Admissible vector fields and related diffusions on infinite dimensional manifolds

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§1 Introduction

The late 1960's saw the remarkable paper of Dalecky and Shnaiderman [DS69] constructing measures on certain infinite dimensional Lie groups via stochastic differential equations (with infinite dimensional noise) and discussing their quasi-invariance properties. Analysis on infinite dimensional spaces was also developing with discussions of differentiability of measures(cf. e.g. [DF91], [SW93]), a topic of increasing importance with continuing work by Professor Dalecky and his co-workers. In both of those advances, the idea of "admissible" derivatives plays an important role: in the curved space case the "derivatives" are given by vector fields. Such ideas are now fundamental to infinite dimensional analysis having been joined with developments in the work on H-differentiability by Gross[Gr67] and on Malliavin calculus[M76], with Driver's work[Dr92] leading the way to recent advances in analysis on path and loop space.

There have been many variations on the notion of "admissibility". Here we will describe that given in [EM96], showing how it leads to the construction of associated diffusions and Markov semi-groups on the state space. We will also show often infinite dimensional stochastic differential equations appear naturally: however in practice the coefficients are not regular enough for existence theorems in their present stage of development, so we have to rely on Dirichlet form theory(cf. [MOT94], [MR92]) to construct our processes. We give a class of concrete examples using results from [ELL 96], [ELL 97] to extend earlier work such as [DR 92].

§2 Admissible vector fields on $\mathcal{C}$-F manifolds

In this section we assume that M is a separable C^1 manifold modeled on a Banach space and equipped with a given Finsler structure τ ([AL78],[Pa66]). Let $TM := \cup_{x\in M} T_xM$ denote the tangent bundle of M.

We write $|v|_x = \tau(v)$ for $v \in T_xM$. Then T_xM equipped with the norm $|\cdot|_x$ is a Banach space. Let f be a $\mathcal{C}^1$ map from M to another Finsler manifold N. We set

$$||df|| := \sup_{x\in M} ||df(x)||_{L(T_xM, T_{f(x)}N)}. \tag{2.1}$$

We write $f \in C_b^1(M;N)$ if $||df|| < \infty$. In particular we write $f \in C_1^1(M;N)$ if $||df|| < 1$. We introduce a pseudo metric ρ_M (Carathéodory metric[Ca69]) on M as follows

$$\rho_M(x,y) := \sup\{f(y) - f(x) : \ f \in C_1^1(M;R)\}. \tag{2.2}$$

Definition 2.1 We say that M is a **Caratheodory-Finsler manifold** (C-F manifold in short) (w.r.t. τ) if ρ_M is an admissible metric on M (i.e., ρ_M generates the original topology on M) and is complete.

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Below we collect some facts and examples concerning C-F manifolds without proof. For details see [EM96].

Proposition 2.2 ([EM96, 2.2]) Suppose that there exists a C-F manifold N and a closed embedding map $J \in C_b^1(M;N)$, then M is a C-F manifold.

The following results was communicated to us by C. J. Atkin.

Proposition 2.3 (Atkin) Suppose that the separable manifold M is modeled on a Banach space with a separable dual, then $\rho_m = d_M$ where d_M is the metric induced by the given Finsler structure. Thus if in addition d_M is complete, then M is a C-F manifold.

Example 2.4 (i) Let M be a separable Banach space with Finsler structure given by the Banach norm. then M is a C-F manifold.

(ii) Let M be a finite dimensional complete Riemannian manifold and the Finsler structure be given by the Riemannian metric. Then by Nash's embedding theorem M has a closed isometric embedding into a Euclidean space $\mathbb{R}^N$. Hence by (i) and proposition 2.2 M is a C-F manifold. See also Remark 2.5 below in this connection.

(iii) It follows directly from proposition 2.2 that any closed submanifold of a C-F manifold is again a C-F manifold.

The manifolds we are interested in will be C-F manifolds by the above results. They will be spaces of maps of a compact metric space S into a C-F manifold M. For example path or loop spaces if $S = [0,1]$ or S^1. Set $E = C(S;M)$ the space of all continuous mappings with the compact open topology. Then E is a C^1 manifold (e.g. see [Pa 66], [Eli 66] for the case of M a Hilbert manifold and [Pe 67], [Ki 72] for more general M). Moreover, spaces of based loops and based paths are closed submanifolds of $C([0,1];M)$. For $\sigma \in E$, the tangent vector space $T_\sigma E$ can be identified with the space of all continuous map $v: S \to TM$ such that $v(s) \in T_{\sigma(s)}M$ for all $s \in S$. A natural Finsler structure on E is given by

$$|v|_\sigma := \sup_{\sigma \in S} |v(s)|_{\sigma(s)}, \ \forall \sigma \in E, \ v \in T_\sigma E. \tag{2.3}$$

One can easily check that $T_\sigma E$ equipped with the norm $|\cdot|_\sigma$ is a Banach space.

Proposition 2.5 (EM 96, 2.6) The mapping space E constructed above is a C-F manifold with respect to the Finsler structure given by (2.3), as are all its closed submanifolds with their induced Finsler structure.

In what follows let $E := C(S;M)$ be the mapping space specified in Proposition 2.5 or a closed submanifold of it. Let $\sigma \in E$ and $v \in T_\sigma E$. For $f \in C^1(E;R)$ we shall write $\partial_v f(\sigma) := df(v)(\sigma)$, the Fréchet derivative of f at σ along the direction v.

For a vector field v on E we write $\partial_v f(\sigma)$ for $\partial_{v(\sigma)} f(\sigma)$. Let $\mathcal{F}C_b^1(E)$ be the space of cylindrical functions in $C_b^1(E;R)$; so they are compositions of evaluations at a finite number of points of S with a C_b^1 function on a product of copies of M with its product Finsler. A monotone class argument shows that $\mathcal{F}C_b^1$ is dense in $L^2(E;\mu)$ for any finite Borel measure μ on E.

We shall always use the Borel σ-algebra of our manifolds. A vector field will be called **measurable** if $\partial_v f$ is measurable for all C^1 functions $f: E \to \mathbb{R}$. From now on we assume a finite measure μ is given on E. As usual we use f to denote the μ-a.s. equivalence class of a measurable function f on E

We need to specify a class of test functions for our Dirichlet forms. The standard example will be $\mathcal{F}C_b^1(E)$ but other choices could sometimes be more useful (e.g. see [EM 96] and below).

Definition 2.6 Let $\mathcal{D}$ be a linear subspace of $C_b^1(E;R) \cap L^2(E,\mu)$ such that $\mathcal{D}$ is dense in $L^2(E,\mu)$. We say that a $\mathcal{B}$-measurable tangent vector field v is $\mathcal{D}$-**Admissible**, if the following three conditions are satisfied.

(i) $f \in \mathcal{D},\ f = 0$ μ-a.e. implies $\partial_v f = 0$ μ-a.e.

(ii)$\partial_v f \in L^2(E,\mu)$ for all $f \in \mathcal{D}$.

(iii)∂_v is a closable operator in $L^2(E,\mu)$.

The above definition goes over to μ-a.s. equivalence classes of vector fields. In our examples the vector fields will only be defined off some μ-null set in E and admissibility will mean admissibility for some (and hence any) measurable extension.

Below is a sufficient condition for v to be $\mathcal{D}$-admissible

Proposition 2.7 ([EM 96, 3.5]) Suppose that $\mathcal{D}$ is an algebra with pointwise multiplication. Let v be a $\mathcal{B}$-measurable vector field such that $\partial_v f \in L^2(E,\mu)$ for all $f \in \mathcal{D}$ and there exists an element div v (called the **divergence** of v) in $L^2(E,\mu)$ satisfying

$$\int_E \partial_v f \mu(d\sigma) = -\int_E f \text{div } v \mu(d\sigma),\ \forall f \in \mathcal{D}, \tag{2.4}$$

then v is $\mathcal{D}$-admissible. In this case we say that v is **strongly $\mathcal{D}$-admissible**

Proposition 2.8 [EM 96, 3.6] Let v be a strongly $\mathcal{D}$-admissible vector field. Then for any bounded element $f \in \mathcal{D}$, fv is again a strongly $\mathcal{D}$-admissible vector field.

Remark 2.9 In the situation of Proposition 2.5 and definition 2.6, let $\mathcal{A}$ be a countable or finite family of $\mathcal{D}$-admissible vector fields. Suppose that

$$\int_E \sum_{v\in\mathcal{A}} |\partial_v f|^2 \mu(d\sigma) < \infty,\ \forall f \in \mathcal{D} \tag{2.5}$$

then one can easily check that the symmetric form $(\mathcal{E},\mathcal{D})$ defined by

$$\mathcal{E}(f,g) = \int_E \sum_{v\in\mathcal{A}} (\partial_v f)(\partial_v g)\mu(d\sigma),\ \forall f,g \in \mathcal{D} \tag{2.6}$$

is closable in $L^2(E,\mu)$ and the closure $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(E,\mu)$. (see [EM 96, 4.1])

The following theorem plays an important role in our further discussion.

Theorem 2.10 In the situation of Remark 2.9 suppose that in addition to (2.5) the following three conditions are also fulfilled.

(i) If $\varphi \in C_b^\infty(R), \varphi(0) = 0$, then $\varphi \circ f \in \mathcal{D}$ for all $f \in \mathcal{D}$.

(ii)If f,g are bounded functions in $\mathcal{D}$, then $fg \in \mathcal{D}$.

(iii)$\mathcal{D}_0 \in \mathcal{D}$ and there exists $\Phi \in L^2(E,\mu)$ such that for all $\varphi \in C_b^1(M,R), s \in S$,

$$\sum_{v\in\mathcal{A}} |d\varphi(\sigma(s))(v_s(\sigma))|^2 \leq ||d\varphi||^2 \Phi^2(\sigma),\ \mu\text{-a.e.} \tag{2.7}$$

where $||d\varphi||$ is defined by (2.1) and

$$\begin{aligned} \mathcal{D}_0 := \{ & f \in \mathcal{F}C_b^1(E) :\ f(\sigma) = \varphi(\sigma(s)) \\ & \text{for all } \sigma \in E,\ \text{for some } \varphi \in C_b^1(M,R) \text{ and some } s \in S\}. \end{aligned} \tag{2.8}$$

Then there exists a diffusion process $(\xi_t)_{t\geq 0}$ on E properly associated with $(\mathcal{E}, D(\mathcal{E}))$. That is, if $(L, D(L))$ is the generator of $(\mathcal{E}, D(\mathcal{E}))$, then

$$E.[f(\xi_t)] \text{ is an } \mathcal{E}\text{-quasi-continuous } \mu\text{-version of } e^{L_t} f \quad \text{for all } f \in L^2(E,\mu).$$

Moreover, $(\xi_t)_{t\geq 0}$ is conservative and hence μ is an invariant measure for e^{L_t}.

The proof of the above theorem goes back to [RS95] but relies heavily on the fact that $E := C(S; M)$ is a C-F manifold. For details see [EM 96, Th. 4.2].

Theorem 2.11 below is a consequence of the above theorem; see [EM 96, 4.8] for its proof.

Theorem 2.11 Let $X : E \times H \to TE$ be measurable with, for μ-almost all σ with $X(\sigma, \cdot) : H \to T_\sigma E$ continuous linear and satisfying $|X(\sigma,h)|_\sigma \leq \Phi(\sigma)||h||_H$ where H is a separable Hillbert space and $\Phi \in L^2(E,\mu)$. Suppose $X(\cdot,h)$ is $\mathcal{D}$-admissible for all $h \in H$.

For $f : E \to R$ in C_b^1 define, for μ-almost all σ, $\nabla f(\sigma) \in H$ by $< \nabla f(\sigma), h >_H = df(X(\sigma,h))$. Set

$$\mathcal{E}_X(f,g) = \int_E < \nabla f(\sigma), \nabla g(\sigma) >_H \mu(d\sigma), \forall f,g \in \mathcal{D}.$$

Then $(\mathcal{E}_X, \mathcal{D})$ is closable in $L^2(E,\mu)$ with closure $(\overline{\mathcal{E}}_X, D(\overline{\mathcal{E}}_X))$ a Dirichlet form. If also conditions (i), (ii) of theorem 2.10 hold then $(\overline{\mathcal{E}}_X, D(\overline{\mathcal{E}}_X))$ is quasi-regular and local, and in particular there is a properly associated diffusion as in the conclusion of Theorem 2.10.

Remark In Theorems 2.10 and 2.11 we could equally say that there exists a diffusion $(\xi_t)_{t\geq 0}$ associated with $(\mathcal{E}, D(\mathcal{E}))$ (i.e. $E.[f(\xi_t)]$ *is an* μ*-version of* $e^{Lt}f$) instead of saying *properly associated*, because by the last assertion of V.Theorem 5.1 of [MR 92] ,which extends an earlier result by Fukushima [Fu 80], association always implies proper association in our context.

§3 Dirichlet forms and diffusions on path spaces

A. Preliminaries

In this section let M be a $\mathcal{C}^\infty n$-dimensional compact Riemannian manifold and $T > 0$. Let $E := \mathcal{C}_{x_0}([0,T]; M)$, a based path space for some fixed point $x_0 \in M$. Then, being a closed submanifold of the C-F manifold $\mathcal{C}([0,T]; M)$, E is a C-F manifold (c.f.Proposition 2.5). For $\sigma \in E$, the tangent space $T_\sigma E$ consists of all continuous paths $v : [0,T] \to TM$ such that $v(0) = 0$ and $v(t) \in T_{\sigma(t)}M$. A natural Finsler structure on $T_\sigma E$ is given for v as above by

$$|v|_\sigma := \sup_{0\leq t\leq T} |v(t)|_{T_{\sigma(t)}M}. \tag{3.1}$$

We choose a Borel measure μ on E to be the law of a M-valued diffusion $\{\xi_t : t \geq 0\}$ starting from x_0 with generator $\frac{1}{2}\Delta + Z$ where Δ is the Laplace-Beltrami operator on M and Z a smooth vector field. In particular if $Z \equiv 0$ then μ is the usual Wiener measure on E which has been considered by many authors.

Let $\{x_t : 0 \leq t \leq T\}$ be the coordinate process on $E(i.e., x_t(\sigma) = \sigma_t)$, and $\mathcal{F}.^{x_0} = \{\mathcal{F}_t^{x_0} : 0 \leq t \leq T\}$ the natural filtration generated by $\{x_t\}$. Then $\{x_t\}$ is a semi-martingale on the filtered probability space $(E, \mathcal{F}_T^{x_0}, \mathcal{F}.^{x_0}, \mu)$.

Let $\hat{\nabla}$ be a smooth linear connection on M of which the dual $\breve{\nabla}$ is a metric connection for the Riemannian structure of M. There is then another vector field A on M with

$$\frac{1}{2}(\Delta + Z)f = \frac{1}{2}trace\hat{\nabla}\mathrm{d}f + Af \tag{3.2}$$

or equivalently

$$\frac{1}{2}(\Delta + Z)f = \frac{1}{2}trace\breve{\nabla}grad f + Af. \tag{3.3}$$

It follows from a result of [ELL 97] that one can always construct a stochastic differential equation

$$\mathrm{d}\xi_t = X(\xi_t) \circ \mathrm{d}B_t + A(\xi_t)\mathrm{d}t \tag{3.4}$$

for some m-dimensional (m is not necessarily equal to n), Brownian motion $(B_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ and some $\mathcal{C}^\infty$ map $X : \mathbb{R}^m \times M \to TM$. with $X(x) \in L(\mathbb{R}^m; T_xM)$ all $x \in M$, in such a way that $\breve{\nabla}$ equals the LeJan-Watanabe connection induced from (3.4), and consequently the law of the solution to $S.D.E.$(3.4) starting from x_0 coincides with the prespecified measure μ.

In what follows we fix an orthogonal frame of $T_{x_0}M$ and hence identify $T_{x_0}M$ as Euclidean space $\mathbb{R}^n$. In this point of view a fibre of the linear frame bundle over $x \in M$ is identified with

$$GL_x(M) \equiv \{u : T_{x_0}M \to T_xM|\ u \text{ is a linear bijection}\},$$

in particular $GL_{x_0}(M) \equiv GL(\mathbb{R}^n)$.

For any smooth connection $\tilde{\nabla}$, there exists a unique horizontal lift $\tilde{\xi}$ of ξ, starting from (x_0, id) ([EI82 13C]), which gives a stochastic parallel translation $\tilde{/\!/}.$ along μ-$a.s.$ paths $\sigma \in E$. There is also a damped parallel translation W. with respect to the connection $\hat{\nabla}$, defined by the covariant equation

$$\frac{\hat{\mathcal{D}}}{\partial t}(W_t v_0) = -\frac{1}{2}\breve{\mathrm{Ric}}^{\#}(W_t v_0) + \breve{\nabla}A(W_t v_0) \tag{3.5}$$

for $v_0 \in T_{x_0}M$. Here $\breve{\mathrm{Ric}}^{\#} : TM \to TM$ is defined by $\breve{\mathrm{Ric}}^{\#}(v) = \mathrm{trace}\breve{R}(v, -)-$, with $\breve{R}$ being the curvature tensor of $\breve{\nabla}$. It is known that ([ELL 97])

$$W_t = E[T_{x_0}\xi_t | \mathcal{F}_T^{x_0}] \quad a.s. \tag{3.6}$$

where $T_{x_0}\xi_t$ is the derivative flow generated by (3.4).

We shall use the following results.

Lemma 3.1 Let W be specified by (3.5). Then

$$\sup_{0\leq t\leq T} |/\!/_t^{-1} W_t|_{H.S.} + \sup_{0\leq t\leq T} |W_t^{-1} /\!/_t|_{H.S.} \in L^p(E, \mu), \quad \forall p \geq 1.$$

Also for any smooth linear connection $\tilde{\nabla}$, we have

$$\sup_{0\leq t\leq T} |/\!/_t^{-1} \tilde{/\!/}_t|_{H.S.} + \sup_{0\leq t\leq T} |\tilde{/\!/}_t^{-1} /\!/_t|_{H.S.} \in L^p(E, \mu), \quad \forall p \geq 1.$$

The proof of Lemma 3.1 will appear elsewhere. Here we mention only that the fact $\sup_{0\le t\le T}|/\!\!/_t^{-1}W_t|_{H.S.}\in L^p(E;\mu)$ can be derived also form the fact that $\sup_{0\le t\le T}|T_{x_0}\xi_t|_{H.S.}\in L^p(E;\mu)$(see e.g. [Li 94]) and (3.6).

B. Damped Dirichlet forms

Let $\mathcal{D}=C_b^1(E;I\!\!R)$. Let H be the Cameron-Martin space $L_0^{2,1}([0,T];\ T_{x_0}M)$ with the norm $|h|_H=\int_0^T(|\dot h_t|^2dt)^{\frac{1}{2}}$ and let $f\in\mathcal{D}$. For μ-a.s. $\sigma\in E$ we have

$$|df(W.h.)(\sigma)|\le ||df||\ |W.h.(\sigma)|_\sigma\le ||df||\Phi(\sigma)T^{\frac{1}{2}}|h|_H,\ \forall h\in H \tag{3.7}$$

where $||df||$ is defined by (2.1) and

$$\Phi(\sigma):=\sup_{0\le t\le T}|W_t(\sigma)|_{H.S.}. \tag{3.8}$$

Hence there exists $\nabla f(\sigma)\in H$ such that

$$<\nabla f(\sigma),h>_H=df(W.h.)(\sigma),\qquad \forall h\in H. \tag{3.9}$$

We call ∇f a **damped gradient of** $\boldsymbol{f}$ (related to the connection $\hat\nabla$ and the drift A). See e.g. [Fa 95] for the history of the notion of damped gradient. More generally, Let S be a $L(H;H)$-valued random variable on $(E,\mathcal{F}_T^{x_0},\mu)$. Then for μ-a.s. $\sigma\in E$, there exists $\nabla^S f(\sigma)\in H$ such that

$$<\nabla^S f(\sigma),h>_H=df(W.(Sh))(\sigma),\qquad \forall h\in H. \tag{3.10}$$

Lemma 3.2 Let ∇f and $\nabla^S f$ be defined as above. Then for μ-a.s. $\sigma\in E$,

$$\nabla^S f(\sigma)=S(\sigma)^*\nabla f(\sigma)$$

where $S(\sigma)^*$ is the adjoint of $S(\sigma)$.

Proof For $h\in H$, we have

$$\begin{aligned}<\nabla^S f(\sigma),h>_H&=df(W.(\sigma)S(\sigma)h)=<\nabla f(\sigma),S(\sigma)h>_H\\&=<S(\sigma)^*\nabla f(\sigma),h>_H.\end{aligned}$$

Definition 3.3 An H-valued random variable k is said to be $\boldsymbol{\mathcal{F}.^{x_0}}$**-adapted**, if it can be regarded as an $\mathcal{F}.^{x_0}$-adapted process $k:\ E\times[0,T]\to T_{x_0}M$. An $L(H;H)$-valued random variable S is said to be $\boldsymbol{\mathcal{F}.^{x_0}}$**-adapted** if Sh is adapted for each $h\in H$.

Lemma 3.4 Let k be an $\mathcal{F}.^{x_0}$-adapted H-valued random variable such that $|k(-)|_H\in L^{2+\delta}(E,\mu)$ for some $\delta>0$. Then the vector field $Wk:=W.(-)k(-)$ is strongly $\mathcal{D}$-admissible in the sense of [EM 96]. If $\hat\nabla$ is metric compatible for some Riemannian metric on M, then it suffices to have $|k(-)|_H\in L^2(E,\mu)$.

Proof Write $v(\sigma)$ for $W.(\sigma)k(\sigma)$. Clearly we have $|v(\sigma)|_\sigma\le\Phi(\sigma)|k(\sigma)|_H$ for μ-a.s. $\sigma\in E$ where Φ is specified by (3.8). Thus $\int|v(\sigma)|_\sigma^2\mu(d\sigma)<\infty$ since by Lemma 3.1 $\Phi\in L^p(E,\mu)$ all $p\ge1$. Moreover, by [EL 96] we have

$$\int_E\partial_v f\ \mu(d\sigma)=\int_E f\left[\int_0^T<W_s\dot k_s,\breve{/\!\!/}_s d\breve B_s>_{T_{\sigma(s)}M}\right]\mu(d\sigma)$$

where $\{\breve{B}_s : \quad 0 \leq s \leq T\}$ is the martingale part of the anti- development of $\{x_s : \quad 0 \leq s \leq T\}$ using $\breve{/\!/}_s$, a Brownian motion on $T_{x_0}M$. One can check that

$$\left(\int_0^T |W_s \dot{k}_s(\sigma)|_{T_{\sigma(s)}M} ds\right)^{\frac{1}{2}} \leq \Phi(\sigma)|k(\sigma)|_H.$$

Hence if we set $\mathrm{div} v = -\int_0^T < W_s \dot{k}_s, \breve{/\!/}_s d\breve{B}_s >_{T_{\sigma(s)}M}$, then div $v \in L^2(E;\mu)$. Thus by Proposition 2.7 v is strongly $\mathcal{D}$ -admissible. The last assertion follows from the fact that if $\hat{\nabla}$ is metric compatible for some Riemannian metric, then $\Phi \in L^\infty(E;\mu)$.

Theorem 3.5 Let S be an $\mathcal{F}.^{x_0}$-adapted $L(H;H)$-valued random variable such that $|S(-)|_{L(H;H)} \in L^{2+\delta}(E,\mu)$ for some $\delta > 0$ and let $\rho \in \mathcal{D}$ and $L \in L(H;H)$. Define

$$\mathcal{E}(f,g) = \int_E < L\nabla^S f, L\nabla^S g >_H \rho^2(\sigma)\mu(d\sigma), \qquad f, g \in \mathcal{D}. \tag{3.11}$$

Then $(\mathcal{E}, \mathcal{D})$ is closable in $L^2(E,\mu)$ and the closure $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a quasi-regular and local Dirichlet form on $L^2(E,\mu)$. In particular there is a diffusion, with μ as invariant measure, properly associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. If $\hat{\nabla}$ is metric compatible for some Riemannian metric on M, then it suffices to have $|S(-)|_{L(H;H)} \in L^2(E,\mu)$.

Proof For $h \in H$ we set $k(-,h) = L^* S(-)h$. By Lemma 3.4 $W.k(-,h)$ is strongly $\mathcal{D}$-admissible. We now define $X(\sigma,h) = \rho(\sigma)W.(\sigma)k(\sigma,h)$. By Proposition 2.8 $X(-,h)$ is strongly $\mathcal{D}$-admissible for each $h \in H$. One can check that X satisfies all the conditions specified in Theorem 2.11. Hence if we define $\nabla^{\rho L^* S} f(\sigma)$ by requiring $< \nabla^{\rho L^* S} f(\sigma), h >_H = df(X(\sigma,h))$ all $h \in H$. Then the form

$$\mathcal{E}_X(f,g) = \int_E < \nabla^{\rho L^* S} f, \nabla^{\rho L^* S} g >_H \mu(d\sigma), \qquad f, g \in \mathcal{D}$$

is closable in $L^2(E,\mu)$, and there is a diffusion associated with its closure. Similarly to the proof of Lemma 3.2 one can show that $\nabla^{\rho L^* S} f = \rho L \nabla^S f$ for $f \in \mathcal{D}$. Therefore $\mathcal{E}_X$ coincides with $\mathcal{E}$ defined by (3.11), which completes the proof.

Corollary 3.6 Let k be an $\mathcal{F}.^{x_0}$-adapted $L(H;H)$- valued random variable such that $k(\sigma)$ is nonnegative definite self-adjoint for μ-a.s. $\sigma \in E$, and $|k(-)|_{L(H;H)} \in L^{1+\delta}(E;\mu)$ for some $\delta > 0$ (if $\hat{\nabla}$ is metric compatible for some Riemannian metric, then δ may be zero). Let $\rho \in \mathcal{D}$. Define

$$\mathcal{E}(f,g) = \int_E < k\nabla f, \nabla g > \rho^2(\sigma)\mu(d\sigma), \qquad f, g \in \mathcal{D}. \tag{3.12}$$

Then all the conclusion of the above theorem holds true.

Proof Let $S = k^{\frac{1}{2}}$ Then we can check that

$$\mathcal{E}(f,g) = \int_E < \nabla^S f, \nabla^S g > \rho^2(\sigma)\mu(d\sigma), \qquad f, g \in \mathcal{D}$$

and S meets the conditions specified in the above theorem. Hence the required conclusion follows.

Remark 3.7 Note that in the above corollary k may be degenerate, e.g. kernel $k(\sigma) \neq \{0\}$ for μ-a.s. $\sigma \in E$.

C. Undamped Dirichlet forms.

Let $\mathcal{D}$ and H be as in Subsection **B**. We now give a version of the integration by parts formula in the original form given by Driver [Dr92] for the torsion skew symmetric case.

Lemma 3.8 (Integration by parts formula)

Let k be an $\mathcal{F}_\cdot^{x_0}$-adapted H-valued process such that

$$|k(-)|_H \leq L^{1+\delta}(E,\mu) \text{ for some } \delta > 0.$$

Then for $f \in D$ we have

$$\begin{aligned}&\int_E df(\widehat{/\!/}.k.)(\sigma)\mu(d\sigma)\\ =&\int_E f\int_0^T < \frac{1}{2}\breve{\mathrm{Ric}}^{\#}(\sigma)\widehat{/\!/}_t k_t - \breve{\nabla}A(\sigma_t)\widehat{/\!/}_t k_t + \widehat{/\!/}_t\dot{k}_t, \breve{/\!/}_t \mathrm{d}\breve{B}_t > \mu(\mathrm{d}\sigma).\end{aligned} \tag{3.13}$$

Proof Let us set $h_t = a_t^{-1}k_t$ where $a_t^{-1} = W_t^{-1}\widehat{/\!/}_t$. Then $W_t h_t = \widehat{/\!/}_t k_t$ and

$$W_t\dot{h}_t = \frac{1}{2}\breve{\mathrm{Ric}}^{\#}(\sigma)\widehat{/\!/}_t k_t - \breve{\nabla}A(\sigma_t)\widehat{/\!/}_t k_t + \widehat{/\!/}_t\dot{k}_t, \tag{3.14}$$

Let

$$C = \sup_{x\in M}|\breve{\mathrm{Ric}}^{\#}(x)|_{O.P.} + \sup_{x\in M}|\breve{\nabla}A(x)|_{O.P.}.$$

Then (3.14) yields

$$\begin{aligned}\left(\int_0^T |W_t\dot{h}_t(\sigma)|^2_{T_{\sigma(t)}M}dt\right)^{\frac{1}{2}} \leq& C\sup_{0\leq t\leq T}|\widehat{/\!/}_t(\sigma)|_{H.S.}\left[\left(\int_0^T |k_t(\sigma)|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T \dot{k}_t(\sigma)|^2 dt\right)^{\frac{1}{2}}\right]\\ \leq& C\sup_{0\leq t\leq T}|\widehat{/\!/}_t(\sigma)|_{H.S.}(1+T^{\frac{1}{2}})|k(\sigma)|_H.\end{aligned}$$

Therefore by the Burkholder-Davis-Gundy inequality and Lemma 3.1 we have for some constant C_1,

$$\begin{aligned}E\left[|\int_0^T \left\langle W_t\dot{h}_t, \breve{/\!/}_t d\breve{B}_t\right\rangle|\right] \leq& C_1 E\left[\left(\int_0^T |W_t\dot{h}_t|^2 dt\right)^{\frac{1}{2}}\right]\\ \leq& C_1\left(E\left[\sup_{0\leq t\leq 1}(\widehat{/\!/}_t)^{\frac{1+\delta}{\delta}}_{H.S.}\right]\right)^{\frac{\delta}{1+\delta}}\left(E\left[|k|_H^{1+\delta}\right]\right)^{\frac{1}{1+\delta}}\\ <&\infty.\end{aligned}$$

Thus all the arguments of [EL 96 Th 3.3] goes through in our case and the corresponding results are available here. In particular we obtain (3.13) by [EL 96, (23)] One can easily check that for μ-a.s. $\sigma \in E$ there exist constants $c(\sigma) > 0$ such that for $f \in \mathcal{D}$

$$|df(\widehat{/\!/}.h.)(\sigma)| \leq \|df\| c(\sigma)|h|_H \qquad \forall h \in H. \tag{3.15}$$

Hence there exists $\mathrm{D}f(\sigma) \in H$ such that

$$< Df(\sigma), h >_H = df(\widehat{/\!/}.h.)(\sigma), \qquad \forall h \in H. \tag{3.16}$$

We call $\mathrm{D}f$ the **gradient of f** (related to the connection $\hat{\nabla}$). More generally, let S be an $L(H,H)$-valued random variable. Then for μ-a.s. $\sigma \in E$ there exists $\mathcal{D}^S f(\sigma) \in H$ such that

$$< D^S f, h >_H = df(\widehat{/\!/}.S.h)(\sigma), \qquad \forall h \in H. \tag{3.17}$$

The following Lemma 3.9 and Theorem 3.10 are consequences of Lemma 3.8. Their proof are similar to those of Lemma 3.4 and Theorem 3.5 respectively. We leave the details to the reader.

Lemma 3.9 Let k be an $\mathcal{F}.^{x_0}$-adapted H-valued random variable such that $|k(\cdot)|_H \in L^{2+\delta}(E,\mu)$ for some $\delta > 0$. Then the vector field $\widehat{/\!/}.k.$ is strongly $\mathcal{D}$-admissible. If $\hat{\nabla}$ is metric compatible with some Riemannian metric on M, then it is sufficient to have $|k(\cdot)|_H \in L^2(E,\mu)$

Theorem 3.10 Let S be an $\mathcal{F}.^{x_0}$-adapted $L(H,H)$-valued random variable such that $|S(\cdot)|_{L(H,H)} \in L^{2+\delta}(E,\mu)$ for some $\delta > 0$, and let $\rho \in \mathcal{D}$ and $L \in L(H,H)$. Define

$$\mathcal{E}(f,g) = \int_E < LD^S f, LD^S g > \rho^2(\sigma)\mu(d\sigma), \qquad \forall f, g \in \mathcal{D} \tag{3.18}$$

Then $(\mathcal{E},\mathcal{D})$ is closable in $L^2(E,\mu)$ and the closure $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a quasi-regular and local Dirichlet form on $L^2(E,\mu)$. In particular there is a diffusion, with μ as invariant measure, properly associated with $(\mathcal{E},\mathcal{D}(\mathcal{E}))$. If $\hat{\nabla}$ is metric for some Riemannian metric on M, then it suffices to have $|S.(-)|_{L_{(H;H)}} \in L^2(E,\mu)$.

Corollary 3.11 Let K be an $\mathcal{F}.^{x_0}$-adapted $L(H,H)$- valued random variable such that $K(\sigma)$ is nonnegative definite self-adjoint for μ-a.s. $\sigma \in E$, and $|K(-)|_{L(H,H)} \in L^{2+\delta}(E,\mu)$ for some $\delta > 0$ (if $\hat{\nabla}$ is metric compatible for some Riemannian metric then δ may be zero). Let $\rho \in \mathcal{D}$. Define

$$\mathcal{E}(f,g) = \int_E < K\mathcal{D}f, \mathcal{D}g > \rho^2(\sigma)\mu(d\sigma), \qquad \forall f, g \in \mathcal{D} \tag{3.19}$$

Then all the conclusion of theorem 3.10 holds true.

Remark 3.12 It is worthwhile to point out that the above results are related to the connection $\hat{\nabla}$ which is not required to be metric compatible as in [DR 92]. Also our conditions for K and ρ in (3.19) are different from those in the literature concerning "gradient type" Dirichlet forms on infinite dimensional spaces (see [R96] and references therein for historical remarks and a recent survey in this connection) where K and ρ are assumed to be strictly positive for μ-a.s. $\sigma \in E$, while in our case we may allow kernel $K(\sigma) \neq \{0\}$ for μ-a.s. $\sigma \in E$ and $\mu\{\rho^2 = 0\} > 0$. But for the price of this degeneracy we have to assume that K is adapted and $\rho \in C_b^1(E)$.

§4 Degenerate case

Let M be a compact C^∞ manifold with dimension n. As before let $E := C_{x_0}([0,T];M)$ be a based path space for some fixed point $x_0 \in M$ and some $T > 0$. In this section we consider a Borel measure μ on E which is the law of a degenerate diffusion $\{\xi_t : t \geq 0\}$ starting from x_0 with a semi-elliptic generator L based on a subbundle I of TM. More precisely I is a smooth subbundle of TM and $\xi. := \xi.(x_0)$ is the solution to some $S.D.E.$

$$d\xi_t = X(\xi_t) \circ dB_t + A(\xi_t)dt \tag{4.1}$$

similar to (3.4) but with Image $X(x) = I_x$. Note that if also $A(x) \subset I_x$ all $x \in M$ then there exists a Borel subset $E_0 \subset E$ such that $\mu(E \backslash E_0) = 0$ and the paths in E_0 can be uniformly approximated by piecewise smooth paths which are tangential to I. Therefore in some case μ may be singular to the usual Wiener measure on E. But it is difficult to find the exact "geometric shape" of E_0, which is perhaps even not a manifold in general.

The map X in (4.1) induces a Riemannian metric $\{<,>_x,\ x \in M\}$ and a metric compatible connection $\breve{\nabla}$ (the LeJan-Watanabe connection) on the subbundle I. Taking any Riemannian metric and metric compatible connection on TM which extend $<,>_x$ and $\breve{\nabla}$ respectively (they will be denoted again by $<,>_x$ and $\breve{\nabla}$ in the sequel), we may construct a horizontal lift $\breve{\xi}$ of ξ, starting from (x_0, id), which gives a stochastic parallel translation $\breve{/\!/}$. The restriction of $\breve{/\!/}_t$ to I_{x_0} maps it isometrically onto I_{x_t}. As in the previous section let $W_t = E[T_{x_0}\xi_t \ |\mathcal{F}_T^{x_0}]$ be the conditional expectation of the derivative flow $T_{x_0}\xi_t$. Then it follows from [ELL 96] that there is a continuous version of $W.$ satisfying the covariant equation:

$$DW_t = \nabla X\left(W_t, X^*(\xi_t)X(\xi_t) \circ dB_t\right) + \left(\nabla A - \frac{1}{2}\breve{\mathrm{Ric}}\right) W_t dt \tag{4.2}$$

using the Levi-Civita connection for $<,>$. Note that $X(\xi_t)dB_t$ can be represented by $\breve{/\!/}_t d\breve{B}_t$ with $\breve{B}$ being the martingale part of the stochastic anti-development of $\{\xi_t :\ 0 \leq t \leq T\}$, hence the solution to (4.2) is $\mathcal{F}_T^{x_0}$-measurable. From (4.2) one can show that

$$\sup_{0 \leq t \leq T} |\breve{/\!/}_t W_t|_{H.S.} + \sup_{0 \leq t \leq T} |W_t^{-1} \breve{/\!/}_t|_{H.S.} \in L^p(E;\mu),\ \forall p \geq 1. \tag{4.3}$$

The proof of (4.3) follows from the same argument used in the proof of Lemma 3.1 and will appear in a separate paper. One can also describe $W.$ using the adjoint object of $\breve{\nabla}$. For details we refer to [ELL 96].

Let $\Omega = C_0([0,T];\mathbb{R}^m)$. The solution to (4.1)(starting from x_0) defines a measurable map $J : \Omega \to E$ with $J(w) = \xi.(w)$ all $w \in \Omega$. Let H^m be the Cameron-Martin space $L_0^{2,1}([0,T];\mathbb{R}^m)$. By the Cameron-Martin Theorem J is differentiable along $h \in H^m$ at a.s. $w \in \Omega$. The differential is given by

$$(dJ(h))_t = T_{x_0}\xi_t \int_0^t (T_{x_0}\xi_s)^{-1} X(\xi_s)\dot{h}_s ds, \quad 0 \leq t \leq T. \tag{4.4}$$

The conditional expectation of the above differential gives a map S from H^m to the space of vector fields on E. Using (4.4) and (4.2) one obtains

$$\begin{aligned}(Sh)_t : &= E\left[(dJ(h))_t|\mathcal{F}_t^{x_0}\right] \\ &= W_t \int_0^t W_s^{-1} X(\sigma_s)\dot{h}_s ds, \quad \mu\text{-a.s..}\end{aligned} \tag{4.5}$$

See [ELL 96] for the proof of (4.5).

We are now in a position to state the following

Theorem 4.1 (i) Let $\mathcal{D} = C_b^1(E;R)$. Then Sh is strongly $\mathcal{D}$-admissible for $h \in H^m$.

(ii) Let $f \in \mathcal{D}$. Then for μ-a.s. $\sigma \in E$ there exists $D^\# f(\sigma) \in H^m$ such that

$$< D^\# f(\sigma), h >_{H^m} = df(Sh)(\sigma), \quad \forall h \in H^m.$$

(iii) The bilinear form

$$\mathcal{E}^\#(f,g) = \int < D^\# f, D^\# g >_{H^m} \mu(d\sigma), \quad f,g \in \mathcal{D}$$

is closable in $L^2(E;\mu)$ and the closure $(\mathcal{E}^\#, D(\mathcal{E}^\#))$ is a quasi-regular local Dirichlet form on $L^2(E;\mu)$. Hence there is a diffusion process properly associated with $(\mathcal{E}^\#, D(\mathcal{E}^\#))$.

Proof (i) Let $h \in H^m$ and write v for Sh. From (4.5) and (4.3) we see that $\partial_v f := df(Sh) \in L^2(E;\mu)$ for all $f \in \mathcal{D}$. Also by [ELL 96. (9)] we have the integration by parts formula:

$$\int_E \partial_v f \mu(d\sigma) = -\int_E f \, div \, v \mu(d\sigma), \quad \forall f \in \mathcal{D}$$

with $div\, v := -\int_0^T < X(\sigma_s)\dot{h}_s, \breve{/\!/}_s d\breve{B}_s >_{T_{\sigma(s)}M}$. Clearly $div\, v \in L^2(E;\mu)$ since $\breve{B}$ is a I_{x_0}-valued Brownian motion. Thus by Proposition 2.7 v is strongly $\mathcal{D}$-admissible.

(ii) For $f \in \mathcal{D}$ and $h \in H^m$ we have by (4.5)

$$|df(Sh)(\sigma)| \leq ||df|| \sup_{0 \leq t \leq T} |Sh(\sigma)|_{T_{\sigma(t)}M} \leq ||df|| |\phi(\sigma)| h|_{H^m} \tag{4.6}$$

with

$$\phi(\sigma) := \sup_{0 \leq s \leq t \leq T} |W_t W_s^{-1}(\sigma)|_{H.S.}. \tag{4.7}$$

In the last step of 4.6 we used the fact that $|X(\sigma_s)\dot{h}_s|_{T_{\sigma(s)}M} \leq |\dot{h}_s|_{R^m}$ since the Riemannian metric on I_x is induced by $X(x)$. By (4.3) we have $\phi(\sigma) < \infty$ for a.s. $\sigma \in E$. Thus the desired conclusion follows from Riesz representation theorem.

(iii) By (4.3) we have $\phi \in L^2(E;\mu)$ for ϕ specified by (4.7). Hence if $X(\sigma, h)$ is defined by $Sh(\sigma)$ for $\sigma \in E$ and $h \in H^m$, then X meets all the conditions stated in Theorem 2.11 from which the desired conclusion follows.

§5 Diffusion on paths of mapping groups

As another example we go back to the sort of situation discussed in [DS 69]. Let A be a compact Lie group with bi-invariant metric and K be a compact n-dimensional manifold. Set $G = H^s(K;A)$, where $s > \frac{n}{2}$, i.e. the Hilbert manifold of maps $g: K \to A$ of class H^s, e.g. see [Pa 68]. Let $\mathcal{A}$ be the Lie algebra of A. Then if $2(r-s) > n$ the inclusion of Sobolev spaces $H^r(K;\mathcal{A}) \to H^s(K;\mathcal{A})$ is Hilbert-Schmidt and there is an associated Gaussian measure and Wiener process $\{W_t : t \geq 0\}$ on $H^s(K;\mathcal{A})$, the tangent space to the Lie group G at the identity. In fact instead of $H^r(K;\mathcal{A})$ we could take any subspace H_0 of $H^s(K;\mathcal{A})$ with Hilbert structure which has a Hilbert-Schmidt inclusion into $H^s(K;\mathcal{A})$. See [Ba 76]. We can therefore consider the left invariant equation on G:

$$dg_t = X_0(g_t) \circ dW_t$$

with g_0 given, where $X_0: H^s(K;\mathcal{A}) \to T_g H^s(K;\mathcal{A})$ is given by left translation just as in [DS 69], see also [El 75]. Solutions exist for all time, for example by the uniform cover technique as used in [El 82] or [BE96] for diffeomorphism groups. Let μ be the law of the solution process restricted to the time interval [0,1], so μ is a Borel measure on $E := C_{g_0}([0,1];G)$.

Set $H = H_0^1([0,1];H_0)$, the Cameron-Martin space of $\{W_t : 0 \le t \le 1\}$. Define $X: E \times H \to TE$ by

$$(X(\sigma)h)_t = TR_{\sigma_t} \int_0^t ad(\sigma_s^{-1})\dot{h}_s ds$$

where $R_g: G \to G$ denotes right multiplication for g in G and $ad(g): H^s(K;\mathcal{A}) \to H^s(K;\mathcal{A})$ is the adjoint representation of g for $g \in G$. (This means that $(ad(g)\alpha)(x) = ad_A(g(x))\alpha(x),\ x \in K,\ \alpha \in H^s(K;\mathcal{A})$ for ad_A the adjoint representation of A on $\mathcal{A}$).

Let $\mathcal{D}$ be the subset of $L^2(E,\mu;R)$ consisting of BC^1 functions using the right invariant Finsler on E:

$$|v|_\sigma = \sup_{0\le t\le 1} ||TR_{\sigma_t}^{-1} v_t||_{H^s(K;\mathcal{A})}, \quad v \in T_\sigma E.$$

Certainly E with the above Finsler structure is a C-F manifold by Atkin's result. Moreover the argument in [EL 96] similarly to that in [DS 69], shows that each $X(\cdot)h$ is strongly $\mathcal{D}$-admissible. However although $X(\sigma): H \to T_\sigma E$ is bounded for each σ it is not clear that it will satisfy the integral condition of Theorem 2.11. To be safe, we can change the state space E to $\widetilde{E} := C_{g_0}([0,1] \times K; A)$ the infinite dimensional Lie group of continuous maps

$$\sigma: [0,1] \times K \to A \text{ with } \sigma(0,x) = g_0(x),\ x \in K$$

i.e. the space of homotopies of g_0. Taking the natural Finsler, $|\ |$ say , we have a C-F manifold again. Let $\widetilde{\mathcal{D}}$ be the corresponding space of BC^1 functions on $\widetilde{E}$. Since E is included in $\widetilde{E}$(as a dense subset and by a C^∞ inclusion), we have an induced probability measure $\widetilde{\mu}$ on $\widetilde{E}$ and X extends to a measurable

$$\widetilde{X}: \widetilde{E} \times H \to T\widetilde{E}.$$

The strong admissibility property is retained under such an extension of state space. Moreover now, by the right invariance of the metric on A,

$$\begin{aligned} |\widetilde{X}(\sigma)h.|_\sigma^2 &= \sup_{0\le t\le 1,\ x\in K} \left| \int_0^t ad_A(\sigma_s(x)^{-1})\dot{h}_s(x)ds \right|_{\mathcal{A}} \\ &\le \int_0^1 \sup_x |\dot{h}_s(x)|_{\mathcal{A}} ds \end{aligned}$$

since ad_A is an orthogonal representation. Thus

$$\begin{aligned} \left|\widetilde{X}(\sigma)h\right|_\sigma &\le \left[\int_0^1 \sup_x |\dot{h}_s(x)|_{\mathcal{A}}^2 ds\right]^{\frac{1}{2}} \\ &\le \text{const.} \left(\int_0^1 |\dot{h}_s|_{H_0}^2 ds\right)^{\frac{1}{2}} \\ &= \text{const.} ||h||_H. \end{aligned}$$

It follows from Theorem 2.11 that the form

$$\mathcal{E}_X(f,f) := \int_{\widetilde{E}} \left\langle df \circ \widetilde{X}(\sigma), df \circ \widetilde{X}(\sigma) \right\rangle_{L(H;\mathbb{R})} \widetilde{\mu}(d\sigma), \quad f \in \widetilde{\mathcal{D}}.$$

is closable with closure a quasi-regular, local, Dirichlet form on $\widetilde{E}$. In particular there is an associated diffusion on $\widetilde{E}$.

Remark (i) We are not claiming that the above example has importance other than that of illustrating the possibilities of the approach described and showing the sort of problems which can arise (e.g. when does the diffusion lie on E ?).

(ii) Instead of using the Hilbert manifolds $H^s(K;\mathcal{A})$ it should be possible to use more general spaces of maps as in [BE96].

(iii) For related constructions of infinite dimensional diffusions see [Ba 80]

(iv) As alternatives to $\mathcal{D}$ and $\widetilde{D}$ we could equally well have used functions cylindrical in the $[0,1]$ variable but BC^1 on $H^s(K;\mathcal{A})$ or $C(K,A)$, or used functions cylindrical on $[0,1]\times K$. It is probably difficult to discover if, or when, they lead to the same Dirichlet forms. For the "classical" case of path spaces, when A is a point, see [Ac 94].

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