DISCRETE ISOPERIMETRIC AND POINCARÉ-TYPE INEQUALITIES *

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Abstract

We study some discrete isoperimetric and Poincaré-type inequalities for product probability measures μ^n on the discrete cube $\{0,1\}^n$ and on the lattice $\mathbf{Z}^n$. In particular we prove sharp lower estimates for the product measures of 'boundaries' of arbitrary sets in the discrete cube. More generally, we characterize those probability distributions μ on $\mathbf{Z}$ which satisfy these inequalities on $\mathbf{Z}^n$. The class of these distributions can be described by a certain class of monotone transforms of the two-sided exponential measure. A similar characterization of distributions on $\mathbf{R}$ which satisfy Poincaré inequalities on the class of convex functions is proved in terms of variances of suprema of linear processes.

1 Introduction

For a number $p \in (0,1)$, let μ_p denote the Bernoulli measure on $\{0,1\}$ with probabilities $q = 1-p$ and p, respectively, and let μ_p^n denote the corresponding product measure on $\{0,1\}^n$. Denote by $s_i(x)$ the 'neighbour' of $x \in \{0,1\}^n$ obtained by changing the i-th coordinate: $[s_i(x)]_j = x_j$, for $j \neq i$, and $[s_i(x)]_i = 1 - x_i$. Furthermore, for any set $A \subset \{0,1\}^n$, define the function h_A on $\{0,1\}^n$ by

$$h_A(x) = \mathrm{card}\{i \leq n : x \in A,\ s_i(x) \notin A\}.$$

Define $C_p = \min(p,q)/\sqrt{pq}$ which ranges in the interval $[0,1]$ and $J(t) = t\sqrt{\log(1/t)}$, $t \geq 0$. The starting point of this paper are the following inequalities for the discrete cube due to M.Talagrand [T]. There exists an absolute constant $K > 0$ such that, for each subset A of $\{0,1\}^n$,

$$\int \sqrt{h_A(x)}\ d\mu_p^n(x) \ \geq\ \frac{C_p}{K}\, J\left(\mu_p^n(A)(1-\mu_p^n(A))\right), \tag{1.1}$$

$$\int \sqrt{h_A(x)}\ d\mu_p^n(x) \ \geq\ \sqrt{2}C_p\, \mu_p^n(A)\,(1-\mu_p^n(A)). \tag{1.2}$$

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Note that by the definition of $J(t)$ (1.2) immediately follows from (1.1) with a weaker bound involving the constant $1/(K\sqrt{\log 4})$.

For a comparison to the continuous case let γ_n denote the canonical Gaussian measure on $\mathbf{R}^n$, and let $\gamma_n^+(A)$ denote the Gaussian surface measure of a subset A of $\mathbf{R}^n$. According to a well-known inequality due to V.N.Sudakov, B.S.Tsirel'son and C.Borel ([ST], [Bor]), we have for any measurable set $A \subset \mathbf{R}^n$

$$\gamma_n^+(A) \geq I(\gamma_n(A)), \tag{1.3}$$

where the (so-called Gaussian isoperimetric) function I is given by $I(t) = \varphi(\Phi^{-1}(t))$, $t \in [0,1]$ (here Φ^{-1} is the inverse distribution function of $\Phi(x) = \gamma_1((-\infty, x])$ and $\varphi(x)$ denotes the density $\exp\{-x^2/2\}/\sqrt{2\pi}$). For the discrete cube, the quantity $\int \sqrt{h_A}\, d\mu_p^n$ measures the interior boundary $\partial_+ A = \{h_A > 0\}$ and thus it can be viewed as a discrete analogue of $\gamma_n^+(A)$. In addition, the expression $J(\mu_p^n(A)(1-\mu_p^n(A)))$ in (1.1) may be replaced by $I(\mu_p^n(A))$ since $c_1 I(t) \leq J(t(1-t)) \leq c_1 I(t)$, for some absolute constants $c_1, c_2 > 0$ and for all $t \in [0,1]$. Therefore, the inequality (1.1) itself may be viewed, up to a multiplicative absolute constant, as a discrete version of the Gaussian isoperimetric inequality (1.3).

Using (1.2), M.Talagrand obtained a quantitative version of Margulis' graph connectivity theorem [M]. G.A.Margulis proved that

$$\mu_p^n(\partial_+ A) \int h_A \, d\mu_p^n$$

is bounded from below by a quantity $c\,(p, \alpha)$ which depends on p and $\alpha = \mu_p^n(A)$ but not on n. It follows from (1.2) via a simple application of the Cauchy-Schwarz inequality to $\int \sqrt{h_A} = \int \sqrt{h_A}\, \mathbf{1}_{\partial_+ A}$ that $c(p,\alpha) = 2C_p^2(\alpha(1-\alpha))^2$. Note that this bound is small when pq is small since $pq \leq C_p^2 \leq 4pq$. We shall prove (c.f. Proposition 3.2) that (1.2) is valid in fact with a constant

$$C_p = \sqrt{\frac{2}{\max(p,q)} - 1}$$

which varies in the interval $[1, \sqrt{3}\,]$. Hence, the bound $c = c(p, \alpha)$ in Margulis' theorem can be chosen *independent* of p.

In Lemma 3.3 we shall provide a lower bound for the optimal constant C_p for inequalities of type (1.1) involving more general functions of $\mu_p^n(A)$ than $J(t(1-t))$, $t(1-t)$ and $I(t)$ discussed above and show e.g. that when p is close to 0 or 1 the optimal constant C_p in (1.1) is aproximately given by $1/\sqrt{\log \frac{1}{pq}}$.

Furthermore, we introduce the outer boundary $\partial_- A = \{h_{\bar{A}} > 0\}$, where $\bar{A} = \{0,1\}^n \setminus A$ denotes the complement of A in $\{0,1\}^n$, and associate to A another function

$$\bar{h}_A = h_A + h_{\bar{A}}.$$

The quantity $\int \sqrt{\bar{h}_A}\, d\mu_p^n$ will serve as a measure of the two-sided boundary $\partial A = \partial_+ A \cup \partial_- A$. Replacing the interior boundary function h_A by the two-sided boundary

$\bar{h}_A$ will change the behaviour of the optimal constant in (1.1) considerably: we prove in Proposition 2.1 that

$$\int \sqrt{\bar{h}_A(x)}\, d\mu_p^n(x) \;\geq\; \frac{1}{\sqrt{2pq}}\, I(\mu_p^n(A)) \tag{1.4}$$

which follows from a general inequality for functions f on an abstract product space $(\Omega, \mathbf{P})$ involving some measure of size of a 'discrete gradient' $\mathrm{D}(f)$ of f like

$$I\left(\int f\, d\mathbf{P}\right) \;\leq\; \int \sqrt{I(f)^2 + 2\,\mathrm{D}(f)^2}\, d\mathbf{P} \tag{1.5}$$

(see Theorem 2.2). Inequalities of this kind had been introduced in [B] for the discrete cube with normalized Bernoulli measure ($p = 1/2$). In that case (1.4) holds with a slightly better constant 2 (instead of $\sqrt{2}$). As we will see, the constant $1/\sqrt{2pq}$ is of correct order in the general case $p \neq 1/2$, and even the factor $1/\sqrt{2}$ turns out to be essential for small p. (See also the paper [BaL] by D.Bakry and M.Ledoux where (1.5) has been established for a large class of operators D.)

Replacing Bernoulli measures on $\{0,1\}$ by probability measures μ on $\mathbf{Z}$, we will characterize more generally those measures μ^n on $\mathbf{Z}^n$ which admit inequalities of type (1.2) with dimension free constants. Since in $\mathbf{Z}$ every point x has the next neighbours $x-1$ and $x+1$, the appropriate definition of the boundary function h_A for any subset A of $\mathbf{Z}^n$ is given by

$$h_A(x) = \mathrm{card}\{i \leq n : x \in A,\ (x + e_i \notin A \text{ or } x - e_i \notin A)\},$$

where $(e_i)_{i \leq n}$ denotes the canonical basis in $\mathbf{R}^n$. We will show that the isoperimetric inequality (1.2) for the product measure μ^n on $\mathbf{Z}^n$ is *equivalent* to the fact that the measure μ satisfies a (one-dimensional) discrete Poincaré inequality on $\mathbf{Z}$

$$\int_{\mathbf{Z}} f^2\, d\mu - \left(\int_{\mathbf{Z}} f\, d\mu\right)^2 \leq C \int_{\mathbf{Z}} |\Delta f(x)|^2\, d\mu(x), \tag{1.6}$$

for the difference operator $\Delta f(x) = f(x+1) - f(x)$. A characterization of such probability measures has recently obtained by J.-H.Lou [Lou]. (See also a joint paper with L.H.Y.Chen [CL].) He proves that μ satisfies (1.6) with some finite C if and only if, for some $c > 0$, and density function $p(x) = \mu(\{x\})$, the linear difference equation

$$\Delta f(x-1)\, p(x-1) - (\Delta f(x) + c f(x))\, p(x) = 0$$

has a strictly monotone solution f such that $\int_{\mathbf{Z}} |f|\, d\mu < +\infty$. Recursively solving (1.6) from a starting point $x_0 \in \mathbf{Z}$ of the support of μ, he obtained the equivalent condition

$$\sup_{x \geq x_0}\, (1 - F_\mu(x)) \sum_{k=x_0}^{x} \frac{1}{p(k)} < +\infty, \qquad \sup_{x \leq x_0 - 1}\, F_\mu(x) \sum_{k=x}^{x_0 - 1} \frac{1}{p(k)} < +\infty, \tag{1.7}$$

where $F_\mu(x) = \mu((-\infty, x])$ denotes distribution function of μ (the support of μ should necessarily be an interval in $\mathbf{Z}$, finite or infinite). J.-H.Lou consideres (1.7) as a discrete analogue of known conditions for a (continuous) measure μ on $\mathbf{R}$ staisfying (1.6) in the class of smooth functions f with Δf replaced by the derivative f' (c.f. [Mu] and Remark 5.3 below and conditions (5.9)–(5.10)). We use another approach based on some special properties of the two-sided exponential distribution with density $\frac{1}{2}\exp(-|x|)$, $x \in \mathbf{R}$, which we denote by ν. We show that the measure ν satisfies certain inequalities related to (1.6), and this has lead us to a different description. Let U_μ denote the non-decreasing left-continuous function U_μ which transforms ν into μ (such a function exists, is unique, and, when $\mu(\mathbf{Z}) = 1$, it represents a step function with values in $\mathbf{Z}$). We prove (c.f. Theorem 4.1) that (1.6) holds for some finite constant C if and only if, for some sufficiently small $a > 0$,

$$\sup_{x \in \mathbf{R}} \, [U_\mu(x+a) - U_\mu(x)] \ \le \ 1.$$

Via such a characterization, we reduce (1.6) to the following property: for some finite constant C and for all $x \in \mathbf{Z}$,

$$F_\mu(x)(1 - F_\mu(x)) \le C\, p(x).$$

We conlude that for such measures μ, some $K > 0$ and any subset A of $\mathbf{Z}^n$, the inequality

$$\int_{\mathbf{Z}^n} \sqrt{h_A(x)} \; d\mu^n(x) \ \ge \ K\, \mu^n(A)\,(1 - \mu^n(A))$$

holds. For example, the measure with density $2^{-|k|}/3$, $k \in \mathbf{Z}$, (with respect to the counting measure on $\mathbf{Z}$) satisfies these conditions. On the other hand, the conditions are not satisfied for example for the Bernoulli measure with atoms at the points 0 and 2 (the above inequality fails for $A = \{-1, 0, 1\}$). The last example suggests to study more general inequalities than (1.6), namely,

$$\int_{\mathbf{R}} f^2\, d\mu - \Big(\int_{\mathbf{R}} f\, d\mu\Big)^2 \le C \int_{\mathbf{R}} (f(x+a) - f(x))^2\, d\mu(x), \tag{1.8}$$

assuming that μ is an arbitrary probability measure on the real line $\mathbf{R}$. When the measure μ is not concentrated on a set of the form $b + a\mathbf{Z}$, given $a > 0$, the inequality (1.8) can not hold for all f since the right-hand side of (1.8) is 0 if f has period a. (This contrasts to (1.6) where the functions f were assumed to be arbitrary). It is reasonable however to investigate (1.8) for the class of monotone functions and to study conditions such that this inequality holds for some tuples (C, a) of positive constants. This is of course a weaker requirement (on μ) than prescribing a in advance. In this sense (1.8) is actually equivalent to the Poincaré inequality

$$\int f^2\, d\mu - \Big(\int f\, d\mu\Big)^2 \le C \int |f'|^2\, d\mu \tag{1.9}$$

on the class of all convex functions f. Somewhat unexpectedly, this turns out to be equivalent to the following property: consider an (almost surely) bounded random process $x(t)$, $t \in T$, of the form

$$x(t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\, \xi_n$$

where ξ_n's are (centered) independent random variables with finite second moment distributed according to μ and where a_n are some functions on an (abstract) set T. Then, for a certain constant C (which depends on μ, only), we have

$$\mathrm{Var}(\sup_t x(t)) \ \leq \ C \sup_t \mathrm{Var}(x(t)) \tag{1.10}$$

(where Var denotes the variance). For the Gaussian measure γ_n, this property (with optimal constant $C = 1$) is well-known since the papers [Bor] and [ST] as a consequence of the isoperimetric inequality (1.3). This is also a direct consequence of the Gaussian Poincaré inequality (1.9) for γ_n. More generally, since the two-sided exponential measure ν admits (1.9), all Lipschitz images of ν (that is, all μ with $\|U_\mu\|_{Lip} < \infty$) satisfy (1.9) and therefore satisfy (1.10). However, the class of *all* probability measures μ satisfying (1.10) turns out to be somewhat larger than the class of *all* Lipschitz images of μ. Using almost the same arguments as those used for the discrete problem (1.6), we prove in Theorem 5.1 that each of the properties (1.8)–(1.10) holds if and only if the map U_μ has a finite modulus, that is, for some $a > 0$,

$$\sup_{x \in \mathbf{R}} \, [U_\mu(x + a) - U_\mu(x)] \ < \ +\infty, \tag{1.11}$$

which is weaker than assuming $\|U_\mu\|_{Lip} < \infty$.

The organization of the paper is as follows. In section 2, we prove (1.4) and extend the result to arbitrary product spaces. The optimal constants in (1.1) and (1.2) (as functions of p) are studied in sections 3 and 6. In section 4, we extend (1.2) to product measures on $\mathbf{Z}^n$, and in section 5, we shall prove the equivalence of (1.8)–(1.11).

2 Gaussian type isoperimetric inequality in abstract product spaces

As noted in [T], the dependence of the right hand side of (1.1) on $\mu_p^n(A)$ given p is of the right order. This is readily seen by checking (1.1) on the sets

$$A_n(a) = \left\{ x \in \{0,1\}^n : \frac{S_n - np}{\sqrt{npq}} \leq a \right\}$$

where $S_n = x_1 + \cdots + x_n$. However, the order of optimal constant in (1.1) (as a function of p) is different: the constant $\min(p,q)/\sqrt{pq}$ which is of order $\sqrt{pq}$ can be replaced by $1/\sqrt{\log(1/(pq))}$ which tends to 0 with p (see sections 3 and 6). For the two-sided boundaries, the constants in (1.1) will change considerably:

Proposition 2.1 *Let $I(t) = \varphi(\Phi^{-1}(t))$, $0 \leq t \leq 1$. For any subset A of $\{0,1\}^n$,*

$$\int \sqrt{\bar{h}_A(x)} \ d\mu_p^n(x) \ \geq \ \frac{1}{\sqrt{2pq}} \, I(\mu_p^n(A)). \tag{2.1}$$

Put $k = \left[np + a\sqrt{npq} \right]$, where $[\,\cdot\,]$ denotes the integer part of a real number. We check (2.1) for the sets $A_n(a)$. Let $\mathbf{1}_A$ denote the indicator function of a set A. Clearly,

$$\bar{h}_{A_n(a)} \;=\; (n-k)\ \mathbf{1}_{\{S_n = k\}} + (k+1)\ \mathbf{1}_{\{S_n = k+1\}},$$

so, by the de Moivré-Laplace local limit theorem,

$$\int \sqrt{\bar{h}_{A_n(a)}}\, d\mu_p^n = \sqrt{n-k}\ \mu_p^n(S_n = k) + \sqrt{k+1}\ \mu_p^n(S_n = k+1) \longrightarrow \frac{\sqrt{p}+\sqrt{q}}{\sqrt{pq}}\,\varphi(a),$$

as $n \to \infty$. On the other hand, by the central limit theorem, $\mu_p^n(A_n(a)) \to \Phi(a)$, hence,

$$\frac{1}{\sqrt{2pq}}\, I(\mu_p^n(A_n(a))) \;\longrightarrow\; \frac{1}{\sqrt{2pq}}\,\varphi(a).$$

Therefore, since $1 \le \sqrt{p} + \sqrt{q} \le \sqrt{2}$, the dependence on $\mu_p^n(A)$ and on p on the right-hand side of (2.1) is sharp.

A remarkable feature of (2.1) however is that it can be extended to arbitrary product measures, and (2.1) follows from a more general and a simpler inequality. We shall use the probabilistic notations

$$\mathbf{E}_\mu f = \mathbf{E}\, f = \int_\Omega f\, d\mu, \qquad \mathrm{Var}_\mu(f) = \mathrm{Var}(f) = \int_\Omega f^2\, d\mu - \left(\int_\Omega f\, d\mu\right)^2,$$

when it is clear from the context which measure spaces (Ω, μ) are used. Let $(\Omega, \mathbf{P})$ be the product of a finite number of probability spaces (Ω_i, μ_i), $i = 1, 2, \cdots, n$. For every (measurable) function f on Ω, we set

$$\mathrm{D}(f) = \sqrt{\sum_{i=1}^{n} \mathrm{Var}_{x_i}(f)}$$

where $\mathrm{Var}_{x_i}(f)$ denotes the variance of f with respect to the i-th coordinate while the remaining of the variables are fixed. The function $\mathrm{D}(f)$ is related to the modulus of the gradient of a smooth function on Euclidean space. Thus, given a measurable set $A \subset \Omega$, one may consider the quantity

$$\mathbf{P}^+(A) = \mathbf{E}\, \mathrm{D}(1_A)$$

as a certain '$\mathbf{P}$-perimeter' of A (the mathematical expectation is taken with respect to $\mathbf{P}$). We shall prove the following result:

Theorem 2.2 *For every measurable function f on Ω with values in* $[0, 1]$,

$$I(\mathbf{E}\, f) \;\le\; \mathbf{E}\sqrt{I(f)^2 + 2\, \mathrm{D}(f)^2}. \tag{2.2}$$

In particular, for any measurable subset A of Ω,

$$\mathbf{P}^+(A) \;\ge\; \frac{1}{\sqrt{2}}\, I(\mathbf{P}(A)). \tag{2.3}$$

The inequality (2.3) reduces to (2.1) choosing $\Omega_i = \{0,1\}$ with distribution μ_p. By (2.3) and as the example of the sets $A_n(a)$ shows, the optimal function I_0 for which

$$\mathbf{P}^+(A) \ \geq \ I_0(\mathbf{P}(A)) \tag{2.4}$$

holds for all product probability spaces $(\Omega, \mathbf{P})$ should satisfy $\frac{1}{\sqrt{2}}\, I \leq I_0 \leq I$. Although we do not know I_0, we know that the factor $1/\sqrt{2}$ in (2.3) and therefore the constant 2 in (2.2) can not be improved. Moreover, $I_0(t)$ should be equivalent to $\frac{1}{\sqrt{2}}\, I(t)$, for t small, and in fact

$$\frac{1}{\sqrt{2}}\, I \leq I_0 \leq J,$$

where the function $J(t) = t\sqrt{\log(1/t)}$ is the same as in (1.1). Indeed, let us return to the case of the discrete cube $\Omega = \{0,1\}^n$ with the measure $\mathbf{P} = \mu_p^n$. In this special case, (2.4) can be rewritten like (2.1) as

$$\sqrt{pq} \int \sqrt{\bar{h}_A(x)}\ d\mu_p^n(x) \ \geq \ I_0(\mu_p^n(A)).$$

For the one-point set $A = \{S_n = 0\}$, this inequality becomes

$$\sqrt{pq}\ (\sqrt{n}p^n + np^{n-1}q) \ \geq \ I_0(p^n).$$

Now, given $t \in (0,1)$, apply this inequality to $p = t^{1/n}$. As $n \to \infty$, we obtain in the limit

$$t\sqrt{\log(1/t)} \ \geq \ I_0(t),$$

for any $t \in (0,1)$. Since $J(t) = t\sqrt{\log(1/t)}$ is equivalent to $\frac{1}{\sqrt{2}}\, I(t)$ near the point $t = 0$, the dependence of the right-hand side of (2.3) in $\mathbf{P}(A)$ is thus sharp when $\mathbf{P}(A)$ is small (or, equivalently, when $\mathbf{P}(A)$ is close to 1).

One may wonder of course whether or not, (2.2) can be sharpened for some special measures, and in particular, when the constant 2 may be improved (e.g., replaced by 1 as the best possible case). Let us apply (2.2) with $n = 1$ to the functions $f = c + \varepsilon g$ with g bounded such that $\mathbf{E}\, g = 0$. If $0 < c < 1$, Taylor's expansion around $\varepsilon = 0$ yields

$$\begin{aligned} \mathbf{E}\sqrt{I(f)^2 + \mathrm{Var}(f)} &= \mathbf{E}\sqrt{I(c+\varepsilon g)^2 + \varepsilon^2 \mathrm{Var}(g)} \\ &= I(c) + \frac{I'(c)}{6\, I(c)^2}\, \mathbf{E}\, g^3\, \varepsilon^3 + O\left(\varepsilon^4\right). \end{aligned}$$

Therefore, validity of the inequality $I(c) = I(\mathbf{E}\, f) \leq \mathbf{E}\sqrt{I(f)^2 + \mathrm{Var}(f)}$ requires that $\mathbf{E}\, g^3 = 0$. But if μ_1 $(= \mathbf{P})$ is different from a Dirac measure in one point, the equality $\mathbf{E}\, g^3 = 0$ holds for all bounded functions g with $\mathbf{E}\, g = 0$ if and only if μ_1 has two support points with probability 1/2 each. Thus inequality (2.2) may hold in product spaces with the factor 2 replaced by 1 only if we restrict ourselves to the discrete cube $\{0,1\}^n$ with the normalized uniform measure. As shown in [B], this is indeed the case.

The proof of (2.2) consists of two main parts: the case $n = 1$ and an induction. First we separate the induction step which has been already shown to hold for some special cases in [B] and [BaL]. For application to different forms of (2.2) in sections 3 and 4, we describe a some more general situation. Assume that Γ_i denote operators defined on subsets of measurable functions on X_i with values in [0,1]. Set, for any function f on Ω, (if it is exists)

$$\Gamma(f) = \sqrt{\sum_{i=1}^{n} \Gamma_i(f_i)^2}$$

where f_i denotes the function of x_i defined via $f_i(x_i) = f(x)$, $x \in \Omega$. The basic assumption on Γ_i is that $\Gamma_i(g)$ represents a non-negative measurable function, and that Γ_i is 'convex' in the following sense: for any measurable function g on $(X_i \times Y, \mu \otimes \nu)$ such that $\Gamma_i(g(x_i, y))$ is defined for ν-almost all y, ν-measurable, and such that $\Gamma_i(\mathbf{E}_\nu g(x_i, y))$ is defined as well, we have

$$\Gamma_i(\mathbf{E}_\nu g(x_i, y)) \ \leq \ \mathbf{E}_\nu \Gamma_i(g(x_i, y)), \tag{2.5}$$

for μ_i-almost all x_i. Under an appropriate continuity assumption, (2.5) will follow from $\Gamma_i(tg + (1-t)h) \leq t\Gamma_i(g) + (1-t)\Gamma_i(h)$, $0 \leq t \leq 1$.

Lemma 2.3 *Let I be a non-negative, Borel measurable function on $[0,1]$. For each $i \leq n$, assume that Γ_i admits the inequality*

$$I(\mathbf{E}_{\mu_i} f) \ \leq \ \mathbf{E}_{\mu_i} \sqrt{I(f)^2 + \Gamma_i(f)^2},$$

for every function f on X_i such that $\Gamma_i(f)$ is defined. Then Γ satisfies the same inequality

$$I(\mathbf{E} f) \ \leq \ \mathbf{E} \sqrt{I(f)^2 + \Gamma(f)^2},$$

for any function f on Ω such that $\Gamma(f)$ is defined (with mathematical expectations taken with respect to $\mathbf{P}$).

The proof does not differ from that of Lemma 2 in [B] and is omitted. In Theorem 2.2, we apply Lemma 2.3 with $\Gamma_i(f) = \sqrt{2\mathrm{Var}(f)}$ (which is a functional rather than an operator) and with the Gaussian isoperimetric function

$$I(t) = \varphi(\Phi^{-1}(t)), \quad 0 \leq t \leq 1.$$

In order to treat the case $n = 1$, we note several elementary properties of this function:

1) I is concave on [0,1] and is symmetric around the point 1/2;

2) I increases on [0,1/2] and decreases on [1/2,1]; $I(0) = I(1) = 1$;

3) I satisfies differential equation $I''I = -1$;

4) the function $I\left(\frac{1}{2}+t\right)^2 + t^2$ is a convex and even function in $|t| \leq 1/2$.

Lemma 2.4 *For every measurable function f on Ω with values in* $[0,1]$,

$$I(\mathbf{E}\,f) - \mathbf{E}\,I(f) \ \leq \ \frac{1}{I(\mathbf{E}\,f)}\ \mathrm{Var}(f). \tag{2.6}$$

Consequently,

$$I(\mathbf{E}\,f) \ \leq \ \mathbf{E}\,\sqrt{I(f)^2 + 2\,\mathrm{Var}(f)}. \tag{2.7}$$

Proof. (A two-point analogue of (2.6) for the function J appears in [T], Lemma 3.2, and we shall borrow some ideas in the step 2 below.) First let us see how (2.6) implies (2.7). We apply (2.6) and Jensen's inequality to get

$$\begin{aligned}(I(\mathbf{E}\,f))^2 - (\mathbf{E}\,I(f))^2 \ &= \ (I(\mathbf{E}\,f) - \mathbf{E}\,I(f))\,(I(\mathbf{E}\,f) + \mathbf{E}\,I(f)) \\ &\leq \ (I(\mathbf{E}\,f) - \mathbf{E}\,I(f))\,2I(\mathbf{E}\,f) \\ &\leq \ 2\,\mathrm{Var}(f).\end{aligned}$$

Hence, $(I(\mathbf{E}\,f))^2 \leq (\mathbf{E}\,I(f))^2 + 2\,\mathrm{Var}(f) \leq \left(\mathbf{E}\,\sqrt{I(f)^2 + 2\,\mathrm{Var}(f)}\,\right)^2$. The last inequality is due to the triangle inequality $(\int u)^2 + (\int v)^2 \leq \left(\int \sqrt{u^2+v^2}\right)^2$ with $u = I(f)$ and $v = \sqrt{2\,\mathrm{Var}(f)}$.

The proof of (2.6) is divided into two parts.

Step 1. We reduce (2.6) to the case $0 \leq f \leq 1/2$. Write an arbitrary function f taking values in [0,1] in the form $f = \frac{1}{2} + \varepsilon g$ where $0 \leq g \leq 1/2$ and $|\varepsilon| = 1$. Since $I(\frac{1}{2}+\varepsilon g) = I(\frac{1}{2}+g)$ and $\mathrm{Var}(f) = \mathbf{E}\,g^2 - (\mathbf{E}\,\varepsilon g)^2$, we can rewrite (2.6) as follows:

$$\left[I^2\left(\frac{1}{2} + \mathbf{E}\,\varepsilon g\right) + (\mathbf{E}\,\varepsilon g)^2\right] - I\left(\frac{1}{2} + \mathbf{E}\,\varepsilon g\right)\mathbf{E}\,I\left(\frac{1}{2} + g\right) \ \leq \ \mathbf{E}\,g^2. \tag{2.8}$$

By properties 1) and 4) above, the left-hand side of (2.8) represents a convex and even function of the variable $x = \mathbf{E}\,\varepsilon g$. Hence the maximal value of the left-hand side provided g is fixed is attained at the constant function $\varepsilon = -1$ (and at $\varepsilon = 1$ which gives the same inequality). Thus, it suffices to establish (2.8) for $f = \frac{1}{2} - g$.

Step 2. Let f take values in [0,1/2]. Fix $c = \mathbf{E}\,f$ and assume that $0 < c \leq 1/2$. By Taylor's formula with integral remainder term, we have

$$I(f) = I(c) + I'(c)\,(f-c) + \int_0^1 I''(t\,c + (1-t)f\,)\,t\,dt\ \ (f-c)^2. \tag{2.9}$$

By the properties 2) and 3), I'' increases in (0,1/2]. Hence, $I''(t\,c + (1-t)f\,) \geq I''(t\,c)$. Using this estimate and integrating (2.9), we obtain

$$\mathbf{E}\,I(f) - I(c) \ \geq \ \mathrm{Var}(f)\int_0^1 I''(t\,c)\ t\,dt \ = \ -\mathrm{Var}(f)\int_0^1 \frac{t}{I(t\,c)}\,dt.$$

Finally note that $I(t\,c) \geq t\,I(c)$. Indeed, the left hand side of this inequality is concave in t and the right hand-side is linear, while the inequality becomes an equality at $t=0$ and $t=1$. This proves Lemma 2.4 and thus Theorem 2.2 is proved, as well.

Remark 2.5 As mentioned, the function $\mathrm{D}(f)$ is related to the modulus of a gradient. Here are two examples:

1. Let each μ_i be two point space $\{-1,1\}$ with the normalized uniform measure. Then $\mathrm{D}(f)$ is discrete gradient (up to the constant $1/4$).

2. Let $\Omega_i = \mathbf{R}^d$ and let $\mu_i = \mu$ be a probability measure with mean 0 and with identity correlation operator. Given a smooth function f on $\mathbf{R}^d$, consider the function f_n on $\Omega = \mathbf{R}^{nd}$ of the form

$$f_n(x_1,\cdots,x_n) = f\left(\frac{x_1+\cdots+x_n}{\sqrt{n}}\right), \quad x_1,\cdots,x_n \in \mathbf{R}^d.$$

If f has bounded first and second partial derivatives, then

$$\mathrm{D}(f_n)\,(x_1,\cdots,x_n) \;=\; \left|\nabla f\left(\frac{x_1+\cdots+x_n}{\sqrt{n}}\right)\right| \left(1+O\left(1/\sqrt{n}\right)\right), \quad \text{as } n\to\infty.$$

The connection of (2.2) and (2.3) with the Gaussian isoperimetric inequality (1.3) is the following: the application of the central limit theorem to the functions f_n in (2.2) yields

$$I(\mathbf{E}_{\gamma_d} f) \;\leq\; \mathbf{E}_{\gamma_d}\sqrt{I(f)^2 + 2\,|\nabla f|^2} \tag{2.10}$$

where the expectations are taken with respect to the Gaussian measure γ_d on $\mathbf{R}^d$. Approximating the indicator function $\mathbf{1}_A$ by smooth functions yields in (2.10)

$$\gamma_d^+(A) \geq \frac{1}{\sqrt{2}}\, I(\gamma_d(A)). \tag{2.11}$$

And, as noted, in the special case where μ is the standard Bernoulli measure on $\{-1,1\}^d$, the factor 2 can be replaced by 1 in (2.2) and (2.10). On this way, one can recover the Gaussian isoperimetric inequality (1.3). But it is unlikely possible to reach (1.3) or even (2.11) via a direct application of the isoperimetric-type inequality (2.3) (in contrast with its functional form (2.2)): the quantity $\mathbf{P}^+(A)$ does not represent the perimeter of A in the 'usual' geometric sense as in (1.3) and (2.11). It could be natural to expect that similarly to the case of smooth functions f_n defined above, for 'regular' sets $A \subset \mathbf{R}^d$ and normalized measure $\mathbf{P}$ on the cube $\left(\{-1,1\}^d\right)^n$, the convergence

$$\lim_{n\to\infty} \mathbf{P}^+\left\{(x_1,\cdots,x_n) : x_i \in \{-1,1\}^d,\ \frac{x_1+\cdots+x_n}{\sqrt{n}} \in A\right\} = \gamma_d^+(A). \tag{2.12}$$

holds. Unexpectedly, (2.12) is false. Already for half-planes $A = \{(x_1,x_2) \in \mathbf{R}^2 : a_1x_1 + a_2x_2 \leq a\}$ with $a_1^2 + a_2^2 = 1$, the limit in (2.12) exists but depends on (a_1,a_2) while the right-hand side does not depend on (a_1,a_2) and is equal to $\varphi(a)$.

Remark 2.6 As another unexpected observation, we would like to mention that Poisson probability measures satisfy inequalities of type (2.10) for moduli of discrete gradients (instead of usual gradients). D.Bakry and M.Ledoux proved in [BaL] that if a probability measure μ on the metric space Ω satisfies, for any ('smooth') function f with values in [0,1],

$$I(\mathbf{E}_\mu f) \ \le \ \mathbf{E}_\mu \sqrt{I(f)^2 + c^2\,|\nabla f|^2},$$

then, every function f on Ω with Lipschitz constant equal to or less than $1/c$ has Gaussian tails and, more precisely, the distribution of f represents a contraction of the canonical Gaussian measure γ_1. In contrast to this property, we shall derive the following corollary from Theorem 2.2.

Denote the Poisson distribution with parameter $\lambda > 0$ by Π_λ. This measure is concentrated on $\mathbf{Z}_+ = \{0, 1, 2, \cdots\}$ and, for every $k \in \mathbf{Z}_+$, $\Pi_\lambda(\{k\}) = e^{-\lambda}\lambda^k/k!$. Note that the tails $\Pi_\lambda(\{i : i \ge k\})$ are 'larger' than Gaussian tails and behave like $\Pi_\lambda(\{k\})$, as $k \to \infty$. Let Π_λ^n denote the corresponding product measure on $\mathbf{Z}_+^n$. For any subset A of $\mathbf{Z}_+^n$, define the function

$$\bar{h}_A(x) = \mathrm{card}\{i \le n : (x \in A,\ x + e_i \notin A) \ \text{ or } \ (x \notin A,\ x + e_i \in A)\},$$

where $(e_i)_{1 \le i \le n}$ denotes the canonical basis in $\mathbf{R}^n$. Put $\partial A = \{x \in \mathbf{R}^n : \bar{h}_A(x) > 0\}$. In the statement below, the difference operator $\Delta f(x) = f(x+1) - f(x)$ acts on the family of all functions f on $\mathbf{Z}_+$.

Proposition 2.7 *Let $I(t) = \varphi(\Phi^{-1}(t))$, $0 \le t \le 1$. For every function f on $\mathbf{Z}_+$ with values in $[0,1]$, we have*

$$I\Big(\int f\,d\Pi_\lambda\Big) \le \int \sqrt{I(f)^2 + 2\lambda\,|\Delta f|^2}\,d\Pi_\lambda. \tag{2.13}$$

Consequently, for any subset A of $\mathbf{Z}_+^n$,

$$\int \sqrt{\bar{h}_A}\,d\Pi_\lambda^n \ \ge \ \frac{1}{\sqrt{2\lambda}}\,I(\Pi_\lambda^n(A)), \tag{2.14}$$

$$\Pi_\lambda^n(\partial A) \ge \frac{1}{\sqrt{2\lambda n}}\,I(\Pi_\lambda^n(A)). \tag{2.15}$$

Proof. By Lemma 2.3, (2.13) extends to n-dimensional case for the operator $\Gamma f(x) = \sqrt{\sum_{i=1}^n (f(x+e_i) - f(x))^2}$. For indicator functions, the n-dimensional variant of (2.13) implies (2.14). To prove (2.13), we apply (2.2) for the discrete cube $\Omega = \{0,1\}^n$ with $p = \lambda/n$ to functions f_n of the form $f_n(x) = f(x_1 + \cdots + x_n)$. Letting $n \to \infty$, it remains to apply Poisson's theorem. At last, (2.14) implies (2.15) since $\bar{h}_A \le \sqrt{n}\,\mathbf{1}_{\partial A}$.

Remark 2.8 Let g be an arbitrary bounded function on $\mathbf{Z}$. Applying (2.13) to functions $f = \frac{1}{2} + \varepsilon g$ with $\varepsilon \to 0$, we get in the limit the Poincaré-type inequality (with suboptimal constant)

$$\mathrm{Var}_{\Pi_\lambda}(g) \le 2\lambda \int |\Delta g|^2\,d\Pi_\lambda. \tag{2.16}$$

In a different way, such inequalities will be studied in section 4. Now let us just note that the optimal constant in (2.16) is equal to λ. Indeed, for any function f on $\mathbf{Z}_+^n$, we have

$$\mathrm{Var}_{\mu_p^n}(f) \le p \int |\Gamma f|^2 \, d\mu_p^n, \tag{2.17}$$

where the operator Γ is the same as in the proof of proposition 2.7 (the inequality (2.17) is obvious for dimension one and extends to multidimensional case by simple induction over n). Applying (2.17) with $p = \lambda/n$ to functions $f_n(x) = g(x_1 + \cdots + x_n)$ and letting $n \to \infty$, we arrive at (2.16). Note that (2.16) turns into an equality for the function $g(x) = x$.

3 Sharp constants in (1.1) and (1.2)

First we show that the factor $\sqrt{2}\min(p,q)/\sqrt{pq}$ can be replaced in (1.2) by a sharper constant, and, as a result, interpretation of (1.2) holds for product measures on the discrete cube with arbitrary marginals. (In the next section, we will give another proof via discrete Poincaré inequalities.) Write $a^+ = \max(a, 0)$. Following [T], for any function f on the discrete cube $\{0,1\}^n$, define

$$Mf(x) = \sqrt{\sum_{i=1}^n ((f(x) - f(s_i(x))^+)^2},$$

so that $M\mathbf{1}_A = \sqrt{h_A}$.

Lemma 3.1 *Let $I(t) = t(1-t)$, for $0 \le t \le 1$. With respect to the measure μ_p on $\{0,1\}$, the optimal constant $c = c_p$ in the inequality*

$$cI(\mathbf{E}\, f) \ \le \ \mathbf{E}\sqrt{c^2 I(f)^2 + M(f)^2} \tag{3.1}$$

is given by

$$c_p \ = \ \min\left\{\sqrt{\frac{2-p}{p}}, \sqrt{\frac{2-q}{q}}\ \right\}. \tag{3.2}$$

Proof. Let $d = 1/c^2$. Since $I(\mathbf{E}\, f) - \mathbf{E}\, I(f) = \mathrm{Var}(f)$, it is better to rewrite (3.1) as

$$\mathrm{Var}(f) \le \mathbf{E}\left[\sqrt{I(f)^2 + dM(f)^2} - I(f)\right].$$

Letting $f(1) = a$, $f(0) = b$, and assuming $a \ge b$, this inequality can be rewritten as

$$pq(a-b)^2 \le p\sqrt{I(a)^2 + d(a-b)^2} - pI(a)$$

which in turn reduces to

$$q^2(a-b)^2 + 2qI(a) \le d. \tag{3.3}$$

To maximize the left-hand side of (3.3), one should take $b = 0$. Therefore,

$$d \ \geq \ \sup_{0 \leq a \leq 1} qa\,(2 - a(2-q)) = \frac{q}{2-q} \tag{3.4}$$

(the maximum is attained at $a = 1/(2-q) \in [0,1]$). Similarly we obtain for $a \leq b$ that $d \geq \frac{p}{2-p}$. This together with (3.4) implies that (3.1) holds for any constant $c > 0$ such that $d = 1/c^2 \geq 1/c_p^2$. Hence, the optimal value is $c = c_p$, and Lemma 3.1 is proved.

As a result, by Lemma 2.3 with $\Gamma(f) = M(f)$, (3.1) extends to the multidimensional case. Thus:

Proposition 3.2 *For any subset A of $\{0,1\}^n$, we have*

$$\int \sqrt{h_A(x)}\ d\mu_p^n(x) \ \geq \ c_p\,\mu_p^n(A)\,(1 - \mu_p^n(A)), \tag{3.5}$$

where c_p is given by (3.2)*. In addition, for any product measure $\mathbf{P}$ on $\{0,1\}^n$ (with possibly different marginals) and for any subset A of $\{0,1\}^n$,*

$$\int \sqrt{h_A(x)}\ d\mathbf{P}(x) \ \geq \ \mathbf{P}(A)\,(1 - \mathbf{P}(A)). \tag{3.6}$$

Note that $1 \leq c_p \leq \sqrt{3}$, for all $p \in [0,1]$, that is, (3.5) is a very slight improvement of (3.6). Inequalities (3.5) and (3.6) lead to a natural question: what is the optimal function I_0 satisfying

$$\int \sqrt{h_A(x)}\ d\mathbf{P}(x) \ \geq \ I_0(\mathbf{P}(A)), \tag{3.7}$$

for all product measures $\mathbf{P}$? Let us apply (3.7) for $n = 1$ to $\mathbf{P} = \mu_p$ and $A = \{1\}$: it yields $p \geq I_0(p)$. Therefore, the estimate $\mathbf{P}(A)\,(1 - \mathbf{P}(A))$ in (3.6) is of the right order when $\mathbf{P}(A)$ is small.

The arguments of Lemma 3.1 are of course valid for arbitrary functions I. Setting $f(1) = a$, $f(0) = b$, $d = 1/c^2$ and assuming $a \geq b$, (3.1) may be written as

$$I(pa + qb) - qI(b) \ \leq \ p\sqrt{I(a^2) + d(a-b)^2}.$$

¿From this, d should satisfy $d \ \geq \ d_p(I)$ where

$$d_p(I) \ = \ \sup_{1 \geq a > b \geq 0} \frac{((I(pa+qb) - qI(b))^+)^2 - p^2 I(a)^2}{p^2(a-b)^2}. \tag{3.8}$$

Repeating these arguments for the case $a < b$ and applying Lemma 2.3, we conclude:

Lemma 3.3 *Let I be an arbitrary non-negative function on $[0,1]$ such that $I(0) = I(1) = 1$. With respect to the measure μ_p^n on $\{0,1\}^n$, the optimal constant $c = c_p(I)$ in the inequality*

$$c\,I(\mathbf{E}\,f) \ \leq \ \mathbf{E}\,\sqrt{c^2 I(f)^2 + M(f)^2} \tag{3.9}$$

is given by

$$c_p(I) = \min\left\{\frac{1}{\sqrt{d_p(I)}}\,,\frac{1}{\sqrt{d_q(I)}}\right\}.$$

Hence, for any subset A of $\{0,1\}^n$, we have with $c = c_p(I)$:

$$\int \sqrt{h_A(x)}\; d\mu_p^n(x) \;\geq\; c\, I(\mu_p^n(A)). \tag{3.10}$$

It seems difficult to compute $c_p(I)$ exactly on the base of (3.8) but its behaviour can be explored in some special cases. But even in this case, one can not guarantee that this constant will be optimal (or, asymptotically optimal) for the inequality (3.10) which is weaker than its functional form (3.9). Neverthless, for the function $I(t) = J(t(1-t))$ where $J(t) = t\sqrt{\log(1/t)}$ is as in (1.1), we have the following:

Proposition 3.4 *Let $I(t) = J(t(1-t))$. $t \in [0,1]$. For some positive K_0 and K_1, and for all $p \in (0,1)$, the optimal constant c in* (3.10) *as well as the optimal constant c in* (3.9) *satisfies*

$$K_0 \frac{1}{\sqrt{\log(1/(pq))}} \leq c \leq K_1 \frac{1}{\sqrt{\log(1/(pq))}}.$$

The lower bound will be obtained in section 6 (the proof is rather routine), and here we will show the upper estimate. Let us test (3.10) on sets $A = \{S_n = 0\}$. Since $h_A = n\mathbf{1}_A$, (3.10) becomes

$$\sqrt{n} q^n \;\geq\; c\, q^n(1-q^n)\sqrt{\log\frac{1}{q^n(1-q^n)}}.$$

Letting $n \to \infty$, we obtain $c \leq 1/\sqrt{\log(1/q)}$. Analogously, testing (3.10) on the sets $A = \{S_n = n\}$, we arrive at $c \leq 1/\sqrt{\log(1/p)}$. Both estimates give $c \leq 1/\sqrt{\log(1/p')}$ where $p' = \min(p,q)$. But p' is of order pq.

Remark 3.5 In [T], M.Talagrand used (and proved) the following Sobolev-type inequalities to derive (1.1) and (1.2):

$$\int |Mf(x)|\, d\mu_p^n(x) \geq K(p)\Psi(\mathrm{Var}(f)) \tag{3.11}$$

where the variance is taken with respect to the measure μ_p^n. If $\Psi(t) = J(t)$, $t \in [0,1/4]$, then (3.11) corresponds to (1.1) (on indicator functions f), and when $\Psi(t) = t$, $t \in [0,1/4]$, it corresponds to (1.2). However, these inequalitites are not additive: the optimal constant $K(p)$ depends on n. In contrast with (3.1) the proof by induction will be rather complicated and might lead to an overestimation of the constant K.

4 Inequalities on the lattice $\mathbf{Z}^n$

For any subset A of $\mathbf{Z}^n$, we introduced the function

$$h_A(x) = \mathrm{card}\{i \leq n : x \in A,\ (x + e_i \notin A \text{ or } x - e_i \notin A)\},$$

where $(e_i)_{i \leq n}$ is the canonical basis in $\mathbf{R}^n$. The interior boundary $\partial_+ A = \{h_A > 0\}$ consists of those points x in A which can leave A at least in one of the n directions. For functions f on $\mathbf{Z}^n$, the corresponding definition of Mf should be

$$Mf(x) = \sqrt{\sum_{i=1}^{n} ((f(x) - f(x - e_i))^+)^2 + ((f(x) - f(x + e_i))^+)^2}.$$

Define also the operator $Df(x) = \sqrt{\sum_{i=1}^{n} (f(x + e_i) - f(x))^2}$ so that in dimension one $Df(x) = |\Delta f(x)| = |f(x+1) - f(x)|$ is the modulus of difference operator. In this section, we will describe the class of all probability measures on $\mathbf{Z}^n$ which admit (1.2)-type inequality with a dimension free constant.

For every probability measure μ on the real line $\mathbf{R}$ with distribution function $F_\mu(x) = \mu((-\infty, x])$, $x \in \mathbf{R}$, define the quantile function

$$F_\mu^{-1}(p) = \inf\{x : F_\mu(x) \geq p\}, \quad p \in (0, 1],$$

which take values in $(-\infty, +\infty]$. As usual, we denote the corresponding product measure on $\mathbf{R}^n$ by μ^n. As in section 1, we denote by ν the measure with density $\frac{1}{2}\exp(-|x|)$, that is, ν is the two-sided exponential distribution. Note that there always exists the (unique) left continuous non-decreasing map $U_\mu : \mathbf{R} \to \mathbf{R}$ which transport ν to μ. Indeed, $U_\mu(x) = F_\mu^{-1}(F_\nu(x))$. If μ is concentrated on $\mathbf{Z}$, then F_μ is a step function, and U_μ takes values in $\mathbf{Z}$.

Theorem 4.1 *Let a probability measure μ be concentrated on $\mathbf{Z}$. Assume that the support of μ represents an interval in $\mathbf{Z}$. The following properties a) -f) are equivalent:*

a) There exists $c > 0$ such that, for every $n \geq 1$ and every subset A of $\mathbf{Z}^n$,

$$\int \sqrt{h_A(x)}\, d\mu^n(x) \ \geq\ c\, \mu^n(A)\,(1 - \mu^n(A)). \tag{4.1}$$

b) There exists $c > 0$ such that, for every $n \geq 1$ and every subset A of $\mathbf{Z}^n$,

$$\mu^n(\partial_+ A) \ \geq\ \frac{c}{\sqrt{n}}\, \mu^n(A)\,(1 - \mu^n(A)). \tag{4.2}$$

c) The inequality (4.1), or the inequality (4.2), hold for $n = 1$ with some $c > 0$.

d) The measure μ admits the following Poincaré-type inequality:

$$c\,\mathrm{Var}_\mu(f) \ \leq\ \mathbf{E}_\mu (Mf)^2 \tag{4.3}$$

where $c > 0$ and where f is an arbitrary function on $\mathbf{Z}$. In this inequality, Mf may be replaced with Df.

e) There exists $c > 0$ *such that* $c\,F_\mu(a)(1 - F_\mu(a)) \leq \mu(\{a\})$, *for all* $a \in \mathbf{Z}$.

f) $\sup_{x\in\mathbf{R}} [U_\mu(x + h) - U_\mu(x)] \leq 1$, *for some* $h > 0$.

The assumption on the support of μ is necessary for all properies $a) - f)$. For $a) - c)$, apply (4.2) with $n = 1$ to the sets $A = (-\infty, a] \cap \mathbf{Z}$ and $A = [a, +\infty) \cap \mathbf{Z}$. To see the necessity for $d)$, assume that $a \in \mathbf{Z}$ is such that $\mu(\{a\}) = 0$ but $p = \mu((-\infty, a-1]) > 0$ and $q = \mu([a + 1, +\infty)) > 0$. Then, for f the indicator function of $(-\infty, a]$, we would have that $Mf = Df = 0$ (μ-almost everywhere) while $\mathrm{Var}_\mu(f) = pq > 0$. The same argument applies to $e)$. At last, it follows from $e)$ that the image $U_\mu(\mathbf{R})$ should be an interval in $\mathbf{Z}$.

Poincaré-type inequalities are well-known to be of additive type, that is, they can be extended to higher dimensions without any loss in constants. However, the multidimensional variant of (4.3) is unlikely to be appropriate to derive isoperimetric-type inequalities like (4.1)–(4.2). To perform the induction step, it is better to work instead with the inequality

$$I(\mathbf{E}_\mu f) \;\leq\; \mathbf{E}_\mu \sqrt{I(f)^2 + dM(f)^2} \tag{4.4}$$

where $I(t) = t(1-t)$ and f takes values in [0,1], as in Lemma 3.1. According to Lemma 2.3, this inequality is additive, as well, and the application of the multidimensional variant of (4.4) to indicator functions yields (4.1) (while the multidimensional variant of (4.3) yields on indicator functions just an estimate for $\int h_A\, d\mu^n$ which is weaker). Furthermore, note that $h_A \leq n\mathbf{1}_{\partial_+ A}$. Hence, (4.2) is a direct consequence of (4.1).

Now we are left with the question whether (4.3) and (4.4) are equivalent in one dimension. If μ is continuous and $Mf = |f'|$, then (4.3) and (4.4) are not equivalent ((4.4) is stronger). However, for the discrete-type gradient, the situation changes considerably. The crucial observation is that $Mf \leq \sqrt{2}$ holds whenever $0 \leq f \leq 1$. Indeed, to see the equivalence of (4.3) and (4.4), first let us apply (4.4) to functions $\frac{1}{2} + \varepsilon f$ with f bounded and such that $\mathbf{E}_\mu f = 0$. We obtain, as $\varepsilon \to 0$,

$$\mathrm{Var}_\mu(f) \leq 2d\,\mathbf{E}_\mu M(f)^2.$$

Hence, (4.4) implies (4.3) with $c = 1/(2d)$. To prove the converse, note that, for all $u \in [0, 1/4]$ and $v \in [0, \sqrt{2}]$,

$$\sqrt{u^2 + dv^2} - u \geq Cv^2, \quad \text{with } \; d = \frac{C}{2} + 2C^2.$$

For $u = I(f)$, $v = Mf$, and for $C = 1/c$, we thus obtain

$$\mathbf{E}_\mu \left[\sqrt{I(f)^2 + dM(f)^2} - I(f)\right] \geq C\,\mathbf{E}_\mu (Mf)^2 \geq \mathrm{Var}_\mu(f) = I(\mathbf{E}_\mu f) - \mathbf{E}_\mu I(f).$$

Hence, (4.3) also implies (4.4) with the constant $d = \frac{1}{2c} + \frac{2}{c^2}$.

Thus, via (4.4) and Lemma 2.3, we have reduced Theorem 4.1 to one dimension. Here it remains to show the implications $d) \Rightarrow e) \Rightarrow f) \Rightarrow d)$, only We start with some preparations.

Lemma 4.2 *If $x \leq z \leq y$, then*

$$\frac{F_\nu(y) - F_\nu(x)}{F_\nu(z)\,(1 - F_\nu(z))} \ \leq\ 2(\exp(y - x) - 1).$$

Proof. Clearly, it suffices to consider the cases $z = x$ and $z = y$, only. These cases are equivalent since ν is symmetric around 0, so let $z = x$. Recall that $F_\nu(u) = \frac{1}{2}\exp(u)$, for $u \leq 0$, and $F_\nu(u) = 1 - \frac{1}{2}\exp(-u)$, for $u \geq 0$. Set $h = y - x$. If $0 \leq x \leq y$, then

$$\frac{F_\nu(y) - F_\nu(x)}{F_\nu(x)\,(1 - F_\nu(x))} = \frac{1 - e^{-h}}{F_\nu(x)} \leq 2(1 - e^{-h}) \leq 2(e^h - 1).$$

If $x \leq y \leq 0$, then

$$\frac{F_\nu(y) - F_\nu(x)}{F_\nu(x)\,(1 - F_\nu(x))} = \frac{e^h - 1}{1 - F_\nu(x)} \leq 2(e^h - 1).$$

Finally, if $x \leq 0 \leq y$, then

$$\frac{F_\nu(y) - F_\nu(x)}{F_\nu(x)\,(1 - F_\nu(x))} = \frac{2e^{-x} - e^{-2x-h} - 1}{1 - F_\nu(x)} \leq 2(2e^{-x} - e^{-2x-h} - 1) \leq 2(e^h - 1),$$

since the last inequality reduces to $2 \leq e^{x+h} + e^{-(x+h)}$.

Lemma 4.3 *Any of the properties a)–d) implies e).*

Proof. First assume (4.2) holds for dimension 1:

$$\mu(\partial_+ A) \ \geq\ c\,\mu(A)\,(1 - \mu(A)).$$

Now apply this inequality to $A = (-\infty, a] \cap \mathbf{Z}$ with $a \in \mathbf{Z}$. We have $\partial_+ A = \{a\}$, therefore,

$$F_\mu(a) - F_\mu(a-1) = \mu(\{a\}) \geq c\,F_\mu(a)\,(1 - F_\mu(a)) \tag{4.5}$$

which is exactly the property *e*). The same inequality can also be derived from (4.3). Indeed, let f be the indicator function of the set $A = (-\infty, a] \cap \mathbf{Z}$, $a \in \mathbf{Z}$. Then $Mf = \mathbf{1}_{\{a\}}$ and, for such an f, (4.3) implies (4.5). The same remark applies to Df, since $Df = Mf$ for non-increasing f.

Lemma 4.4 *Property e) implies f).*

Proof. Assume that (4.5) holds and define the function $U_\mu^{-1} : \mathbf{R} \to [-\infty, +\infty]$, using the identity

$$\{t : U_\mu(t) \leq a\} = (-\infty, U_\mu^{-1}(a)], \quad a \in \mathbf{R}.$$

Since U_μ is left continuous, we may define the function U_μ^{-1} via $U_\mu^{-1}(a) = F_\nu^{-1}(F_\mu(a))$. ¿From this definition, $F_\mu(a) = F_\nu(U_\mu^{-1}(a))$, for all $a \in \mathbf{R}$. With $x = U_\mu^{-1}(a-1)$, $z = y = U_\mu^{-1}(a)$, one can rewrite (4.5) as follows:

$$F_\nu(y) - F_\nu(x) \geq c\,F_\nu(z)\,(1 - F_\nu(z)).$$

Consequently, by Lemma 4.2, $c \leq 2\,[\exp(y-x)-1]$, provided $x<+\infty$ or $y>-\infty$. That is,

$$U_\mu^{-1}(a) - U_\mu^{-1}(a-1) \geq \log\left(1+\frac{c}{2}\right),$$

for all $a \in \mathbf{R}$ such that $U_\mu^{-1}(a-1) < +\infty$ or $U_\mu^{-1}(a) > -\infty$. Anyway, for all $a \in \mathbf{R}$,

$$U_\mu^{-1}(a) \geq U_\mu^{-1}(a-1) + \log\left(1+\frac{c}{2}\right). \tag{4.6}$$

Since, by definition of U_μ^{-1},

$$(-\infty, x] \subset \{t : U_\mu(t) \leq U_\mu(x)\} = (-\infty, U_\mu^{-1}(U_\mu(x))],$$

we get $U_\mu^{-1}(U_\mu(x)) \geq x$, for all $x \in \mathbf{R}$. Therefore, putting in (4.6) $a = U_\mu(x)+1$ and $h = \log(1+\frac{c}{2})$, we obtain

$$U_\mu^{-1}(U_\mu(x)+1) \geq U_\mu^{-1}(U_\mu(x)) + h \geq x+h.$$

The definition of U_μ^{-1} implies $U_\mu(U_\mu^{-1}(a)) \leq a$, for all $a \in \mathbf{R}$, where, if necessary, we understand the values $U_\mu(-\infty)$ and $U_\mu(+\infty)$ in the usual (limit) sense. Thus

$$U_\mu(x+h) \leq U_\mu(U_\mu^{-1}(U_\mu(x)+1)) \leq U_\mu(x)+1,$$

and therefore, $\sup_{x\in\mathbf{R}} [U_\mu(x+h) - U_\mu(x)] \leq 1$ which proves Lemma 4.4.

For the part $f) \Rightarrow d)$, first we study some properties of the measure ν.

Lemma 4.5 *For any non-negative measurable function f on $\mathbf{R}$, and for all $h \in \mathbf{R}$, we have*

$$e^{-|h|}\,\mathbf{E}_\nu f(x) \ \leq\ \mathbf{E}_\nu f(x+h) \ \leq\ e^{|h|}\,\mathbf{E}_\nu f(x),$$

$$e^{-2|h|}\,\mathrm{Var}_\nu f(x) \leq \mathrm{Var}_\nu f(x+h) \leq e^{2|h|}\,\mathrm{Var}_\nu f(x).$$

Proof. The density p_ν of ν satisfies $p_\nu(x+h)/p_\nu(x) \leq e^{|h|}$.

The second inequality of Lemma 4.5 will be only used to prove Lemma 4.7 below. So, while referring to Lemma 4.5 in other places, we mean the first inequality. The following two lemmas will be used in the proof of Lemma 4.8 so that to establish (4.3) for the operators M and D, respectively.

Lemma 4.6 *For any $a \in \mathbf{R}$ and $h>0$, there exists $c = c(a,h) > 0$ such that, for every non-decreasing function f on $\mathbf{R}$ with $f(0)=0$,*

$$c\,\mathbf{E}_\nu f(x)^2 \ \leq\ \mathbf{E}_\nu \left[(f(x)-f(x-h))^2\mathbf{1}_{\{x>a\}} + (f(x+h)-f(x))^2\mathbf{1}_{\{x\leq a-h\}}\right].$$

Proof. By Lemma 4.5, the above inequality holds with $c(a,h) = e^{-2|a|}c(0,h)$, so we may only consider the case $a=0$. First we show, that for any non-decreasing function f such that $f \equiv 0$ on $(-\infty,0]$,

$$\mathbf{E}_\nu f(x+h) \geq (2-e^{-h})\,\mathbf{E}_\nu f(x). \tag{4.7}$$

Indeed, this inequality is linear in f, so it suffices to check it on indicator functions $f = \mathbf{1}_{(a,+\infty)}$, $a \geq 0$ (since every non-decreasing, left continuous function f vanishing on $(-\infty, 0]$ can be represented as a mixture of such indicators). For $f = \mathbf{1}_{(a,+\infty)}$, (4.7) takes the form

$$1 - F_\nu(a-h) \geq (2 - e^{-h})\,(1 - F_\nu(a)), \quad a, h \geq 0.$$

When $a \geq h$, this is simply $e^h + e^{-h} \geq 2$. When $0 \leq a \leq h$, it becomes

$$1 - \frac{1}{2}\, e^{a-h} \geq \frac{1}{2}\,(2 - e^{-h})\, e^{-a}$$

which is also evident (with equality for $a = 0$). Let us now rewrite (4.7) in the following way:

$$(1 - e^{-h})\, \mathbf{E}_\nu f(x) \leq \mathbf{E}_\nu (f(x+h) - f(x)) \tag{4.8}$$

When $f \equiv 0$ on $(-\infty, 0]$, one can also apply (4.8) to f^2 so that, by Cauchy-Schwarz inequality and by Lemma 4.5,

$$\begin{aligned}(1 - e^{-h})\,\|f\|_2^2 \ = \ (1 - e^{-h})\,\mathbf{E}_\nu f(x)^2 \ &\leq \ \mathbf{E}_\nu (f(x+h) - f(x))\,(f(x+h) + f(x)) \\ &\leq \ \|f(x+h) - f(x)\|_2\, \|f(x+h) + f(x)\|_2 \\ &\leq \ \|f(x+h) - f(x)\|_2 \left(e^{h/2}\|f\|_2 + \|f\|_2\right).\end{aligned}$$

Hence,

$$\left(\frac{1 - e^{-h}}{1 + e^{h/2}}\right)^2 \mathbf{E}_\nu f(x)^2 \ \leq \ \mathbf{E}_\nu (f(x+h) - f(x))^2.$$

By Lemma 4.5, and since $f(x) - f(x-h) = 0$ for $x \leq 0$,

$$c(h)\, \mathbf{E}_\nu f(x)^2 \ \leq \ \mathbf{E}_\nu (f(x) - f(x-h))^2\, \mathbf{1}_{\{x>0\}} \tag{4.9}$$

where $c(h) = e^{-h} \left(\frac{1-e^{-h}}{1+e^{h/2}}\right)^2$. By symmetry, for any non-decreasing function f such that $f \equiv 0$ on $[0, +\infty)$, we get as well

$$c(h)\, \mathbf{E}_\nu f(x)^2 \ \leq \ \mathbf{E}_\nu (f(x+h) - f(x))^2\, \mathbf{1}_{\{x<0\}}. \tag{4.10}$$

Consider now the general case. Let $f_0 = f\, \mathbf{1}_{(-\infty,0]}$ and $f_1 = f\, \mathbf{1}_{[0,+\infty)}$. Applying (4.9) to f_1 and (4.10) to f_0, summing these inequalities and noting that, for all $x \in \mathbf{R}$,

$$f_1(x+h) - f_1(x) \leq f(x+h) - f(x), \quad \text{and} \quad f_0(x) - f_0(x-h) \leq f(x) - f(x-h),$$

we obtain

$$c(h)\, \mathbf{E}_\nu f(x)^2 \leq \mathbf{E}_\nu (f(x) - f(x-h))^2 \mathbf{1}_{\{x>0\}} + \mathbf{E}_\nu (f(x+h) - f(x))^2 \mathbf{1}_{\{x<0\}}.$$

But, again by Lemma 4.5,

$$\begin{aligned}\mathbf{E}_\nu (f(x+h) - f(x))^2 \mathbf{1}_{\{-h<x<0\}} \ &\leq \ e^h\, \mathbf{E}_\nu (f(x) - f(x-h))^2 \mathbf{1}_{\{0<x<h\}} \\ &\leq \ e^h\, \mathbf{E}_\nu (f(x) - f(x-h))^2 \mathbf{1}_{\{x>0\}},\end{aligned}$$

so,

$$\begin{aligned} c(h)\,\mathbf{E}_\nu f(x)^2 &\le (1+e^h)\,\mathbf{E}_\nu (f(x)-f(x-h))^2 \mathbf{1}_{\{x>0\}} \\ &\quad + \mathbf{E}_\nu (f(x+h)-f(x))^2 \mathbf{1}_{\{x\le -h\}}. \end{aligned}$$

This yields the desired result with $c(0,h) = c(h)/(1+e^h)$.

Lemma 4.7 *For any real $a > b$, there exists a constant $c = c(a,b) > 0$ such that, for every non-decreasing function f,*

$$c\,\mathrm{Var}_\nu(f(x)) \le \mathbf{E}_\nu (f(x+a) - f(x+b))^2.$$

Proof. Due to Lemma 4.5, it suffices to consider the case $a = h > 0$, $b = -h$. In addition, one may assume that $f(0) = 0$. Then, apply Lemma 4.6 with $a = 0$ and note that $f(x) - f(x-h) \le f(x+h) - f(x-h)$, $f(x+h) - f(x) \le f(x+h) - f(x-h)$.

The last step in the proof of Theorem 4.1 is the following:

Lemma 4.8 *Property f) implies property d).*

Proof. Let $h > 0$ be such that $U_\mu(x+h) \le U_\mu(x) + 1$, for all $x \in \mathbf{R}$. Let f be a non-decreasing function on $\mathbf{Z}$. Applying Lemma 4.7 with $a = h$ and $b = 0$ to the function $f(U_\mu)$, we obtain

$$\begin{aligned} c\,\mathrm{Var}_\mu(f(y)) = c\,\mathrm{Var}_\nu(f(U_\mu(x))) &\le \mathbf{E}_\nu (f(U_\mu(x+h)) - f(U_\mu(x)))^2 \\ &\le \mathbf{E}_\nu (f(U_\mu(x)+1) - f(U_\mu(x)))^2 \\ &= \mathbf{E}_\mu (f(y+1) - f(y))^2. \end{aligned}$$

That is, $c\,\mathrm{Var}_\mu(f) \le \mathbf{E}_\mu (Df)^2$, and (4.3) is thus proved for D defined on the class of non-decreasing functions f. Assume that μ is not a Dirac measure: let for definiteness $p = \mu(\{0\}) > 0$ and $p_1 = \mu(\{1\}) > 0$. Then, for any function $f \in L^2(\mathbf{Z},\mu)$ with $f(0) = 0$, we get by the Cauchy-Schwarz inequality,

$$(\mathbf{E}_\mu f)^2 = (\mathbf{E}_\mu f\,\mathbf{1}_{\{x\ne 0\}})^2 \le (1-p)\,\mathbf{E}_\mu f^2.$$

Hence, $\mathrm{Var}_\mu(f) \ge p\,\mathbf{E}_\mu f^2$. For a non-decreasing f with $f(0) = 0$, we arrive at

$$cp\,\mathbf{E}_\mu f(y)^2 \le \mathbf{E}_\mu (Df(y))^2. \tag{4.11}$$

To treat the case of the operator M, let $a \in \mathbf{R}$ be the maximal solution to equation $U_\mu(a) = 0$ (such an a exists since the function U_μ is left continuous and $\mu([1,+\infty)) > 0$). Then, for all $x > a$, we have $y = U_\mu(x) \ge 1$, while, for $x \le a-h$, we have $y = U_\mu(x) \le 0$. In the same way, applying Lemma 4.6 to these values of a and h, we obtain, for any non-decreasing f with $f(0) = 0$, that

$$c\,\mathbf{E}_\mu f(y)^2 \le \mathbf{E}_\mu (Df(y-1))^2 \mathbf{1}_{\{y\ge 1\}} + \mathbf{E}_\mu (Df(y))^2 \mathbf{1}_{\{y\le 0\}}. \tag{4.12}$$

The second expectation on the right contains the term $(Df(0))^2 p$ which can be estimated from above by $\frac{p}{p_1}\,\mathbf{E}_\mu (Df(y-1))^2 \mathbf{1}_{\{y\geq 1\}}$. Thus, we get from (4.12)

$$c_1\,\mathbf{E}_\mu f(y)^2 \ \leq\ \mathbf{E}_\mu (Df(y-1))^2 \mathbf{1}_{\{y\geq 1\}} + \mathbf{E}_\mu (Df(y))^2 \mathbf{1}_{\{y\leq -1\}}, \tag{4.13}$$

with the constant $c_1 = c/(1+\frac{p}{p_1})$. As a result, (4.11) and (4.13) hold in the class of non-decreasing functions f with $f(0)=0$.

Now we extend (4.11) and (4.13) to an arbitrary function f with $f(0)=0$. Define the function g as follows. Let $g(0)=0$,

$$g(y) = \sum_{i=1}^{y} |f(i)-f(i-1)|, \qquad \text{for } \ y\geq 1,$$
$$g(y) = -\sum_{i=y}^{-1} |f(i)-f(i+1)|, \qquad \text{for } \ y\leq -1.$$

Then, for all $y\in\mathbf{Z}$, $|g(y)|\geq |f(y)|$ and $g(y+1)-g(y) = |f(y+1)-f(y)|$. Since g is non-decreasing and $g(0)=0$, by (4.11), we get

$$cp\,\mathbf{E}_\mu f(y)^2 \leq cp\,\mathbf{E}_\mu g(y)^2 \leq\ \mathbf{E}_\mu (Dg(y))^2 = \mathbf{E}_\mu (Df(y))^2.$$

By similar arguments, (4.13) holds for f, too. Thus, $cp\,\mathrm{Var}_\mu(f) \leq \mathbf{E}_\mu (Df)^2$, and, since the condition $f(0)=0$ can be assumed in (4.3), (4.3) is proved in general for the operator D.

Now we prove (4.3) for the operator M using (4.13). First let us observe that, for all $y\in\mathbf{Z}$,

$$(Df(y))^2 \leq (Mf(y))^2 + (Mf(y+1))^2.$$

Hence, by (4.13),

$$\begin{aligned} c_1\,\mathrm{Var}_\mu(f) \ \leq\ c_1\,\mathbf{E}_\mu f(y)^2 \ &\leq\ \mathbf{E}_\mu \left[(Mf(y))^2\mathbf{1}_{\{y\geq 1\}} + (Mf(y-1))^2\mathbf{1}_{\{y\geq 1\}}\right] \\ &+\ \mathbf{E}_\mu \left[(Mf(y))^2\mathbf{1}_{\{y\leq -1\}} + (Mf(y+1))^2\mathbf{1}_{\{y\leq -1\}}\right]. \end{aligned}$$

Therefore, it suffices to show that the inequalities

$$\mathbf{E}_\mu (Mf(y-1))^2 \mathbf{1}_{\{y\geq 1\}} \leq \alpha\,\mathbf{E}_\mu (Mf(y))^2 \mathbf{1}_{\{y\geq 0\}}, \tag{4.14}$$

$$\mathbf{E}_\mu (Mf(y+1))^2 \mathbf{1}_{\{y\leq -1\}} \leq \beta\,\mathbf{E}_\mu (Mf(y))^2 \mathbf{1}_{\{y\leq 0\}} \tag{4.15}$$

hold with some α and β not depending on f. In fact, (4.14)–(4.15) are valid for an arbitrary non-negative function V and thus for $V=(Mf)^2$ in particular. Indeed, since we have proved (4.3) for the operator D, we can use the property (4.5) (as its consequence according to Lemma 4.3). ¿From (4.5), for any $y\geq 0$ integer,

$$\mu(\{y\}) \geq c\,F_\mu(y)\,(1-F_\mu(y)) \geq c\,F_\mu(0)\,\mu(\{y+1\}).$$

Hence,

$$\mathbf{E}_{\mu} V(y-1)\,\mathbf{1}_{\{y\geq 1\}} \;=\; \sum_{y=0}^{\infty} V(y)\mu(\{y+1\}) \;\leq\; \frac{1}{c\,F_{\mu}(0)}\,\mathbf{E}_{\mu} V(y)\,\mathbf{1}_{\{y\geq 0\}}.$$

Analogously, for $y \leq -1$,

$$\mu(\{y\}) \geq c\,F_{\mu}(y)\,(1-F_{\mu}(y)) \geq c\,\mu(\{y-1\})\,(1-F_{\mu}(-1)).$$

Hence,

$$\mathbf{E}_{\mu} V(y+1)\,\mathbf{1}_{\{y\leq -2\}} \;=\; \sum_{y=-\infty}^{-1} V(y)\mu(\{y-1\}) \;\leq\; \frac{1}{c\,(1-F_{\mu}(-1))}\,\mathbf{E}_{\mu} V(y)\,\mathbf{1}_{\{y\leq -1\}}.$$

In addition,

$$\mathbf{E}_{\mu} V(y+1)\,\mathbf{1}_{\{y=-1\}} \;=\; V(0)\,\mu(\{-1\}) \;\leq\; \frac{1}{p}\,\mathbf{E}_{\mu} V(y)\,\mathbf{1}_{\{y=0\}}.$$

Since $F_{\mu}(0) \geq p > 0$ and $1 - F_{\mu}(-1) \geq p > 0$, we get (4.14) and (4.15). Thus Lemma 4.8 and Theorem 4.1 are proved.

5 Discrete Poincaré inequalities on the real line

As announced before, we shall prove in this section the following result. Let μ be an arbitrary probability measure on the real line $\mathbf{R}$. Recall that $U_{\mu} : \mathbf{R} \to \mathbf{R}$ denotes the left continuous non-decreasing map which transforms the two-sided exponential distribution ν into μ.

Theorem 5.1 *The following properties are equivalent:*

a) There exist $h > 0$ and C such that, for every non-decreasing function f on $\mathbf{R}$,

$$\int f^2\,d\mu - \Big(\int f\,d\mu\Big)^2 \leq C \int |f(x+h) - f(x)|^2\,d\mu(x). \tag{5.1}$$

b) There exists C such that, for every differentiable convex function f on $\mathbf{R}$,

$$\int f^2\,d\mu - \Big(\int f\,d\mu\Big)^2 \leq C \int |f'|^2\,d\mu(x). \tag{5.2}$$

c) There exists C such that, for all $n \geq 1$ and every convex function f on $\mathbf{R}^n$ with Lipschitz constant at most 1,

$$\int_{\mathbf{R}^n} f^2\,d\mu^n - \Big(\int_{\mathbf{R}^n} f\,d\mu^n\Big)^2 \leq C. \tag{5.3}$$

d) For some (or, equivalently, for all) $h > 0$,

$$\sup_{x\in\mathbf{R}}\,[U_{\mu}(x+h) - U_{\mu}(x)] \;<\; +\infty.$$

Proof. $a) \Rightarrow d)$. Let F_μ be distribution function of μ. If we apply (5.1) to indicator functions $f = \mathbf{1}_{(a,+\infty)}$, we get

$$F_\mu(a) - F_\mu(a-h) \le C F_\mu(a)(1 - F_\mu(a)).$$

This is exactly (4.5) with $c = 1/C$ and $h = 1$. The same arguments as in the proof of Lemma 4.4 ensure that, for all $x \in \mathbf{R}$,

$$U_\mu\left(x + \log \frac{1}{2C}\right) - U_\mu(x) \le h.$$

$d) \Rightarrow a)$. Since (5.1) holds for ν (apply Lemma (4.7) to $a = h$ and $b = 0$), we get (5.1) for μ with $h^* = \sup_{x \in \mathbf{R}} [U_\mu(x+h) - U_\mu(x)]$ which replaces h.

$d) \Rightarrow b)$. Again apply Lemma (4.7), but now to $a = 0$ and $b = -h$: for any non-decreasing function g, we have

$$\mathrm{Var}_\nu(g(x)) \ \le \ C\mathbf{E}_\nu(g(x) - g(x-h))^2. \tag{5.4}$$

Assuming that f is non-decreasing, apply now (5.4) to $g = f(U_\mu)$. We then obtain

$$\mathrm{Var}_\mu(f(y)) \ \le \ C\mathbf{E}_\mu(f(y) - f(y-h^*))^2$$

with h^* defined in the previous step. Since f is convex and non-decreasing,

$$0 \le f(y) - f(y-h^*) \le h^* f'(y).$$

Hence, (5.2) holds (with constant $C\,(h^*)^2$ and parameter h^*) for all convex and non-decreasing functions f. Similarly, it holds for all convex and non-increasing functions f. Consider now the third case: f is non-increasing on an interval $(-\infty, a]$ and is non-decreasing on $[a, +\infty)$. One may assume that $f(a) = 0$. As already proved, (5.2) holds for functions $f_0 = f\mathbf{1}_{[a,+\infty)}$ and $f_1 = f\mathbf{1}_{(-\infty,a]}$ since these functions are convex and monotone. Writing $a_0 = \mathbf{E}_\mu f_0$, $a_1 = \mathbf{E}_\mu f_1$, and noting that $f = f_0 + f_1$, we obtain

$$\begin{aligned}
\mathrm{Var}_\mu(f) = \|f - \mathbf{E}_\mu f\|_2 \ &\le \ \|f_0 - a_0\|_2 + \|f_1 - a_1\|_2 \\
&\le \ \sqrt{2C}\,\|f_0'\|_2 + \sqrt{2C}\,\|f_1'\|_2 \\
&\le \ 2\sqrt{C}\,\|f'\|_2,
\end{aligned}$$

where $\|\cdot\|_2$ denotes the norm in the space $L^2(\mu)$. In the last inequality we applied $\sqrt{u_0} + \sqrt{u_1} \le \sqrt{2(u_0+u_1)}$ to $u_0 = \mathbf{E}_\mu(f_0')^2$ and $u_1 = \mathbf{E}_\mu(f_1')^2$. Thus, (5.2) is proved in the general case.

$b) \Rightarrow c)$. Apply the multidimensional variant of (5.2) to f.

$c) \Rightarrow d)$. One can apply (5.3) to the functions $f_n(x) = \max(x_1, \dots, x_n)$ and $f_n(x) = -\min(x_1, \dots, x_n)$, for arbitrary n. This suffices to prove $d)$, see [BH] for a proof.

Remark 5.2 That any probability measure μ on $\mathbf{R}$ with a compact support satisfies the 'convex' Poincaré inequality (5.2) is known, see e.g. [L]. The simplest argument of getting (5.2) for such measures is the following: for any f convex and differentiable, we have

$$(f(x) - f(y))^2 \leq (f'(x)^2 + f'(y)^2)\,(x-y)^2.$$

Hence, if $\mu([a,b]) = 1$,

$$\mathrm{Var}_\mu(f) = \frac{1}{2}\int_a^b \int_a^b (f(x) - f(y))^2\, d\mu(x)d\mu(y) \ \leq\ (b-a)^2 \int_a^b f'(x)^2\, d\mu(x).$$

Remark 5.3 It is interesting to compare (5.2) with the 'usual' Poincaré-type inequality when no assumption on f (except absolute continuity) is made. A characterization of probability measures μ which satisfy (5.2) for all f has been found by M.Artola, G.Talenti and G.Tomaselli in the class of absolutely continuous μ, and by B.Muckenhoupt [Mu] who found a short proof and extended their result to arbitrary μ. More precisely, given (not necessary finite) Borel measures ν and λ on $[0,+\infty)$, they studied, instead of Poincaré-type inequality on the real line, a related Hardy-type inequality

$$\int_0^{+\infty} |f(x)|^p\, d\nu(x) \ \leq\ C \int_0^{+\infty} |f'(x)|^p\, d\lambda(x) \tag{5.5}$$

where f denotes an absolutely continuous function such that $f(0) = 0$ ($1 \leq p < +\infty$, with the usual modifications for $p = +\infty$). It turned out that (5.5) holds for some finite constant C if and only if

$$\sup_{x>0} \nu^{1/p}([x,+\infty)) \left[\int_0^x \left(\frac{d\lambda(t)}{dt}\right)^{-p'/p} dt\right]^{1/p'} \ <\ +\infty \tag{5.6}$$

where $p' = \frac{p}{p-1}$ and where $\frac{d\lambda(x)}{dx}$ denotes the density of the absolutely continuous component of ν. In order to connect (5.6) to the Poincaré-type inequality, consider in (5.5) the case where $\nu = \lambda$ and where $p = 2$. Then (5.6) takes the form

$$\sup_{x>0} \nu([x,+\infty)) \int_0^x \left(\frac{d\nu(t)}{dt}\right)^{-1} dt \ <\ +\infty. \tag{5.7}$$

On the other hand, for any probability measure μ on $\mathbf{R}$ (for simplicity, with median 0), the Poincaré inequality

$$\mathrm{Var}_\mu(f) \leq C \int_{\mathbf{R}} |f'(x)|^2\, d\mu(x) \tag{5.8}$$

is equivalent (up to a constant) to Hardy's inequalities (5.5) for restrictions of μ to half-axes $[0,+\infty)$ and $(-\infty,0]$. To be more precise, consider measures defined by

$$\mu_+(A) = \mu(A), \quad \mu_-(A) = \mu(-A), \quad A \subset [0,+\infty).$$

If (5.5) holds for $\nu = \mu_+$ and $\nu = \mu_-$ with a constant C, then we easily obtain (5.8) with the same constant. Conversely, assume that (5.8) holds and let f be a square-integrable absolutely continuous function on $[0, +\infty)$ such that $f(0) = 0$. Put $f(x) = 0$, for $x < 0$. Then, by Cauchy-Schwarz' inequality,

$$(\mathbf{E}_\mu f)^2 = (\mathbf{E}_\mu f\, \mathbf{1}_{[0,+\infty)})^2 \le \frac{1}{2}\,\mathbf{E}_\mu f^2.$$

Hence, $\mathbf{E}_\mu f^2 \le 2\mathrm{Var}_\mu(f)$, and thus (5.5) holds for μ_+ with constant $2C$. Similarly, (5.5) holds for μ_-. Therefore, the measure μ admits the Poincaré-type inequality (5.8) if and only if (5.7) holds for $\nu = \mu_+$ and for $\nu = \mu_-$, that is, if and only if

$$\sup_{x>0}\, \mu([x, +\infty)) \int_0^x \left(\frac{d\mu(t)}{dt}\right)^{-1} dt \ < \ +\infty, \tag{5.9}$$

$$\sup_{x<0}\, \mu((-\infty, x]) \int_x^0 \left(\frac{d\mu(t)}{dt}\right)^{-1} dt \ < \ +\infty. \tag{5.10}$$

What are the measures μ satisfying (5.9) and (5.10)? Necessarily, μ should have a non-trivial absolutely continuous component positive on the interval support of μ. Compactness of the support is not sufficient: the measure μ with density $|x|$ on the interval $[-1, 1]$ provides a counter-example while μ satisfies (5.2) according to Remark 2. Assume that an absolutely continuous probability measure μ (with median 0) has a density which is positive a.e. on the interval support. Then, the conditions (5.9) and (5.10) may be rewritten in terms of the map U_μ, which transforms the two-sided exponential measure ν into μ, as follows:

$$\sup_{x>0}\ e^{-x} \int_0^x [U_\mu'(t)^2 + U_\mu'(-t)^2]\, e^t\, dt \ < \ +\infty.$$

6 Proof of Proposition 3.4

Throughout this section, let $J(x) = x\sqrt{\log(1/x)}$ and let $I(x) = J(x(1-x))$, for $0 \le x \le 1$. Fix $p \in (0, 1)$. Let $q = 1 - p$. Since (3.9) is stronger than (3.10), and therefore the optimal constant c in (3.9) is not larger than the optimal constant c in (3.10), we need to prove the lower estimate for c from (3.9). That is, it remains to show that the inequality

$$I(pa + qb) - qI(b) \ \le \ p\sqrt{I(a^2) + d(a-b)^2} \tag{6.1}$$

holds, for all $1 \ge a > b \ge 0$, with some constant d such that $d \le K \log(1/(pq))$. Set

$$L(a, b) = I(pa + qb) - (pI(a) + qI(b)).$$

The above inequality may be rewritten, in terms of L, as

$$\frac{L}{pI(a)} \le \sqrt{1 + t^2} - 1, \quad \text{where} \quad t = \frac{\sqrt{d}\,(a-b)}{I(a)}.$$

Since $\frac{1}{3}\min(t,t^2) \le \sqrt{1+t^2}-1 \le \min(t,t^2)$, the proof of (6.1) is a consequence of the

$$3\,\frac{L}{p(a-b)} \;\le\; \sqrt{d}, \tag{6.2}$$

and

$$3\,\frac{LI(a)}{p(a-b)^2} \;\le\; d. \tag{6.3}$$

We start with some auxiliary properties of the functions I and J.

Lemma 6.1 *$J(x) \le 1$, for all $x \in [0,1]$.*

Proof. We have $J(0) = J(1) = 0$. Let $0 < x < 1$. Then,

$$J'(x) = \sqrt{\log\frac{1}{x}} - \frac{1}{2\sqrt{\log\frac{1}{x}}} = 0 \tag{6.4}$$

if and only if $x = 1/\sqrt{e}$. Hence, $J(x) \le J(1/\sqrt{e}) = 1/\sqrt{2e} < 1$.

Lemma 6.2 *a) The function I increases in $[0,1/2]$ and decreases in $[1/2,1]$.*
b) I is concave on $[0,1]$.
c) $\frac{1}{3} \le -I''(x)I(x) \le 1$, for all $x \in (0.1)$.

Proof. Set $y = x(1-x) \in [0,1/4]$. Since J increases in $[0,1/\sqrt{e}\,]$ and $1/\sqrt{e} \ge 1/4$, we get $I'(x) = J'(y)(1-2x) \ge 0$, for all $0 < x \le 1/2$. This proves $a)$ since I is symmetric around $1/2$. The property $b)$ follows from $c)$. To prove $c)$, we obtain from (6.4):

$$J''(x) = -\frac{1}{2x\sqrt{\log\frac{1}{x}}} - \frac{1}{4x\log\frac{1}{x}\sqrt{\log\frac{1}{x}}}\ .$$

Hence,

$$-J''(x)J(x) = \frac{1}{2} + \frac{1}{4\log\frac{1}{x}}\,, \qquad J'(x)J(x) = x\log\frac{1}{x} - \frac{1}{2}x.$$

From the definition $y = x(1-x)$, we get $(y')^2 = 1-4y$, $y'' = -2$, therefore,

$$\begin{aligned} -I''(x)I(x) &= -[J''(y)(1-4y) - 2J'(y)]\ J(y) \\ &= \left(\frac{1}{2} + \frac{1}{4\log\frac{1}{y}}\right)(1-4y) + \left(2y\log\frac{1}{y} - y\right) \\ &= \frac{1}{2} + \frac{1}{4\log\frac{1}{y}} - y\left(3 - 2\log\frac{1}{y} + \frac{1}{\log\frac{1}{y}}\right). \end{aligned}$$

Thus, introducing the variable $t = \log\frac{1}{y} \in [\,\log 4, +\infty)$, we have

$$-t\,I''(x)I(x) = \frac{1}{4} + \frac{1}{2}\,t + e^{-t}\,(2t^2 - 3t - 1).$$

Hence, $-I''(x)I(x) \le 1$ if and only if

$$u(t) = \frac{1}{4} - \frac{1}{2}\,t + e^{-t}\,(2t^2 - 3t - 1) \le 0.$$

Introduce the function $v(t) = e^{-t}\,(2t^2 - 3t - 1)$. We have $v'(t) = -e^{-t}\,(2t-1)(t-3)$, hence, v increases in $[\frac{1}{2}, 3]$ and decreases in $[3, +\infty)$. Hence, $v(t) \le v(3) = 8\,e^{-3}$, for all $t \ge \log 4$. Therefore,

$$u(t) \le \frac{1}{4} - \frac{1}{2}\,\log 4 + 8\,e^{-3} = -0.045\ldots < 0.$$

For the lower estimate, note that $-I''(x)I(x) \ge \frac{1}{3}$ if and only if

$$w(t) = \frac{1}{4} + \frac{1}{6}\,t + v(t) \ge 0.$$

Since $v(+\infty) = 0$ and $v(\log 4) < 0$, we have, for all $t \ge \log 4$,

$$v(t) \ge v(\log 4) = 2\,\log^2 2 - \frac{3}{2}\,\log 2 - \frac{1}{4}.$$

Hence, $w(t) \ge w(\log 4) = 2\,\log^2 2 - \frac{7}{6}\,\log 2 = 0.152\ldots > 0$. Lemma 6.2 is proved.

Lemma 6.3 *For all $a \in (0,1)$,*

$$Q_p(a) \;\equiv\; \frac{I^2(pa) - p^2 I^2(a)}{(pa)^2} \;\le\; 3\log\frac{1}{pq}.$$

Proof. Writing $1 - pa = (1-a) + qa$, we have by definition

$$\begin{aligned}
Q_p(a) &= (1-pa)^2 \log\frac{1}{pa(1-pa)} - (1-a)^2 \log\frac{1}{a(1-a)} \\
&= \Big((1-a)^2 + 2qa(1-a) + a^2\Big)\log\frac{1}{pa(1-pa)} - (1-a)^2\log\frac{1}{a(1-a)} \\
&= (1-pa)^2\log\frac{1}{p} + (1-a)^2\log\frac{1-a}{1-pa} + \Big(2qa(1-a) + a^2\Big)\log\frac{1}{a(1-pa)}.
\end{aligned}$$

The first term can be estimated by $\log\frac{1}{p}$ and the second is negative since $1-a < 1-pa$. For the third term, we use $1 - pa \ge q$ and $2qa(1-a) + a^2 \le 2a$, bounding it by

$$2a\log\frac{1}{qa} \le 2\max\left(1, \log\frac{1}{q}\right) \le 2\log\frac{1}{pq}$$

since $1 < \log\frac{1}{pq}$. Collecting these estimates, we obtain

$$Q_p(a) \le \log\frac{1}{p} + 2\log\frac{1}{pq} \le 3\log\frac{1}{pq}.$$

Lemma 6.4 *For all $a \in (0,1]$,*

$$R_p(a) \;\equiv\; \frac{I(pa) - pI(a)}{a} \;\le\; 4pq\sqrt{\log\frac{1}{pq}}.$$

Proof. We have, by definition of I, and writing $1 - pa = (1-a) + qa$,

$$\begin{aligned}
\frac{1}{p}\,R_p(a) &= (1-pa)\sqrt{\log\frac{1}{pa(1-pa)}} - (1-a)\sqrt{\log\frac{1}{a(1-a)}} \\
&= (1-a)\left[\sqrt{\log\frac{1}{pa(1-pa)}} - \sqrt{\log\frac{1}{a(1-a)}}\,\right] + qa\sqrt{\log\frac{1}{pa(1-pa)}} \\
&= (1-a)\,\frac{\log\frac{1-a}{p(1-pa)}}{\sqrt{\log\frac{1}{pa(1-pa)}} + \sqrt{\log\frac{1}{a(1-a)}}} + qa\sqrt{\log\frac{1}{pa(1-pa)}}.
\end{aligned}$$

For the second term, we use $1 - pa \ge q$ and $J(a) \le 1 \le \sqrt{\log\frac{1}{pq}}$ (by Lemma 6.1) to get

$$qa\sqrt{\log\frac{1}{pa(1-pa)}} \le qa\left[\sqrt{\log\frac{1}{pq}} + \sqrt{\log\frac{1}{a}}\right] \le q\sqrt{\log\frac{1}{pq}} + qJ(a) \le 2q\sqrt{\log\frac{1}{pq}}.$$

The first term is non-positive when $a \ge 1/(1+p)$ since then $\frac{1}{pa(1-pa)} \le 1$. So consider the case $a \le 1/(1+p)$. Since $(1-a)/(1-pa) \le 1$, the first term can be bounded by

$$(1-a)\,\frac{\log\frac{1}{p}}{\sqrt{\log\frac{1}{p}}} \le \sqrt{\log\frac{1}{pq}} \le 2q\sqrt{\log\frac{1}{pq}},$$

the last inequality being valid for $q \ge 1/2$. Thus in this case, we obtain the desired estimate

$$\frac{1}{p}\,R_p(a) \;\le\; 4q\sqrt{\log\frac{1}{pq}}.$$

Assume $q \le 1/2$. We need to estimate the first term in a different way. Again one may assume that $a \le 1/(1+p)$ when the first term is non-negative. First of all,

$$(1-a)\,\frac{\log\frac{1-a}{p(1-pa)}}{\sqrt{\log\frac{1}{pa(1-pa)}} + \sqrt{\log\frac{1}{a(1-a)}}} \;\le\; \log\frac{1-a}{p(1-pa)},$$

since $a(1-a) \ge 1/4$ and $pa(1-pa) \ge 1/4$. Write

$$\frac{1-a}{p(1-pa)} = 1 + q\,\frac{1-(1+p)\,a}{p(1-pa)} \le 1 + 2q,$$

since $p \ge 1/2$ and $1 - (1+p)\,a \le 1 - pa$. Therefore,

$$\log\frac{1-a}{p(1-pa)} \le \log(1+2q) \le 2q \le 2q\sqrt{\log\frac{1}{pq}}.$$

Using the estimate for the second term, we again obtain the desired conclusion. Thus Lemma 6.4 is proved.

Lemma 6.5 *For all* $1 > a > b \geq 0$,

$$\frac{L(a,b)}{(a-b)^2} \ \leq \ \frac{pq}{2I(a)} + 3\,\frac{L(a,0)}{a^2}.$$

Proof. By Taylor's formula we get: for all $x \in [0,1]$ and $c \in (0,1)$,

$$I(x) = I(c) + I'(c)\,(x-c) + \int_0^1 I''(t\,c + (1-t)\,x\,)\,t\,dt \ \ (x-c)^2.$$

Writing there $c = pa + qb$ and $x = a$, $x = b$, and averaging with weights p and q, we get a general identity:

$$\frac{L(a,b)}{(a-b)^2} \ = \ pq\left[q\int_0^1 -I''(t\,c + (1-t)\,a\,)\,t\,dt + p\int_0^1 -I''(t\,c + (1-t)\,b\,)\,t\,dt\right]. \quad (6.5)$$

By Lemma 6.2c, $-I'' \leq 1/I$, hence,

$$\frac{L(a,b)}{(a-b)^2} \ \leq \ pq\left[q\int_0^1 \frac{1}{I(t\,c + (1-t)\,a\,)}\,t\,dt + p\int_0^1 \frac{1}{I(t\,c + (1-t)\,b\,)}\,t\,dt\right]. \qquad (6.6)$$

By Lemma 6.2a, the function $1/I$ decreases in [0,1/2] and increases in [1/2,1]. Consequently, the maximum of $1/I$ on every interval $[\alpha, \beta]$ is attained at one of the endpoints of the interval. Therefore, for all $x \in [\alpha, \beta]$,

$$\frac{1}{I(x)} \leq \frac{1}{I(\alpha)} + \frac{1}{I(\beta)}.$$

Let t and a in (6.6) be fixed and let b vary in $[0,a]$. Then, $x = t\,c + (1-t)\,a$ varies in $[\alpha,\beta] = [t\,pa + (1-t)\,a, a]$, and applying the above inequality, we get

$$\frac{1}{I(t\,c + (1-t)\,a)} \leq \frac{1}{I(a)} + \frac{1}{I(t\,pa + (1-t)\,a)}.$$

In the same way,

$$\frac{1}{I(t\,c + (1-t)\,b)} \leq \frac{1}{I(a)} + \frac{1}{I(t\,pa)}.$$

Using these estimates in (6.6) yields

$$\frac{L(a,b)}{(a-b)^2} \ \leq \ \frac{pq}{2I(a)} + pq\left[q\int_0^1 \frac{1}{I(t\,pa + (1-t)\,a)}\,t\,dt + p\int_0^1 \frac{1}{I(t\,pa)}\,t\,dt\right].$$

Applying the inequality $1/I \leq -3I''$, from Lemma 6.2c, we get

$$\frac{L(a,b)}{(a-b)^2} \ \leq \ \frac{pq}{2I(a)} + 3\,pq\left[q\int_0^1 -I''(t\,pa + (1-t)\,a)\,t\,dt + p\int_0^1 -I''(t\,pa)\,t\,dt\right].$$

But the second term (without the constant 3) corresponds to the right hand side of (6.5) with $b = 0$ and is equal to $L(a,0)/a^2$. Thus Lemma 6.4 is proved.

Proof of Proposition 3.4. We start with (6.3). By Lemma 6.5,

$$3\,\frac{LI(a)}{p(a-b)^2} \;\le\; \frac{3q}{2} + 9\,\frac{L(a,0)I(a)}{pa^2}. \tag{6.7}$$

The function I is concave (Lemma 6.2a) and $I(0) = 0$. Hence, $I(pa) \ge pI(a)$. Therefore,

$$L(a,0) = I(pa) - pI(a) = \frac{I^2(pa) - p^2I^2(a)}{I(pa) + pI(a)} \le \frac{I^2(pa) - p^2I^2(a)}{2pI(a)} = \frac{(pa)^2 Q_p(a)}{2pI(a)}$$

where $Q_p(a)$ has been defined in Lemma 6.3. Applying this Lemma, we arrive at

$$\frac{L(a,0)I(a)}{pa^2} \;\le\; \frac{3}{2}\log\frac{1}{pq}, \tag{6.8}$$

and by (6.7),

$$3\,\frac{LI(a)}{p(a-b)^2} \;\le\; \frac{3q}{2} + \frac{27}{2}\log\frac{1}{pq} \;\le\; 15\log\frac{1}{pq},$$

since $\frac{3}{2} \le \frac{3}{2}\log\frac{1}{pq}$. Thus, (6.3) holds with $d = 15\log\frac{1}{pq}$.

To establish (6.2), consider, for $1 \ge a \ge b \ge 0$, the inequality

$$L(a,b) = I(pa+qb) - (pI(a) + qI(b)) \;\le\; C(p)\,(a-b). \tag{6.9}$$

Since I is concave, for any $c \in (a,b)$, the left hand side of (1.9) is a convex function on the segment

$$\Delta(c) = \{(a,b) : pa + qb = c, 1 \ge a \ge b \ge 0\}$$

while the right hand side of (6.9) is a linear function. Therefore, (6.9) holds for all $(a,b) \in \Delta(c)$ if and only if it holds for the endpoints of the segment. Note that for each of these endpoints with coordinates (a,b), either $b = 0$, or $a = 1$. When $b = 0$, (6.9) yields

$$I(pa) - pI(a) \le C_0(p)\,a \tag{6.10}$$

with $C_0(p) = C(p)$, and for $a = 1$, it yields, after a formal change of the variable b to a,

$$I(qa) - qI(a) \le C_0(q)\,a$$

where again $C_0(q) = C(p)$. Thus, the optimal constants $C(p)$ in (6.9) are connected with the optimal constants $C_0(p)$ in (6.10) via

$$C(p) = \min(C_0(p), C_0(q)).$$

By Lemma 6.4, $C_0(p) \le 4pq\sqrt{\log\frac{1}{pq}}$. Hence, $C(p) \le 4pq\sqrt{\log\frac{1}{pq}}$. Therefore,

$$3\,\frac{L}{p(a-b)} \le 12\sqrt{\log\frac{1}{pq}}.$$

As a result, (6.2) holds with $d = 144 \log \frac{1}{pq}$. Thus Proposition 3.4 is proved with $K_0 = \frac{1}{12}$.

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