PROPERLY DISCONTINUOUS GROUPS OF AFFINE TRANSFORMATIONS WITH ORTHOGONAL LINEAR PART

GROUPES PROPREMENT DISCONTINUS DE TRANSFORMATIONS AFFINES AVEC PARTIES LINÉAIRES ORTHOGONALES

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Abstract. Let Γ be a subgroup of the group of all affine transformations of a real affine space A of finite dimension. Suppose Γ acts properly discontinuously on A . We determine which orthogonal groups can occur as Zariski closures of the linear part of Γ . As an application of the methods we shall prove Auslander's conjecture for affine spaces of dimension at most 6.

Résumé Soit Γ un sous-groupe du groupe de toutes les transformations affines d'un espace réel affine A de dimension finie. Supposons que l'opération de Γ sur A soit proprement discontinue. Alors nous déterminons quels sont les groupes orthogonaux qui peuvent s'obtenir comme clôture de Zariski de la partie linéaire de Γ . Les méthodes utilisées nous permettent de démontrer la conjecture d'Auslander pour les espaces affines de dimension ≤ 6 .

Version française abrégée

Soit A un espace affine réel de dimension finie et soit $\mathrm{Aff}A$ le groupe de toutes les transformations affines de A . Un sous–groupe Γ de $\mathrm{Aff}A$ est appelé *proprement discontinu* si pour chaque compact K dans A l'ensemble $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ est fini. Le groupe Γ est appelé *crystallographique* si de plus l'espace $\Gamma\backslash A$ des orbites est compact. La conjecture suivante est due à L. Auslander.

$(*)$ *Tout groupe crystallographique est virtuellement résoluble.*

Ici nous utilisons la terminologie suivante: Un groupe est dit d'avoir une propriété *virtuellement* s'il y a un sous–groupe d'indice fini qui possède la propriété correspondante.

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Dans le cas où Γ est un groupe de transformations euclidiennes, ce qui veut dire que Γ préserve une metrique qui est invariante par translations et qui est induite par une forme quadratique positive définie, alors la conjecture est vraie d'après un théorème de Bieberbach [4]. En fait, Bieberbach a démontré que chaque groupe de transformations euclidiennes qui est proprement discontinu est virtuellement abélien [3]. J. Milnor [12] a posé la question suivante.

$(**)$ *Est–ce qu'il est vrai que chaque sous–groupe proprement discontinu de* $\mathrm{Aff}A$ *est virtuellement résoluble.*

Voici des versions équivalentes de ces questions en géométrie différentielle. Est–il vrai que chaque variété affinément plate complète M a un groupe fondamental qui est virtuellement résoluble (Milnor) Est–il vrai si de plus M est compacte (Auslander)

G.A. Margulis [10], [11] a donné une réponse négative à la question de Milnor en construisant un sous–groupe libre Γ de $\mathrm{Aff}A$ où $\dim A = 3$ et Γ préserve une forme quadratique de signature (2,1). La conjecture d'Auslander a été démontrée dans un nombre de cas, voir [7], [8], [9], [6], [5], [13], [14] et a été démontrée en général seulement pour les espaces affines de dimension ≤ 3 .

Cette note continue les recherches commencées dans [2]. Pour énoncer les résultats principaux il nous faut les notations suivantes. Le groupe $\mathrm{Aff}A$ est le produit semi–direct de l'espace vectoriel V des translations de A et du groupe linéaire $GL(V)$. On a alors un homomorphisme canonique $\ell : \mathrm{Aff}A \to GL(V)$. Disons que $g \in \mathrm{Aff}A$ préserve une forme quadratique B sur V si $\ell(g)$ est dans le groupe orthogonal $O(B)$.

Théorème A *Soit* $\dim A = 2n+1$ *et soit* Γ *un sous–groupe proprement discontinu de* $\mathrm{Aff}A$ *préservant une forme quadratique* B *de signature* $(n+1, n)$ *. Alors* $\ell(\Gamma)$ *n'est pas Zariski dense dans* $O(B)$ *si* n *est pair.*

En contraste avec ce résultat, nous avons le résultat suivant pour n impair.

Théorème B *Soit* $\dim A = 2n+1$ *,* n *impair, et soit* $(n+1, n)$ *la signature de la forme quadratique* B *. Alors il existe un sous–groupe* Γ *de* $\mathrm{Aff}A$ *qui est libre et proprement discontinu et tel que* $\ell(\Gamma)$ *est Zariski dense dans* $O(B)$ *.*

Pour les autres signatures cette dichotomie disparaît.

Théorème C *Soit* $\dim A = p+q$ *et soit* Γ *un sous–groupe proprement discontinu de* $\mathrm{Aff}A$ *qui préserve une forme quadratique* B *ayant la signature* (p, q) *où* $|p-q| \neq 1$ *. Alors* $\ell(\Gamma)$ *n'est pas Zariski dense dans* $O(B)$ *.*

Les méthodes des démonstrations nous permettent de démontrer la conjecture d'Auslander pour les espaces affines de dimension ≤ 6 .

Nos méthodes sont basées sur les idées de [10], [11] et [2]. Elles sont essentiellement différentes de celles qui figurent dans [7], [8], [9], [6], [5], [13] et [14].

Introduction

Let A be a real affine space of finite dimension and let $\mathrm{Aff}A$ denote the group of all affine transformations of A . A subgroup Γ of $\mathrm{Aff}A$ is said to act *properly discontinuosly* on A if for every compact subset K of A the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite. If in addition the orbit space $\Gamma \backslash A$ is compact then Γ will be called *crystallographic.* L. Auslander [1] stated the following conjecture.

$(*)$ *Every crystallographic group is virtually solvable.*

Here we use the following terminology: A group is said to have a certain property *virtually* if it contains a subgroup of finite index with that property.

In case Γ is a group of Euclidean transformations, i.e. Γ preserves a translation invariant metric on A induced by a positive definite quadratic form, the conjecture holds by a classical theorem due to Bieberbach [4]. In fact, Bieberbach also proved that every properly discontinuous group of Euclidean transformations is virtually abelian [3]. J. Milnor [12] asked the following more general question.

$(**)$ *Is every properly discontinuous subgroup* Γ *of* $\mathrm{Aff}A$ *virtually solvable?*

There are differential geometric versions of these questions as follows. Milnor's question is equivalent to asking if every affinely flat complete manifold M has a virtually solvable fundamental group. Auslander's conjecture states that this is the case if M is furthermore compact.

G.A. Margulis [10], [11] answered Milnor's question negatively by constructing a free subgroup Γ of $\mathrm{Aff}A$ for $\dim A = 3$ where Γ actually preserves a quadratic form of signature (2,1). Auslander's conjecture has been proven in a number of cases, see [7], [8], [9], [6], [5], [13], [14], and is known to hold in general only for affine spaces of dimension at most 3.

The present paper continues the work on the Auslander conjecture started in [2]. To state the main results let us first recall that for $\dim A = n$ the group $\mathrm{Aff}A$ is the semidirect product of the vectorspace V of translations of A and the general linear group $GL(V)$. We thus have a natural homomorphism $\ell : \mathrm{Aff}A \to GL(V)$ which associates with every affine map g its *linear part* $\ell(g)$. We say that $g \in \mathrm{Aff} A$ preserves a quadratic form B on V if $\ell(g)$ is contained in the corresponding orthogonal group $O(B)$.

Theorem A *Let Γ be a properly discontinuous subgroup of Aff$A$, $\dim A = 2n+1$, preserving a quadratic form B of signature $(n+1, n)$. Then $\ell(\Gamma)$ is not Zariski dense in $O(B)$ if n is even.*

On the contrary, for odd n we have the following result.

Theorem B *If the quadratic form B has signature $(n+1, n)$ and n is odd, then there is a free subgroup Γ of Aff$A$, $\dim A = 2n+1$, acting properly discontinuously on A such that $\ell(\Gamma)$ is Zariski dense in $O(B)$.*

For the case of other signatures this dichotomy disappears.

Theorem C *Let Γ be a properly discontinuous subgroup of Aff$A$, $\dim A = p+q$, preserving a quadratic form B of signature (p, q) with $|p-q| \neq 1$. Then $\ell(\Gamma)$ is not Zariski dense in $O(B)$.*

The methods of proof allow us to conclude that the Auslander conjecture is true for affine spaces of dimension at most 6 .

Our methods are essentially different from those used in [7], [8], [9], [6], [5] [13] and [14]. They are based on the ideas of [10], [11] and [2].

1. Some inequalities

With a linear automorphism h of a finite dimensional real vector space V we associate the following subspaces A_h^+ , A_h^- and A_h^0 of V where A_h^+ (A_h^-, A_h^0) is the largest h–invariant subspace W of V such that all the eigenvalues of h on $W \otimes_{\mathbb{R}} \mathbb{C}$ are of absolute value > 1 (respectively < 1 , respectively $= 1$). Put $A_h^{00} = \{v \in V \mid hv = v\}$. Clearly $A_h^{00} \subset A_h^0$ and

$$V = A_h^+ \oplus A_h^- \oplus A_h^0.$$

Put $B_h^+ = A_h^+ \oplus A_h^0$ and $B_h^- = A_h^- \oplus A_h^0$. For an affine map g we put $A_g^\pm := A_{\ell(g)}^\pm$, $A_g^0 := A_{\ell(g)}^0$ and $A_g^{00} := A_{\ell(g)}^{00}$.

Now let B be a quadratic form of signature (p, q) , $p \geq q$, on the vector space V corresponding to our affine space A . So $\dim V = \dim A = p+q$. Let G be the subgroup of those elements $g \in \mathrm{Aff} A$ which preserve B , i.e. such that $\ell(g) \in O(B)$. An element $g \in G$ is called *hyperbolic* if $\dim A_g^0 = p-q$ and $\dim A_g^{00} \leq 1$. Note that $\dim A_g^{00} \geq 1$ for every element g of a properly discontinuous torsion free subgroup Γ of Aff$A$. We shall henceforth assume that $\dim A_g^{00} \geq 1$ for every $g \in \Gamma$. This implies the same inequality for every g of the Zariski closure of Γ and thus settles the cases "$p-q$ even" of theorem C.

For a hyperbolic element $g \in \mathrm{Aff}A$ the unique g–invariant line will be called the *axis* C_g of g . Put $D_g^0 = C_g + A_g^0$, $D_g^{\pm} = D_g^0 + A_g^{\pm}$. Two hyperbolic elements g and h of $\mathrm{Aff}A$ are called *transversal* if $A_g^+ \cap B_h^- = A_g^- \cap B_h^+ = \{0\}$ — which we abbreviate by $A_g^{\pm} \cap B_h^{\mp} = \{0\}$ — and similarly $A_h^{\pm} \cap B_g^{\mp} = \{0\}$.

Let d be a Euclidean metric on $\mathbb{R}^{p+q}$. Then we say that a hyperbolic element g is ε *–hyperbolic* if $d(A_g^+, A_g^-) \geq \varepsilon$. Two transversal elements g, h are called ε*–transversal* if $d\left(A_g^{\pm}, B_h^{\mp}\right) \geq \varepsilon$ and $d\left(A_h^{\pm}, B_g^{\mp}\right) \geq \varepsilon$. Here the distance $d(W_1, W_2)$ of two vector subspaces W_1 , W_2 with $W_1 \cap W_2 = \{0\}$ is the minimum of the distances $d(w_1, w_2)$, $w_i \in W_i$, $d(w_i, 0) = 1$.

The main result of [2] implies that if the group $\ell(G)$ is Zariski dense in $O(B)$ then so is the set of its hyperbolic elements. For $\ell(g) \in O(B)$ let λ_g be the maximum of $|\lambda|$ where λ is an eigenvalue of $\ell(g)$ with $|\lambda| < 1$.

We shall use the following notion similar to one in [10]. Let $k > 1$ be a natural number and let two functions f_1 , f_2 be given on some subset E of the direct product of k copies of G . We say that f_1 is dominated by f_2 and write $f_1 \ll f_2$ if for every $\varepsilon > 0$ there is a $C(\varepsilon) > 0$ such that $f_1(g_1, \ldots, g_k) \leq C(\varepsilon) f_2(g_1, \ldots, g_k)$ whenever $(g_1, \ldots, g_k) \in E$ and the elements $g_i, i = 1, \ldots, k,$ are ε–hyperbolic and pairwise ε–transversal and $\lambda_{g_i} \leq C(\varepsilon)^{-1}$. If $f_1 \ll f_2$ and $f_2 \ll f_1$ we write $f_1 \sim f_2$ and say that f_1 and f_2 are equivalent.

Lemma 1. 1) $\lambda_g \lambda_h \sim \lambda_{gh}$
2) $d\left(A_{gh}^+, A_g^+\right) \sim \lambda_g$
3) $d\left(A_{gh}^-, A_h^-\right) \sim \lambda_h$

Lemma 2. *For every natural number* m *and every* $\varepsilon > 0$ *there are positive constants* a *and* δ *such that if* $g_1, \ldots, g_m$ *are* ε*–hyperbolic and pairwise* ε*–transversal elements with* $\lambda_{g_i} < a^{-1}$ *for every* $i = 1, \ldots, m$ *, then*
1) *every element* $\neq e$ *of the group* $\langle g_1, \ldots, g_m \rangle$ *is* δ*–hyperbolic,*
2) *any two nonidentity elements* g, h *of* $\langle g_1, \ldots, g_m \rangle$ *are* δ*–transversal unless some positive power of* g *is the inverse of some positive power of* h *,*
3) *the group* $\langle g_1, \ldots, g_m \rangle$ *is the free group with basis* $\{g_1, \ldots, g_m\}$ *.*
4) *the function* $g \longmapsto d(q_o, D_g^0)$ *on* $\langle g_1, \ldots, g_m \rangle$ *is bounded for every point* $q_0 \in A$ *.*

2. The case $p = q + 1$

For this case we define the *signed translational part* $\alpha(g) \in \mathbb{R}$ of a hyperbolic element $g \in G$ as follows. Its length $|\alpha(g)|$ is the square root of $B(gx-x, gx-x)$ for any point x of the axis C_g of g. The sign of $\alpha(g)$, also called the *sign of* g, is obtained by giving C_g an orientation $\leq$ which will be defined shortly. Then $\alpha(g) \geq 0$ iff $gx \geq x$ for $x \in C_g$. The orientation of C_g is defined as follows. Note that all the maximal isotropic subspaces W of $V = \mathbb{R}^{p+q}$ can be oriented in a coherent way and also all the spaces $W^\perp$ for maximal isotropic W. Hence every line in the $q+1$–dimensional space $W^\perp$ not contained in the q–dimensional subspace W can be oriented. This applies in particular for the line A_g^0 parallel to C_g which is contained in $W^\perp = B_g^+$ but not in the maximal isotropic subspace $W = A_g^+$.

Lemma 3. $\alpha(g^{-1}) = (-1)^p \alpha(g)$.

Lemma 4. *If $g, h \in G$ are two hyperbolic transversal elements of different sign then there exists a compact subset K of A and an infinite set of pairs (m, n) such that $g^m h^n K \cap K \neq \emptyset$.*

If now $\ell(\Gamma)$ is Zariski dense in $O(B)$ then we can choose two elements g, h in Γ of different sign in view of lemma 3 and the results of [2] provided p is odd. On the other hand the group $\langle g, h\rangle$ is free by lemma 2 which is impossible if Γ acts properly discontinuously on A. This proves theorem A.

For the proof of theorem B we use

Lemma 5. $\alpha(g\,h) \sim \alpha(g) + \alpha(h)$.

This implies for p even and an appropriate constant a for $g_1, \dots, g_m$ in G as in lemma 2 that

$$|\alpha(g)| \sim w(g)$$

where $w(g)$ is the word length of g in the free group with basis $g_1, \dots, g_m$. On the other hand, for every point $x \in A$ we have

$$d(x, gx) \geq c(x) \cdot |\alpha(g)|.$$

We then obtain for p even a subgroup Γ of G which acts properly discontinuously on A and such that $\ell(\Gamma)$ is Zariski dense in $O(B)$.

3 The case $p-q \geq 2$

Let g,h be two hyperbolic transversal elements of G . Put $A^0_{g,h} = B^+_g \cap B^-_h$. Think of this space as close to A^0_{gh} , in view of lemma 1. We have the projections

$$\begin{array}{llllllll} & \pi^+_g & : & B^+_g & \longrightarrow & A^0_g & \text{along} & A^+_g \\ \text{and} & \pi^-_h & : & B^-_h & \longrightarrow & A^0_h & \text{along} & A^-_h . \end{array}$$

The restrictions $\sigma(g,h) = \pi^+_g \mid A^0_{g,h}$ and $\tau(g,h) = \pi^-_h \mid A^0_{g,h}$ are isometries with respect to the positive definite forms induced by B .

The following lemma 6 plays in the proof of lemma 7 a role similar to that of elements of opposite sign in lemma 4. For every hyperbolic element $g \in G$ let v_g be a vector of A^{00}_g with $B(v_g, v_g) = 1$ and $B(gx - x, v_g) > 0$ for every $x \in A$. So $|\alpha(g)| v_g = gx - x$ for $x \in C_g$.

Lemma 6. *Let Γ be a subgroup of G such that $\ell(\Gamma)$ is Zariski dense in $O(B)$. Then there are hyperbolic and pairwise transversal elements $g_1, \dots, g_m$ in Γ and for every $\varepsilon > 0$ there is a $\beta = \beta(\varepsilon) > 0$ with the following property: If g is an ε -hyperbolic element ε -transversal to every $g_i, i = 1, \dots, m$, and $x \in D^0_g$ then there is an $i, 1 \leq i \leq m$, such that*

$$B\left(v_{g_i}, \tau(g,g_i)\sigma(g,g_i)^{-1}(gx - x)\right) \leq -\beta.$$

Now suppose the $m+1$ elements $g, g_1, \dots, g_m$ are as in lemma 6 and they together also satisfy the conditions of lemma 2. Fix a point $x \in A$ and put $\Gamma^* = \langle g_1, \dots, g_m \rangle$.

Lemma 7. *There is a compact subset K of A and a constant c such that $d(x, \Gamma^* g^i K) \leq c$ for every $i \in \mathbb{N}$.*

In particular the group $\langle g, g_1, \dots, g_m \rangle$ does not act properly discontinuously on A since this group is free. Theorem C is proved using lemma 7 as follows. Choose hyperbolic and pairwise transversal elements $g_1, \dots, g_m$ in our group Γ as in lemma 6. Then there is an element $g \in \Gamma$ which is hyperbolic and transversal to $g_1, \dots, g_m$ by the results of [2]. Then raise these elements to sufficiently high powers which does not destroy the properties of lemma 6 but makes them satisfy those of lemma 2. Now lemma 7 shows that Γ does not act properly discontinuosly.

References

[1] L. Auslander, *The structure of compact locally affine manifolds*, Topology **3** (1964), 131–139.

[2] H. Abels, G.A. Margulis, G.A. Soifer, *Semigroups containing proximal linear maps*, Israel J. of Math. **91** (1995), 1–30.

[3] L. Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume (Erste Abhandlung)*, Math. Ann. **70** (1911), 297–336.

[4] ______, *Über die Bewegungsgruppen der Euklidischen Räume (Zweite Abhandlung). Die Gruppen mit einem endlichen Fundamentalbereich*, Math. Ann. **72** (1912), 400–412.

[5] T. Drumm, *Fundamental polyhedra for Margulis space–times*, Topology **31** (1992), 677–683.

[6] T. Drumm and W. Goldman, *Complete flat Lorentz 3–manifolds with free fundamental group*, Int. J. Math **1** (1990), 149–161.

[7] D. Fried and W.M. Goldman, *Three–dimensional affine crystallographic groups*, Adv. in Math. **47** (1983), 1–49.

[8] W.M. Goldman and Y. Kamishima, *The fundamental group of a compact flat Lorentz space form is virtually polycyclic*, J. of Diff. Geom. **19** (1984), 233–240.

[9] F. Grunewald and G.A. Margulis, *Transitive and quasitransitive actions of affine groups preserving a generalized Lorentz–structure*, J. of Geometry and Physics **5** (1988), 493–531.

[10] G.A. Margulis, *Free properly discontinuous groups of affine transformations*, Dokl. Akad. Nauk SSSR **272** (1983), 937–940.

[11] ______, *Complete affine locally flat manifolds with a free fundamental group*, Journal of Soviet Mathematics **134** (1987), 129–139, translated from Zapiski NS LOMI **134**, (1984), 190–205.

[12] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Adv. in Math. **25** (1977), 178–187.

[13] G.A. Soifer, *Affine discontinuous groups*, In L.A. Bokut', M. Hazewinkel, Yu. Reshetnyak (eds): Algebra and Analysis, Proc. of the first Siberian school, Lake Baikal, Amer. Math. Soc. (1992).

[14] G. Tomanov, *The virtual solvability of the fundamental group of a generalized Lorentz space form*, J. Diff. Geom. **32** (1990), 539–547.

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