r_{n+1})$ be a left unimodular vector. Then there are elements $t_1, \dots, t_{n-1} \in R$, such that $(r_1 + t_1 r_n, \dots, r_{n-1} + t_{n-1} r_n, r_{n+1} + t r_n)$ is unimodular for any $t \in R$.

If we assume that a row $(r_1, \dots, r_n)$ is unimodular then the absolute stable rank condition is transformed into stable rank condition [Ba1],[V1]. Absolute stable rank condition satisfies the usual properties, namely for every ideal $I \lhd R$ it may be lifted to R/I, and if $n \geq m$, then ASR_m implies ASR_n. Finally, it is well-known that if the dimension of the maximal spectrum $\mathrm{dimMax}(R)$ is $n-2$, then both the conditions ASR_n and SR_n are fulfilled [EO],[S2],[MVkV].

§ 2. Weight diagrams and basic representations

Let us fix an order on Φ, and let Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the sets of positive, negative, and fundamental roots respectively. Our numbering of the fundamental roots follows that of [Bu1]. By $\overline{\omega}_1, \dots, \overline{\omega}_l$ one denotes the corresponding fundamental weights. Let $W = W(\Phi)$ be the Weyl group of the root system Φ, i.e., the group generated by the set of fundamental reflections $w_{\alpha_1}, \dots, w_{\alpha_l}$.

Recall that an irreducible representation π of the complex semi-simple Lie algebra L is called *basic* [M] if the Weyl group $W = W(\Phi)$ acts transitively on the set $\Lambda(\pi)$ of non-zero weights of the representation π. This is equivalent to saying that if for any two non-zero weights λ, ν their difference is a fundamental root $\alpha = \lambda - \nu$, then $w_\alpha \lambda = \nu$ for the corresponding fundamental reflection $w_\alpha \in W$.

In this paper we can restrict ourselves to the basic representations without zero weight. In this case, $\Lambda(\pi)$ is the set of all weights of π, and the weights $\Lambda(\pi)$ form the one Weyl orbit. Such representations are called *microweight* or *minuscule* representations, and the list of these representations is very well-known (see [Bu2]).

To each complex representation π of a simple Lie algebra L of type Φ there corresponds a representation π of the Chevalley group $G = G(\Phi, R)$ on the free R–module $V = V_R = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ ([M],[Sb]). We identify G with its image $\pi(G) = \mathrm{G}_\pi(\Phi, R)$ under this representation and omit the symbol π in the action of G on V. Thus, for an $x \in \mathrm{G}$ and $v \in V$ we write xv for $\pi(x)v$. If we want to specify that the group $G = G(\Phi, R)$ is considered in the basic representation π with the highest weight μ, the notation $(G(\Phi, R), \mu)$ is used. In the sequel, μ always stands for the highest weight of representation.

Decompose the module V into the direct sum of its weight submodules

$$V = \oplus \sum V^\lambda, \quad \lambda \in \Lambda(\pi).$$

It follows from the definition of microweight representation that all V^λ, $\lambda \in \Lambda(\pi)$ are one-dimensional. H. Matsumoto [M, Lemma 2.3] has shown that there is a special base of weight vectors $v^\lambda \in V^\lambda$, $\lambda \in \Lambda(\pi)$, such that the action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is described by the following simple formulae:

$$\begin{array}{llll} \textbf{i.} & \text{if} \quad \lambda\in\Lambda^*(\pi),\ \lambda+\alpha\notin\Lambda(\pi), & \text{then} & x_\alpha(\xi)v^\lambda=v^\lambda; \qquad (*)\\ \textbf{ii.} & \text{if} \quad \lambda,\lambda+\alpha\in\Lambda^*(\pi), & \text{then} & x_\alpha(\xi)v^\lambda=v^\lambda\pm\xi v^{\lambda+\alpha}; \end{array}$$

For any $v\in V$ in the chosen base, we have $v=\sum c_\lambda v^\lambda$, $\lambda\in\Lambda(\pi)$, and Matsumoto's lemma provides explicit formulae for the action of $x_\alpha(\xi)$ on v and on its coordinates c_λ.

Now if $g\in G=G(\Phi,R)$, then the ν-th column $g_{*,\nu}$ of the corresponding matrix $\pi(g)\in(G(\Phi,R),\mu)$, where μ is the highest weight of π, consists of the coefficients in the expansion of $\pi(g)e^\nu$ with respect to e^λ, $\lambda\in\Lambda(\pi)$. We may conceive any element $g\in G$ as a *matrix* $g=g_{\lambda,\nu}$, where λ and ν range over all the weights of the representation π. Then the columns above are obtained by freezing the second index in such a matrix. These columns may be identified with the corresponding elements of V. Analogously, the rows $g_{\lambda,*}$ are obtained by freezing the first index and correspond to the vectors from the dual module V^*.

As a rule, in stability questions all the calculations use only one (say, the first) column of the matrix. We denote by $\lambda(g)$ the λ–th coordinate of the first column $g_{\lambda\mu}$, $\lambda\in\Lambda(\pi)$, of the matrix g. As we know from the formulae $(*)$, one can very efficiently perform calculations with such columns (and rows) and calculate $x_\alpha(\xi)g$.

Moreover, we can obtain further simplification of calculations, using the machinery of weight diagrams (see [PSV],[S2],[Va1-Va3],[PR], [Ho], etc., for the detailed description of these diagrams and references). Let us recall here the corresponding definitions.

It is well-known that a choice of a fundamental system Π defines a partial order of the weight lattice $P(\Phi)$ as follows: $\lambda\geq\nu$ if and only if $\lambda-\nu$ is a linear combination of the fundamental roots with non-negative integral coefficients. Let us associate with a representation π a graph which is in fact the Hasse diagram of the set $\Lambda(\pi)$ of its weights with respect to the above order.

We construct a labeled graph in the following way. Its vertices correspond to the weights $\lambda\in\Lambda(\pi)$ of the representation π, and the vertex corresponding to λ is actually marked by λ (usually these labels are omitted).

We read the diagram from right to left and from bottom to top, which means that a larger weight tends to stand to the left of and higher than a smaller one, with the landscape orientation being primary. The leftmost vertex corresponds to the highest weight μ of a representation.

The vertices corresponding to the weights $\lambda,\nu\in\Lambda(\pi)$ are joined by a bond marked α_i (or simply i) if and only if their difference $\lambda-\nu=\alpha_i\in\Pi$ is a fundamental root. We draw the diagrams in such a way that the marks on the opposite sides of a parallelogram are equal, and, in that case, at least one of them is omitted. Thus, all paths of minimal length connecting two vertices have the same sum of labels and correspond to one root.

Now one may conceive a vector $v\in V$ as such a weight diagram which has an element of R attached to every node. A standard weight vector e^λ has 1 in the λ-th node and zeros elsewhere; an arbitrary vector v has its λ-th coordinate v^λ with respect to this weight base as the label at the λ-th node. The above-mentioned Matsumoto's lemma gives a very simple rule describing what happens with such a vector v under the action of $x_\alpha(\xi)$. If π is minuscule and the root $\alpha = \alpha_i$ is fundamental, then $x_\alpha(\xi)$ adds or subtracts (*always* adds for a clever choice of the weight base) ξv^λ to v^ν along each edge labeled with i. For other roots one merely has to trace all paths in the diagram which have the same labels at their edges as the root α in its linear expansion with respect to the fundamental roots. For example, if $\alpha = 2\alpha_1 + \alpha_2$, one has to look at the paths which have the labels 1,1,2, in any order (the order of the labels on such a path starting in λ together with the structure constants of the Lie algebra is responsible for the *sign* with which $x_\alpha(\xi)$ acts on v^λ).

In this paper we can use only microweight representations; that is we do not care about weight diagrams with zero weights. The details of how to construct and operate with weight diagrams in case of presence of zero weight can be found in [Va1],[PSV]. Weight diagrams for the aims of stability of K_1 and K_2–functors were introduced by M.Stein in [S2].

§ 3. Stability theorems for $\mathrm{E}_6 \to \mathrm{E}_7$

Stability of the K_1–functor is closely related to the fact known as the Chevalley–Matsumoto decomposition theorem (see [Ch],[M],[S2]). Let us formulate the particular case of this theorem used for the stability problem.

Consider a basic representation π of the group $G(\Phi, R)$ with the highest weight μ. Denote by $\alpha_k \in \Pi$ the fundamental root, such that $\mu - \alpha_k$ is a weight, and by Δ the subsystem in Φ generated by all fundamental roots except α_k. Further, let $\Sigma = \Phi \setminus \Delta$, $\Sigma^+ = \Phi^+ \cap \Sigma$, $\Sigma^- = \Phi^- \cap \Sigma$ and

$$U(\Sigma, R) = \langle x_\alpha(\xi), \alpha \in \Sigma^+, t \in R\rangle,$$

$$V(\Sigma, R) = \langle x_\alpha(\xi), \alpha \in \Sigma^-, t \in R, \rangle.$$

Now take a matrix $g \in G(\Phi, R), \mu)$ and suppose that $g_{\mu\mu}$ is an invertible element in R. Then, the Chevalley–Matsumoto theorem states that g can be expressed in the form

$$g = vg_1u,$$

where $v \in V(\Sigma, R)$, $u \in U(\Sigma, R)$, $g_1 \in T(\Phi, R)G(\Delta, R)$ and all the factors are uniquely determined. Moreover, if $g_{\mu\mu} = 1$ then $g_1 \in G(\Delta, R)$.

The set Φ is the disjoint union of sets:

$$\Phi = \Sigma^+ \cup \Delta \cup \Sigma^-,$$

and, in other words, the Chevalley–Matsumoto theorem says that if $g_{\mu\mu}$ is invertible, then an element g of Chevalley group $G(\Phi, R)$ can be expressed as a product of the element from a Levy factor of the proper parabolic subgroup $G(\Delta, R)$ and two factors from the unipotent radicals of this parabolic subgroup and its opposite ([Va1]).

Thus, the Chevalley–Matsumoto theorem yields, that getting by elementary transformations a unit of the ring in the left corner of the matrix $g \in (G(\Phi, R), \mu)$, we get the surjective stability of the K_1–functor for the embedding $\Delta \to \Phi$, where Δ and Φ are of the same type.

Therefore, the problem is to transform a matrix $g \in (G(\Phi, R), \mu)$ using multiplications by elements $e \in (E(\Phi, R), \mu)$ to a matrix with the invertible element $g_{\mu\mu}$.

Let us agree to use the notation $e \in E(\Delta \to \Phi, R)$ for the element $e \in E(\Delta, R) \subset G(\Phi, R)$ and the notation $e \in (E(\Delta \to \Phi, R), \mu)$ if we want to specify the representation.

Denote by $Eq(\Phi, \mu, \nu)$ a set of equations which are satisfied by the elements of the ν–th column of the matrix $g \in (G(\Phi, R), \mu)$. In other words, this is a generating set of the ideal of all such equations. Since for microweight representations all columns of a matrix are equivalent, we can suppress ν in the notation and write $Eq(\Phi, \mu)$. Thus, $Eq(\Phi, \mu)$ is the set of equations which determine the orbit of the highest weight vector v^μ in the representation with the highest weight μ. It is known (see [Li],[Va4]) that this set consists of quadratic equations which are characteristic-free, and, therefore, they are the same for rings and fields.

Now we can define three types of vectors. Denote by $Um_\mu(R, \Phi)$ the set of unimodular columns (rows) of the length $n = \dim \pi$; by $Um'_\mu(R, \Phi)$ the set of unimodular columns (rows) of the length n which satisfy the set of equations $Eq(\Phi, \mu)$; and by $Um''_\mu(R, \mu)$ the set of unimodular columns (rows) of the length n which can be completed up to a matrix $g \in (G(\Phi, R), \mu)$. It is clear that $Um''_\mu(R, \Phi) \subset Um'_\mu(R, \Phi) \subset Um_\mu(R, \Phi)$. If $R = K$ is a field then, obviously, $Um''_\mu(K, \Phi) = Um'_\mu(K, \Phi)$, but, in fact, the difference between $Um'_\mu(R, \Phi)$ and $Um''_\mu(R, \Phi)$ is measured by the corresponding K–functor.

The following result can be derived from the proof of the Chevalley-Matsumoto theorem.

Proposition 1. *Let $(g_1, ..., g_n) \in Um'_\mu(R, \Phi)$ and $g_1 \in R^*$. There exists an element $e \in (E(\Phi, R), \mu)$, such that $(eg)_i = 0$, $i \neq 1$.*

Consider now the weight diagram of the type (E_7, ω_7), i.e., the diagram of the microweight representation of $G(E_7, R)$ with the highest weight ω_7 (see Figure 3). Let us number the weights of the representation according to Figure 4.

Lemma 1. *Let $(g_1, \dots, g_i, \dots, g_{-i}, \dots, g_{-1}) \in Um'_{\omega_7}(R, E_7)$, $i = 1, \dots, 28$. Suppose $g_1 = 0$. Then $(g_1, \dots, g_{28}) \in Um'_{\omega_6}(R, E_6)$.*

Proof. The proof immediately follows from the full description of the sets $Eq(\Phi, \mu)$ in [As],[Co],[Va4]. Here we consider some particular approach.

The algebraically independent equations of the set $Eq(E_7, \omega_7)$ can be obtained explicitly by using the Chevalley-Matsumoto theorem.

Take an element $g \in G(\mathrm{E}_7, R))$ with the invertible entry $g_{\mu\mu}$. There exists an element $e \in E(\mathrm{E}_7, R)$, $e = \prod x_\alpha(\xi)$, $\alpha \in \mathrm{E}_7$, $\xi \in R$, such that $eg \in G(\mathrm{E}_6, R)$. This means that $(eg)_{\lambda\mu} = 0$ for all $\lambda \neq \mu$. In particular, the matrix entries of eg corresponding to the weights λ such that $\mu - \lambda$ are not roots, should also be zeroes. Using diagrams 3 and 4 we see that these are exactly $(eg)_{i\mu}$, $i = -28, \dots, -1$. It remains to notice, that multiplying g by e, we can get zeroes only for $(eg)_{i\mu}$, $i = 1, \dots, 28$, which means that all other matrix entries are zeroes according to the defining equations.

Straightforward calculations of the entries $(eg)_{i\mu}$, $i = -28, \dots, -1$ by formulae $(*)$ and Figure 3 yield that up to the choice of signs, which is immaterial in this case (and which can be determined using the algorithm described in [Va3]), the set of algebraically independent equations in $Eq(\mathrm{E}_7, \omega_7)$ is the following

$$t_1t_{-28} \pm t_8t_5 \pm t_9t_4 \pm t_{10}t_3 \pm t_{11}t_2 \pm t_7t_6 = 0$$

$$t_1t_{-27} \pm t_{12}t_6 \pm t_{13}t_5 \pm t_{14}t_4 \pm t_{15}t_3 \pm t_{16}t_2 = 0$$

$$t_1t_{-26} \pm t_{12}t_8 \pm t_{13}t_7 \pm t_{17}t_4 \pm t_{18}t_3 \pm t_{19}t_2 = 0$$

$$t_1t_{-25} \pm t_{12}t_9 \pm t_{14}t_7 \pm t_{20}t_3 \pm t_{17}t_5 \pm t_{21}t_2 = 0$$

$$t_1t_{-24} \pm t_{12}t_{10} \pm t_{15}t_7 \pm t_{18}t_5 \pm t_{20}t_4 \pm t_{22}t_2 = 0$$

$$t_1t_{-23} \pm t_{12}t_{11} \pm t_{16}t_7 \pm t_{19}t_5 \pm t_{21}t_4 \pm t_{22}t_3 = 0$$

$$t_1t_{-22} \pm t_{13}t_9 \pm t_{14}t_8 \pm t_{17}t_6 \pm t_{23}t_3 \pm t_{24}t_2 = 0$$

$$t_1t_{-21} \pm t_{13}t_{10} \pm t_{15}t_8 \pm t_{18}t_6 \pm t_{23}t_4 \pm t_{25}t_2 = 0$$

$$t_1t_{-20} \pm t_{13}t_{11} \pm t_{16}t_8 \pm t_{24}t_4 \pm t_{25}t_3 \pm t_{19}t_6 = 0$$

$$t_1t_{-19} \pm t_{14}t_{10} \pm t_{15}t_9 \pm t_{20}t_6 \pm t_{23}t_5 \pm t_{26}t_2 = 0$$

$$t_1t_{-18} \pm t_{14}t_{11} \pm t_{16}t_9 \pm t_{21}t_6 \pm t_{24}t_5 \pm t_{26}t_3 = 0$$

$$t_1t_{-17} \pm t_{15}t_{11} \pm t_{16}t_{10} \pm t_{22}t_6 \pm t_{25}t_5 \pm t_{26}t_4 = 0$$

$$t_1t_{-16} \pm t_{17}t_{10} \pm t_{18}t_9 \pm t_{20}t_8 \pm t_{23}t_7 \pm t_{27}t_2 = 0$$

$$t_1t_{-15} \pm t_{17}t_{11} \pm t_{19}t_9 \pm t_{21}t_8 \pm t_{24}t_7 \pm t_{27}t_3 = 0$$

$$t_1t_{-14} \pm t_{18}t_{11} \pm t_{19}t_{10} \pm t_{22}t_8 \pm t_{25}t_7 \pm t_{27}t_4 = 0$$

$$t_1t_{-13} \pm t_{20}t_{11} \pm t_{21}t_{10} \pm t_{22}t_9 \pm t_{26}t_7 \pm t_{27}t_5 = 0$$

$$t_1t_{-12} \pm t_{23}t_{11} \pm t_{24}t_{10} \pm t_{25}t_9 \pm t_{26}t_8 \pm t_{27}t_6 = 0$$

$$t_1t_{-11} \pm t_{17}t_{15} \pm t_{18}t_{14} \pm t_{20}t_{13} \pm t_{28}t_2 \pm t_{23}t_{12} = 0$$

$$t_1t_{-10} \pm t_{16}t_{17} \pm t_{19}t_{14} \pm t_{21}t_{13} \pm t_{24}t_{12} \pm t_{28}t_3 = 0$$

$$t_1t_{-9} \pm t_{18}t_{16} \pm t_{19}t_{15} \pm t_{22}t_{13} \pm t_{25}t_{12} \pm t_{28}t_4 = 0$$

$$t_1t_{-8} \pm t_{20}t_{16} \pm t_{21}t_{15} \pm t_{22}t_{14} \pm t_{26}t_{12} \pm t_{28}t_5 = 0$$

$$t_1t_{-7} \pm t_{23}t_{16} \pm t_{24}t_{15} \pm t_{25}t_{14} \pm t_{26}t_{13} \pm t_{28}t_6 = 0$$

$$t_1t_{-6} \pm t_{20}t_{19} \pm t_{21}t_{18} \pm t_{22}t_{17} \pm t_{27}t_{12} \pm t_{28}t_7 = 0$$

$$t_1t_{-5} \pm t_{23}t_{19} \pm t_{24}t_{18} \pm t_{25}t_{17} \pm t_{27}t_{13} \pm t_{28}t_8 = 0$$

$$t_1t_{-4} \pm t_{23}t_{21} \pm t_{24}t_{20} \pm t_{26}t_{17} \pm t_{27}t_{14} \pm t_{28}t_9 = 0$$

$$t_1t_{-3} \pm t_{23}t_{22} \pm t_{25}t_{20} \pm t_{26}t_{18} \pm t_{28}t_{10} \pm t_{27}t_{15} = 0$$

$$t_1t_{-2} \pm t_{24}t_{22} \pm t_{25}t_{21} \pm t_{26}t_{19} \pm t_{27}t_{16} \pm t_{28}t_{11} = 0$$

This agrees with the results of [Se], [LS], that all equations of microweight representations come from the systems of type D_n. For modulo t_1 the equations above coincide with the set $Eq(\mathrm{E}_6, w_6)$ (see [CC],[Va3],[Va4]) – they are exactly 27 quadratic equations defining the orbit of the highest weight vector for the minimal representation of E_6.

Consider the diagram of microweight representation (E_6, ω_6)

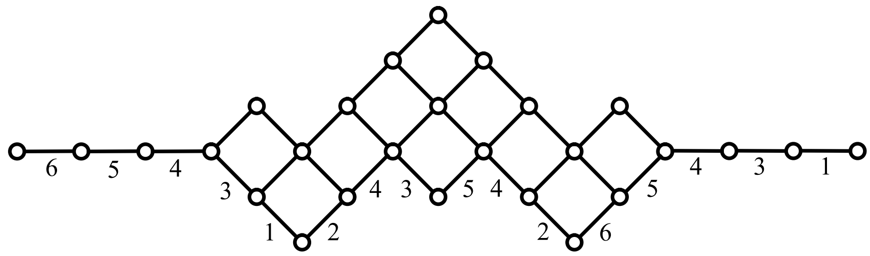

Figure 1. (E_6, ω_6)

with the following numbering of weights

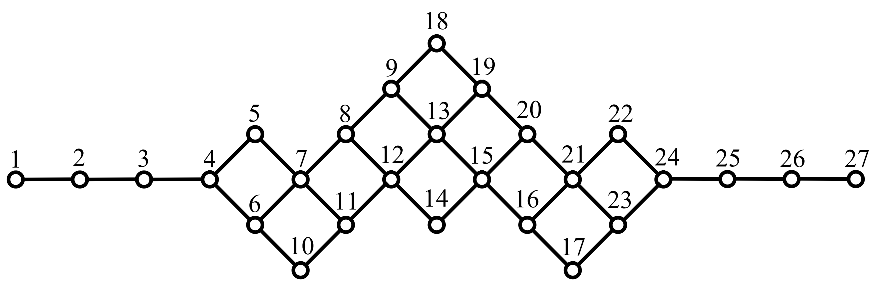

Figure 2. (E_6, ω_6)

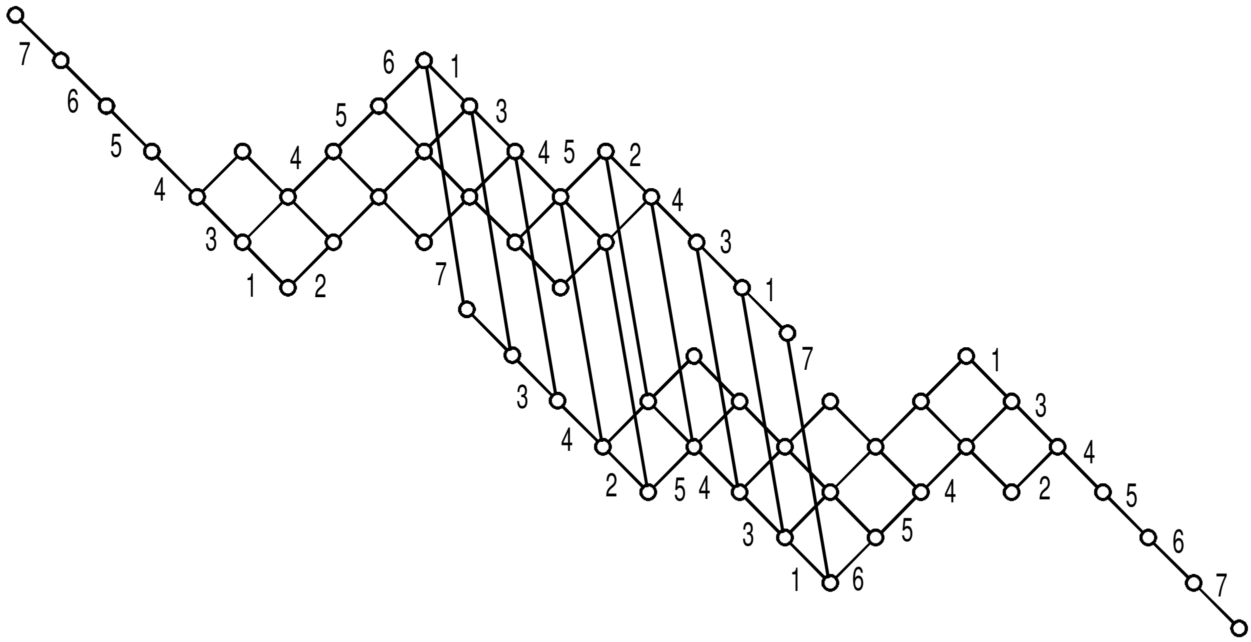

Figure 3. $(\mathrm{E}_7, \overline{\omega}_1)$

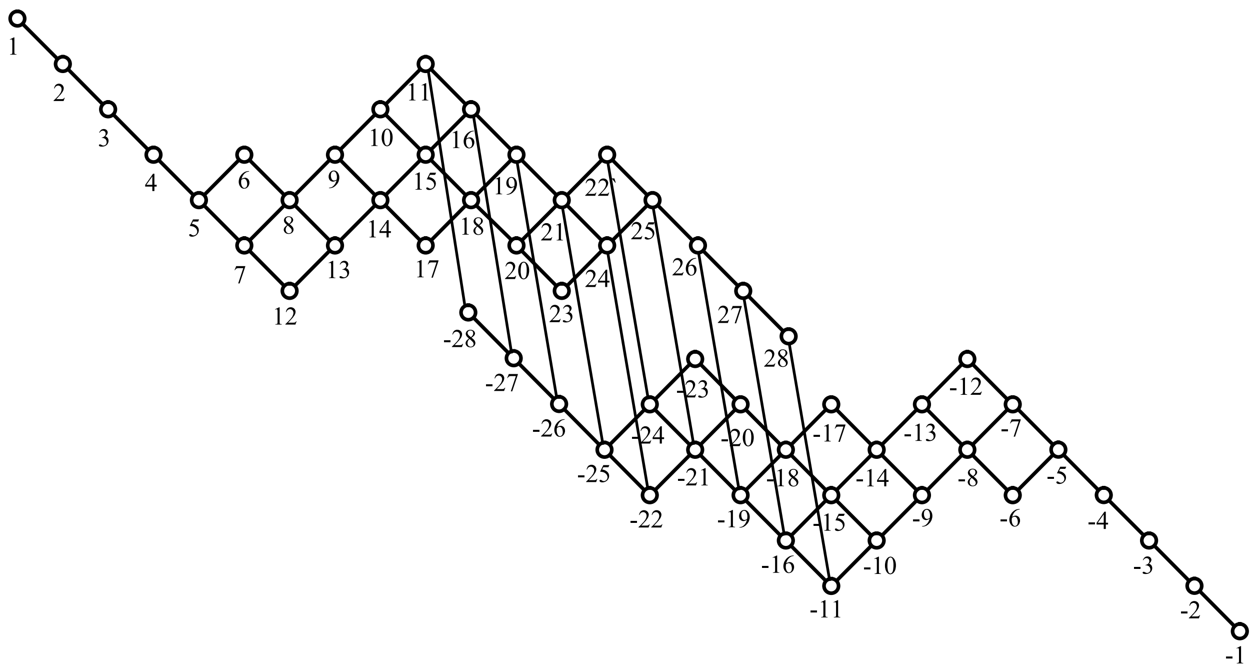

Figure 4. (E_7, ω_7)

Lemma 2. *Let the ring R satisfy the condition* $\mathrm{dimMax}(R) \le 4$. *Then for any row $a = (a_{\lambda_1}, \dots, \alpha_{\lambda_{27}}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$ there exists $e \in (E(\mathrm{E}_6, R), \omega_6)$, such that $\langle (ae)_{\lambda_1}, (ae)_{\lambda_{18}} \rangle = R$.*

Proof. Denote by Γ the set of weights λ_i, such that $\mu - \lambda_i \notin \Phi$. It is easy to see that these are the weights λ_i, $i = 18, \dots, 27$.

Consider $a = (a_{\lambda_1}, \dots, \alpha_{\lambda_n}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$. Let us choose in each irreducible component A_i of the space MaxSpec (R) a maximal ideal $\mathfrak{u}_i$ which does not belong to other components, and set $\mathfrak{u} = \prod \mathfrak{u}_i$. The ring $R/\mathfrak{u}$ is semilocal and we can find an element $e \in (E(\mathrm{E}_6, R), \omega_6)$, such that $(ae)_{\lambda_1} \equiv 1(mod\ \mathfrak{u})$. Since $a = (a_{\lambda_1}, \dots, \alpha_{\lambda_{27}}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$, using Proposition 1 we can choose e, such that $(ae)_{\lambda_i} \equiv 0(mod\ \mathfrak{u})$, $i \ne 1$. Let $ae = a_1$. Applying the condition ASR_6 to the subrow $((a_1)_{\lambda_1}, \dots, (a_1)_{\lambda_4}, (a_1)_{\lambda_6}, (a_1)_{\lambda_{10}})$ of the row a_1, we can find $e_1 \in E(A_5 \to \mathrm{E}_6, R), \omega_1)$, such that the row $((e_1a_1)_{\lambda_2}, \dots, (e_1a_1)_{\lambda_{27}})$ is unimodular. Since $\alpha_2 \notin A_5 \to \mathrm{E}_6$ we have

$$(e_1a_1)_{\lambda_i} \equiv 0(mod\ \mathfrak{u}), \quad i = 5, 7, 8, 9, 11, \dots, 27.$$

Let $e_1a_1 = a_2$. Set $\mathfrak{a} = \langle (a_2)_{\lambda_i} \rangle$, $\lambda_i \in \Gamma$. The row $((a_2)_{\lambda_2}, \dots, (a_2)_{\lambda_{17}}$ is unimodular modulo ideal $\mathfrak{a}$. Thus, there exist $t_2, \dots, t_{17} \in R$, such that $t_2(a_2)_{\lambda_2} + \dots + t_{17}(a_2)_{\lambda_{17}} \equiv 1 - (a_2)_{\lambda_1}(mod\ \mathfrak{a})$. Since $\mu - \lambda_i \in \Phi$, $i = 2, \dots, 17$, we can find $e_2 \in E(\mathrm{E}_6, R), \omega_6)$ preserving $\mathfrak{a}$, such that $(a_2e_2)_{\lambda_1} \equiv 1(mod\ \mathfrak{a})$. Therefore, the row $((a_2e_2)_{\lambda_1}, (a_2e_2)_{\lambda_{18}}, \dots, (a_2e_2)_{\lambda_{27}})$ is unimodular. Besides, $(a_2e_2)_{\lambda_i} \equiv 0(mod\ \mathfrak{u})$, $i = 18, \dots, 27$. Set $\langle (e_2a_2)_{\lambda_1} \rangle = \mathfrak{m}$ and consider $R/\mathfrak{m}$. Since $\mathfrak{m} + \mathfrak{a} = R$ where $\mathfrak{a} \subset \mathfrak{u}$, standard argumentation with the multiplicative system $S = R \setminus \cup \mathfrak{u}_i$ ([Ba1],[V1],[V2]) shows that $\mathrm{dimMax}(R)/m \le 3$. Let $e_2a_2 = a_3$. The row $((a_3)_{\lambda_{18}}, \dots, (a_3)_{\lambda_{27}})$ is unimodular modulo $\mathfrak{m}$. It remains to mention that Theorem 2.1 of [S2] (case $\mathrm{D}_{n-1} \to \mathrm{D}_n$) does not require that a row $(a_1, \dots, a_n)$ belongs to $Um'_{\omega_1}(\mathrm{D}_n, R)$, but only that the group $G(\mathrm{D}_n, R)$ acts on $Um_{\omega_1}(\mathrm{D}_n, R)$. Since $\mathrm{dimMax}R = 3$ implies ASR_5, there exists $e_3 \in (E(\mathrm{D}_5 \to \mathrm{E}_6, R), \omega_1)$, such that $(e_3a_3)_{\lambda_{18}} \equiv 1(mod\ \mathfrak{m})$. At last, obviously, $(e_3a_3)_{\lambda_1} = (a_3)_{\lambda_1}$. Therefore, $\langle (e_3a_3)_{\lambda_1}, (e_3a_3)_{\lambda_{18}} \rangle = R$.

Theorem 1. *Let* $\mathrm{dimMax}(R) \le 4$. *Then the homomorphism*

$$\nu \colon K_1(\mathrm{E}_6, R) \to K_1(\mathrm{E}_7, R)$$

is surjective.

Proof. Let $g \in (G(\mathrm{E}_7, R), \omega_7)$. Then $(\lambda_1(g), \dots, \lambda_{-1}(g)) \in Um''_{\omega_7}(\mathrm{E}_7, R)$. Let

$$\mathfrak{a} = \langle (\lambda_6(g), \lambda_8(g), \dots, \lambda_{11}(g), \lambda_{13}(g), \dots, \lambda_{28}(g), \lambda_{-28}(g), \dots, \lambda_{-1}(g) \rangle.$$

The row $(\lambda_1(g), \dots, \lambda_5(g), \lambda_7(g), \lambda_{12}(g))$ is unimodular modulo $\mathfrak{a}$, and using SR_7 we can find $e \in (E(\mathrm{A}_6 \to E_7), R), \omega_1)$, such that $(\lambda_2(eg), \dots, \lambda_5(eg), \lambda_7(eg))$ is unimodular modulo $\mathfrak{a}$. Since e preserves the ideal $\mathfrak{a}$, the row $(\lambda_2(eg), \dots, \lambda_{-1}(eg))$ is unimodular.

Set $eg = g_1$. Now let $\mathfrak{a} = \langle \lambda_{-28}(g_1), \dots, \lambda_1(g_1)\rangle$. Then $\langle \lambda_2(g_1), \dots, \lambda_{28}(g_1)\rangle + \mathfrak{a} = R$. Thus, there exist $t_2, \dots, t_{28} \in R$, such that $\sum \lambda_i(g_1)t_i \equiv 1 - \lambda_1(g_1)(mod\, \mathfrak{a})$, $i = 2, \dots, 28$. Denote by Γ the set of weights λ_i, $i = 2, \dots, 28$. Let us set

$$e_1 = \prod x_{\beta_i}(t_i), \qquad \beta_i = \mu - \lambda_i, \quad \lambda_i \in \Gamma.$$

Applying formulae $(*)$ we have

$$\lambda_1(e_1g_1) = \lambda_1(g_1) + \sum_{i=2}^{28} \lambda_i(g_1)t_i + u,$$

where $u \in \mathfrak{a}$. Thus, $\lambda_1(e_1g_1) \equiv 1(mod\ \mathfrak{a})$ and the row

$$(\lambda_1(e_1g_1), \lambda_{28}(e_1g_1), \dots, \lambda_{-1}(e_1g_1))$$

is unimodular.

Let $g_2 = e_1g_1$ and $\mathfrak{a} = \langle \lambda_1(g_2), \lambda_{-1}(g_2)\rangle$. The row $(\lambda_{-28}(g_2), \dots, \lambda_{-2}(g_2))$ is unimodular modulo $\mathfrak{a}$. From the proof of surjective stability of the K_1–functor for the embedding $\mathrm{D}_5 \to \mathrm{E}_6$ (see [P1]) it follows that there exists $e_2 \in (E(\mathrm{E}_6 \to \mathrm{E}_7, R), \omega_6)$, such that $(\lambda_{-28}(e_2g_2), \lambda_{-11}(e_2g_2), \dots, \lambda_{-2}(e_2g_2))$ is unimodular modulo $\mathfrak{a}$. Since e_2 preserves $\mathfrak{a}$, the row

$$(\lambda_1(e_2g_2), \lambda_{-28}(e_2g_2), \lambda_{-11}(e_2g_2) \dots, \lambda_{-2}(e_2g_2), \lambda_{-1}(e_2g_2))$$

is unimodular. Then, the row

$$(\lambda_1(e_2g_2), \dots, \lambda_{11}(e_2g_2), \lambda_{28}(e_2g_2), \lambda_{-28}(e_2g_2), \lambda_{-11}(e_2g_2), \dots, \lambda_{-1}(e_2g_2))$$

is also unimodular.

Let $g_3 = e_2g_2$ and $\mathfrak{a} = \langle \lambda_1(g_3), \lambda_{11}(g_3), \lambda_{-28}(g_3)\rangle$. Take the row

$$(\lambda_{28}(g_3), \lambda_{-11}(g_3), \dots, \lambda_{-1}(g_3)).$$

Arguing as in [S2] and using the condition ASR_6, we can obtain $e_3 \in (E(\mathrm{D}_6 \to \mathrm{E}_7, R), \omega_1)$, such that

$$\lambda_{28}(e_3g_3) \equiv 1(mod\ \mathfrak{a}), \quad \lambda_{-11}(e_3g_3) \equiv 0(mod\ \mathfrak{a}), \dots, \lambda_{-2}(e_3g_3) \equiv 0(mod\ \mathfrak{a}).$$

Since e_3 preserves $\mathfrak{a}$, the row $(\lambda_1(e_3g_3), \dots, \lambda_{11}(e_3g_3), \lambda_{-28}(e_3g_3), \lambda_{28}(e_3g_3))$ is unimodular.

Let $g_4 = e_3g_3$. Let us apply condition ASR_6 to the elements

$$\lambda_1(g_4), \dots, \lambda_{11}(g_4), \lambda_{-28}(g_4).$$

There exist $t_7, \dots, t_{11} \in R$, such that every maximal ideal, containing the ideal

$$\langle \lambda_7(g_4) + t_7\lambda_{-28}(g_4), \dots, \lambda_{11}(g_4) + t_{11}\lambda_{-28}(g_4)\rangle,$$

also contains $\langle \lambda_7(g_4), \dots, \lambda_{-28}(g_4)\rangle$. Let $e_4 \in (E(\mathrm{A}_5 \to \mathrm{E}_7, R), \omega_1)$ be the corresponding element making these transformations. Since $\langle \lambda_1(g_4), \dots, \lambda_6(g_4)\rangle = \langle \lambda_1(e_4g_4), \dots, \lambda_6(e_4g_4)\rangle$, every maximal ideal $\mathfrak{n}$ containing $\langle \lambda_1(g_5), \dots, \lambda_4(g_5)\rangle$, where $g_5 = e_4g_4$, also contains the ideal $\mathfrak{a} = \langle \lambda_1(g_4), \dots, \lambda_{11}(g_4), \lambda_{-28}(g_4)\rangle$. Besides, $\lambda_{28}(g_5) \equiv 1(mod\ \mathfrak{a})$. Therefore, the row

$$(\lambda_1(g_5), \dots, \lambda_{11}(g_5), \lambda_{28}(g_5))$$

is unimodular. Thus, the row $(\lambda_1(g_5), \dots, \lambda_{28}(g_5))$ is unimodular.

Denote $\mathfrak{a} = \langle \lambda_1(g_5)\rangle$. By Lemma 1 we have

$$\lambda_2(\overline{g}_5), \dots, \lambda_{28}(\overline{g}_{28}) \in Um'_{\omega_6}(\mathrm{E}_6, R/\mathfrak{a}),$$

where $\overline{g}_i \in R/\mathfrak{a}$. Lemma 2 provides that there exists $e_5 \in (E(\mathrm{E}_6 \to \mathrm{E}_7, R), \omega_6)$, such that the row $(\lambda_2(e_5g_5), \lambda_{11}(e_5g_5))$ is unimodular modulo $\mathfrak{a}$. Since $\alpha_7 \notin \mathrm{E}_6 \to \mathrm{E}_7$, the element e_5 preserves $\mathfrak{a}$. Therefore, we have the unimodular row

$$(\lambda_1(e_5g_5), \lambda_2(e_5g_5), \lambda_{11}(e_5g_5)).$$

Let $g_6 = e_5g_5$. It is clear that there exists $e_6 \in (E(\mathrm{E}_7, R), \omega_7)$, such that $\lambda_3(e_6g_6) = 1$.

Now we introduce the condition on a ring which generalizes Vaserstein's condition used in the proof of surjective stability of the K_1–functor for the case of the ortogonal groups $B_{n-1} \to B_n$, $D_{n-1} \to D_n$ (see [Va2]).

Recall how Vaserstein's condition looks for the particular case $\Phi = D_n$. A ring R satisfies the condition V_n if for any unimodular row

$$(a_1, \dots, a_n, a_{-n}, \dots, a_{-1})$$

with elements in R there exist elements $t_1, \dots, t_n, t_{-n}, \dots .t_{-1}$ from R, such that

1. $\sum_{i=1}^{-1} t_ia_i = 1$,
2. $(t_1, \dots, t_n, t_{-n}, \dots, t_{-1}) \in Um'_{\omega_1}(D_n, R)$.

The second condition means that elements t_i, $i = 1, \dots, -1$ satisfy the equality $\sum_{i=1}^{n} t_it_{-i} = 0$.

Suppose that a ring R satisfies both the conditions SR_n and V_{n-1}. Then the proof of surjective stability for $D_{n-1} \to D_n$ goes as follows (compare [Va2]).

Consider the diagram of the representation (D_n, ω_1) with the natural numbering of weights

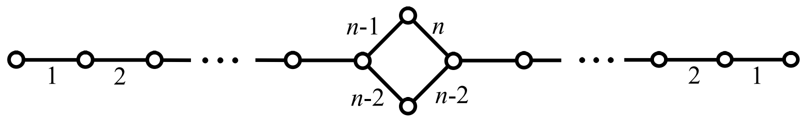

FIGURE 5. (D_n, ω_1)

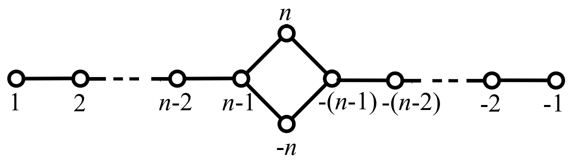

FIGURE 6. (D_n, ω_1)

Let $g \in (G(D_n, R), \omega_1)$ and $(g_1, \dots, g_n, g_{-n}, \dots, g_{-1})$ be its first row. Using the condition SR_n we can find elements $t_2, \dots, t_n \in R$, such that the row

$$(g'_2, \dots, g'_n, g'_{-n}, \dots, g'_{-1})$$

is unimodular, where

$$g'_i = g_i + t_i g_1, \quad i = 2, \dots, n,$$
$$g'_i = g_i, \quad i = -n, \dots, -2,$$
$$g'_{-1} = g_{-1} + \sum_{i=2}^{n} t_i g_{-i}.$$

Let $e \in (E(D_n, R), \omega_1)$ make these transformations and denote $g' = ge$. Analogously, using SR_n one can choose $t'_2, \dots, t'_n \in R$, such that the row

$$(g''_2, \dots, g''_n, g'_{-n}, \dots, g''_{-2})$$

is unimodular, where

$$g''_i = g'_i, \quad i = 2, \dots, n,$$
$$g''_i = g'_i + t'_i g_{-1}, \quad i = -n, \dots, -2,$$

Let element $e_1 \in (E(D_n, R), \omega_1)$ make these transformations and denote $g'' = g' e_1$. Applying V_{n-1} to the unimodular row

$$(g''_2, \dots, g''_n, g''_{-n}, \dots, g''_{-2})$$

one can find $t''_2, \dots, t''_n, t''_{-n}, \dots, t''_{-2}$, such that

1. $\sum_{i=2}^{-2} t''_i g''_i = 1 - g''_1$,
2. $(t''_2, \dots, t''_n, t''_{-n}, \dots, t''_{-2}) \in Um'_{\omega_1}(D_{n-1}, R)$.

Let $\beta_i = \omega_1 - \lambda_i$, $i = 2, \dots, -2$ and set

$$e_2 = \sum_{i=2}^{-2} x_{-\beta_i}(t_i).$$

Denote $g''' = g'' e_2$. Calculating g_1''', we have

$$g_1''' = g_1'' + \sum_{i=2}^{n} t_i'' g_i'' + \sum_{i=2}^{n} (g_{-i}'' + t_i'' g_{-1}'') t_{-i}'' =$$

$$g_1'' + \sum_{i=2}^{-2} t_i'' g_i'' + g_1'' \sum_{i=2}^{n} (t_i' t_{-i}'') = g_1'' + 1 - g_1'' = 1,$$

which finishes the proof.

Let n be the dimension of the basic representation π with the highest weight μ.

Definition. *We say that the ring R satisfies the condition $V_\mu(\Phi, R)$ if for each unimodular vector $(r_1, \dots, r_n)$ there exist elements $t_1, \dots, t_n \in R$, such that*
1.$\sum_{i=1}^n t_i r_i = 1$.
2. $(t_1, \dots, t_n) \in Um'_\mu(\Phi, R)$.

For the case $\Phi = D_n$ and $\mu = \omega_1$, this condition is converted to V_n. The following theorem combines the condition used by M.Stein and the generalized L.Vaserstein condition for the particular case of $E_6 \to E_7$ embedding.

Theorem 2. *Let the ring R satisfy the conditions ASR_6 and $V_{\omega_6}(E_6, R)$. Then the homomorphism*

$$\nu \colon K_1(E_6, R) \to K_1(E_7, R)$$

is surjective.

Proof. Let $g \in (G(E_7, R), \omega_7)$. Then $(\lambda_1(g), \dots, \lambda_{-1}(g)) \in Um''_{\omega_7}(E_7, R)$. Analogously to Theorem 1, one can get a unimodular row

$$(\lambda_1(g), \lambda_{-28}(g), \dots, \lambda_{-1}(g)),$$

using the condition SR_7. Moreover, the row

$$(\lambda_1(g), \dots, \lambda_6(g), \lambda_{-28}(g), \dots, \lambda_{-1}(g))$$

is also unimodular. Let $\mathfrak{a} = \langle \lambda_i(g) \rangle$, $i = 1, \dots, 6, -28, \dots, -7$. Applying SR_6 to the elements $\lambda_{-6}(g), \dots, \lambda_{-1}(g)$, one can find $e \in (E(A_5 \to E_7, R), \omega_1)$ preserving the ideal $\mathfrak{a}$, such that $(\lambda_{-6}(eg), \dots, \lambda_{-2}(eg))$ is unimodular modulo $\mathfrak{a}$. Then the row $(\lambda_1(eg), \dots, \lambda_6(eg), \lambda_{-28}(eg), \dots, \lambda_{-2}(eg))$ is unimodular.

Set $eg = g_1$ and $\mathfrak{a} = \langle \lambda_1(g_1), \dots, \lambda_6(g_1) \rangle$. Using the condition $V_{\omega_6}(\mathrm{E}_6, R)$ in respect to the row $(\lambda_{-28}(g_1), \dots, \lambda_{-2}(g_2))$, one can find $(t_2, \dots, t_{28}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$, such that

$$\sum_{i=2}^{28} t_i r_i = 1 - \lambda_{-1}(g_1)(mod\ \mathfrak{a}).$$

Let $\beta_i = \lambda_i - \lambda_1$, $i = -28, \dots, -1$ and

$$e_1 = \prod_{i=2}^{-28} x_{-\beta_i}(t_i).$$

Computing $\lambda_{-1}(g_2)$ where $g_2 = e_1 g_1$, we have

$$\lambda_{-1}(g_2) = \lambda_{-1}(g_1) + t_2\lambda_{-2}(g_1) + \cdots + t_{28}\lambda_{-28}(g_1) + \sum_{i=2}^{28} u_i t_i + v_1 g_1,$$

where $u_i = u_i(t_i)$, $v_1 = v(t_i)$.

Straightforward computations using formulae $(*)$ yield that $u_i = 0$, since

$$(t_2, \dots, t_{28}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$$

and $v_1 = 0$ as a linear combination of u_i.

Therefore,

$$\lambda_{-1}(g_2) = \lambda_{-1}(g_1) + \sum_{i=2}^{28} t_i \lambda_{-i}(g_1) + a = 1 - \lambda_{-1}(g_1) + \lambda_{-1}(g_1) + a = 1 + a,$$

where $a \in \mathfrak{a}$. Since the ideal $\langle \lambda_1(g_2), \dots, \lambda_6(g_2) \rangle$ is still $\mathfrak{a}$, then the row

$$(\lambda_1(g_2), \dots, \lambda_6(g_2), \lambda_{-1}(g_2))$$

is unimodular. It follows immediately, that the row

$$(\lambda_1(g_2), \dots, \lambda_{11}(g_2), \lambda_{-28}(g_2), \lambda_1(g_2))$$

is unimodular. Replacing g_2 by appropriate $e_2 g_2 = g_3$, $e_2 \in (E(\mathrm{D}_6 \to \mathrm{E}_7, R), \omega_1)$, we find that the row

$$(\lambda_1(g_3), \dots, \lambda_{11}(g_3), \lambda_{-28}(g_3), \lambda_{28}(g_3))$$

is unimodular and

$$\lambda_i(g_3) \equiv 0(mod\ \mathfrak{a}), \quad i = -11, \dots, -2,$$

where $\mathfrak{a} = \langle \lambda_1(g_3), \dots, \lambda_{11}(g_3), \lambda_{-28}(g_3) \rangle$. Now, arguing as in Theorem 1 and applying ASR_6, we can find $e_3 \in (E(\mathrm{D}_6 \to \mathrm{E}_7, R), \omega_1)$, such that the row

$$(\lambda_1(e_3g_3), \dots, \lambda_{11}(e_3g_3), \lambda_{-28}(e_3g_3))$$

is unimodular. Let $g_4 = e_3g_3$. Using SR_6, one can get a unimodular row

$$(\lambda_2(g_5), \dots, \lambda_{11}(g_5), \lambda_{28}(g_5))$$

where $g_5 = e_4g_4$ and $e_4 \in (E(\mathrm{A}_5 \to \mathrm{E}_7, R), \omega_1)$.

Consider now the unimodular row $(\lambda_2(g_5), \dots, \lambda_{28}(g_5))$. Using the condition $V_{\omega_6}(\mathrm{E}_6, R)$, we can find $(t_2, \dots, t_{28}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$, such that

$$\sum_{i=2}^{28} t_i \lambda_i(g_5) = 1 - \lambda_1(g_1).$$

Take the element $e_5 \in (E(\mathrm{E}_7, R), \omega_7)$, which adds all the elements of

$$(\lambda_2(g_5), \dots, \lambda_{28}(g_5))$$

to the element $\lambda_1(g_5)$ with the coefficients t_i.

Then $g_1(e_5g_5) = 1$ since $(t_2, \dots, t_{28}) \in Um'_{\omega_6}(\mathrm{E}_6, R)$, and e_5 adds all the elements $\lambda_i(g_5)$, $i = -28, \dots, -1$ to $\lambda_1(g_5)$ with zero coefficients.

In the conclusion we formulate the theorem concerning the stability of the K_1-functor for the embedding $\mathrm{D}_5 \to \mathrm{E}_6$. The proof uses arguments, similar to those in Theorem 2 and the description of the sets $Eq(\mathrm{D}_5, \omega_5)$ and $Eq(\mathrm{E}_6, \omega_6)$.

Theorem 3. *Let the ring R satisfy the conditions SR_5 and $V_{\omega_5}(\mathrm{D}_5, R)$. Then the homomorphism*

$$\nu\colon K_1(\mathrm{D}_5, R) \to K_1(\mathrm{E}_6, R)$$

is surjective.

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