A family of extra-special 2-groups

Alexander A.Sapozhenko and Andrzej Szczepański *

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Let Γ be a Bieberbach group. By definition it is the torsion - free group given by a short exact sequence

$$(*)\ 1 \to Z^n \to \Gamma \to G \to 1,$$

where G is a finite group (called the holonomy group of Γ) and Z^n is a free abelian group of rank n which is a maximal abelian subgroup of Γ. Let $T : G \to GL(n, Z)$ be the representation which the group Γ defines on Z^n by conjugations. The rank of Γ is n. It is well known (see [1]) that Γ is isomorphic to a uniform discrete subgroup of the group $E(n)$ of the rigid motions of the n - dimensional Euclidean space $\mathbf{R^n}$. The orbit space $\mathbf{R^n}/\Gamma = M$ is a closed flat manifold, for short a flat manifold, with fundamental group Γ. In this note we shall define the sequence of extra - special 2-groups which are the epimorphic image of some Bieberbach groups. For the definition and properties of extra-special groups see [2] page 179.

Let Γ_{2n} be a (2n+1)-dimensional generalised **Hantzsche - Wendt** Bieberbach group defined by the short exact sequence

$$1 \to Z^{2n+1} \to \Gamma_{2n} \to (Z_2)^{2n} \to 1,$$

where $n \geq 1$ and the action of $(Z_2)^{2n}$ on Z^{2n+1} is defined by the $((2n+1) \times (2n+1))$ matrices:

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$$A_i = \left[\begin{array}{ccccccccc} -1 & 0 & ... & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & -1 & ... & 0 & 0 & 0 & ... & 0 & 0 \\ ... \\ 0 & 0 & ... & -1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 & -1 & ... & 0 & 0 \\ ... \\ 0 & 0 & ... & 0 & 0 & 0 & ... & -1 & 0 \\ 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & -1 \end{array}\right], 1 \leq i \leq 2n,$$

the "1" being in the i-th row and i-th column.
Moreover, the cocycle $s^\star \in H^2((Z_2)^{2n}, Z^{2n+1}) = H^1((Z_2)^{2n}, \mathbf{R^{2n+1}}/Z^{2n+1})$ correspondin the above exact sequence is given by the function $s : (Z_2)^{2n} \to \mathbf{R^{2n+1}}/Z^{2n+1}$; and

$$s(A_i) = u_i/2 + u_{i+1}/2 \text{ for } 1 \geq i \geq 2n$$

where $u_1, u_2, ..., u_n$ denotes a canonical base of Z^n. In [4] it was proven that Γ_{2n} is a Bieberbach group.

Let $T_{2n} \subset Z^{2n+1}$ be a subgroup generated by the following elements:

$$2u_1, (u_1 + (-u_2)), (u_2 + (-u_3)), ..., (u_{2n} + (-u_{2n+1})).$$

It is easy to see that T_{2n} is a subgroup of Z^{2n+1} of index 2. We shall prove that 2-groups $\mathcal{G}_{2n} = \Gamma_{2n}/T_{2n}, n \geq 1$, are extra-special 2-groups with known number of the involutions. From definition we have the following short exact sequence of finite groups:

$$0 \to Z_2 \to \mathcal{G}_{2n} \to (Z_2)^{2n} \to 0.$$

Hence we have $\mid \mathcal{G}_{2n} \mid = 2^{2n+1}$. For $n = 1$ $\mathcal{G}_2$ is the quaternion group of order 8, (see [3] the proof of the Proposition 3.5.). Let $\tau_{2n} = \sqrt{2}\cos\frac{\pi}{4}(2n+1)$. We have:

Theorem 1 *For each n, $\mathcal{G}_{2n}$ is an extra-special 2-group with exactly $2^{2n} + \tau_{2n}2^n - 1$ involutions.*

Proof: We shall consider Γ_{2n} as a subgroup of the isometry group $E(2n+1)$ of (2n+1)-dimensional euclidean space $\mathbf{R^{2n+1}}$, (see [4]). First we observe that $\mathcal{G}_{2n}$ has a presentation with $2n$ generators $x_1, x_2, ..., x_{2n}$ satisfying the following relations:

$$\forall i, x_i^2 = z, [x_i, x_j] = \begin{cases} z & \text{for } \mid i-j \mid = 1 \\ 0 & \text{otherwise} \end{cases}, z^2 = 1.$$

Here x_i corresponds to the coset represented by $(A_i, u_i/2 + u_{i+1}/2)$in Γ_{2n}/T_{2n}. Moreover, z corresponds to the coset represented by u_1.

We now go on to prove that $\mathcal{G}_{2n}$ is an extra-special 2-group. Let $Z(\mathcal{G}_{2n})$ denote the center and $[\mathcal{G}_{2n}, \mathcal{G}_{2n}]$ a commutator subgroup of $\mathcal{G}_{2n}$. By definition of extra-special group it suffices to prove that $Z(\mathcal{G}_{2n}) = Z_2 = [\mathcal{G}_{2n}, \mathcal{G}_{2n}]$ (see [2] page 179). Let us show that $Z(\mathcal{G}_{2n}) = < z >$. In fact, the inclusion $< z > \subset Z(\mathcal{G}_{2n})$ follows from the definition of Γ_{2n} and straightforward calculations. For the proof of the second inclusion $Z(\mathcal{G}_{2n}) \subset < z >$ we shall consider the following cases:

$$(1) \quad x_1 x_2 ... x_{2n} \notin Z(\mathcal{G}_{2n}).$$

In fact by induction one can prove that this element is represented in Γ_{2n}/T_{2n} by

$$(A_1 A_2 ... A_{2n}, u_1/2 - u_{2n+1}/2).$$

Hence an elementary calculation shows that $x_1 x_1 x_2 ... x_{2n} \neq x_1 x_2 ... x_{2n} x_1$.

(2) Any other product of $x_i (i = 1, 2, ..., 2n)$ different from $z \notin Z(\mathcal{G}_{2n})$. In fact let y be such an element. We can assume that $y = x_{i_1} x_{i_2} ... x_{i_k}$ where $1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq 2n$ and $k < 2n$. Hence there exists $1 \leq r \leq 2n$, $r \neq i_1, i_2, ..., i_k$. From the above relations one can observe that $x_r y \neq y x_r$ in $\mathcal{G}_{2n}$. This proves that $\mathcal{G}_{2n}$ are extra-special groups.

Let us start to prove the second part of the theorem. Let $i(2n)$ denote the number of involutions in $\mathcal{G}_{2n}$. We have $2n$ elements $1, 2, ..., 2n$, where n is a positive integer. We consider sequences

$$(i_1, i_2, ..., i_s) \quad (**)$$

such that $1 \leq i_1 < i_2 < ... < i_s \leq 2n$, $1 \leq s \leq 2n$.
We say that $(**)$ has l-breaks if the set

$$\{j \in \{1, 2, ..., s-1\} \mid i_{j+1} - i_j \geq 2\}$$

has exactly l elements.

From the presentation of the group $\mathcal{G}_{2n}$ one can observe that the number $p(2n)$ of all sequences $(**)$ with l-breakes, where l is odd , satisfies the following formula

$$i(2n) = 2p(2n) + 1.$$

Let us define $f(2n, s, k)$ to be the number of sequences $(**)$ with s elements and with k-breaks. By straightforward calculations we have

$$f(2n, s, k) = \sum_{l=k}^{2n-s} (2n - s - l + 1) \binom{s-1}{k} \binom{l-1}{k-1}.$$

From this we have

$$p(2n) = \sum_{q=1}^{\lfloor n/2 \rfloor} \sum_{s=2q}^{2n-2q+1} \sum_{l=2q-1}^{2n-s} (2n - s - l + 1) \binom{s-1}{2q-1} \binom{l-1}{2q-2}.$$

By summing over l we get

$$p(2n) = \sum_{q=1}^{\lfloor n/2 \rfloor} \sum_{s=2q}^{2n-2q+1} \binom{s-1}{2q-1} \binom{2n-s+1}{2q}.$$

From the property of extra-special 2-groups (see [2] page 193) we know that $p(2n)$ is equal to $2^{2n-1} + 2^{n-1} - 1$ or $2^{2n-1} - 2^{n-1} - 1$.

We now go on to prove that

$$p(2n) = 2^{2n-1} + \tau_{2n} 2^{n-1} - 1 \quad (***).$$

We shall use the generating function techniques. Let $\bar{p}(m) = p(2n), \quad r(m) = 2^{m-1} + \sqrt{2} \cos \frac{\pi}{4}(m+1) \cdot 2^{\frac{m}{2}-1} - 1$. Let $P(x) = \sum_m \bar{p}(m) x^m$ and $R(x) = \sum_m r(m) x^m$ be the corresponding generating functions. It is sufficient to show that $P(x) = R(x)$. We have

$$P(x) = \sum_{m=0}^{\infty} x^m \sum_{q=1}^{\lfloor m/4 \rfloor} \sum_{s=2q}^{m-2q+1} \binom{s-1}{2q-1} \binom{m-s+1}{2q}$$

$$= \sum_{q=1}^{\infty} \sum_{s=2q}^{\infty} \binom{s-1}{2q-1} x^{s-1} \sum_{m=0}^{\infty} \binom{m-s+1}{2q} x^{m-s+1}.$$

Using the identity $\sum_i \binom{i}{j} x^i = x^j(1-x)^{-j-1}$ (see [5] page 15 formula (1.5.9)), we obtain

$$P(x) = \sum_{q=1}^{\infty} x^{4q-1}(1-x)^{-(4q+1)} = \frac{x^3}{(1-x)(1-2x)(1-2x+2x^2)}.$$

On the other hand,

$$R(x) = \sum_{m=0}^{\infty} \left(2^{m-1} + \left(\sqrt{2}\cos(m+1)\frac{\pi}{4}\right) \cdot 2^{\frac{m}{2}-1} - 1\right) x^m$$

$$= \frac{1}{2}\sum_m (2x)^m + \frac{1}{2\sqrt{2}}\sum_m \left(e^{i\frac{\pi}{4}(m+1)} + e^{-i\frac{\pi}{4}(m+1)}\right)(\sqrt{2}x)^m - \sum_m x^m$$

$$= \frac{1}{2}\cdot\frac{1}{1-2x} + \frac{e^{\frac{\pi}{4}i}}{2\sqrt{2}}\cdot\frac{1}{1-1e^{\frac{\pi}{4}i}x} + \frac{e^{-\frac{\pi}{4}i}}{2\sqrt{2}}\cdot\frac{1}{1-\sqrt{2}e^{-\frac{\pi}{4}i}x} - \frac{1}{1-x}$$

$$= \frac{1}{2}\cdot\frac{1}{1-2x} + \frac{1}{2}\cdot\frac{1-2x}{1-2x+2x^2} - \frac{1}{1-x}$$

$$= \frac{x^3}{(1-x)(1-2x)(1-2x+2x^2)}.$$

This proves the formula $(***)$ and completes the proof of the theorem.

□

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References

[1] L.S.Charlap. *Bieberbach groups and flat manifolds*, (Springer, 1986).

[2] L.Dornhoff. *Group representation theory*, part A, (Marcel Dekker, 1971).

[3] H.Hiller, Z.Marciniak, C.H.Sah, A.Szczepański. Holonomy of flat manifolds with $b_1 = 0$ II. *Quart.J.Math.Oxford* (2), **38** (1987), 213 - 220.

[4] A.Szczepański. Aspherical manifolds with the Q - homology of a sphere. *Mathematatika*, **30** (1983), 291 - 294.

[5] H.S.Wilf. *generatingfunctionology*, Academic Press, 1994.

Departament of Computational
Mathematics and Cybernetics
Moscow State University
Vorobiovy Gory 119899
Moscow Russia
E-mail: mathcyb@msu.su

Institute of Mathematics
University of Gdańsk
ul.Wita Stwosza 57
80 - 952 Gdańsk
Poland
E - mail : matas @ paula.univ.gda.pl