ESTIMATES FOR THE RAPID DECAY OF CONCENTRATION FUNCTIONS OF n-FOLD CONVOLUTIONS

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ABSTRACT. We estimate the concentration functions of n–fold convolutions of one-dimensional probability measures. The Kolmogorov–Rogozin inequality implies that for non-degenerate distributions these functions decrease at least as $O(n^{-1/2})$. On the other hand, Esseen (1968) has shown that this rate is $o(n^{-1/2})$ iff the distribution has an infinite second moment. This statement was sharpened by Morozova (1977). Theorem 1 of this paper provides an improvement of Morozova's result. Moreover, we present more general estimates which imply the rates $o(n^{-1/2})$.

1. Introduction

Let us first introduce some notation. Let $\mathfrak{F}$ denote the set of probability distributions defined on the Borel σ-field of subsets of the real line $\mathbf{R}$, Φ_B the normal distribution with zero mean and variance B^2 $(B \geqslant 0)$, $\mathcal{L}(\xi) \in \mathfrak{F}$ the distribution of a random variable ξ, and $\mathbf{I}\{A\}$ the indicator function of an event A. For $F \in \mathfrak{F}$ the concentration function is defined by the equality $Q(F, b) = \sup_x F\big\{[x, x+b]\big\}$, $b \geqslant 0$. For $F, H \in \mathfrak{F}$ we denote the corresponding distribution functions by $F(x)$, $H(x)$; write $\rho(F,H) = \sup_x \big|F(x) - H(x)\big|$ for the Kolmogorov distance. Products and powers of measures will be understood in the convolution sense: $FH = F * H$, $H^m = H^{m*}$. For $F = \mathcal{L}(\xi) \in \mathfrak{F}$ we shall use the notation $\overline{F} = \mathcal{L}(-\xi)$ and $\widetilde{F} = F\overline{F}$. The distribution $\widetilde{F}$ is called symmetrized. Let $E_a \in \mathfrak{F}$ be the distribution concentrated at a point $a \in \mathbf{R}$, $E = E_0$. By $c(\,\cdot\,)$, $c_1(\,\cdot\,)$, $c_2(\,\cdot\,), \dots$ we shall denote positive quantities depending on the arguments only. The symbols $c, c_1, c_2, \dots$ will be used for absolute

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positive constants. Note that $c(\,\cdot\,)$, c can be different in different (or even in the same) formulas. We shall write $f \asymp g$ if $f \leqslant cg$ and $g \leqslant cf$. We denote by $\lceil\,\cdot\,\rceil$ the integer part of a number. The ends of proofs will be marked by $\square$.

This paper deals with the classical question about the rate of decay of $Q(F^n,\,b)$, $F \in \mathfrak{F}$, $b \geqslant 0$, as $n \to \infty$. Most of the known general bounds for $Q(F^n,\,b)$ decreases as $O(n^{-1/2})$. The Kolmogorov–Rogozin inequality (Rogozin (1961)) states that

$$Q(F^n,\,b) \leqslant \frac{c}{\sqrt{n\,\bigl(\,1 - Q(F,\,b)\,\bigr)}}\,. \tag{1.1}$$

Esseen (1968) (see Petrov (1976), inequality (2.7) of Chapter III) strengthened (1.1) to

$$Q(F^n,\,b) \leqslant \frac{c}{\sqrt{n\,D(\,\widetilde{F},\,b)}}\,, \tag{1.2}$$

where

$$D(F,\,b) = \int_{-\infty}^{\infty} \min\bigl\{x^2 b^{-2},\,1\bigr\}\,F\{dx\}, \qquad F \in \mathfrak{F}, \quad b > 0. \tag{1.3}$$

We can rewrite $D(F,\,b)$ in the form

$$D(F,\,b) = K(F,\,b) + G(F,\,b), \tag{1.4}$$

where

$$K(F,\,b) = b^{-2} \int\limits_{|x| \leqslant b} x^2\,F\{dx\} \qquad \text{and} \qquad G(F,\,b) = F\bigl\{\{x : |x| > b\}\bigr\}. \tag{1.5}$$

Esseen (1968) showed that $Q(F^n,\,b) = o(n^{-1/2})$ as $n \to \infty$ iff the distribution F has an infinite second moment. Concrete bounds of order $o(n^{-1/2})$ can be found in Esseen (1968), Mukhin (1972, 1973, 1976a, 1976b, 1989), Morozova (1977), Hall (1983), Griffin, Jain and Pruitt (1984), Suchkov and Ushakov (1989). However, most of these bounds have been proved assuming additional regularity-type conditions (see, e.g., (1.6)).

Arak (1981), Lemma 3.2, has found a powerful generalization of the Kolmogorov–Rogozin inequality (1.1) (see as well Arak and Zaitsev (1988), Theorem 4.2 of Chapter II). The right-hand side of Arak's inequality is expressed in terms of the distance of the support of F to a neighbourhood of the projection of a finite piece of a multidimensional lattice on the real line. This inequality may provide estimates of order $o(n^{-1/2})$.

The exact rate of decay of $Q(F^n,\,b)$ under the additional condition

$$\limsup_{y \to \infty} \frac{G(F,\,y)}{K(F,\,y)} < \infty \tag{1.6}$$

was found independently by Hall (1983) and by Griffin, Jain and Pruitt (1984). They have proved that

$$c_1(F,\,b) \leqslant a_n\, Q(F^n,\,b) \leqslant c_2(F,\,b), \qquad \text{for} \quad n \geqslant c_3(F,\,b), \tag{1.7}$$

where a_n is the solution of the equation

$$D(F,\,a_n) = \frac{1}{n}. \tag{1.8}$$

The condition (1.6) is equivalent to the existence of a_n and b_n such that the sequence $(S_n - b_n)/a_n$ $\big(\mathcal{L}(S_n) = F^n\big)$ is stochastically compact, i.e., it is a tight sequence and all its limit laws are non-degenerate (see Feller (1967)).

The relation (1.7) extends Theorem 5.1 of Esseen (1968): if a distribution F belongs to the domain of normal attraction of a stable law with an exponent α, $0<\alpha<2$, then

$$c_4(F,\,b) \leqslant n^{1/\alpha}\, Q(F^n,\,b) \leqslant c_5(F,\,b). \tag{1.9}$$

Furthermore, if F belongs to the corresponding (non-normal) domain of attraction, then

$$c_6(F,\,b,\,\varepsilon)\, n^{-1/(\alpha-\varepsilon)} \leqslant Q(F^n,\,b) \leqslant c_7(F,\,b,\,\varepsilon)\, n^{-1/(\alpha+\varepsilon)}, \qquad 0<\varepsilon<\alpha \tag{1.10}$$

(see Esseen (1968), Theorem 5.2).

In this paper we shall prove the following theorem.

Theorem 1. *Let $F \in \mathfrak{F}$, $b \geqslant 0$, $\delta > 0$. Then*

$$Q(F^n,\,b) \leqslant \frac{c_8(F,b,\delta)}{n\sqrt{D(\widetilde{F},\,\delta\sqrt{n})}}. \tag{1.11}$$

Remark. See (2.47) for an explicit form of $c_8(F,b,\delta)$.

Theorem 1 refines a result of Morozova (1977) stating that

$$Q(F^n,\,b) \leqslant \frac{c_9(F,b,\delta)}{n\sqrt{K(\widetilde{F},\,\delta\sqrt{n})}} = c_9(F,b,\delta)\left(n\,\delta^{-2}\int\limits_{|x|\leqslant\delta\sqrt{n}} x^2\,\widetilde{F}\{dx\}\right)^{-1/2}. \tag{1.12}$$

Indeed, according to (1.4), the inequality (1.12) follows from Theorem 1. On the other hand, it is clear that there exist F such that $G\big(\widetilde{F},\,\delta\sqrt{n}\big) \leqslant D\big(\widetilde{F},\,\delta\sqrt{n}\big)$ is essentially larger than $K\big(\widetilde{F},\,\delta\sqrt{n}\big)$. If the second moment of F is infinite, then each of the relations (1.11) and (1.12) implies Esseen's bound $Q(F^n,\,b) = o(n^{-1/2})$ with an explicit rate of decrease. Below we shall prove the following fact.

Proposition 1. *There exists an* $F \in \mathfrak{F}$ *with infinite second moment and a strictly increasing sequence of positive integers* k_j, $j \geqslant 1$, *such that*

$$Q(F^{k_j}, b) \geqslant \frac{c}{k_j \sqrt{K(\widetilde{F}, \sqrt{k_j})}}, \qquad j = 1, 2, \dots, \qquad b \geqslant 0, \tag{1.13}$$

and

$$K\big(\widetilde{F}, \lceil k_j/2 \rceil\big) = o\Big(D\big(\widetilde{F}, \lceil k_j/2 \rceil\big)\Big) \qquad \text{as} \quad j \to \infty. \tag{1.14}$$

Thus, there exist distributions for which even the inequality (1.12) is sharp and for which the inequality (1.11) is stronger than (1.12). Note that in (1.13) and (1.14) we consider the behaviour of $K\big(\widetilde{F}, \cdot\big)$ along different sequences of arguments: $l_j = \sqrt{k_j}$ and $m_j = \lceil k_j/2 \rceil$.

However, the inequality (1.11) does not provide exact rates of decay of $Q(F^n, b)$ for all F since the estimates of Theorem 1 decrease not faster than $O(n^{-1})$. In particular, bounds of the form $O(n^{-1/\alpha})$, $0 < \alpha < 2$, (which can be obtained by an application of (1.7) to concrete distributions F, see (1.9)) yield faster rates than $O(n^{-1})$ (when $\alpha < 1$). The loss of precision is due to the fact that in the proof of Theorem 1 we use the inequality (1.2) which is far from to be uniformly optimal. Theorem 2 and Corollaries 1 and 2 below contain general estimates which may provide bounds decreasing more rapidly than $O(n^{-1})$. Theorem 1 is, in fact, a consequence of Corollary 2.

One should emphasize that if the condition (1.6) (which is essentially used in the proof of (1.7)) is satisfied, then the inequality (1.11) of Theorem 1 can be easily derived from (1.12) (see (1.4) and Griffin (1983), Lemma 2.9). Thus, the estimate of Theorem 1 is better than the already known bound (1.12) in the case when the quantity $G\big(\widetilde{F}, \delta\sqrt{n}\big)$ is essentially larger than $K\big(\widetilde{F}, \delta\sqrt{n}\big)$, i.e., roughly speaking, when the condition (1.6) is not fulfilled.

Theorem 2. *Assume that a distribution* $F \in \mathfrak{F}$ *is represented in the form*

$$F = (1-p)U + pV, \qquad U, V \in \mathfrak{F}, \quad 0 < p < 1. \tag{1.15}$$

Let X *be a random variable with* $\mathcal{L}(X) = U$. *Suppose that*

$$0 < \sigma^2 = \mathbf{E}\, X^2 < \infty, \qquad \mathbf{E}\, X = 0, \tag{1.16}$$

and

$$b \geqslant \frac{\varkappa_n}{\sigma^2}, \qquad \text{where} \quad \varkappa_n = \mathbf{E}\, X^2 \min\big\{ |X|, \sigma\sqrt{n} \big\}. \tag{1.17}$$

Let r, s *be integers,* $0 \leqslant r < s \leqslant n$. *Then*

$$\begin{aligned} Q(F^n, b) \leqslant \mathbf{E}\, \frac{cb}{\sigma\sqrt{n-\mu}}\, Q\big(V^\mu, \sigma\sqrt{n-\mu}\big)\, \mathbf{I}\big\{ r \leqslant \mu < s \big\} \\ + \mathbf{E}\, Q\big(U^{n-\mu} V^\mu, b\big)\, \mathbf{I}\big\{ \mu \geqslant s \big\} + \min\Big\{ 1, \frac{cb}{\sigma\sqrt{n-r}} \Big\}\, \mathbf{P}\,\big\{ \mu < r \big\}, \end{aligned} \tag{1.18}$$

where μ *is a random variable having binomial distribution with parameters* n *and* p. *If, in addition,* $r \leqslant np$, *then*

$$Q(F^n,\, b) \leqslant \frac{c\,b}{\sigma\sqrt{n\,(1-p)}}\; Q\big(V^r,\, \sigma\sqrt{n}\big) + \min\Big\{\,1,\ \frac{c\,b}{\sigma\sqrt{n-r}}\,\Big\} \\ \times \max\Big\{\,\exp\Big\{-\frac{(np-r)^2}{4np(1-p)}\Big\},\ \exp\Big\{-\frac{np-r}{4\,\max\{\,p, (1-p)\,\}}\Big\}\Big\}. \tag{1.19}$$

Taking in (1.19) $r = \lceil np/2 \rceil$, and, then, $p = {}^1\!/_2$, we obtain

Corollary 1. *Under the conditions of Theorem 2*

$$Q(F^n,\, b) \leqslant \frac{c\,b}{\sigma\sqrt{n\,(1-p)}}\; \Big(Q\big(V^{\lceil np/2 \rceil},\, \sigma\sqrt{n}\big) + \exp\Big\{-\frac{np}{8\,\max\{\,p, (1-p)\,\}}\Big\}\Big). \tag{1.20}$$

Corollary 2. *Let the conditions of Theorem 2 be satisfied with* $p = {}^1\!/_2$. *Then*

$$Q(F^n,\, b) \leqslant \frac{c\,b}{\sigma\sqrt{n}}\; \Big(Q\big(V^{\lceil n/4 \rceil},\, \sigma\sqrt{n}\big) + e^{-n/8}\Big). \tag{1.21}$$

The moment restrictions in Theorem 2 are imposed on the distribution U only. The distribution V is arbitrary and therefore the initial distribution F may have arbitrarily bad moment properties.

Theorem 2 and Corollaries 1 and 2 provide in some sence "multiplicative inequalities" for concentration functions of convolutions. Corollary 2 reduces the problem of estimation of $Q(F^n,\, b)$ to that of the estimation of $Q\big(V^{\lceil n/4 \rceil},\, \sigma\sqrt{n}\big)$. The trivial inequality

$$Q\big(V^{\lceil n/4 \rceil},\, \sigma\sqrt{n}\big) \leqslant \Big(1 + \frac{\sigma\sqrt{n}}{b}\Big)\, Q(V^{\lceil n/4 \rceil},\, b) \tag{1.22}$$

(see (2.13)) together with (1.21) can return the problem to the initial stage (in particular, when $Q(F^n,\, b)$ and $Q(V^n,\, b)$ decrease with some negative power of n). However, the inequality (1.22) may lead to a loss of precision. For instance, if V is concentrated on a one-dimensional lattice with step size $2\sigma\sqrt{n}$, we have

$$Q\big(V^{\lceil n/4 \rceil},\, \sigma\sqrt{n}\big) = Q(V^{\lceil n/4 \rceil},\, b), \qquad \text{for} \quad 0 \leqslant b \leqslant \sigma\sqrt{n}. \tag{1.23}$$

Our proofs are based on estimates in the Central Limit Theorem and on elementary properties of concentration functions. In this respect our proofs differ from most of the previous papers, where Esseen's (1968) method of characteristic functions had been extensively used.

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2. Proofs

Let $\mathcal{G}$ be the class of functions $g:\mathbf{R}\mapsto[0,\infty)$ such that

a) $g(x)$ is even, positive for $x\neq 0$, and does not decrease for $x\geqslant 0$;
b) the function $x/g(x)$ is non-decreasing for $x>0$.

Examples of functions in $\mathcal{G}$ are given by $g(x)=|x|^\alpha$, $0<\alpha\leqslant 1$.

Lemma 1 (see Petrov (1976), Theorem 5 of Chapter V). *Let ξ_i, $i=1,\dots,n$, be independent random variables, $\mathbf{E}\,\xi_i=0$, $W=\mathcal{L}(\xi_1+\dots+\xi_n)$, $g\in\mathcal{G}$,*

$$\beta_i(g)=\mathbf{E}\,\xi_i^2\,g(\xi_i),\qquad \beta(g)=\sum_{i=1}^n\beta_i(g)<\infty,\qquad B^2=\sum_{i=1}^n\mathbf{E}\,\xi_i^2>0. \tag{2.1}$$

Then

$$\rho(W,\Phi_{B})\leqslant\frac{c_{10}\,\beta(g)}{B^2\,g(B)}. \tag{2.2}$$

Lemma 2 (Zaitsev (1992)). *Let the conditions of Lemma 1 be satisfied and*

$$\lambda=\lambda(g)=\min\Big\{B,\ \frac{\beta(g)}{B\,g(B)}\Big\}. \tag{2.3}$$

Then, for any $H\in\mathfrak{F}$, we have

$$\rho(WH,\Phi_{B}H)\leqslant\frac{c\lambda}{B}\,Q(H,B), \tag{2.4}$$

$$Q(WH,\lambda)\asymp Q(\Phi_{B}H,\lambda)\asymp\frac{\lambda}{B}\,Q(H,B). \tag{2.5}$$

Lemma 2 slightly generalizes Lemma 1 which follows from (2.4) with $H=E$. For $g(x)\equiv|x|$ Lemma 2 was proved by Zaitsev (1987).

Below we shall use the following refinement of (2.5).

Lemma 3. *Let the conditions of Lemma 1 be satisfied and*

$$g_0(x)=\min\big\{|x|,B\big\},\quad b\geqslant\frac{\beta(g_0)}{B^2},\quad \Lambda=\Lambda(b)=\min\big\{B,\ b\big\}. \tag{2.6}$$

Then, for any $H\in\mathfrak{F}$, we have

$$Q(WH,\Lambda)\asymp Q(\Phi_{B}H,\Lambda)\asymp\frac{\Lambda}{B}\,Q(H,B) \tag{2.7}$$

and

$$Q(WH,\ b)\leqslant\frac{c\,b}{B}\,Q(H,B). \tag{2.8}$$

The relation (2.5) of Lemma 2 easily follows from Lemma 3. Indeed, if $g \in \mathcal{G}$, then

$$\begin{aligned}\beta_i(g) &= \mathbf{E}\,\xi_i^2\, g(\xi_i)\,\mathbf{I}\big\{\,|\xi_i| \leqslant B\,\big\} + \mathbf{E}\,\xi_i^2\, g(\xi_i)\,\mathbf{I}\big\{\,|\xi_i| > B\,\big\}\\ &\geqslant g(B)\,B^{-1}\,\mathbf{E}\,|\xi_i|^3\,\mathbf{I}\big\{\,|\xi_i| \leqslant B\,\big\} + g(B)\,\mathbf{E}\,\xi_i^2\,\mathbf{I}\big\{\,|\xi_i| > B\,\big\}\\ &= \frac{g(B)}{g_0(B)}\,\mathbf{E}\,\xi_i^2\, g_0(\xi_i) = \frac{g(B)}{g_0(B)}\,\beta_i(g_0). \end{aligned} \tag{2.9}$$

Hence,

$$\beta(g) \geqslant \frac{g(B)}{g_0(B)}\,\beta(g_0). \tag{2.10}$$

Therefore, $\lambda(g) = \Lambda(b)$ with $b = \dfrac{\beta(g)}{B\,g(B)} = \dfrac{\beta(g)\,g_0(B)}{B^2 g(B)} \geqslant \dfrac{\beta(g_0)}{B^2}$.

Note that (2.10) implies that $\lambda(g) \geqslant \lambda(g_0)$, for any $g \in \mathcal{G}$. Thus, if we apply the inequality (2.4) to the function $g(x) \equiv g_0(x)$ only, we do not loose in generality. The use of another function $g \in \mathcal{G}$ only weakens this statement. Note that a similar reasoning about Lemma 1 has been carried out by Studnev (1965).

For the proof of Lemma 3 we need the following well-known simple properties of concentration functions, which are valid for any $F, H \in \mathfrak{F}$ and $\gamma, \gamma_1, \gamma_2 > 0$:

$$Q(FH, \gamma) \leqslant \min\big\{\,Q(F,\gamma),\, Q(H,\gamma)\,\big\}; \tag{2.11}$$

$$Q(FH,\, \gamma_1 + \gamma_2) \geqslant Q(F,\gamma_1)\,Q(H,\gamma_2); \tag{2.12}$$

$$Q(F,\gamma_1) \leqslant \big(1 + \lceil \gamma_1/\gamma_2 \rceil\big)\,Q(F,\gamma_2) \tag{2.13}$$

(see, e.g., Hengartner and Theodorescu (1973)). Note that the proof of Lemma 3 is similar to the proof of Lemma 2 in Zaitsev (1992).

Proof of Lemma 3. Set $c_{11} = 4\,c_{10}\sqrt{2\pi e}$, and let at first $B \geqslant c_{11}\,b$. Choose a point y_0 so that $H\big\{\,[y_0,\, y_0 + B]\,\big\} = Q(H, B)$. The existence of y_0 follows from Theorem 1.1.8 of Hengartner and Theodorescu (1973). Using Lemma 1 with $g = g_0$, we obtain that

$$\begin{aligned} Q(WH,\, c_{11}\,b) &\geqslant WH\big\{\,[y_0,\, y_0 + c_{11}\,b]\,\big\} \geqslant \int_{y_0}^{y_0+B} W\big\{\,[y_0 - y, y_0 + c_{11}\,b - y]\,\big\}\,H\{dy\}\\ &\geqslant H\big\{\,[y_0,\, y_0 + B]\,\big\} \inf_{y \in [y_0, y_0+B]} W\big\{\,[y_0 - y,\, y_0 + c_{11}\,b - y]\,\big\}\\ &\geqslant Q(H,B)\left(\inf_{y \in [y_0, y_0+B]} \Phi_B\big\{\,[y_0 - y,\, y_0 + c_{11}\,b - y]\,\big\} - \frac{2\,c_{10}\,\beta(g_0)}{B^3}\right)\\ &\geqslant Q(H,B)\left(\frac{c_{11}\,b}{\sqrt{2\pi\,e}\,B} - \frac{2\,c_{10}\,b}{B}\right) = \frac{2\,c_{10}\,b}{B}\,Q(H,B). \end{aligned} \tag{2.14}$$

Similarly one can prove that

$$Q(\Phi_{B} H,\, c_{11}\, b) \geqslant \frac{4\, c_{10}\, b}{B}\, Q(H, B). \tag{2.15}$$

From the conditions of Lemma 3 it follows that

$$\lambda(g_0) \leqslant \frac{\beta(g_0)}{B^2} \leqslant b. \tag{2.16}$$

Applying Lemma 2 with $g = g_0$, and using (2.13), (2.16), we see that

$$Q(\Phi_{B} H,\, c_{11}\, b) \leqslant \frac{c\, b}{B}\, Q(H, B), \tag{2.17}$$

$$Q(W H,\, c_{11}\, b) \leqslant \frac{c\, b}{B}\, Q(H, B). \tag{2.18}$$

Now (2.7) for $B \geqslant c_{11}\, b$ follows from (2.13)–(2.15), (2.17) and (2.18) since in this case $\Lambda \asymp b$.

Let $B < c_{11}\, b$. Then, using the Chebyshev inequality, one can easily verify that

$$W\{(-2\,B,\, 2\,B)\} \geqslant 3/4, \qquad \Phi_{B}\{(-2\,B,\, 2\,B)\} \geqslant 3/4. \tag{2.19}$$

By an application of (2.12) we thus obtain that

$$Q(W H, 5B) \geqslant Q(W, 4B)\, Q(H, B) \geqslant 3\, Q(H, B)/4, \tag{2.20}$$

$$Q(\Phi_{B} H, 5\, B) \geqslant 3\, Q(H, B)/4. \tag{2.21}$$

Now (2.7) for $B < c_{11}\, b$ follows from (2.11), (2.13), (2.20) and (2.21) since in this case $\Lambda \asymp B$.

Inequality (2.8) is an obvious consequence of (2.7) and (2.13). $\square$

Proof of Theorem 2. It is known that F^n can be written in the form

$$F^n = \mathcal{L}(S_n), \qquad \text{where} \quad S_n = \sum_{i=1}^{n} (1 - \mu_i)\, \xi_i + \mu_i\, \eta_i, \tag{2.22}$$

and ξ_i, η_i, μ_i are mutually independent random variables with

$$\mathcal{L}(\xi_i) = U, \qquad \mathcal{L}(\eta_i) = V, \qquad \mathcal{L}(\mu_i) = (1-p)\, E + p\, E_1. \tag{2.23}$$

Define $\mu = \sum\limits_{i=1}^{n} \mu_i$. Obviously, μ has binomial distribution with parameters n and p. Given a fixed value of μ, the random variable S_n has the conditional distribution $U^{n-\mu}\, V^{\mu}$. Hence, for any $x \in \mathbf{R}$ we have

$$\begin{aligned} \mathbf{P}\big\{ S_n \in \big[x,\, x+b\big] \big\} &= \mathbf{E}\; \mathbf{I}\big\{ S_n \in \big[x,\, x+b\big] \big\} \\ &= \mathbf{E}\; \mathbf{E}\big\{ \mathbf{I}\big\{ S_n \in \big[x,\, x+b\big] \big\} \,\big|\, \mu \big\} \\ &\leqslant \mathbf{E}\, Q\big(U^{n-\mu}\, V^{\mu},\, b\big). \end{aligned} \tag{2.24}$$

Therefore,

$$Q(F^n,b) \leqslant \mathbf{E}\, Q\big(U^{n-\mu}V^{\mu},\, b\big). \tag{2.25}$$

In view of $\varkappa_{n-\mu} \leqslant \varkappa_n$ and applying inequality (2.8) of Lemma 3 with $W = U^{n-\mu}$, $H = V^{\mu}$, $B = \sigma\sqrt{n-\mu}$, we see that

$$\begin{aligned}\mathbf{E}\, Q\big(U^{n-\mu}V^{\mu},\, b\big) \leqslant \mathbf{E}\, \frac{c\,b}{\sigma\sqrt{n-\mu}}\, Q\big(V^{\mu},\, \sigma\sqrt{n-\mu}\big)\, \mathbf{I}\big\{\, r \leqslant \mu < s\,\big\} \\ + \mathbf{E}\, Q\big(U^{n-\mu}V^{\mu},\, b\big)\, \mathbf{I}\big\{\, \mu \geqslant s\,\big\} + \mathbf{E}\, Q\big(U^{n-\mu}V^{\mu},\, b\big)\, \mathbf{I}\big\{\, \mu < r\,\big\}\end{aligned} \tag{2.26}$$

and

$$\mathbf{E}\, Q\big(U^{n-\mu}V^{\mu},\, b\big)\, \mathbf{I}\big\{\, \mu < r\,\big\} \leqslant \min\Big\{\, 1,\ \frac{c\,b}{\sigma\sqrt{n-r}}\,\Big\}\, \mathbf{P}\,\big\{\, \mu < r\,\big\}. \tag{2.27}$$

The inequality (1.18) now follows from (2.25)–(2.27).

Using the relations $\mathbf{P}\,\big\{\, \mu = n\,\big\} = p^n$, (1.18) with $s = n$ and (2.11), we obtain

$$\begin{aligned}Q(F^n,\, b) \leqslant \frac{c\,b}{\sigma}\, Q\big(V^r,\, \sigma\sqrt{n}\big)\, \mathbf{E}\, \frac{1}{\sqrt{n-\mu}}\, \mathbf{I}\big\{\, r \leqslant \mu < n\,\big\} \\ + p^n\, Q\big(V^n,\, b\big) + \min\Big\{\, 1,\ \frac{c\,b}{\sigma\sqrt{n-r}}\,\Big\}\, \mathbf{P}\,\big\{\, \mu < r\,\big\}.\end{aligned} \tag{2.28}$$

Applying the Hölder inequality, we derive that

$$\begin{aligned}\mathbf{E}\, \frac{1}{\sqrt{n-\mu}}\, \mathbf{I}\big\{\, r \leqslant \mu < n\,\big\} &\leqslant c\, \mathbf{E}\, \frac{1}{\sqrt{n-\mu+1}} \\ &\leqslant c\, \Big(\, \mathbf{E}\, \frac{1}{n-\mu+1}\,\Big)^{1/2} \leqslant \frac{c}{\sqrt{n\,(1-p)}}.\end{aligned} \tag{2.29}$$

Using (1.16) and (1.17), it is easy to see that

$$\begin{aligned}\varkappa_n \geqslant \varkappa_1 &= \mathbf{E}\,|X|^3\, \mathbf{I}\big\{\, |X| \leqslant \sigma\,\big\} + \sigma\, \mathbf{E}\,|X|^2\, \mathbf{I}\big\{\, |X| > \sigma\,\big\} \\ &\geqslant \Big(\, \mathbf{E}\,|X|^2\, \mathbf{I}\big\{\, |X| \leqslant \sigma\,\big\}\Big)^{3/2} + \Big(\, \mathbf{E}\,|X|^2\, \mathbf{I}\big\{\, |X| > \sigma\,\big\}\Big)^{3/2} \geqslant c\sigma^3.\end{aligned} \tag{2.30}$$

The relations (1.17) and (2.30) together imply

$$b \geqslant \varkappa_n\, \sigma^{-2} \geqslant c\sigma. \tag{2.31}$$

According to (2.11), (2.13) and (2.31), we have

$$Q\big(V^n,\, b\big) \leqslant Q\big(V^r,\, b\big) \leqslant \Big(\, \frac{b}{\sigma\sqrt{n}} + 1\Big)\, Q\big(V^r,\, \sigma\sqrt{n}\big) \leqslant \frac{c\,b}{\sigma}\, Q\big(V^r,\, \sigma\sqrt{n}\big). \tag{2.32}$$

Moreover,

$$p^n = \big(1 - (1-p)\big)^n \leqslant e^{-n\,(1-p)} \leqslant \frac{c}{\sqrt{n\,(1-p)}}. \tag{2.33}$$

Finally, by Bernstein's inequality,

$$\mathbf{P}\,\big\{\, \mu < r\,\big\} \leqslant \max\Big\{\, \exp\Big\{ -\frac{(n\,p-r)^2}{4\,n\,p\,(1-p)}\,\Big\},\ \exp\Big\{ -\frac{n\,p-r}{4\,\max\{\, p,\,(1-p)\,\}}\,\Big\}\,\Big\} \tag{2.34}$$

(see Petrov (1976), Theorem 17 of Chapter III, or Arak and Zaitsev (1988), Theorem 4.1 of Chapter I). The inequality (1.19) can be easily derived from (2.28), (2.29) and (2.32)–(2.34). □

Lemma 4. *Let the conditions of Theorem 2 be satisfied with* $p = {}^1\!/_2$. *Then, for any* $\delta > 0$,

$$Q(F^n, b) \leqslant \frac{c\, b\, (\delta + \sigma)}{\delta\, \sigma\, n \sqrt{D(\widetilde{V}, \delta\sqrt{n})}}. \tag{2.35}$$

Proof. In view of (2.31), we may assume without loss of generality that $n \geqslant 4$. Using Corollary 2, (1.2), (1.3) and (2.13), we obtain

$$\begin{aligned} Q(F^n, b) &\leqslant \frac{c\, b}{\sigma\sqrt{n}} \left(Q\big(V^{\lceil n/4 \rceil}, \sigma\sqrt{n}\big) + e^{-n/8} \right) \\ &\leqslant \frac{c\, b}{\sigma\sqrt{n}} \left(\big(1 + \sigma\, \delta^{-1}\big)\, Q\big(V^{\lceil n/4 \rceil}, \delta\sqrt{n}\big) + e^{-n/8} \right) \\ &\leqslant \frac{c\, b}{\sigma\sqrt{n}} \left(\frac{1 + \sigma\, \delta^{-1}}{\sqrt{n\, D(\widetilde{V}, \delta\sqrt{n})}} + e^{-n/8} \right) \\ &\leqslant \frac{c\, b\, (\delta + \sigma)}{\delta\, \sigma\, n \sqrt{D(\widetilde{V}, \delta\sqrt{n})}}. \qquad \square \end{aligned} \tag{2.36}$$

Proof of Theorem 1. Without loss of generality we may assume that the distribution F is non-degenerate. Then there exist a number $q = q(F) \in \big(0, {}^1\!/_3\big]$ such that one can represent F as a mixture

$$F = (1 - q)\, V_1 + q\, V_2, \qquad V_1,\, V_2 \in \mathfrak{F}, \tag{2.37}$$

so that the distribution V_1 is non-degenerate, and

$$\varkappa \overset{\text{def}}{=} \int_{-\infty}^{\infty} |x|^3\, V_1\{dx\} < \infty. \tag{2.38}$$

Without loss of generality we may assume furthermore that

$$\int_{-\infty}^{\infty} x\, V_1\{dx\} = 0. \tag{2.39}$$

Write

$$\sigma^2 \overset{\text{def}}{=} \int_{-\infty}^{\infty} x^2\, V_1\{dx\} > 0. \tag{2.40}$$

Define

$$V = (1 - 2\, q)\, V_1 + 2\, q\, V_2 \in \mathfrak{F}. \tag{2.41}$$

From (2.37) and (2.41) it follows that F can be represented in the form (1.15) with $U = V_1$, $p = {}^1\!/_2$ and V defined in (2.41). According to (2.37) and (2.41),

$$\begin{aligned} \widetilde{F} &= (1 - q)^2\, V_1 \overline{V}_1 + q\, (1 - q) \big(V_1 \overline{V}_2 + V_2 \overline{V}_1 \big) + q^2\, V_2 \overline{V}_2, \\ \widetilde{V} &= (1 - 2\, q)^2\, V_1 \overline{V}_1 + 2\, q\, (1 - 2\, q) \big(V_1 \overline{V}_2 + V_2 \overline{V}_1 \big) + 4\, q^2\, V_2 \overline{V}_2. \end{aligned} \tag{2.42}$$

Using the relations (2.42) and $1-q\leqslant 2\,(1-2\,q)$ for $q\in\big(0,{}^1\!/_3\big]$, we easily obtain that $\widetilde F\{dx\}\leqslant 4\,\widetilde V\{dx\}$ and, hence,

$$D\big(\widetilde F,\,\delta\sqrt{n}\big)\leqslant 4\,D\big(\widetilde V,\,\delta\sqrt{n}\big). \tag{2.43}$$

Obviously,

$$\varkappa=\int_{-\infty}^{\infty}|x|^3\,V_1\{dx\}\geqslant \varkappa_n\overset{\mathrm{def}}{=}\int_{-\infty}^{\infty}x^2\,\min\big\{|x|,\,\sigma\sqrt{n}\big\}\,V_1\{dx\}. \tag{2.44}$$

Applying Lemma 4 and using (2.40), (2.43) and (2.44), we see that

$$Q(F^n,\,b)\leqslant\frac{c\,b\,(\delta+\sigma)}{\delta\,\sigma\,n\sqrt{D\big(\widetilde F,\,\delta\sqrt{n}\big)}}\,,\qquad\text{for}\quad b\geqslant\frac{\varkappa_n}{\sigma^2}\,. \tag{2.45}$$

The inequality (2.45) with $b=\dfrac{\varkappa_n}{\sigma^2}$ and the monotonicity of the concentration functions imply that

$$Q(F^n,\,b)\leqslant\frac{c\,\varkappa_n\,(\delta+\sigma)}{\delta\,\sigma^3\,n\sqrt{D\big(\widetilde F,\,\delta\sqrt{n}\big)}}\,,\qquad\text{for}\quad 0\leqslant b<\frac{\varkappa_n}{\sigma^2}\,. \tag{2.46}$$

Combining (2.45) and (2.46) and using (2.44), we obtain that

$$Q(F^n,\,b)\leqslant\frac{c\,(b+\varkappa_n\,\sigma^{-2})\,(\delta+\sigma)}{\delta\,\sigma\,n\sqrt{D\big(\widetilde F,\,\delta\sqrt{n}\big)}}\leqslant\frac{c\,(b+\varkappa\sigma^{-2})\,(\delta+\sigma)}{\delta\,\sigma\,n\sqrt{D\big(\widetilde F,\,\delta\sqrt{n}\big)}}\,,\qquad\text{for}\quad b\geqslant 0. \tag{2.47}$$
$\square$

Proof of Proposition 1. We shall construct the sequence of positive integers k_j, $j\geqslant 1$, inductively as follows. First choose

$$k_1=1\quad\text{and}\quad k_2=2. \tag{2.48}$$

Suppose $k_1,\,k_2,\dots,k_{j-1}$ being already constructed, for $j\geqslant 3$. Introduce the distribution

$$F_j=\sum_{m=1}^{j-1}\frac{a_j}{k_m}\,E_{-k_m}+\sum_{m=1}^{j-1}\frac{a_j}{k_m}\,E_{k_m}\in\mathfrak{F},\qquad a_j=\Big(\sum_{m=1}^{j-1}\frac{2}{k_m}\Big)^{-1}, \tag{2.49}$$

Put

$$\sigma_j^2=\int_{-\infty}^{\infty}x^2\,F_j\{dx\}. \tag{2.50}$$

The distribution F_j has bounded support and zero mean and it is lattice distributed with the maximal step size 1 (see (2.48) and (2.49)). According to the Local Limit Theorem, see Petrov (1976), Theorem 1 of Chapter VII, we have

$$\Big|\,\sigma_j\,\sqrt{k}\,F_j^k\{\{0\}\} - \frac{1}{\sqrt{2\,\pi}}\,\Big| \xrightarrow[k\to\infty]{} 0. \tag{2.51}$$

Obviously, $\dfrac{1}{3} < \dfrac{1}{\sqrt{2\,\pi}}$. In view of (2.51), we can choose k_j to be large enough so that

$$F_j^{k_j}\{\{0\}\} \geqslant \frac{1}{3\,\sigma_j\,\sqrt{k_j}} \tag{2.52}$$

and

$$k_j \geqslant 4\,k_{j-1}^2 > 2\,k_{j-1}. \tag{2.53}$$

By virtue of (2.48) and (2.53), we have

$$k_m \geqslant k_s\,2^{m-s}, \qquad s = 1, 2, \dots; \qquad m = s,\ s+1, \dots. \tag{2.54}$$

Now define the distribution F as the mixture

$$F = \sum_{m=1}^{\infty} \frac{C}{k_m}\,E_{-k_m} + \sum_{m=1}^{\infty} \frac{C}{k_m}\,E_{k_m}, \qquad C = \Big(\sum_{m=1}^{\infty} \frac{2}{k_m}\Big)^{-1}. \tag{2.55}$$

By (2.48), (2.54) and (2.55), it follows that the distribution $F \in \mathfrak{F}$ has an infinite second moment and

$$\frac{1}{4} \leqslant C \leqslant \frac{1}{3}. \tag{2.56}$$

According to (2.49), (2.53), (2.55) and (2.56), we can rewrite the distribution F in the form

$$F = (1 - p_j)\,F_j + p_j\,V_j, \qquad j = 3, 4, \dots, \tag{2.57}$$

where

$$V_j = \sum_{m=j}^{\infty} \frac{r_j}{k_m}\,E_{-k_m} + \sum_{m=j}^{\infty} \frac{r_j}{k_m}\,E_{k_m} \in \mathfrak{F}, \qquad r_j = \Big(\sum_{m=j}^{\infty} \frac{2}{k_m}\Big)^{-1} \tag{2.58}$$

and

$$p_j = \frac{C}{r_j} \leqslant \frac{4\,C}{k_j} \leqslant \frac{1}{3}. \tag{2.59}$$

Using (2.56) and (2.59), we obtain

$$(1 - p_j)^{k_j} \geqslant c, \qquad j = 3, 4, \dots. \tag{2.60}$$

From (2.52), (2.57) and (2.60) it follows that

$$F^{k_j}\bigl\{\{0\}\bigr\} \geqslant (1-p_j)^{k_j} F_j^{k_j}\bigl\{\{0\}\bigr\} \geqslant \frac{c}{\sigma_j\sqrt{k_j}}, \qquad j=3,4,\dots. \tag{2.61}$$

Taking into account (2.55) and (2.57), it is easy to see that

$$\widetilde{F} = F^2 = (1-p_j)^2 F_j^2 + 2p_j(1-p_j)V_jF_j + p_j^2V_j^2, \qquad j=3,4,\dots. \tag{2.62}$$

According to (1.5), (2.49), (2.50), (2.53), (2.59) and (2.62), we have

$$\sigma_j^2 = \frac{1}{2}\int_{-\infty}^{\infty} x^2\,F_j^2\{dx\} = \frac{1}{2}\int_{|x|\leqslant 2k_{j-1}} x^2\,F_j^2\{dx\} \leqslant \frac{1}{2(1-p_j)^2}\int_{|x|\leqslant\sqrt{k_j}} x^2\,\widetilde{F}\{dx\}$$

$$\leqslant \frac{9}{8}\int_{|x|\leqslant\sqrt{k_j}} x^2\,\widetilde{F}\{dx\} = c\,k_j\,K\bigl(\widetilde{F},\,\sqrt{k_j}\bigr). \tag{2.63}$$

Using (2.61) and (2.63), we see that, for all $b\geqslant 0$,

$$Q(F^{k_j},b) \geqslant F^{k_j}\bigl\{\{0\}\bigr\} \geqslant \frac{c}{k_j\sqrt{K(\widetilde{F},\,\sqrt{k_j})}}, \qquad j=3,4,\dots. \tag{2.64}$$

This proves the inequality (1.13) (for $j=1,2$ it obviously follows from (1.5), (2.48), (2.55) and (2.56)).

From (2.49), (2.53), (2.54) and (2.58) one can easily derive that, for $j=3,4,\dots$,

$$(V_jF_j)\bigl\{\{x:|x|>k_j/2\}\bigr\} = (V_j^2)\bigl\{\{x:|x|>k_j/2\}\cup\{0\}\bigr\} = 1. \tag{2.65}$$

It is evident that the relations (2.49), (2.50) and (2.54) imply that

$$\sigma_j^2 \leqslant c\,k_{j-1}. \tag{2.66}$$

Using (1.5), (2.53), (2.62), (2.63), (2.65) and (2.66), we see that

$$K\bigl(\widetilde{F},\,k_j/2\bigr) = \frac{4(1-p_j)^2}{k_j^2}\int_{|x|\leqslant k_j/2} x^2\,F_j^2\{dx\} \leqslant \frac{c\,\sigma_j^2}{k_j^2} \leqslant \frac{c\,k_{j-1}}{k_j^2} \leqslant \frac{c}{k_j^{3/2}}. \tag{2.67}$$

On the other hand, by (1.5), (2.56), (2.58), (2.59), (2.62) and (2.65),

$$G\bigl(\widetilde{F},\,k_j/2\bigr) \geqslant 2p_j(1-p_j) \geqslant cp_j \geqslant \frac{c}{r_j} \geqslant \frac{c}{k_j}. \tag{2.68}$$

Obviously, the inequalities (2.67) and (2.68) imply that

$$K\bigl(\widetilde{F},\,k_j/2\bigr) = o\Bigl(G\bigl(\widetilde{F},\,k_j/2\bigr)\Bigr) \qquad \text{as} \quad j\to\infty. \tag{2.69}$$

The relation (1.14) now follows from (1.4), (1.5) and (2.69). $\square$

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