Vector fields on mapping spaces and related Dirichlet forms and diffusions

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1 Introduction

A. Let μ be a Radon measure on an infinite dimensional smooth manifold E. Associated to μ there are various additional structures on E. This is seen from the example of Gaussian spaces where E is a separable Banach space inducing an abstract Wiener space structure on E, from the example of path and loop spaces on finite dimensional Riemannian manifolds with measures induced by Brownian motions and Brownian bridges which are usefully analyzed using special "tangent spaces" [18], from the notions of "differentiability" of measures leading to classes of "admissible" vector fields describing the directions in which μ can be differentiated [4], and from very general considerations [11]. Here we describe a class of vector fields determined by μ and the differential structure of E which also have a claim to be called "admissible" but are defined in terms of Dirichlet form theory rather than differentiability of μ. Finite or suitably bounded countable families of such vector fields are shown to give rise to quasi-regular Dirichlet forms on E with their associated diffusion processes, Markovian semigroups, and infinitesimal generators.

The ideas are valid for general separable metrizable manifolds but an adequately rich class of differentiable "test" functions is needed. Such would be assured if E were modeled on a space admitting smooth partitions of unity with bounded derivatives. However this is not so for spaces of continuous paths (such as classical Wiener space) and for such mapping spaces it is often convenient to use cylindrical functions. On the other hand we wish to include such cases as iterated path spaces (paths on path spaces) and other examples

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of spaces of maps into infinite dimensional manifolds. To do this we introduce in Section 2 the notion of a Caratheodory-Finsler (C-F) manifold: a class of Finsler manifolds possessing a rich enough family of "test" functions. Closed submanifolds of separable Banach spaces, with induced Finsler structure are C-F manifolds, as is the space of continuous maps of a compact metric space into a C-F manifold. In this way we are able to give a unified treatment which covers and extends the existing results on path and loop spaces. We are also able to give a more detailed analysis of the structures involved, e.g. see the final parts of sections 3 and 4.

We go on to summarize our results in more detail. In particular subsection C,D which follow are summaries of what will be proved in sections 4 and 3 respectively.

B. Suppose μ is a finite measure and let $\mathcal{D}$ be a dense linear subspace of $L^2(E,\mu)$ consisting of C^1 maps $f : M \to R$. A Borel measurable vector field v on E is **$\mathcal{D}$-admissible** if

(i) $f \in \mathcal{D}$, $f = 0$ μ-a.e. implies $\partial_v f = 0$ μ-a.e.,

(ii) $\partial_v f \in L^2(E,\mu)$ for all $f \in \mathcal{D}$,

and

(iii) ∂_v, with domain $\mathcal{D}$, is a closable operator in $L^2(E,\mu)$.

Here $\partial_v f : E \to R$ is the Frechet derivative of f in the direction v, i.e., $\partial_v f(\sigma) = df(v(\sigma))$, $\sigma \in E$.

If v satisfies (i) and (ii) and possesses a divergence, i.e., there exists $\operatorname{div} v$ in $L^2(E,\mu)$ such that

$$\int_E \partial_v f(\sigma)\mu(d\sigma) = -\int_E f(\sigma) \operatorname{div} v(\sigma)\mu(d\sigma)$$

for all $f \in \mathcal{D}$, then v will be said to be **strongly $\mathcal{D}$-admissible.** Strong admissibility implies admissibility if $\mathcal{D}$ is an algebra.

C. Now let E be a closed submanifold (e.g. the manifold itself) of the manifold of continuous maps of a compact metric space S into a C-F manifold M and let $\mathcal{A}$ be a countable or finite family of $\mathcal{D}$-admissible vector fields on E such that

$$\int_E \sum_{v\in\mathcal{A}} |\partial_v f(\sigma)|^2 \mu(d\sigma) < \infty$$

for all $f \in \mathcal{D}$. Then the form $(\mathcal{E},\mathcal{D})$ given by

$$\mathcal{E}(f,g) = \int_E \sum_{v\in\mathcal{A}} \partial_v f(\sigma)\partial_v g(\sigma)\mu(d\sigma)$$

for $f, g \in \mathcal{D}$, is closable in $L^2(E, \mu)$ with closure a Dirichlet form. If in addition

(i) $\mathcal{D}$ is an algebra with pointwise multiplication

(ii) if $\varphi \in C_b^\infty(R)$ with $\varphi(0) = 0$ then $\varphi \circ f \in \mathcal{D}$ whenever $f \in \mathcal{D}$,

(iii) $\mathcal{D}$ contains all functions f of the form $f(\sigma) = \varphi(\sigma(s))$ for some $s \in S$ and $\varphi \in C_b^1(M; R)$

and

(iv) there exists $\Phi \in L^2(E, \mu)$ such that for all $\varphi \in C_b^1(M; R)$ and $s \in S$

$$\sum_{v \in \mathcal{A}} |d\varphi(v(\sigma)_s)|^2 \leq ||d\varphi||^2 \Phi^2(\sigma) \quad \mu\text{–a.e.}$$

where $||d\varphi||$ is defined from the Finsler norm on M, then the closure $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular (in the sense of [2], [22]) and local so that there is an associated diffusion process $(\xi_t)_{t \geq 0}$ on E which is conservative, has μ as invariant measure and has generator L where $\mathcal{E}(f, g) = \int_E (-Lf) g \, d\mu$ for all $f \in D(L), g \in D(\mathcal{E})$. Note that, intuitively, $L = \sum_{v \in \mathcal{A}} -\partial_v^* \partial_v$. This $(\mathcal{E}, D(\mathcal{E}))$ is in fact a special case of the square field operator forms of Röckner and Schmuland [26].

D. The strongly $\mathcal{D}$-admissible vector fields form a particularly nice class. For example if v is strongly $\mathcal{D}$-admissible and $\mathcal{D}$ is an algebra then fv is strongly $\mathcal{D}$-admissible for every bounded $f \in D$. Moreover if V is the space of all strongly $\mathcal{D}$-admissible v with

$$||v||_V := \int_E |v(\sigma)|_\sigma^2 \mu(d\sigma) + \int_E |\operatorname{div} v(\sigma)|^2 \mu(d\sigma) < \infty$$

then $(V, || \, ||_V)$ is a Banach space, where $|v(\sigma)|_\sigma$ is the Finsler norm on $T_\sigma E$. One consequence of this result is to have a simple way to go from nonanticipative vector fields on path and loop spaces to a wide class of anticipative ones (cf. subsection E.(c) below). We mention also that when E is a linear space the well admissible elements (cf. [22] II. Def.3.2) form a special subclass of strongly admissible vector fields, i.e. constant valued ones in the flat case.

E. We conclude this introduction by pointing out some applications of the above results to path spaces, of which the details will be discussed elsewhere, together with further examples.

(a) Let M be a compact Riemannian manifold, take $S = [0, 1]$, and for fixed $x_0 \in M$ choose μ to be the law of a diffusion $\{X_t : t \geq 0\}$ on M starting at x_0 with generator $\frac{1}{2}\triangle + Z$ where $\triangle$ is the Laplace-Beltrami operator on M and Z a smooth vector field. Let $\mathcal{D}$ be the space $\mathcal{F}C^\infty$ of smooth cylindrical functions on E. Let $\widetilde{\nabla}'$ be an affine connection on M whose adjoint $\widetilde{\nabla}$ is a metric connection for the Riemannian structure of M. There is then another vector field $\tilde{Z}$ on M with

$$\frac{1}{2}\triangle f + Zf = \frac{1}{2}\mathrm{trace}\widetilde{\nabla}' df + \tilde{Z}f.$$

For μ almost all paths σ in E, we can define $\widetilde{W}_t^{\tilde{Z}}(\sigma) : T_{x_0}M \to T_{\sigma(t)}M$ by the covariant equation

$$\frac{\widetilde{D}'}{\partial t}(\widetilde{W}_t^{\tilde{Z}}(\sigma)(v_0)) = -\frac{1}{2}\widetilde{\mathrm{Ric}}^{\#}(\widetilde{W}_t^{\tilde{Z}}(\sigma)(v_0)) + \widetilde{\nabla}\tilde{Z}(\widetilde{W}_t^{\tilde{Z}}(\sigma)(v_0));$$

this is a "damped parallel translation". Let H be the Cameron-Martin space $L_0^{2,1}([0,1];T_{x_0}M)$ of paths with values in $T_{x_0}M$. For $h \in H$ define a vector field v^h on E by

$$v^h(\sigma)_t = \widetilde{W}_t^{\tilde{Z}}(\sigma)(h_t).$$

It follows from [13], (essentially from Driver's integration by parts formulae [6] in the torsion skew-symmetric case), that v^h is strongly $\mathcal{D}$-admissible for each $h \in H$. Define $X : E \times H \to TE$ by $X(\sigma, h) = v^h(\sigma)$. Then $X(\sigma,\cdot) : H \to T_\sigma E$ is continuous linear for μ-a.e. σ and so there is a "gradient operator" on $\mathcal{D}$ defined by

$$< \nabla f(\sigma), e >_H = df(X(\sigma, e)) \quad \mathrm{all} e \in H \quad \mu\mathrm{-a.e.}\sigma$$

If $\mathcal{A} = \{v^{e_j} : j \geq 1\}$ for $\{e_j\}_{j=1}^\infty$ an orthonormal base of H then (i) to (iv) of subsection C are satisfied and we obtain a diffusion on the space of paths on M, generalizing the construction by Driver and Röckner [[7]]. Indeed the form defined by $\mathcal{A}$ can generally well be written

$$\mathcal{E}(f,g) = \int_E < \nabla f(\sigma), \nabla g(\sigma) >_H \ \mu(d\sigma).$$

Similarly the construction of a diffuson in the loop space by Albeverio, Leandre, and Röckner [1] can be generalized this way.

Remark We could have used $//_t$ in place of $\widetilde{W}_t^{\tilde{Z}}$, i.e. parallel translation rather than damped parallel translation, to get the corresponding results.

(b) For S, M, μ and $\mathcal{D}$ as in (a), take the Levi-Civita connection of M. Let AV be the space of vector fields on E of the form

$$v_t(\sigma) = //_t(\sigma)\{\int_0^t Q_s(\sigma)dB_s(\sigma) + \int_0^t \dot{h}_s(\sigma)ds\}, \quad 0 \leq t \leq 1$$

where the Brownian motion $\{B_t : t \geq 0\}$ is the martingale part of the stochastic anti-development of our diffusion process while $\{Q_s : 0 \leq s \leq 1\}$, $\{\dot{h}_s : 0 \leq s \leq 1\}$ are predictable process with values in the skew symmetric $n \times n$-matrices and in $I\!R^n$ respectively, ($n = \dim M$), such that

$$||v||^2_{AV} := \int_E \{ \int_0^1 tr Q^*_s(\sigma) Q_s(\sigma)\, ds + \int_0^1 |\dot{h}_s(\sigma)|^2\, ds \} \mu(d\sigma) < \infty.$$

These are **adapted vector fields** in the sense of Driver [7]. Then AV is a Hilbert space with its obvious inner product (using uniqueness of the semi-martingale decomposition of such v). According to [7], each $v \in AV$ is strongly $\mathcal{D}$-admissible. Applying Doob's inequality and Schwartz's inequality respectively to the martingale part and the finite variation part of such v, one can check that $\int_E |v|^2_\sigma \mu(d\sigma) \leq C||v||^2_{AV}$ some constant C and all $v \in AV$, this together with the norm estimate of Lemma 3.7 in [7] shows that AV is a closed subspace of V of subsection D. Now let $\mathcal{A}$ be a finite or countable family of AV such that $\sum_{v\in\mathcal{A}} ||v||^2_{AV} < \infty$, then (i) to (iv) in Subsection C are satisfied and we get diffusion processes driven by adapted vector fields, though there is in general no "gradient operator" to be defined in this case.

Remark. The vector fields in (a) actually are elements of AV, see [13].

(c) Using subsection D above if $v_i \in AV$ and $f_i \in \mathcal{D}$ for $i = 1$ to n then $\sum_{i=1}^n f_i v_i$ is strongly $\mathcal{D}$-admissible. We can therefore take limits in the space $(V, ||\ ||_V)$ to obtain a wide class of anticipating vector fields which have divergences, including the classes described by Nualart and Pardoux, by Leandre and by Fang [14].

2 Caratheodory-Finsler manifolds

Throughout this section let M be a separable C^1 manifold modeled on a Banach space and equipped with a given Finsler structure τ([3], [24]). (In what follows we shall call such manifold a **Finsler Manifold**.) $TM := \cup_{x\in M} T_x M$ denotes the tangent bundle of M.

We write $|v|_x := \tau(v)$ for $v \in T_x M$. $T_x M$ equipped with the norm $|\cdot|_x$ is then a Banach space. Let f be a C^1 map from M to another Finsler manifold N. We set

$$||df|| := \sup_{x\in M} ||df(x)||_{L(T_x M, T_{f(x)} N)} \tag{2.1}$$

We write $f \in C^1_b(M; N)$ if $||df|| < \infty$. In particular we write $f \in C^1_1(M; N)$ if $||df|| \leq 1$. Recall that for a piecewise C^1 map $\sigma : [0, 1] \to M$, the length

$l(\sigma)$ of σ (w.r.t. τ) is defined by

$$l(\sigma) := \int |\dot{\sigma}(s)|_{\sigma(s)} ds \tag{2.2}$$

where $\dot{\sigma}$ denotes the tangent vector of σ. The corresponding metric d_M induced by τ is then defined by

$$d_M(x,y) := \inf\{l(\sigma) : \sigma : [0,1] \to M \text{ is piecewise } C^1 \text{ and } \sigma(0) = x, \sigma(1) = y\} \tag{2.3}$$

(with the convention that $\inf \emptyset = \infty$) for $x, y \in M$. Note that d_M is an admissible metric on M, i.e. d_M generates the original topology of M [3]. (Note also that we allow the distance between two points to be infinite in our definition of distance.)

For our purpose we introduce a pseudo metric ρ_M (Carathéodory metric [5]) as follows

$$\rho_M(x,y) := \sup\{f(y) - f(x) : f \in C_1^1(M;R)\} \tag{2.4}$$

Definition 2.1 *We say that M is a* **Caratheodory-Finsler manifold** *(C-F manifold in short) (w.r.t. τ) if ρ_M is an admissible metric on M and is complete.*

Proposition 2.2 *Suppose that there exists a C-F manifold N and a closed embedding map $J \in C_b^1(M;N)$, then M is a C-F manifold.*

Proof. Let ρ_N be the Carathéodory metric on N. If we set $d_{J(N)}(x,y) = \rho_N(J(x), J(y))$ for $x, y \in M$, then $d_{J(N)}$ is an admissible metric on M and is complete. Let $\lambda = ||dJ||$. If $f \in C_1^1(N;R)$, then $\lambda^{-1} f \circ J \in C_1^1(M;R)$. Therefore for $x, y \in M$,

$$\begin{aligned} \lambda^{-1} d_{J(N)}(x,y) &= \sup\{\lambda^{-1} f \circ J(y) - \lambda^{-1} f \circ J(x) : f \in C_1^1(N;R)\} \\ &\leq \sup\{f(y) - f(x) : f \in C_1^1(M;R)\} = \rho_M(x,y). \end{aligned}$$

Thus the proof is completed by showing that $\rho_M \leq d_M$. The last assertion is proved in the next lemma

Lemma 2.3 *For any $x, y \in M$ we have*

$$\rho_M(x,y) \leq d_M(x,y).$$

Proof. Without loss of generality we assume $d_M(x,y)<\infty$. Then for $\varepsilon>0$ we can find a piecewise C^1 map $\sigma:[0,1]\to M$ with $\sigma(0)=x, \sigma(1)=y$, such that $l(\sigma)\le d_M(x,y)+\varepsilon$. Thus for any $f\in C^1_1(M;R)$, we have

$$\begin{aligned} f(y)-f(x) &= \textstyle\int_0^1 \frac{d}{ds} f(\sigma(s))\,ds = \int_0^1 df(\dot\sigma(s))\,ds \\ &\le \textstyle\int_0^1 |\dot\sigma(s)|_{\sigma(s)}\,ds = l(\sigma)\le d_M(x,y)+\varepsilon. \end{aligned}$$

Hence $\rho_M(x,y)\le d_M(x,y)$ since ε is arbitrary.

Example 2.4 *(i) Let M be a separable Banach space with Finsler structure given by the Banach norm. Then M is a C-F manifold.*

Proof. Let $x,y\in M$, we set $\sigma(t)=x+t(y-x)$. Then one can easily check that $l(\sigma)=||x-y||_E$. Hence $d_M(x,y)\le ||x-y||_E$. On the other hand by the Hahn-Banach Theorem we can find $\varphi\in M^*$ such that $||\varphi||_{M^*}=1$ and $\varphi(y-x)=||y-x||_E$. Hence $\rho_M(x,y)\ge\varphi(y)-\varphi(x)=||x-y||_E$.Consequently taking Lemma 2.3 into account we have

$$\rho_M(x,y)=||x-y||_E=d_M(x,y).$$

(ii) Let M be a finite dimensional complete Riemannian manifold and the Finsler structure be given by the Riemannian metric. Then by Nash's embedding Theorem M is closely and isometrically embedded into a Euclidean space $I\!R^N$. Hence by (i) and Proposition 2.2 M is a C-F manifold. See also Remark 2.5 below in this connection.

(iii) It follows directly from Proposition 2.2 that any closed submanifold of a C-F manifold is again a C-F manifolds.

Remark 2.5 *We are grateful to C.J. Atkin who kindly communicated to us the following result.*

Suppose that the separable manifold M is modelled on a Banach space with a separable dual, then $d_M=\rho_M$. (2.5)

Thus according to Atkin's result, any complete separable Finsler manifold modeled on a C^1 smooth Banach space, in particular any finite dimensional complete Finsler manifold, or any complete separable Finsler manifold modeled on a Hilbert space, is a C-F manifold.

Plenty of examples of infinite dimensional C-F manifolds come from mapping spaces over a given manifold which we are going to discuss now. Let M be a C-F manifold. Let S be a compact metric space. We set $E:=C(S;M)$, all the continuous mappings from S to M. Note that if $S=[0,1]$, then E is the path space over M. If $S=S^1$, then E is the loop space over M. We give E the compact -open topology [19], then E is separable because the compact-open topology on $C(S;[0,1]^{I\!N})$ is separable and M is homeomorphic to a F_δ subset of $[0,1]^{I\!N}$. It is known that E is a C^1 manifold modeled on a Banach space. In case that M is modeled on a Hilbert space, the differential structure may be constructed by employing the exponential maps between TM and M e.g. see [9]. In the general case the corresponding differential structure requires a more delicate construction. For details we refer to [25] and [20]. For $\sigma \in E$, the tangent vector space $T_\sigma E$ can be identified with the space of all continuous maps $v : S \to TM$ such that $v(s) \in T_{\sigma(s)}M$ for all $s \in S$. A natural Finsler structure on E is given by

$$|v|_\sigma := \sup_{s\in S} |v(s)|_{\sigma(s)}, \forall \sigma \in E, v \in T_\sigma E. \tag{2.6}$$

One can easily check that $T_\sigma E$ equipped with the norm $|\cdot|_\sigma$ is a Banach space.

Proposition 2.6 *The mapping space E constructed above is a C-F manifold with respect to the Finsler structure given by (2.6), as are all its closed submanifolds with their induced Finsler structure.*

Proof. Let ρ_M be the Caratheodory metric on M. We define

$$\bar\rho_M(\sigma,\sigma') := \sup_{s\in S} \rho_M(\sigma(s),\sigma'(s)), \quad \forall \sigma,\sigma' \in E. \tag{2.7}$$

Clearly $\bar\rho_M$ is a complete metric on E since so is ρ_M on M. Moreover, $\bar\rho_M$ generates the topology of uniform convergence which coincides the compact-open topology on E ([19]). Hence $\bar\rho_M$ is admissible on E. For $s \in S$ and $\varphi \in C^1_1(M;R)$, if we set $f(\sigma) = \varphi(\sigma(s))$ all $\sigma \in E$, then it follows from (2.6) that $f \in C^1_1(E;R)$. Therefore for $\sigma,\sigma' \in E$

$$\begin{aligned} \bar\rho_M(\sigma,\sigma') &= \sup_{s\in S} \sup\{\varphi(\sigma'(s) - \varphi(\sigma(s))) : \varphi \in C^1_1(M;R)\} \\ &\le \sup\{f(\sigma') - f(\sigma) : f \in C^1_1(E;R)\} := \rho_E(\sigma,\sigma'), \end{aligned} \tag{2.8}$$

which together with Lemma 2.3 and Proposition 2.2 proves the proposition.

Remark 2.7 *Proposition 2.6 allows us to conclude that submanifolds of $C([0,1];M)$ such as the space of based loops can be covered by our treatment as will be spaces of paths and loops on the Hilbert manifolds $\mathcal{D}^s$ of those diffeomorphisms of a compact n-dimensional manifold in the Soblev class $H^s, s > \frac{2}{n}+1$, e.g. see [12], where $\mathcal{D}^s$ is given a right invariant Riemannian metric.*

The following result will be useful in the subsequent section.

Proposition 2.8 *Let M be a C-F manifold. Then there exists a countable family $\{f_j\}_{j\in I\!N} \subset C_1^1(M;R)$ such that for all $x, y \in M$*

$$\rho_M(x,y) = \sup_{j\in I\!N}[f_j(y) - f_j(x)]. \tag{2.9}$$

Proof. Let $\{x_l\}_{l\in I\!N}$ be a countable dense subset of M. For each pair (x_l, x_m) we can find a sequence $\{f_{l,m,n}\}_{n\in I\!N} \subset C_1^1(M;R)$ such that

$$\rho_M(x_l, x_m) = \sup_n [f_{l,m,n}(x_m) - f_{l,m,n}(x_l)]$$

Rearrange $\{f_{l,m,n}\}$ by $\{f_j\}_{j\in I\!N}$. Then $\{f_j\}_{j\in I\!N}$ is as desired.

3 Admissible vector fields

Throughout this section let $E := C(S;M)$ be the mapping space specified in Proposition 2.6 or a closed submanifold of it. Let $\sigma \in E$ and $v \in T_\sigma E$. For $f \in C^1(E;R)$ we shall write $\partial_v f(\sigma) := df(v(\sigma))$, the Fréchet derivative of f at σ along the direction v. If v is a tangent vector field on E, i.e., $v(\sigma) \in T_\sigma E$ for all $\sigma \in E$, then the notation $\partial_v f(\sigma)$ will stands for $\partial_{v(\sigma)} f(\sigma)$. We shall say that f is a cylindrical function, written by $f \in \mathcal{F}C_b^1(E)$, if

$$f(\sigma) = \varphi(\sigma(s_1), \sigma(s_2), \cdots, \sigma(s_n)), \forall \sigma \in E, \tag{3.1}$$

for some subset $(s_1, s_2, \cdots, s_n) \subset S$ and some bounded $\varphi \in C^1(M^n;R)$ satisfying

$$\sup\{|d^i\varphi(x_1, x_2, \cdots, x_n)|_{L(T_xM,R)} : (x_1, x_2, \cdots, x_n) \in M^n\} < \infty,\ \forall 1 \le i \le n,$$

where $d^i\varphi$ denotes the differential of φ with respect to the ith variable. Clearly $\mathcal{F}C_b^1(E) \subset C_b^1(E;R)$. Hence $\partial_v f$ is well defined for a tangent vector field v on E and cylindrical function f. One can check that if f is given by (3.1), then $\partial_v f = \sum_{i=1}^n d^i\varphi(v_{t_i})$, more precisely

$$\partial_v f(\sigma) = \sum_{i=1}^n d^i\varphi(\sigma(s_1), \sigma(s_2), \cdots, \sigma(s_n))(v_{t_i}(\sigma)), \forall \sigma \in E \tag{3.2}$$

for all tangent vectors v. In particular, the evaluation of the right hand side of (3.2) is independent of the expression (3.1) since so is the definition of $\partial_v f$.

We denote by $\mathcal{B}$ the Borel sets on E. A vector field v is said to be $\mathcal{B}$-measurable if $\partial_v f$ is $\mathcal{B}$-measurable for all $f \in C^1(E;R)$.

From now on we assume that a finite measure μ is given on $(E, \mathcal{B})$. For notational convenience, we shall use the same symbol f for the μ-equivalence class determined by a function f. With this convention $\mathcal{F}C_b^1$ can be viewed as a subspace of $L^2(E,\mu)$ in such a way that if $f, g \in \mathcal{F}C_b^1, f = g$ μ-a.e., then f and g are regarded as the same element of $L^2(E,\mu)$. Note that by (2.4), (2.7) and the proof of Proposition 2.6 one can check that $\mathcal{F}C_b^1$ separates the points of E, therefore by monotone class argument $\mathcal{F}C_b^1$ is dense in $L^2(E,\mu)$.

Definition 3.1 *Let $\mathcal{D}$ be a linear subspace of $C_b^1(E;R) \cap L^2(E,\mu)$ such that $\mathcal{D}$ is dense in $L^2(E,\mu)$. We say that a $\mathcal{B}$-measurable tangent vector field v is $\mathcal{D}$-**admissible**, if the following three conditions are satisfied.*

(i) $f \in \mathcal{D}, f = 0$ μ-a.e implies $\partial_v f = 0$ μ-a.e.

(ii) $\partial_v f \in L^2(E,\mu)$ for all $f \in \mathcal{D}$.

(iii) ∂_v is a closable operator in $L^2(E,\mu)$.

Remark 3.2 *(i) Condition (i) and (ii) above ensure that ∂_v is a densely defined operator on $L^2(E,\mu)$. Hence the statement of condition (iii) above is meaningful.*

(ii) Let v be a $\mathcal{D}$-admissible vector field and $v' = v$ μ-a.e. then v' is again a $\mathcal{D}$-admissible vector field and $\partial_{v'}$ as a linear operator on $L^2(E,\mu)$ coincides with ∂_v. In practice we shall deal with a vector field v which is defined on $E \setminus N$ for some μ-null set N. In this case v is said to be $\mathcal{D}$-admissible if there exists a $\mathcal{D}$-admissible vector field $\bar{v}$ on E such that $\bar{v} = v$ on $E \setminus N$. By the above reason ∂_v being a linear operator does not depend on the particular choice of $\bar{v}$.

(iii) $\mathcal{D}$ may be interpreted as the space of test functions for tangent vector fields on E. In the literature of path spaces and loop spaces over finite dimensional Riemannian manifolds, most authors consider only the case that $\mathcal{D} = \mathcal{F}C_b^1(E)$. But as a matter of fact one can also take $\mathcal{D}$ to be all the bounded functions in $C_b^1(E;R)$, or even take $\mathcal{D}$ to be all the L^2-integrable functions in $C_b^1(E;R)$. On the other hand one can also take $\mathcal{D}$ to be smaller than $\mathcal{F}C_b^1(E)$ for different purposes. For example in case that M itself is a mapping space, then the space of all cylindrical functions over cylindrical mappings might be a good candidate for $\mathcal{D}$.

In what follows we fix a linear subspace $\mathcal{D}$ specified in Definition 3.1. Denote by $(\cdot,\cdot)$ the inner product of $L^2(E,\mu)$. Let v be a $\mathcal{D}$-admissible vector field and let $\bar{\partial_v}$ with domain $D(\bar{\partial_v})$ be the closure of $(\partial_v, \mathcal{D})$ in $L^2(E,\mu)$. We set

$$\begin{aligned}&\mathcal{E}_v(f,g) = (\bar{\partial}_v f, \bar{\partial}_v g), \quad \forall f,g \in D(\bar{\partial}_v\\ &D(\mathcal{E}_v) = D(\bar{\partial}_v)\end{aligned} \tag{3.3}$$

Then $(\mathcal{E}, D(\mathcal{E}_v))$ is a symmetric closed form on $L^2(E,\mu)$. In fact we have the following criterion for v to be $\mathcal{D}$-admissible.

Proposition 3.3 *A $\mathcal{B}$-measurable vector field v is $\mathcal{D}$-admissible if and only if there is a symmetric closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E,\mu)$ such that $D(\mathcal{E}) \supset \mathcal{D}$ and for some constant $C > 0$,*

$$C^{-1}\mathcal{E}(f,f) \le ||\partial_v f||^2_{L^2(E,\mu)} \le C\mathcal{E}(f,f) \tag{3.4}$$

for all $f \in \mathcal{D}$.

Proof. Suppose that (3.4) holds. From the right hand side inequality of (3.4) we see that $\partial_v f$ is in $L^2(E,\mu)$ and the corresponding L^2-norm is controlled by $C\mathcal{E}(f,f)$. in particular if $f = 0$ μ-a.e., then $||\partial_v f||^2_{L^2} \le C\mathcal{E}(f,f) = 0$, which implies $\partial_v f = 0$ μ-a.e.. Thus 3.1 (i) and (ii) are satisfied. To verify 3.1 (iii) we define $\mathcal{E}_v(f,g) = (\partial_v f, \partial_v g)$ for $f,g \in \mathcal{D}$. Then (3.4) implies that the bilinear form $(\mathcal{E}_v, \mathcal{D})$ is closable in $L^2(E,\mu)$. Hence ∂_v is closable in $L^2(W,\mu)$, verifying 3.1 (iii). Conversely suppose that v is $\mathcal{D}$-admissible, then the symmetric closed form defined by (3.3) satisfies (3.4) with constant 1.

Proposition 3.4 *Let v be a $\mathcal{D}$-admissible vector field and ξ be a real-valued $\mathcal{B}$-measurable function on E. Suppose that there exists a positive constant C such that $C^{-1} \le |\xi| \le C$, then ξv is $\mathcal{D}$-admissible.*

Proof. Clearly ξv satisfies 3.1 (i) and (ii). But by the assumption ξv satisfies also

$$C^{-1}\mathcal{E}_\mu(f,f) \le ||\partial_{\xi v} f||^2_{L^2(E,\mu)} \le C\mathcal{E}_v(f,f), \forall f \in \mathcal{D}$$

with $(\mathcal{E}_v, D(\mathcal{E}_v))$ being defined by (3.3).

Below is a sufficient condition for v to be $\mathcal{D}$-admissible.

Proposition 3.5 *Suppose that $\mathcal{D}$ is an algebra with pointwise multiplication. Let v be a $\mathcal{B}$-measurable vector field such that $\partial_v f \in L^2(E,\mu)$ for all $f \in \mathcal{D}$ and there exists an element* $\operatorname{div} v$ *(called the* **divergence** *of v) in $L^2(E,\mu)$ satisfying*

$$\int_E \partial_v f \mu(d\sigma) = -\int_E f \operatorname{div} v \mu(d\sigma), \quad \forall f \in \mathcal{D}. \tag{3.5}$$

Then v is $\mathcal{D}$-admissible.

Proof. Let us define

$$\partial_v^* f = -\partial_v f - f \operatorname{div} v, \quad \forall f \in \mathcal{D} \tag{3.6}$$

One can check that (3.5) implies

$$(\partial_v f, g) = (f, \partial_v^* g), \quad \forall f, g \in \mathcal{D}. \tag{3.7}$$

Thus $f = 0$ μ-a.e. implies $\partial_v f = 0$ μ-a.e. since $\mathcal{D}$ is dense in $L^2(E, \mu)$. Moreover, (3.7) shows that the adjoint operator of ∂_v is densely defined in $L^2(E, \mu)$. Hence ∂_v is closable.

In the remainder of this section we assume that $\mathcal{D}$ is an algebra. We shall say that v is **strongly $\mathcal{D}$-admissible** if v has a divergence $\operatorname{div} v$ specified by the above proposition, with $\operatorname{div} v$ in $L^2(E, \mu)$ and $\partial_v f \in L^2(E, \mu)$ for all $f \in \mathcal{D}$.

Proposition 3.6 *Let v be a strongly $\mathcal{D}$-admissible vector field. Then for any bounded element $f \in \mathcal{D}$, fv is again a strongly $\mathcal{D}$-admissible vector field.*

Proof. Let fv be as in the Proposition. We define

$$\operatorname{div}(fv) = +\partial_v f + f \operatorname{div} v$$

Then $\operatorname{div}(fv) \in L^2(E, \mu)$ and

$$\int_E \partial_{fv} g \mu(d\sigma) = -\int_E g \operatorname{div}(fv) \mu(d\sigma), \quad \forall g \in \mathcal{D}.$$

Hence the assertion follows by definition.

Let us denote by V all the strongly $\mathcal{D}$-admissible vector fields v such that

$$||v||_V^2 := \int_E |v(\sigma)|_\sigma^2 \mu(d\sigma) + \int_E |\operatorname{div} v|^2 \mu(d\sigma) < \infty \tag{3.8}$$

We assume that V is not empty.

Theorem 3.7 *V equipped with $||\cdot||_V$ is a Banach space.*

Proof. It is easy to check $(V, ||\cdot||_V)$ is a normed linear space. We need only to check that V is complete with respect to the norm $||\cdot||_V$. To this end let $v_n, n \geq 1$, be a Cauchy sequence in $(V, ||\cdot||_V)$. By taking a subsequence if necessary, we may assume that

$||v_{n+1} - v_n||_V \le 2^{-n}$ for all n. Let $C = \mu(E)^{\frac{1}{2}}$. By Schwartz's inequality we have $\int_E |v_{n+1} - v_n|_\sigma \mu(d\sigma) \le C2^{-n}$. Therefore

$$\int \sum_{n=1}^{\infty} |v_{n+1} - v_n| \mu(d\sigma) < \infty \tag{3.9}$$

which implies

$$\sum_{n=1}^{\infty} |v_{n+1}(\sigma) - v_n(\sigma)|_\sigma < \infty \quad \mu\text{−a.e. } \sigma \in E. \tag{3.10}$$

Note that each tangent space $T_\sigma E$ is a Banach space with respect to the norm $|\ |_\sigma$. Hence the following $\mathcal{B}$-measurable vector field v is well defined.

$$\begin{array}{ll} v(\sigma) = v_1(\sigma) + \sum_{n=1}^{\infty}(v_{n+1}(\sigma) - v_n(\sigma)), & \text{if (3.10) is true} \\ v(\sigma) = 0, & \text{otherwise} \end{array} \tag{3.11}$$

Moreover, by Fatou's lemma one can easily check that

$$\lim_{n\to\infty} \int |v_n(\sigma) - v(\sigma)|_\sigma^2 \mu(d\sigma) = 0 \tag{3.12}$$

Similarly we can find a $\mathcal{B}$-measurable function $\operatorname{div} v$ such that

$$\lim_{n\to\infty} \int |\operatorname{div} v_n(\sigma) - \operatorname{div} v(\sigma)|^2 \mu(d\sigma) = 0 \tag{3.13}$$

Now (3.12) implies that $\int_E |v(\sigma)|_\sigma^2 \mu(d\sigma) < \infty$ and hence $\partial_v f \in L^2(E, \mu)$ for $f \in \mathcal{D}$. Moreover, we have

$$\begin{aligned} &|\textstyle\int_E |\partial_{v_n} f - \partial_v f| \mu(d\sigma) \\ &\le ||df|| \textstyle\int_E |v_n(\sigma) - v(\sigma)|_\sigma \mu(d\sigma) \to 0 \quad (n \to \infty) \end{aligned}$$

and

$$\begin{aligned} &\textstyle\int_E |f \operatorname{div} v_n - f \operatorname{div} v| \mu(d\sigma) \\ &\le (\textstyle\int f^2 \mu(d\sigma))^{\frac{1}{2}} (\textstyle\int |\operatorname{div} v_n - \operatorname{div} v|^2 \mu(d\sigma))^{\frac{1}{2}} \to 0 \quad (n \to \infty). \end{aligned}$$

Consequently, since each $\operatorname{div} v_n$ is the divergence of v_n, we see that v and $\operatorname{div} v$ satisfy (3.5), i.e., $\operatorname{div} v$ is the divergence of v. This fact together with (3.12), (3.13) imply that $v \in V$ and $||v_n - v||_V \to 0$.

Remark 3.8 *If we use L^p-norm instead of L^2-norm in (3.9), then $(V, ||\cdot||_V)$ is again a normed linear space. Similar to the above proof one can show that if V is not empty, then $(V, ||\cdot||_V)$ is a Banach space whenever $p \ge 2$.*

We now assume that for each $\sigma \in E$ there is a Hilbert space $H(\sigma)$ continuously embedded in $T_\sigma E$. For example, each $T_\sigma E$ an abstract Wiener space, or the case that E is modeled on a Hilbert space and the Finsler structure is given by the Hilbert norm. In the latter case we have $H(\sigma) = T_\sigma E$. Assume that there exists a $\mathcal{B}$-measurable function $\Phi \geq 1$ such that $|v|_\sigma \leq \Phi(\sigma)|v|_{H(\sigma)}$ for all $v \in H(\sigma)$.

Theorem 3.9 *Let VH be all the strongly $\mathcal{D}$-admissible vector fields v with $v(\sigma) \in H(\sigma)$ almost all σ and satisfying (3.8) and (3.14) below*

$$||v||^2_{VH} := \int_E |v(\sigma)|^2_{H(\sigma)} \Phi^2(\sigma)\mu(d\sigma) + \int_E |\operatorname{div} v|^2 \mu(d\sigma) < \infty \tag{3.14}$$

Assume that VH is not empty and define for $v_1, v_2 \in VH$,

$$(v_1, v_2)_{VH} := \int_E (v_1, v_2)_{H(\sigma)} \Phi^2(\sigma)\mu(d\sigma) + \int (\operatorname{div} v_1)(\operatorname{div} v_2)\mu(d\sigma) \tag{3.15}$$

Then VH equipped with $(\cdot, \cdot)_{VH}$ is a Hilbert space.

Proof. Clearly $(VH, (\cdot, \cdot)_{VH})$ is a pre-Hilbert space. As in Theorem 3.6 we can show that VH is complete with the norm $|| \cdot ||_{VH}$.

In practice some spaces of vector fields are obtained as image of a Hilbert space H under a linear map τ. In this case we may induce a Hilbert structure for those vector fields which are in image of τ.

Proposition 3.10 *Let H be a Hilbert space and $\tau : H \to V$ be a linear map such that there exists a constant C satisfying*

$$||\tau h||^2_V \leq C||h||^2_H, \forall h \in H. \tag{3.16}$$

Let $\ker\tau = \{h \in H | \tau h = 0\}$ *and* $\ker\tau^\perp$ *be the orthogonal space of* $\ker\tau$. *Define*

$$V(\tau H) = \{v \in V | v = \tau h \text{ for some } h \in \ker\tau^\perp\} \tag{3.17}$$

$$(v_1, v_2)_{\tau H} = (h_1, h_2)_H \tag{3.18}$$

for $v_1 = \tau h_1, v_2 = \tau h_2$ with $h_1, h_2 \in$ $\ker\tau^\perp$

Then $V(\tau H)$ with inner product $(\cdot, \cdot)_{\tau H}$ is a Hilbert space. In particular, if $\tau : H \to V$ is a continuous injective linear map. Then $V(\tau H)$ is isomorphic to H.

We omit the proof because it is an easy exercise.

Example 3.11 *Let H be a Hilbert space and $\tau : H \to V$ be a linear map, such that there exists a $\Phi \in L^2(E,\mu)$ satisfying*

$$|\tau h|_\sigma \leq \Phi(\sigma)||h||_H, \quad \mu-\text{a.e.}, \ \forall h \in H. \tag{3.19}$$

Then

$$\int |\tau h|_\sigma^2 \mu(d\sigma) \leq ||h||_H^2 ||\Phi||^2_{L^2(E,\mu)}, \quad \forall h \in H.$$

If in addition

$$\int |\operatorname{div}(\tau h)|^2 \mu(d\sigma) \leq C||h||_H^2 \text{ some constant } C$$

then condition (3.16) is fulfilled and $V(\tau H)$ specified by (3.17) is a Hilbert space. In this case we define $\tau(\sigma) : H \to T_\sigma E$ by $\tau(\sigma)h = \tau h(\sigma)$. Assume in addition that $\tau(\sigma)$ is injective for μ-a.e. $\sigma \in E$. Let

$$H(\sigma) = \{v(\sigma) \in T_\sigma E \mid v(\sigma) = \tau h(\sigma) \quad \text{for some } h \in H\} \tag{3.20}$$

Then $H(\sigma)$ is isomorphic to H for μ-a.e. $\sigma \in E$. Hence $V(\tau H)$ coincides with VH specified in Theorem 3.9 (Note that we may always assume that Φ in (3.19) is not less than 1).

4 Construction of diffusion processes

All the assumptions and notations are the same as in Proposition 2.6 and Definition 3.1.

Lemma 4.1 *In the situation of Proposition 2.6 and Definition 3.1. Let $\mathcal{A}$ be a countable or finite family of $\mathcal{D}$-admissible vector fields. Suppose that*

$$\int_E \sum_{v\in\mathcal{A}} |\partial_v f|^2 \mu(d\sigma) < \infty, \ \forall f \in \mathcal{D}. \tag{4.1}$$

Then the symmetric form $(\mathcal{E}, \mathcal{D})$ defined by

$$\mathcal{E}(f,g) = \int_E \sum_{v\in\mathcal{A}} \partial_v f)(\partial_v g)\mu(d\sigma), \quad \forall f,g \in \mathcal{D} \tag{4.2}$$

is closable in $L^2(E,\mu)$ and the closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form on $L^2(E,\mu)$.

Proof. For $v \in \mathcal{A}$ we set $\mathcal{E}_v(f,g) = \int_E (\partial_v f)\partial_v g)\mu(d\sigma)$.It follows from 3.1 (iii) that $(\mathcal{E}_v, \mathcal{D})$ is closable in $L^2(E,\mu)$. Clearly $\mathcal{E}(f,g) = \sum_{v\in\mathcal{A}} \mathcal{E}_v(f,g)$, hence condition (4.1) and the denseness of $\mathcal{D}$ implies that $(\mathcal{E},\mathcal{D})$ is closable (cf. e.g. [22] I. Prop.3.7). The fact that the closure $(\mathcal{E}, D(\mathcal{E}))$ of $(\mathcal{E},\mathcal{D})$ is a Dirichlet form follows from the chain rule and e.g. [22] I.Prop.4.10.

Let $(\mathcal{E}, D(\mathcal{E}))$ be the Dirichlet from constructed in the above lemma. By the theory of Dirichlet forms there exists a unique self-adjoint operator L with domain $D(L)$ on $L^2(E,\mu)$ satisfying

$$D(L) \subset D(\mathcal{E})$$

and

$$\mathcal{E}(f,g) = (-Lf, g), \ \forall f \in D(L), g \in D(\mathcal{E}). \tag{4.3}$$

$(L, D(L))$ is called the generator of $(\mathcal{E}, D(\mathcal{E}))$. Intuitively we may think that $L = \sum_{v\in V} -\partial_v^*\partial_v$, which will rigorously hold e.g. in the case $\partial_v f \in D(\partial_v^*)$ for all $v \in \mathcal{A}$ and $\sum \partial_v^* \partial_v f$ converges in $L^2(E,\mu)$.

Theorem 4.2 *In the situation of Lemma 4.1, suppose that in addition to (4.1) the following three conditions are also fulfilled.*

(i) If $\varphi \in C_b^\infty(R), \varphi(0) = 0$, then $\varphi \circ f \in \mathcal{D}$ for all $f \in \mathcal{D}$.

(ii) If f, g are bounded functions in $\mathcal{D}$, then $fg \in \mathcal{D}$.

(iii) $\mathcal{D}_0 \subset \mathcal{D}$ and there exists $\Phi \in L^2(E,\mu)$ such that for all $\varphi \in C_b^1(M;R), s \in S$,

$$\sum_{v\in\mathcal{A}} |d\varphi(\sigma(s))(v_s(\sigma))|^2 \le ||d\varphi||^2 \Phi^2(\sigma), \mu\text{−a.e.} \tag{4.4}$$

where $||d\varphi||$ is defined by (2.1) and

$$\begin{aligned} \mathcal{D}_0 := \{ & f \in \mathcal{F}C_b^1(E) : f(\sigma) = \varphi(\sigma(s)) \\ & \text{for all } \sigma \in E, \text{ for some } \varphi \in C_b^1(M;R) \text{ and some } s \in S\} \end{aligned} \tag{4.5}$$

Then there exists a diffusion process $(\xi_t)_{t\ge0}$ on E associated with $(\mathcal{E}, D(\mathcal{E}))$. That is, if $(L, D(L))$ is the generator of $(\mathcal{E}, D(\mathcal{E}))$, then

$$E.[f(\xi_t)] \ \ \textit{is an } \ \mu\textit{-version of } \ e^{Lt}f \ \ \textit{for all } \ f \in L^2(E,\mu). \tag{4.6}$$

Moreover, $(\xi_t)_{t\ge0}$ is conservative and hence μ is an invariant measure for e^{Lt}.

Remark 4.3 *By IV Theorem 5.1 of [22] the existence of* $(\xi_t)_{t\geq 0}$ *satisfying (4.6) always implies that* $(\mathcal{E}, D(\mathcal{E}))$ *is quasi-regular and* $(\xi_t)_{t\geq 0}$ *is properly associated with* $(\mathcal{E}, D(\mathcal{E}))$*. The latter assertion, which extends Theorem 4.3.3 of [15] (see also Theorem 4.2.3 of [16]) by relaxing the regularity of* $(\mathcal{E}, D(\mathcal{E}))$*, means that (4.6) is automatically strenthened by*

$$E.[f(\xi_t)] \text{ is an } \mathcal{E}\text{-quasi-continuous } \mu\text{-version of } e^{Lt}f \quad \text{for all } f \in L^2(E,\mu). \tag{4.6'}$$

The proof of the above theorem relies on the fact that E is a C-F manifold and is split into several steps. Our strategy is to show that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form and is local, and hence the desired conclusion follows from [22] IV.Th.3.5 and V.Th. 1.11. We shall follow the argument of [26] §3 and [8] to check the quasi-regularity of $(\mathcal{E}, D(\mathcal{E}))$. The experts may find that $(\mathcal{E}, D(\mathcal{E}))$ is in fact a special case of [26] Th. 3.4. For later use we recall that a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular if (cf. [22] IV. Def.3.1):

(Q.1) There exists an $\mathcal{E}$-nest $(F_k)_{k\geq I\!N}$ consisting of compact sets.

(Q.2) There exists an $\mathcal{E}_1$-dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous μ-versions.

(Q.3) There exists $u_n \in D(E), n \in I\!N$, having $\mathcal{E}$-quasi-continuous μ-version $\tilde{u}_n, n \in I\!N$ and an $\mathcal{E}$-exceptional set $N \subset W$ such that $\{\tilde{u}_n | n \in I\!N\}$ separates the points of $W \setminus N$.

See also [21], [23] and [27] for the notion of quasi-regularity in more general contexts.

Let us set by $\Gamma(f,g) = \sum_{v\in V}(\partial_v f)(\partial_v g)$ for $f, g \in \mathcal{D}$. By (4.1) and the Cauchy-Schwartz inequality $\Gamma(f,g) \in L^1(E,m)$. By (4.2) we have

$$\mathcal{E}(f,g) = \int_E \Gamma(f,g)\mu(d\sigma) \tag{4.7}$$

which implies that $\Gamma : \mathcal{D} \times \mathcal{D} \to L^1(E,\mu)$ is a continuous bilinear map with respect to the product topology on $\mathcal{D} \times \mathcal{D}$ induced by the $\mathcal{E}_1$-norm $(\mathcal{E}_1(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2})$ on $\mathcal{D}$. Therefore Γ extends to a continuous bilinear map on $D(\mathcal{E}) \times D(\mathcal{E})$ which we shall denote again by Γ. Clearly (4.7) holds for all $f, g \in D(\mathcal{E})$.

Lemma 4.4 *For* $f, g \in D(\mathcal{E})$*, we have*

$$\begin{aligned} \Gamma(f \vee g, f \vee g) &\leq \Gamma(f,f) \vee \Gamma(g,g) \\ \Gamma(f \wedge g, f \wedge g) &\leq \Gamma(f,f) \vee \Gamma(g,g) \end{aligned}$$

Proof. One can easily check that if φ is a smooth function on R with $\varphi(0) = 0$ and $|\varphi'(x)| \leq 1$, then

$$|\Gamma(f, \varphi(g))| \leq \Gamma(f, g)|, \quad \forall f, g \in D(\mathcal{E}).$$

Hence the desired assertion follows from [26] Lemma 3.2. (See also [22] IV.Lemma 4.1)

Let ρ_M be the Caratheodory metric on M and $\bar{\rho}_M$ be its lift to E defined by (2.7). We set

$$\rho(\sigma, \sigma') = \bar{\rho}_M(\sigma, \sigma') \wedge 1, \forall \sigma, \sigma' \in E. \tag{4.8}$$

The following lemma is crucial and the hypothesis that M is C-F plays an important role in it.

Lemma 4.5 *(i) ρ is a bounded complete metric on E and it generates the original topology of E.*

(ii) Let $\sigma' \in E$. Then $\rho(\cdot, \sigma') \in D(\mathcal{E})$ and

$$\Gamma(\rho(\cdot, \sigma'), \rho(\cdot, \sigma')) \leq \Phi^2(\cdot) \quad \mu\text{−a.e.} \tag{4.9}$$

where Φ is specified by (4.4)

Proof. (i) follows directly from the proof of Proposition 2.6. For proving (ii) we take an odd and increasing function $\varphi \in C_b^\infty(R)$ such that $|\varphi| \leq 2, \varphi' \leq 1, \varphi'' \leq 0$ on $[0, \infty)$ and $\varphi(x) = x$ for $x \in [-1, 1]$. Let $\{f_j\}_{j \in I\!N}$ be a countable subset of $C_1^1(M; R)$ satisfying (2.9). Let $\{s_i\}_{i \in I\!N}$ be a countable dense subset of S. We set for $\sigma \in E$.

$$a_{ij}(\sigma) = \varphi(f_j(\sigma(s_i)) - f_j(\sigma'(s_i)),$$
$$\rho_i(\sigma) = \sup\nolimits_{j \in I\!N} \varphi(f_j \sigma(s_i)) - f_j(\sigma'(s_i)) \wedge 1.$$

By (2.9) we have $\rho_i(\sigma) = \rho_M(\sigma(s_i),\ \sigma'(s_i)) \wedge 1$. Therefore it follows from (2.7), (4.8) and the denseness of $\{s_i\}_{i \in I\!N}$ that $\sup_{i \in I\!N} \rho_i(\sigma) = \rho(\sigma, \sigma')$. Clearly $a_{ij} \in \mathcal{D}_0 \subset D(\mathcal{E})$ and by (4.4) we have $\Gamma(a_{ij}, a_{ij}) \leq \Phi^2$ μ-a.e.. If we define for fixed i,

$$b_n(\sigma) = \sup_{j \leq n} |a_{ij}(\sigma)| \wedge 1, n \geq 1,$$

then $b_n \in D(\mathcal{E})$ and Lemma 4.4 yields $\Gamma(b_n, b_n) \leq \Phi^2$ μ-a.e for all $n \geq 1$. It in turn implies that

$$\sup_{n \geq 1} \mathcal{E}_1(b_n, b_n) < \infty.$$

Therefore by the Banach-Saks theorem there exists a subsequence b_{n_k} whose averages $\frac{1}{l}\sum_{k=1}^{l} b_{n_k} := g_l$ converge strongly in the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$. But b_n converges to ρ_i pointwise. Therefore $\rho_i \in D(\mathcal{E})$ and g_l converges to ρ_i in $\mathcal{E}_1$-norm. Note that by Minkowski's inequality we have $\Gamma(g_l, g_l) \le \Phi^2$ μ-a.e. for all $l \ge 1$. Since $\Gamma : D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(E, \mu)$ is continuous and L^1-convergence implies μ-a.e. convergence for a subsequence, $\Gamma(\rho_i, \rho_i) \le \Phi^2$ μ-a.e.. We now define

$$h_n(\sigma) = \sup_{i \le n} \rho_i(\sigma), \quad \forall \sigma \in E.$$

Applying Lemma 4.4 again and repeating the above argument we see that $\rho(\cdot, \sigma') = \lim_{n\to\infty} h_n \in D(\mathcal{E})$ and (4.9) holds.

Lemma 4.6 *$(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form.*

Proof. We need to show that $(\mathcal{E}, D(\mathcal{E}))$ satisfies (Q.1)$-$(Q.3). Let $\{\sigma^j\}_{j\in I\!N}$ be a countable dense subset of E. Then it follows from Lemma 4.5 (i) that $\rho(\cdot, \sigma^j)$ is continuous and $\{\rho(\cdot, \sigma^j) : j \in I\!N\}$ separates the points of E. Moreover, by Lemma 4.5 (ii) $\rho(\cdot, \sigma^i)$ is in $D(\mathcal{E})$. Hence (Q.3) is fulfilled. Also (Q.2) is fulfilled because $\mathcal{D}$ is dense in $D(\mathcal{E})$. It remains to check (Q.1). To this end we set

$$a_n := \inf_{1\le j\le n} \rho(\cdot, \sigma^j), \quad \forall n \in I\!N.$$

It follows from Lemma 4.5(ii) and Lemma 4.4 that $a_n \in D(\mathcal{E})$ and $\Gamma(a_n, a_n) \le \Phi^2$ μ-a.e. for all n. Therefore repeating the argument used in Lemma 4.5 (ii) we see that there exists a subsequence a_{n_k} whose averages $\frac{1}{l}\sum_{k=1}^{l} a_{n_k} := Y_l$ converge to zero pointwise and each Y_l is continuous, hence by [MR 92, III 3.1] there exists a subsequence $Y_{l_j}, j \in I\!N$, converging to zero quasi-uniformly. That is, there is an $\mathcal{E}$-nest $(F_k)_{k\in I\!N}$ such that $Y_{l_j} \downarrow 0$ uniformly on each F_k. We now fix $k \in I\!N$. Let $\delta > 0$ be arbitrary. We can choose N so that $Y_{l_N}(\sigma) \le \delta$ for all $\sigma \in F_k$. Then

$$\inf_{1\le j\le l_N} \rho(\sigma, \sigma^j) \le \delta, \quad \forall \sigma \in F_k$$

or equivalently,

$$F_k \subset \cup_{j=1}^{l_N} \{\sigma \in E : \rho(\sigma, \sigma^j) \le \delta\}.$$

Thus F_k is totally bounded and consequently, since ρ is a complete admissible metric on E, F_k is compact, verifying (Q.1).

Lemma 4.7 *Let F_1 and F_2 be two closed subsets of E such that*

$$\rho(F_1, F_2) := \inf\{\rho(\sigma, \sigma') : \sigma \in F_1, \sigma' \in F_2\} > 0, \tag{4.10}$$

where ρ is defined by (4.8). Then there exist continuous functions $f_1, f_2 \in D(\mathcal{E})$ such that $\mathrm{supp}[f_1] \cap \mathrm{supp}[f_2] = \emptyset$ *and $f_1(\sigma) = 1$ for $\sigma \in F_1, f_2(\sigma) = 1$ for $\sigma \in F_2$.*

Proof. Let $\alpha = \rho(F_1, F_2)$. Let $\{\sigma^j\}_{j \in I\!N}$ be a countable dense subset of E. Set

$$g_n = \inf\{\rho(\cdot, \sigma^j) : 1 \le j \le n, \rho(\sigma^j, F_1) \ge \frac{\alpha}{3}\}.$$

Similar to the proof of the above lemma we see that $g := \lim_{n\to\infty} g_n \in D(\mathcal{E})$. Clearly $g(\sigma) = 0$ if $\rho(\sigma, 1F_1) \ge \frac{\alpha}{3}$ and $g(\sigma) \ge \frac{\alpha}{3}$ if $\sigma \in F_1$. We now define $f_1 = (\frac{3}{\alpha} f) \wedge 1$. Define f_2 similarly with F_2 in place of F_1. Then f_1, f_2 are desired.

Lemma 4.8 *$(\mathcal{E}, D(\mathcal{E}))$ is local.*

Proof. Let Γ be specified as in (4.7). By the chain rule it is easy to check that for bounded $g_1, g_2, f_1 \in \mathcal{D}$,

$$\Gamma(g_1 f_1, g_2) = g_1 \Gamma(f_1, g_2) + f_1 \Gamma(g_1, g_2). \tag{4.11}$$

Since Γ is a continuous map from $D(\mathcal{E}) \times D(\mathcal{E})$ to $L^1(E, \mu)$, equation (4.11) extends to all bounded $g_1, g_2, f_1 \in D(\mathcal{E})$. Let now $g_1, g_2 \in D(\mathcal{E})$ such that $\mathrm{supp}[g_1] \cap \mathrm{supp}[g_2] = \emptyset$ and $\mathrm{supp}[g_1], \mathrm{supp}[g_2]$ compact. Without loss of generality we may assume g_1, g_2 are bounded. Let f_1, f_2 be specified by the above lemma with $F_1 = \mathrm{supp}[g_1], F_2 = \mathrm{supp}[g_2]$. Then applying (4.11) we see that

$$\Gamma(g_1, g_2) = \Gamma(f_1 g_1, f_2 g_2) = 0 \quad \mu\text{−a.e.}$$

Hence $\mathcal{E}(g_1, g_2) = 0$.

Proof of Theorem 4.2.

The existence of the diffusions $(\xi_t)_{t\ge 0}$ satisfying (4.6) follows from Lemmas 4.6, 4.8 and [22] V.Theorem 1.11. The last assertion of the theorem follows from the fact that $1 \in D(\mathcal{E})$ and $\mathcal{E}(1,1) = 1$.

Corollary 4.9 *Let $X : E \times H \to TE$ be measurable with, for μ-almost all σ, $X(\sigma, \cdot) : H \to T_\sigma E$ continuous linear and satisfying $|X(\sigma, h)|_\sigma \le \Phi(\sigma) \|h\|_H$ where H is a seperable Hilbert space and $\Phi \in L^2(E, \mu)$. Suppose $X(\cdot, h)$ is $\mathcal{D}$-admissible for all $h \in H$.*

For $f : E \to \mathbf{R}$ in C_b^1 define, for μ-almost all σ, $\nabla f(\sigma) \in H$ by $< \nabla f(\sigma), h >_H = df(X(\sigma, h))$. Set

$$\mathcal{E}_X(f, g) = (f, g) = \int_E < \nabla f(\sigma), \nabla g(\sigma) >_H \mu(d\sigma), \quad \forall f, g \in \mathcal{D}.$$

Then $(\mathcal{E}_X, \mathcal{D})$ is closable in $L^2(E,\mu)$ with closure $(\bar{\mathcal{E}}_X, D(\bar{\mathcal{E}}_X))$ a Dirichlet form. If also conditions (i), (ii) of Theorem 4.2 hold then $(\bar{\mathcal{E}}_X, D(\bar{\mathcal{E}}_X))$ is quasi-regular and local, and in particular there is an associated diffusion as in the conclusion of Theorem 4.2.

Proof. Let $\{e_i\}$ be an orthonormal base for H. Then

$$\begin{array}{rcl} <\nabla f(\sigma), \nabla g(\sigma)>_H & = & \sum_i <\nabla f(\sigma), e_i><\nabla g(\sigma), e_i> \\ & = & \sum_i \partial_{v^i} f(\sigma) \partial_{v^i} g(\sigma) \end{array}, \qquad (4.12)$$

where $v^i = X(\cdot, e_i)$. Now

$$\|\nabla f(\sigma)\| = \|df \circ X(\sigma, \cdot)\|_{H^*} \le |df|_\sigma \Phi(\sigma).$$

Therefore

(i) $\|\nabla f(\cdot)\|_H \in L^2(E,\mu)$ if f is C^1_b and so in particular $\mathcal{E}_X$ is defined on $\mathcal{D}$, and

(ii) $\mathcal{E} \int_E |\partial_{v^i} f|^2 \mu(d\sigma) = \int_E \|\nabla f(\sigma)\|^2_H \mu(d\sigma) < \infty, \forall f \in \mathcal{D}$.

Thus the first part of the theorem holds by lemma 4.1. Moreover for any $\phi \in C^1_b(M; \mathbf{R})$ and $s \in S$, if $\rho_s : E \to M$ is $\rho_s(\sigma) = \sigma(s)$, then

$$\begin{array}{rcl} \sum_i |d\phi(v^i(\sigma)_s)|^2 & = & \sum_i <\nabla(\phi \circ \rho_s)(\sigma), e_i>^2_H \\ & = & \|\nabla(\phi \circ \rho_s)(\sigma)\|^2_H \\ & \le & |d(\phi \circ \rho_s)(\sigma)\|^2_\sigma \Phi(\sigma) \\ & \le & \|d\phi\|^2 \Phi(\sigma) \end{array}$$

since $\|d\rho_s\| = 1$, each $s \in S$. Thus the conditions of Theorem 4.2 for the form

$$(f,g) \to \int_E \sum_i (\partial_{v^i} f)(\partial_{v^i} g) \mu(d\sigma), \quad f, g \in \mathcal{D}$$

are met. But this form is just $\mathcal{E}_x$ by (4.12).

Remark 4.10 *: Let now τ be a linear map of H into the space of Borel measurable vector fields on M, e.g., suppose $\tau(h)(\sigma) = X(\sigma, h)$ for X as in the Corollary. Let $f : M \to \mathbf{R}$ be such that 'derivatives' $df(\tau(h)) : M \to \mathbf{R}$ are defined in some way and give a linear map*

$$df \circ \tau : H \to L^0(M, \mu; \mathbf{R})$$

$$h \to df(\tau(h)(\cdot))$$

where L^0 refers to equivalence classes of measurable maps and we give it the topology of convergence in measure. Then it is immediate from Itô's *regularization theorem [17] that there exists a 'gradient' vector field $\nabla f \in L^0(M, \mu; H)$ satisfying*

$$< \nabla f(\sigma), h >_H = df(\tau(h)(\sigma)) \text{ all } h \in H, \ \mu\text{-almost all } \sigma$$

if and only if $df \circ \tau$ is continuous in the Sazonov topology of H. In particular if $df \circ \tau$ maps into $L^2(M, \mu, \mathbf{R})$ it has to be Hilbert-Schmidt for a gradient to exist.

Using this remark we can see that there is in general no gradient operator associated to the Hilbert space AV of adapted vector fields described in $\S E(b)$ of the introduction. Indeed take E to be classical Wiener space $C([0,1]; \mathbf{R}^n)$ with μ its Wiener measure. Let $f : E \to \mathbf{R}$ be evaluation of the first coodinate at time 1. Consider the Hilbert space G of vector fields on E of the form $v(\sigma)_t = \int_0^t \alpha_s(\sigma) d\sigma_s, \ 0 \leq t \leq 1$, where $\alpha_s : E \to L(\mathbf{R}^n; \mathbf{R}^n)$ is adapted and

$$|v|_G := \int_E \int_0^1 |\alpha_s(\sigma)|^2 \ ds \ \mu(d\sigma)$$

Using the Hilbert-Schmidt norm on $L(\mathbf{R}^n; \mathbf{R}^n)$. There G is just the space of square integral martingales and the martingale representation theorem implies that the map $v \to v_1$ is an isometry ρ_1 of G onto $L^2(E, \mathcal{F}_1, \mu; \mathbf{R}^n)$ for $\mathcal{F}_1$ the σ-algebra generated by $\sigma \to \sigma(1)$. If follows that ρ_1 sends the closed subspace G_0 of G consisting of those v. for which $\alpha_s(\sigma)$ is skew symmetric onto any infinite dimensional closed subspace of $L^2(E, \mathcal{F}_1, \mu; \mathbf{R}^n)$. Composing ρ_1 with co-ordinate projections we obtain maps $\rho_1^k : G_0 \to L^2(E, \mathcal{F}_1, \mu; \mathbf{R})$, $k = 1$ to n, not all of which can be Hilbert-Schmidt. By symmetry ρ_1^1 is not Hilbert-Schmidt. However $\partial_v f = \rho_1^1(v)$, and G_0 is a closed subspace of AV.

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