Ergodicity for the stochastic dynamics of quasi–invariant measures with applications to Gibbs states

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Abstract

The convex set $\mathcal{M}^a$ of quasi–invariant measures on a locally convex space E with given "shift"–Radon-Nikodym derivatives (i.e., cocycles) $a = (a_{tk})_{k\in K_0, t\in\mathbb{R}}$ is analyzed. The extreme points of $\mathcal{M}^a$ are characterized and proved to be non-empty. A specification (of lattice type) is constructed so that $\mathcal{M}^a$ coincides with the set of the corresponding Gibbs states. As a consequence, via a well-known method due to Dynkin–Föllmer a unique representation of an arbitrary element in $\mathcal{M}^a$ in terms of extreme ones is derived. Furthermore, the corresponding classical Dirichlet forms $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ and their associated semigroups $(T_t^\nu)_{t>0}$ on $L^2(E;\nu)$ are discussed. Under a mild positivity condition it is shown that $\nu \in \mathcal{M}^a$ is extreme if and only if $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible or equivalently, $(T_t^\nu)_{t>0}$ is ergodic. This implies time–ergodicity of associated diffusions. Applications to Gibbs states of classical and quantum lattice models as well as those occuring in Euclidean quantum field theory are presented. In particular, it is proved that the stochastic quantization of a Guerra–Rosen–Simon Gibbs state on $\mathcal{D}'(\mathbb{R}^2)$ in *infinite volume* with polynomial interaction is ergodic, if the Gibbs state is extreme (i.e., is a pure phase) which solves a long-standing open problem.

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Contents

1 Introduction

In this paper we analyze the L^2-ergodicity of stochastic dynamics intrinsicly associated with arbitrary *quasi–invariant* measures. Though such analysis is certainly of its own interest, since quasi-invariant measures on topological vector spaces have been studied intensively for more than thirty years (cf. e.g. [GeV64], [PrYo75], [Roy75], [RoyYo76], [AH-K77], [AR90a,b], and the references therein as well as below), this work is stronlgy directed towards and motivated by applications to statistical physics and Euclidean quantum field theory. One of our main applications gives a proof of the L^2–ergodicity of the stochastic quantization (cf. [PaW81]) of an extreme Guerra–Rosen–Simon Euclidean $P(\Phi)_2$–quantum Gibbs state *in infinite volume.* This solves a long-standing open problem left unsettled e.g. in [J-LMi85], [Mi86], [BoCMi87], [AR89a,b, 91], [RZ92] and [GaGo95] (see below for details).

Let us describe our results more precisely. We start with the abstract part. The applications will be summarized subsequently.

Let E be a "nice" locally convex (Hausdorff) topological vector space over $\mathbb{R}$ (e.g. a separable Banach space or the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^d)$) and H a separable real Hilbert space such that

$$E' \subset H' \equiv H \subset E \qquad \text{densely and continuously.}$$

Let $K_0 \subset E'$ be an orthonormal basis of H separating the points of E. For each $k \in K_0$ let $(t,z) \mapsto a_{tk}(z)$ be given measurable real functions on $\mathbb{R} \times E$ and let $\mathcal{M}^a$ denote the set of all probability measures on E such that the shifted measures $\nu(\cdot + tk)$ have Radon–Nikodym derivatives

$$\frac{d\nu(z+tk)}{d\nu(z)} = a_{tk}(z) \qquad , \ z \in E \ , \tag{1.1}$$

w.r.t ν for all $k \in K_0$, $t \in \mathbb{R}$ (cf. Subsections 2.1, 2.2 below, in particular assumptions (A.1), (A.2)). Suppose $\mathcal{M}^a \neq \emptyset$. We have that $\#\mathcal{M}^a > 1$ in general, unless E is finite dimensional (cf. Remark 2.11 (ii) resp. Section 4). We are interested in the following five questions about the convex set $\mathcal{M}^a$ resp. the set of its extreme points $\mathcal{M}^a_{ex}$.

(1) Is $\mathcal{M}^a_{ex} \neq \emptyset$?

(2) Can we characterize $\mathcal{M}^a_{ex}$?

(3) Can we prove an integral representation of an arbitrary element in $\mathcal{M}^a$ in terms of extreme elements?

(4) Can $\mathcal{M}^a$ be described as Gibbs states w.r.t. a *natural, explicit* specification (in the sense of [Fö75], [P76], [G88])?

(5) If for $\nu \in \mathcal{M}^a$ we denote the corresponding ("maximal") classical Dirichlet form on $L^2(E;\nu)$ by $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ (cf. [AR90a] and Subsection 3.2 below), can we describe the *irreducible* ones among them (i.e., the ones with their associated L^2-semigroups being *ergodic* ?

In Subsection 2.3 we answer question (2) positively and prove among other things that $\nu \in \mathcal{M}^a$ is extreme if and only if it is K_0–*ergodic*, i.e., ergodic w.r.t. the shifts given by K_0 (cf. Theorem 2.10 (i) resp. Definition 2.2).

Under the technical condition (A.3) on $a := (a_{tk})_{k\in K_0, t\in\mathbb{R}}$ (cf. Subsection 2.2) we treat questions (1), (3), and (4) in Subsection 2.4. Condition (A.3) can always be "made" to hold in all applications we are aiming at, by changing a_{tk}, $k \in K_0$, $t \in \mathbb{R}$, correspondingly, but without changing $\mathcal{M}^a$ or at least the *face* thereof (i.e., a convex subset of $\mathcal{M}^a$ with extreme points contained in $\mathcal{M}^a_{ex}$) that we are interested in (cf. Remark 2.6 (iii) and Section 4). In Subsection 2.4 we first identify $\mathcal{M}^a$ as the Gibbs states of a natural specification, explicitly given in terms of the "shift"–Radon–Nikodym densities a_{tk}, $k \in K_0$, $t \in \mathbb{R}$ (sometimes also called *cocycles*, cf. [Roy75] and Remark 2.1 (ii) below). There is an underlying lattice structure given by the "sites" in K_0 (cf. (2.16), (2.20), Proposition 2.13, and Theorem 2.14). Applying a well-known method due to Dynkin–Föllmer (cf. [D78], [Fö75]) we then also obtain positive answers to questions (1) and (3) (see Corollary 2.19 below). The corresponding *Dynkin–Föllmer–Martin boundary* E_a, of course, only depends on $a = (a_{tk})_{k\in K_0, t\in\mathbb{R}}$, and the representation is unique.

Before we discuss question (5) let us emphasize that (at least if $H = E$) a characterization of K_0–ergodic elements in the set $\mathcal{M}^{K_0}$ of all probability measures on E, which are quasi–invariant w.r.t. all directions in K_0, is well-known and beautifully described by A.V. Skorohod in [Sk74, §23]. In [Sk74, §23] also a representation of an arbitrary element in $\mathcal{M}^{K_0}$ in terms of K_0-ergodic ones is derived. However, both results are quite different from ours. First of all in contrast to $\mathcal{M}^a$, the set of extreme points of the convex set $\mathcal{M}^{K_0}$ is always empty, and the representation in [Sk74, §23] is "highly" non-unique (cf. Remark 2.11 (i) and the beginning of Subsection 2.4 below). It is exactly the restriction to all those measures in $\mathcal{M}^{K_0}$ with fixed "shift"–Radon–Nikodym densities a_{tk}, $k \in K_0$, $t \in \mathbb{R}$, (i.e., the restriction to consider $\mathcal{M}^a$) that enables us to construct the said specification and to apply the Dynkin–Föllmer method, as well as to give concrete applications to statistical physics and Euclidean quantum field theory which were impossible to obtain previously.

In Subsection 3.3 we give an answer to question (5) under the positivity condition (A.4) specified in Subsection 3.1 (cf. also Remark 3.1 (ii)). We prove that for $\nu \in \mathcal{M}^a$ the corresponding ("maximal") classical Dirichlet form $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ on $L^2(E;\nu)$ is irreducible or equivalently the associated strongly continuous operator semigroup $(T_t^\nu)_{t>0}$ on $L^2(E;\nu)$ is *ergodic* if and only if $\nu \in \mathcal{M}^a_{ex}$ (cf. Theorem 3.3 and Proposition 3.2).

The equivalence of the irreducibility of $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ to "$\nu \in \mathcal{M}^a$ being K_0–ergodic", is under some stronger assumptions (e.g. the existence of the corresponding *logarithmic derivative* and an a-priori stronger positivity condition) already essentially contained in [AH-K77, Theorem 3.5] (see also [AH-K75a]). Though all our applications in Section 4 could only be obtained by establishing the link to "extremality", i.e., by the equivalence of the irreducibility to "ν being extreme" in $\mathcal{M}^a$, this classical result by R. Høegh-Krohn and one of the authors is an important intermediate step and was in fact the starting point as well as one of the main motivations of the present work.

We emphasize that the smooth bounded cylinder functions $\mathcal{F}C_b^\infty$ on E in general might not be a dense subset of $D(\mathcal{E}_\nu)$ w.r.t. the norm $\mathcal{E}_{\nu,1}^{\frac{1}{2}} := [\mathcal{E}_\nu + (\ ,\ )_{L^2(E;\nu)}]^{\frac{1}{2}}$ (cf. Remark 3.6 (iii)). So, $(T_t^\nu)_{t>0}$ might not describe the transition probabilities of a nice stochastic process (e.g. a diffusion) on E. But if $D(\mathcal{E}_\nu^\circ)$ denotes the closure of $\mathcal{F}C_b^\infty$ w.r.t. $\mathcal{E}_{\nu,1}^{\frac{1}{2}}$, then the semigroup $(\mathring{T}_t^\nu)_{t>0}$ associated to $(\mathcal{E}_\nu, D(\mathcal{E}_\nu^\circ))$ (i.e., the restriction of $\mathcal{E}_\nu$ to $D(\mathcal{E}_\nu^\circ)$) does so (cf. e.g. [MR92, Subsection 4.3]). But since for $\nu \in \mathcal{M}^a_{ex}$ by Theorem 3.3 we know that $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible, hence so is $(\mathcal{E}_\nu, D(\mathcal{E}_\nu^\circ))$; therefore, $(\mathring{T}_t^\nu)_{t>0}$ is ergodic and the corresponding diffusion $\mathbb{M}$ is time ergodic when started with ν. All that is discussed in detail in Subsection 3.4 below. $(\mathring{T}_t^\nu)_{t>0}$ (as well as $\mathbb{M}$) is called the *stochastic dynamics* (or the *stochastic quantization* of ν).

We also discuss the relation with our previous related work [AKR95b] which in certain cases is quite "orthogonal" to the present one. We refer to

Remark 3.6 (ii) where we discuss an example with $E = \mathbb{R}^1$ in detail, and at the same time we show that the positivity condition (A.4) in Theorem 3.3 cannot be dropped.

Two more results in the abstract part are worth mentioning here: the strengthening of the ergodicity (resp. uniqueness) result [BR95, Theorem 6.2] in Remark 3.6 (iv) below; and the representation of the Dirichlet form $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ in terms of irreducible ones (cf. Theorem 3.7). The latter result generalizes [AH-K77, Theorem 3.4] to our setting. Finally, we emphasize that no smoothness assumptions on the "shift"–Radon–Nikodym densities are imposed in the entire abstract part. In particular, $\nu \in \mathcal{M}^a$ might not have a logarithmic derivative (i.e., might not have an integration by parts formula) as they had in [AH-K75a, 77], [BR95] or [AKR95b].

Now let us describe our applications in Section 4. The idea is for a given set $\mathcal{G}$ of *Gibbs measures* (or *states*) of a classical or quantum lattice system in statistical mechanics resp. a continuous system in Euclidean quantum field theory, to identify functions a_{tk}, $k \in K_0$, $t \in \mathbb{R}$, such that $\mathcal{G} = \mathcal{M}^a$ or at least such that $\mathcal{G}$ coincides with a face in $\mathcal{M}^a$. We recall that such Gibbs states in $\mathcal{G}$ are usually given through a specification describing their local characteristics (which except for the case in Subsection 4.1 is entirely different from the one constructed in Subsection 2.4) and that, in general, the convex set $\mathcal{G}$ can contain more than one element. Though describing Gibbs states alternatively in terms of certain "shift"–Radon–Nikodym derivatives can be quite hard in concrete cases, the idea to do so is, of course, not new (cf. e.g. [Roy75, 77] or [Fr77]). But the characterization of the extreme Gibbs states in terms of K_0–ergodicity resp. the irreducibility of the corresponding classical Dirichlet forms $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ resp. the L^2–ergodicity of the associated stochastic dynamics seems to have not been realized before. We would like to emphasize that since $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ "heavily" depends on the Gibbs measure $\nu \in \mathcal{G}$, at first sight one would not believe that one can recognize the extremality of ν in $\mathcal{G}$ from $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$. This a-priori disbelieve was, in fact, the cause of some criticism about the usefulness of the approach to study Gibbs states via Dirichlet forms. The present work shows that this criticism is unjustified. The decisive common underlying entity, linking the various Dirichlet forms $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ for different $\nu \in \mathcal{G}$, is the collection of joint "shift"–Radon–Nikodym densities $(a_{tk})_{k \in K_0, t \in \mathbb{R}}$.

In Subsection 4.1 we show that for a large class of *classical lattice models* with finite–dimensional *linear* spin spaces, the corresponding set of Gibbs measures can be identified as $\mathcal{M}^a$ for certain explicit a_{tk}, $k \in K_0$, $t \in \mathbb{R}$ (cf. (4.3)). (In this special case, however, the latter is more or less known.) Then all our results from the abstract part in Sections 2, 3 apply to give a new characterization of the extreme Gibbs states and new irreducibility results for Dirichlet forms resp. ergodicity properties for the corresponding stochastic dynamics. Since *no temperedness conditions* are imposed, this strictly generalizes the corresponding results in [AKR95b, Section 5]. The latter in turn were an extension of the classical work by R.A. Holley and D.W. Stroock in the case of the Ising model (cf. [HoSt76]) to (non-compact) linear spin spaces (cf also Remark 4.5 (i), (iv) below).

In Subsection 4.2 results analogous to those in Subsection 4.1 are proven for *quantum lattice systems*, i.e., certain systems with *infinite-dimensional linear* spin spaces. In this case also the equality "$\mathcal{G} = \mathcal{M}^a$" (for a_{tk} as in (4.10) – (4.12)) appears to be new. Again no temperedness conditions are required.

In Subsection 4.3 we present our *main* application which lies in *Euclidean quantum field theory*. In this case $\mathcal{G}$ is the set of all Guerra–Rosen–Simon *infinite volume* Gibbs states on $\mathcal{D}'(\mathbb{R}^2)$ with polynomial (self–) interaction (see Remark 4.8 for other types of interactions). We recall all necessary definitions at the beginning of Subsection 4.3. In this case the equality "$\mathcal{G} = \mathcal{M}^a$" (for a_{tk} as in (4.26) – (4.28)) or at least a variant of it (cf. Remark 4.12 (i)) was announced in [S74, Theorem X.14] to be contained in [Fr74, 77]. But unfortunately in neither of these references a complete proof was given and, in fact, no such statement was made explicitly. E.g. only very special elements in $\mathcal{G}$ were proved to be in $\mathcal{M}^a$. This was, however, improved considerably later in [FrS77] where it was shown that every *ultra-regular* (see ibidem for the definition) element in $\mathcal{G}$ belongs to $\mathcal{M}^a$. In this paper we show that the announcement [S74, Theorem X.14] was fully justified. More precisely, we give a complete proof of "$\mathcal{G} = \mathcal{M}^a_{loc,\mu_0}$" where $\mathcal{M}^a_{loc,\mu_0}$ denotes the set of all elements in $\mathcal{M}^a$ which are locally equivalent to the free Euclidean field μ_0. This is sufficient for our purpose since $\mathcal{M}^a_{loc,\mu_0}$ is a face in $\mathcal{M}^a$ (cf. Lemma 4.13). The proof of the corresponding Theorem 4.11 is rather involved and makes full use of the results by one of the authors in [R85, 86]. Again *no* temperedness (or ultra-regularity) conditions are necessary. As a consequence all our results from Sections 2, 3 apply (cf. Theorem 4.14). In particular, the stochastic dynamics/quantization of an *extreme* Guerra–Rosen–Simon Gibbs state is ergodic. This problem has remained unsolved since the work of [J-

LMi85] on the stochastic quantization of *finite volume* $P(\Phi)_2$–measures. In the *finite* volume case ergodicity is, however, well-known (cf. Remark 4.15 (vi) below). It should also be noted that both the specification $(\Pi_{>n})_{n\in\mathbb{N}}$ and the family of kernels $(\Pi_k)_{k\in K_0}$ defined in (2.20), (2.16) resp., provide a completely new description of the Guerra–Rosen–Simon Gibbs states in terms of explicit regular conditional probabilities on $\mathbb{R}^n$, $n\in\mathbb{N}$, resp. $\mathbb{R}$ (cf. Remark 4.15 (i)).

At this point we would like to mention that all approaches to the stochastic quantization of $P(\Phi)_2$–measures ν so far (cf. the references above) only *reconstruct* ν (thus only partly fulfill the programme suggested in [PaW81]), since ν is used in the construction. In fact, in the infinite volume case so far only the Dirichlet form approach was succesful (cf. [AR89 a,b; 91]) and other approaches (e.g. using stochastic differential equations) failed. The reason is that the $P(\Phi)_2$–measure or, more precisely, their logarithmic derivatives are too singular.

In the final Remark 4.15 in part (vii) we also include a discussion about the ergodicity of the stochastic quantization of so-called time–zero fields on $\mathcal{S}'(\mathbb{R}^1)$ (without cut-offs). Here ergodicity follows easily from the well-known result about the existence of a mass gap by J. Glimm and A. Jaffe (cf. [GlJ81]), provided the corresponding space-time field satisfies the global Markov property.

Finally, we would like to stress that in all the applications of Section 4 a representation as in Corollary 2.19 is well-known to hold by the Dynkin–Föllmer method resp. [R86] in the (continuum) $P(\Phi)_2$–case, applied, however, to the defining specification rather than the totally different one, constructed in Subsection 2.4 of this paper (the only subsection where the technical condition (A.3) is, in fact, used).

The results of Subsections 2.3, 2.4 in a preliminary form were written up in [Ki95], a Diploma-thesis supervised by the last named author. All results of this paper were presented in a talk in the Kansai – Seminar (Kyoto), at this particular instance held in Osaka in October 1996. The three authors would like to take the opportunity to express their sincere gratitude for the warm-hearted hospitality of their Japanese hosts at this occasion, in particular M. Fukushima and S. Watanabe.

2 Integral representation and extremality for quasi–invariant measures with given “shift”–Radon–Nikodym densities

2.1 Preliminaries

Let E be a locally convex topological vector space which is Hausdorff and (for simplicity) a *Souslin space* (i.e., the continuous image of a complete separable metric space). Let $\mathcal{B}(E)$ denote its Borel σ–algebra, $\mathcal{B}_b(E)$ the set of all bounded $\mathcal{B}(E)$–measurable functions $f : E \to \mathbb{R}$ and $M_1(E)$ the set of all probability measures on $(E, \mathcal{B}(E))$. Recall that $\mu \in M_1(E)$ is called *k–quasi–invariant*, $k \in E$, if the (“shift”–)Radon–Nikodym densities

$$\tilde{a}^{\mu}_{tk} := \frac{d(\mu \circ \tau_{tk}^{-1})}{d\mu} , \tag{2.1}$$

exist for all $t \in \mathbb{R}$. Here for $k \in E$, $\tau_k(z) := z - k$, $z \in E$. Clearly, $\tilde{a}^{\mu}_{tk} > 0$ μ–a.e. for all $t \in \mathbb{R}$. For $K_0 \subset E$ define $\mathcal{M}^{K_0}$ to be the set of all $\nu \in M_1(E)$ which are k–quasi–invariant for all $k \in K_0$.

Remark 2.1 (i) $\mathcal{M}^{K_0}$ is convex. Indeed, if μ_1, $\mu_2 \in \mathcal{M}^{K_0}$ and $\alpha \in]0, 1[$, then μ_1, μ_2 are both absolutely continuous w.r.t. $\mu := \alpha\ \mu_1 + (1 - \alpha)\mu_2$. Hence for $k \in K_0$, $t \in \mathbb{R}$,

$$\tilde{a}^{\mu}_{tk} = \alpha\ \tilde{a}^{\mu_1}_{tk}\ \frac{d\mu_1}{d\mu} + (1 - \alpha)\ \tilde{a}^{\mu_2}_{tk}\ \frac{d\mu_2}{d\mu}$$

(as is easily seen).

(ii) If K denotes the linear hull of K_0, then it is straightforward to check that

$$\mathcal{M}^{K_0} = \mathcal{M}^{K} ,$$

and that for $\mu \in \mathcal{M}^K$; $k_1, \ldots, k_n \in K$

$$\begin{aligned} \tilde{a}^{\mu}_{k_1+\cdots+k_n}(z) \quad &= \quad \tilde{a}^{\mu}_{k_1}(z) \cdot \tilde{a}^{\mu}_{k_2}(z + k_1) \cdot \tilde{a}^{\mu}_{k_3}(z + k_1 + k_2) \ldots \\ &\quad \cdot \tilde{a}^{\mu}_{k_n}(z + k_1 + \cdots + k_{n-1}) \text{ for } \mu\text{-a.e. } z \in E \end{aligned} \tag{2.2}$$

(which is just the *cocycle property*).

(iii) Let k, $k_n \in E$, $n \in \mathbb{N}$, such that $k_n \underset{n\to\infty}{\longrightarrow} k$ in E and let $\mu \in \mathcal{M}^{\{k_n|n\in\mathbb{N}\}}$ such that $\left\{\tilde{a}^{\mu}_{tk_n}|n \in \mathbb{N}\right\}$ is uniformly μ–integrable for each $t \in \mathbb{R}$. Then obviously $\mu \in \mathcal{M}^{\{k\}}$.

(iv) We note that the set of extreme points of the convex set $\mathcal{M}^{K_0}$ is empty. Indeed, any $\mu \in \mathcal{M}^{K_0}$ is a non–trivial convex combination $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ with μ_1, $\mu_2 \in \mathcal{M}^{K_0}$, $\mu_1 \neq \mu_2$, if

$$\begin{aligned} \mu_1 := &\quad (1_A + \mu(E \setminus A)) \cdot \mu \ , \\ \mu_2 := &\quad (1_{E\setminus A} + \mu(A)) \cdot \mu \ , \end{aligned}$$

where $A \in \mathcal{B}(E)$ such that $\mu(A) \in]0, 1[$. Since μ cannot be a Dirac–measure such a set A exists. Note that μ_1, μ_2 are in fact in $\mathcal{M}^{K_0}$ since they are both equivalent to μ. But $\mu_1 \neq \mu_2$, since $\mu_1(A) = \mu(A)(2 - \mu(A)) \neq \mu(A)^2 = \mu_2(A)$. However, if we define (as in [Sk 74, § 23]) $\mathcal{M}^{K_0}_{ex,s}$ to be the set of all elements in $\mathcal{M}^{K_0}$ which cannot be written as a non–trivial convex combination of two *mutually singular* measures in $\mathcal{M}^{K_0}$, then, as we shall see below, $\mathcal{M}^{K_0}_{ex,s} \neq \emptyset$.

We recall the following standard notions:

Definition 2.2 Let $K_0 \subset E$ and $\nu \in \mathcal{M}^{K_0}$.

(i) A $\mathcal{B}(E)$-measurable function $f : E \to \mathbb{R}$ is called K_0–*invariant* w.r.t. ν if for all $k \in K_0$ and $t \in \mathbb{R}$

$$f(z + tk) = f(z) \text{ for } \nu\text{–a.e. } z \in E \ .$$

(ii) ν is called K_0–*ergodic* if every $f \in \mathcal{B}_b(E)$, which is K_0–invariant w.r.t. ν, is ν–a.e. constant.

Note that the K_0–invariance w.r.t. $\nu \in \mathcal{M}^{K_0}$ of f implies this property for any g which coincides with f ν–a.e. .

Definition 2.3 (cf. [Ku 82]) Let $k \in E$ and $\mu \in \mathcal{M}^{\{k\}}$. Define

$$\sigma^{\mu}_k(A) := \int_{\mathbb{R}} \mu \circ \tau^{-1}_{tk}(A)\, dt \ , \ A \in \mathcal{B}(E) \ .$$

(where dt denotes Lebesgue measure on $\mathbb{R}$).

Let $k \in E$ and for $\mu \in \mathcal{M}^{\{k\}}$. Let $l \in E'$ such that ${}_{E'}\langle l, k\rangle_E = 1$. Then $\sigma_k^\mu \circ l^{-1} = dt$, in particular, σ_k^μ is σ-finite and $\sigma_k^\mu = dt$ if $E = \mathbb{R}^1$. Furthermore, clearly,

$$\sigma_k^\mu \circ \tau_{tk}^{-1} = \sigma_k^\mu \text{ for all } t \in \mathbb{R} \tag{2.3}$$

and σ_k^μ is equivalent to μ, i.e., there exists $\tilde{\rho}_k^\mu : E \to]0, \infty[$ such that

$$\mu = \tilde{\rho}_k^\mu \cdot \sigma_k^\mu \ , \tag{2.4}$$

and $\tilde{\rho}_k^\mu(z) = (\int_{\mathbb{R}} \tilde{a}_{tk}^\mu(z)\, dt)^{-1}$ for μ-a.e. $z \in E$, provided $(t, z) \mapsto \tilde{a}_{tk}^\mu(z)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(E)$-measurable.

For later use we need the following (technical) result:

Proposition 2.4 *Let $k \in E$ and $\mu \in \mathcal{M}^{\{k\}}$.*

(i) Then there exists a $\mathcal{B}(E)$–measurable μ–version ρ_k^μ of $\tilde{\rho}_k^\mu$ such that

$$\rho_k^\mu(z) > 0 \ \textit{and} \ \int_{\mathbb{R}} \rho_k^\mu(z + tk)\, dt = 1 \ \textit{for} \text{ all } z \in E \ . \tag{2.5}$$

In particular,

$$a_{tk}^\mu(z) := \frac{\rho_k^\mu(z + tk)}{\rho_k^\mu(z)} \, , \ z \in E \, , \ t \in \mathbb{R} \ , \tag{2.6}$$

is a μ–version of $\tilde{a}_{tk}^\mu$ for all $t \in \mathbb{R}$, which is jointly measurable in t, z such that

$$\rho_k^\mu(z) = \left(\int_{\mathbb{R}} a_{tk}^\mu(z)\, dt \right)^{-1} \quad \textit{for} \text{ all } z \in E \ . \tag{2.7}$$

(ii) Suppose that for any compact set $C \subset \mathbb{R}$

$$\int_C \tilde{a}_{tk}^\mu(z)^{-1}\, dt < \infty \quad \textit{for } \mu - \textit{a.e.} \ \ z \in E \ .$$

Then ρ_k^μ in (i) can be constructed in such a way that additionally, for any compact set $C \subset \mathbb{R}$

$$\int_C \rho_k^\mu(z + tk)^{-1}\, dt < \infty \quad \textit{for} \text{ all } z \in E \ ,$$

thus

$$\int_C a_{tk}^\mu(z)^{-1}\, dt < \infty \quad \textit{for} \text{ all } z \in E \ ,$$

where a_{tk}^μ is given as in (2.6).

Proof. (i): Define

$$N := \left\{ z \in E \mid \int_{\mathbb{R}} \tilde{\rho}_k^\mu(z+tk)\,dt \neq 1 \right\} .$$

Then $N \in \mathcal{B}(E)$ and

$$\mu(N) = 0 . \tag{2.8}$$

Indeed, for all $f \in \mathcal{B}_b(E)$ by Fubini's theorem and (2.3), (2.4)

$$\begin{aligned} & \int f(z) \int_{\mathbb{R}} \tilde{\rho}_k^\mu(z+tk)\,dt\ \sigma_k^\mu(dz) \\ \underset{(2.3)}{=} \quad & \int_{\mathbb{R}} \int f(z-tk)\ \tilde{\rho}_k^\mu(z)\ \sigma_k^\mu(dz)\,dt \\ \underset{(2.4)}{=} \quad & \int_{\mathbb{R}} \int f(z-tk)\ \mu(dz)\,dt \\ = \quad & \int f(z)\ \sigma_k^\mu(dz) . \end{aligned}$$

Hence, $\sigma_k^\mu(N) = 0$ and thus $\mu(N) = 0$. Furthermore,

$$N + tk = N \text{ for all } t \in \mathbb{R} , \tag{2.9}$$

hence

$$E \setminus N + tk = E \setminus N \text{ for all } t \in \mathbb{R} . \tag{2.10}$$

Let E_k be a closed subspace of E such that

$$E = k \cdot \mathbb{R} \oplus E_k ,$$

hence for $z \in E$, $z = k \cdot s + x$ with unique $s \in \mathbb{R}$, $x \in E_k$. Let $\sigma : \mathbb{R} \to]0,\infty[$ be a fixed probability density (e.g. $\sigma(s) := (2\pi)^{-1/2}\exp(-s^2/2), s \in \mathbb{R}$) and define the $\mathcal{B}(E)$–measurable function

$$\rho_k^\mu(z) := \begin{cases} \tilde{\rho}_k^\mu(z) & , \text{ if } z \in E \setminus N \\ \sigma(s) & , \text{ if } z \in N ,\ z = x + sk \text{ with } x \in E_k ,\ s \in \mathbb{R} . \end{cases}$$

Then $\rho_k^\mu = \tilde{\rho}_k^\mu$ μ–a.e. by (2.8) and because of (2.9), (2.10), ρ_k^μ satisfies (2.5). If $a_{tk}^\mu(z)$ is given by (2.6) it is obviously jointly measurable in t, z and satisfies

(2.7) because of (2.5). Furthermore, for all $t \in \mathbb{R}$ and all $f \in \mathcal{B}_b(E)$ by (2.4), (2.3)

$$\begin{aligned}
\int f\ \tilde{a}^{\mu}_{tk}\, d\mu &= \int f(z - tk)\ \mu\,(dz) \\
&\underset{(2.4)}{=} \int f(z - tk)\ \rho^{\mu}_{k}(z)\ \sigma^{\mu}_{k}\,(dz) \\
&\underset{(2.3)}{=} \int f(z)\ \rho^{\mu}_{k}(z + tk)\ \sigma^{\mu}_{k}\,(dz) \\
&\underset{(2.4)}{=} \int f(z)\ \frac{\rho^{\mu}_{k}(z + tk)}{\rho^{\mu}_{k}(z)}\ \mu\,(dz)\ .
\end{aligned}$$

Hence, $\tilde{a}^{\mu}_{tk}(z) = \frac{\rho^{\mu}_{k}(z+tk)}{\rho^{\mu}_{k}(z)}$ for μ–a.e. $z \in E$. $\square$

(ii): By the preceding calculation we see that also

$$\tilde{a}^{\mu}_{tk}(z) = \frac{\tilde{\rho}^{\mu}_{k}(z + tk)}{\tilde{\rho}^{\mu}_{k}(z)} \quad \text{for } \mu\text{-a.e. } z \in E$$

for all $t \in \mathbb{R}$. Hence if we define

$$N_1 := \{z \in E \mid \int_{-n}^{n} \tilde{\rho}^{\mu}_{k}(z + sk)^{-1} ds = \infty \text{ for some } n \in \mathbb{N}\}\ ,$$

then by assumption, $\mu(N_1) = 0$ and obviously $N_1 + tk = N_1$, hence $E \setminus N_1 + tk = E \setminus N_1$ for all $t \in \mathbb{R}$. Replacing N in the proof of (i) by $N \cap N_1$, the first half of the assertion follows. The second part is then immediate. $\square$

2.2 The "set–up"

Consider the following assumptions:

(A.1) There exists a separable real Hilbert space $(H, \langle\ ,\ \rangle_H)$ densely and continuously embedded into E (hence $E' \subset H' \equiv H \subset E$ densely and continuously, where E' denotes the dual of E and H' is identified with H via the Riesz isomorphism).

(A.2) Let $K_0 \subset E'(\subset H \subset E)$ such that K_0 is an orthonormal basis of H which separates the points of E, and for $k \in K_0$ let $(t, z) \mapsto a_{tk}(z)$ be given $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(E)$-measurable functions taking values in $]0, \infty[$ such that

$$\begin{aligned}
\mathcal{M}^a &:= \{\nu \in M_1(E) \big| \nu \circ \tau_{sk}^{-1} = a_{tk} \cdot \nu \text{ for all } k \in K_0,\ t \in \mathbb{R}\} \\
&\neq \emptyset\ .
\end{aligned}$$

Here $a := (a_{tk})_{k\in K_0,\, t\in\mathbb{R}}$.

(A.3) For a_{tk}, $k \in K_0$, $t \in \mathbb{R}$, as in (A.2) and

$$\rho_k(z) := \Big(\int_{\mathbb{R}} a_{tk}(z)\,dt\Big)^{-1} \quad , \; z \in E \; , \tag{2.11}$$

the following holds for all $k \in K_0$, $z \in E$:

$$0 < \rho_k(z) < \infty, \quad \int_{\mathbb{R}} \rho_k(z+tk)\,dt = 1, \text{ and}$$
$$a_{tk}(z) = \frac{\rho_k(z+tk)}{\rho_k(z)} \quad \text{for all } t \in \mathbb{R}. \tag{2.12}$$

Note that, if (A.1) holds, by [AR91, Lemma 5.6] an orthonormal basis $K_0 \subset E'$ of H separating the points of E always exists, but that $\mathcal{M}^a$ might be empty. Note also that (A.1) implies that the restriction of the dualization ${}_{E'}\langle\ ,\ \rangle_E : E' \times E \to \mathbb{R}$ to $E' \times H$ coincides with $\langle\ ,\ \rangle_H$ and that by (A.2) and [Sch73, Lemma 18, p. 108] the σ–algebra $\sigma(K_0)$ generated by K_0 coincides with $\mathcal{B}(E)$. As a consequence of the latter we immediately obtain:

Lemma 2.5 *Let $f \in \mathcal{B}_b(E)$. Then there exists $\tilde{f} \in \mathcal{B}_b(\mathbb{R}^\infty)$ (where $\mathbb{R}^\infty$ is equipped with the product topology) such that*

$$f(z) = \tilde{f}(({}_{E'}\langle k, z\rangle_E)_{k\in K_0}) \text{ for all } z \in E \; .$$

Remark 2.6 (i) $\mathcal{M}^a$ in (A.2) is obviously convex. Let $\mathcal{M}^a_{ex}$ denote the set of its extreme points.

(ii) Suppose (A.1) – (A.3) hold. Then it is straightforward to check that for all $\nu \in \mathcal{M}^a$, $k \in K_0$,

$$\nu = \rho_k \cdot \sigma_k^\nu$$

with σ_k^ν as in Definition 2.3.

(iii) Suppose that (A.1) holds and let $K_0 \subset E'$ be an orthonormal basis of H separating the points of E such that there exists $\mu \in \mathcal{M}^{K_0}$. Then by Proposition 2.4 (i) we can construct versions a_{tk} from $\tilde{a}^\mu_{tk}$, $k \in K_0$, $t \in \mathbb{R}$, so that (A.2), (A.3) hold. The reader should realize that $\mathcal{M}^a$ then depends on this particular choice of a_{tk} (for the fixed measure μ). But this will cause no problems in the applications in Section 4 below, since all measures in question will be "locally equivalent" to μ and all a_{tk} will be measurable w.r.t. the corresponding "local σ-algebras". Also in case $E = H = \mathbb{R}^d$ this does not impose any restrictions because if K_0 denotes any linear basis of $\mathbb{R}^d$ each measure in $\mathcal{M}^{K_0}$ is equivalent to Lebesgue measure. (The latter follows e.g. as in Lemma 2.17 (iv) below or is easily proved directly).

(iv) Suppose $\rho_k : E \to]0, \infty[$ $\mathcal{B}(E)$-measurable, $k \in K_0$, are given such that $\int_{\mathbb{R}} \rho_k(z + tk)\, dt = 1$ for all $z \in E$, $k \in K_0$, we can define corresponding a_{tk} by (2.12). Then clearly (2.11) holds. We shall encounter this situation in the applications below (cf. Subsection 4.1).

2.3 Characterization of extreme points

In this subsection we assume that (A.1), (A.2) hold. We want to prove various characterizations of $\mathcal{M}^a_{ex}$ and $\mathcal{M}^{K_0}_{ex,s}$. Similarly as in [Sk74, § 23] we introduce the following sub–σ–algebras of $\mathcal{B}(E)$:

$$\mathcal{F}^n := \sigma\{k_i \mid i > n\}\,,\ n \in \mathbb{N};\ \mathcal{F}^\infty := \bigcap_{n \in \mathbb{N}} \mathcal{F}^n\ . \tag{2.13}$$

Furthermore, for a sub–σ–algebra $\mathcal{A}$ of $\mathcal{B}(E)$ and a probability measure ν on $(E, \mathcal{B}(E))$ let $\mathcal{A}^\nu$ denote the smallest σ–algebra containing $\mathcal{A}$ and all ν–zero sets in $\mathcal{B}(E)$. Since $\mathcal{F}^\infty$ is the countable intersection of decreasing σ–algebras, we have

$$(\mathcal{F}^\infty)^\nu = \bigcap_{n \in \mathbb{N}} (\mathcal{F}^n)^\nu\ . \tag{2.14}$$

We shall use this below without further notice.

Lemma 2.7 *Let $\nu \in \mathcal{M}^{K_0}$, $f \in \mathcal{B}_b(E)$. Then f is K_0–invariant w.r.t. ν if and only if f is $(\mathcal{F}^\infty)^\nu$–measurable.*

Proof. Suppose f is $(\mathcal{F}^\infty)^\nu$ measurable. Then by Lemma 2.5 for every $n \in \mathbb{N}$ there exists $g_n \in \mathcal{B}_b(\mathbb{R}^\infty)$ such that

$$f = g_n\left(({}_{E'}\langle k_i, \cdot\rangle_E)_{i>n}\right)\ \nu\text{–a.e.}\quad .$$

Let $m \in \mathbb{N}$. Then for all $n \geq m$, $t \in \mathbb{R}$,

$$f(z + tk_m) = g_n\left(({}_{E'}\langle k_i, z\rangle_E + t\langle k_i, k_m\rangle_H)_{i>n}\right) = f(z)$$

for ν–a.e. $z \in E$, since $\langle k_i, k_m\rangle_H = 0$ for all $i > n$. Hence f is K_0–invariant w.r.t. ν.

Suppose f is K_0–invariant w.r.t. ν. By Lemma 2.5

$$f = g\left(({}_{E'}\langle k_i, \cdot\rangle_E)_{i\in\mathbb{N}}\right) \quad \nu\text{–a.e.}$$

for some $g \in \mathcal{B}_b(\mathbb{R}^\infty)$. By assumption on f for all $t \in \mathbb{R}$ we hence have that for ν–a.e. $z \in E$

$$\begin{aligned} g\left(({}_{E'}\langle k_i, z\rangle_E)_{i\in\mathbb{N}}\right) &= g\left(({}_{E'}\langle k_i, z\rangle_E + t\langle k_i, k_1\rangle_H)_{i\in\mathbb{N}}\right) \\ &= g\left({}_{E'}\langle k_1, z\rangle_E + t\ ,\ {}_{E'}\langle k_2, z\rangle_E\ ,\ {}_{E'}\langle k_3, z\rangle_E\ ,\dots\right)\ . \end{aligned}$$

Consequently, choosing $t := {}_{E'}\langle k_1, z'\rangle_E$, $z' \in E$, we obtain that

$$\begin{aligned} \int \Big|\ & g\left({}_{E'}\langle k_1, z\rangle_E + {}_{E'}\langle k_1, z'\rangle_E\ ,\ {}_{E'}\langle k_2, z\rangle_E\ ,\dots\right) \\ & -g\left({}_{E'}\langle k_1, z\rangle_E\ ,\ {}_{E'}\langle k_2, z\rangle_E\ ,\dots\right)\Big|\ \sigma^\nu_{k_1}(dz) = 0 \end{aligned} \tag{2.15}$$

for all $z' \in E$. But ${}_{E'}\langle k_1, z\rangle_E + {}_{E'}\langle k_1, z'\rangle_E = {}_{E'}\langle k_1, z' + {}_{E'}\langle k_1, z\rangle_E\ k_1\rangle_E$. Hence integrating (2.15) over z' w.r.t. $\sigma^\nu_{k_1}$ and using that $\sigma^\nu_{k_1}$ is invariant under τ_{sk_1} for all $s \in \mathbb{R}$ we get by Fubini's theorem that

$$\begin{aligned} 0 \ = \ \int\!\!\int & |g\left({}_{E'}\langle k_1, z'\rangle_E\ ,\ {}_{E'}\langle k_2, z\rangle_E\ ,\ {}_{E'}\langle k_3, z\rangle_E\ ,\dots\right) \\ & -g\left({}_{E'}\langle k_1, z\rangle_E\ ,\ {}_{E'}\langle k_2, z\rangle_E\ ,\dots\right)|\ \sigma^\nu_{k_1}(dz')\ \sigma^\nu_{k_1}(dz)\ . \end{aligned}$$

Again by Fubini's theorem we conclude that there exists $z' \in E$ such that for ν–a.e. $z \in E$

$$g\left({}_{E'}\langle k_1, z\rangle_E\ ,\ {}_{E'}\langle k_2, z\rangle_E\ ,\dots\right) = g\left({}_{E'}\langle k_1, z'\rangle_E\ ,\ {}_{E'}\langle k_2, z\rangle_E\ ,\ {}_{E'}\langle k_3, z\rangle_E\ ,\dots\right)\ .$$

Therefore, f is $(\mathcal{F}^2)^\nu$–measurable. Iterating (by induction) we obtain that f is $\bigcap_{n\in\mathbb{N}}(\mathcal{F}^n)^\nu = (\mathcal{F}^\infty)^\nu$–measurable. $\square$

As an immediate consequence of Lemma 2.7 we obtain the following two lemmas.

Lemma 2.8 *Let $\nu \in \mathcal{M}^{K_0}$. Then ν is K_0–ergodic if and only if ν is trivial on $\mathcal{F}^\infty$ (i.e., $\nu(A) \in \{0,1\}$ for all $A \in \mathcal{F}^\infty$).*

Proof. Suppose ν is K_0–ergodic and let $A \in \mathcal{F}^\infty$. Then 1_A is K_0–invariant w.r.t. ν by Lemma 2.7. Hence by assumption on ν, 1_A is constant ν–a.e., hence $\nu(A) \in \{0,1\}$.

Suppose ν is trivial on $\mathcal{F}^\infty$ and let $f \in \mathcal{B}_\lfloor(\mathcal{E})$ be K_0–invariant w.r.t. ν. By Lemma 2.7 it follows that f is $(\mathcal{F}^\infty)^\nu$–measurable. Hence by assumption on ν, f is constant ν–a.e. Consequently, ν is K_0–ergodic w.r.t. ν. $\square$

Lemma 2.9 *Let $\nu \in \mathcal{M}^{K_0}$ and $g \in \mathcal{B}_b(E)$ such that $\int g\, d\nu = 1$.*

(i) $g \cdot \nu \in \mathcal{M}^{K_0}$ if g is $(\mathcal{F}^\infty)^\nu$–measurable.

(ii) If $\nu \in \mathcal{M}^a$, then $g \cdot \nu \in \mathcal{M}^a$, if and only if g is $(\mathcal{F}^\infty)^\nu$–measurable.

Proof. Suppose that $g \in (\mathcal{F}^\infty)^\nu$–measurable. Then by Lemma 2.7 g is K_0–invariant, hence for all $f \in \mathcal{B}_\lfloor(\mathcal{E})$, $t \in \mathbb{R}$, $k \in K_0$,

$$\begin{aligned} \int f(z - tk)\, g(z)\, \nu(dz) &= \int f(z - tk)\, g(z - tk)\, \nu(dz) \\ &= \int f(z)\, g(z)\, \tilde{a}^\nu_{tk}(z)\, \nu(dz)\ . \end{aligned}$$

Consequently, $g \cdot \nu \in \mathcal{M}^{K_0}$ and, if $\nu \in \mathcal{M}^a$, replacing $\tilde{a}^\nu_{tk}$ by a_{tk} in the last equality we see that also $g \cdot \nu \in \mathcal{M}^a$. This proves (i) and one half of (ii).

Suppose $g \cdot \nu \in \mathcal{M}^a$. Let $f \in \mathcal{B}_b(E)$, $t \in \mathbb{R}$, $k \in K_0$. Then

$$\begin{aligned} \int f(z)\, g(z + tk)\, a_{tk}(z)\, \nu(dz) &= \int f(z - tk)\, g(z)\, \nu(dz) \\ &= \int f(z)\, a_{tk}(z)\, g(z)\, \nu(dz)\ . \end{aligned}$$

Since $a_{tk} > 0$, we conclude that $g(z + tk) = g(z)$ for ν–a.e. $z \in E$. Thus g is K_0–invariant w.r.t. ν, hence $(\mathcal{F}^\infty)^\nu$–measurable by Lemma 2.7 and the remaining part of (ii) is proven. $\square$

As a simple consequence of Lemmas 2.7 – 2.9 we obtain the following characterization of $\mathcal{M}^{K_0}_{ex,s}$ resp. $\mathcal{M}^a_{ex}$.

Theorem 2.10 *(i) Let $\nu \in \mathcal{M}^a$. Then $\nu \in \mathcal{M}^a_{ex}$ if and only if ν is K_0–ergodic.*

(ii) Let $\nu \in \mathcal{M}^{K_0}$. Then $\nu \in \mathcal{M}^{K_0}_{ex,s}$ if and only if ν is K_0–ergodic.

Proof. (ii): Suppose that ν is not K_0–ergodic. Then ν is not trivial on $(\mathcal{F}^\infty)^\nu$ by Lemma 2.8. Hence there exists $A \in \mathcal{F}^\infty$ such that $0 < \nu(A) < 1$. Let $\alpha \in]0,1[$ and define $\nu_1 := \alpha^{-1}\, 1_A \cdot \nu$; $\nu_2 := (1-\alpha)^{-1} 1_{E\setminus A} \cdot \nu$. Then ν_1, ν_2 are mutually singular, $\nu = \alpha\, \nu_1 + (1-\alpha)\, \nu_2$, and $\nu_1, \nu_2 \in \mathcal{M}^{K_0}$ by Lemma 2.9. Hence $\nu \notin \mathcal{M}^{K_0}_{ex,s}$.

Suppose ν is K_0–ergodic and let $\nu = \alpha\nu_1 + (1-\alpha)\nu_2$ for some mutually singular ν_1, $\nu_2 \in \mathcal{M}^{K_0}$, $\alpha \in [0,1]$. Let $A \in \mathcal{B}(E)$ such that $\nu_1(A) = 1$, $\nu_2(A) = 0$. Then for all $k \in K_0$, $t \in \mathbb{R}$,

$$\int 1_A(z - tk)\, \nu_i(dz) = \begin{cases} 1 \text{ if } i = 1\ , \\ 0 \text{ if } i = 2\ . \end{cases}$$

Hence A is K_0–invariant w.r.t. ν_1 and ν_2, thus w.r.t. ν. By assumption and Lemma 2.8 it follows that $\nu(A) \in \{0,1\}$, hence $\alpha \in \{0,1\}$, i.e., $\nu \in \mathcal{M}_{ex,s}^{K_0}$.

(i): Let $\nu \in \mathcal{M}_{ex}^a$. Then it follows that ν is K_0–ergodic by the same arguments as in the first half of the proof of (i) because ν_1, $\nu_2 \in \mathcal{M}^a$ by the last part of Lemma 2.9.

Suppose now that ν is K_0–ergodic and let $\nu_1, \nu_2 \in \mathcal{M}^a$, $\alpha \in]0,1[$ such that $\nu = \alpha\, \nu_1 + (1-\alpha)\, \nu_2$. Then the Radon–Nikodym density $g := \frac{d\nu_1}{d\nu}$ exists and $g \in (\mathcal{F}^\infty)^\nu$ by Lemma 2.9. Hence g is K_0–invariant w.r.t. ν by Lemma 2.7. By assumption on ν, g must then be constant, i.e., equal to 1, ν–a.e.. Consequently, $\nu = \nu_1$, thus $\nu = \nu_2$ as well. Hence $\nu \in \mathcal{M}_{ex}^a$. $\square$

Remark 2.11 *(i)* Theorem 2.10 (ii) and Lemmas 2.7, 2.8 in the special case, where $H = E$, are well-known. They were first proven in [Sk74, § 23] by slightly different techniques. We emphasize, however, that of course $\mathcal{M}^a$ and $\mathcal{M}_{ex}^a$ are in general *much smaller* than $\mathcal{M}^{K_0}$, $\mathcal{M}_{ex,s}^{K_0}$ respectively. Therefore, in view of part (ii), Theorem 2.10 (i) is quite unexpected. It is this latter part which we shall solely use below and which is crucial for this paper.

(ii) Suppose $E = H = \mathbb{R}^d$, and let K_0 be a linear basis of $\mathbb{R}^d$. Then by Remark 2.6 (iii) any $\nu \in \mathcal{M}^a$ is equivalent to Lebesgue measure, hence K_0-ergodic. Consequently, by Theorem 2.10 (i) in this case $\mathcal{M}^a = \mathcal{M}_{ex}^a$ and thus $\#\mathcal{M}^a = 1$.

2.4 Existence of extreme points and the representation theorem

Assume (A.1) – (A.3) hold. In this subsection we prove that any $\nu \in \mathcal{M}^a$ can be uniquely represented as an integral in terms of its extreme points. A representation of $\mathcal{M}^{K_0}$ in terms of $\mathcal{M}_{ex,s}^{K_0}$ was proved by A.V. Skorohod in [Sk 74, § 23]. However, Skorohod's representations are ("highly") non–unique (as follows from Corollary 2.19 below). Our method of proof is entirely different from [Sk74, § 23] and based on the construction of a (kind of) *local specification* in the sense of [P76] such that $\mathcal{M}^a$ coincides with the corresponding set of Gibbs measures. Then we can use a general representation result due to E.B. Dynkin and H. Föllmer ([Dy78], [Fö75], see, in particular, the excellent exposition by C. Preston in [P76, Proof of Theorem 2.2]).

Define for $k \in K_0$

$$P_k(z) := {}_{E'}\langle k, z\rangle_E \; k\,,\ z \in E\,,\ \text{and}\ \ \pi_k := id_E - P_k\ .$$

Clearly, $\pi_k(E)$ is a closed subspace of E and $E = k \cdot \mathbb{R} \oplus \pi_k(E)$. Define for $k \in K_0$ the probability kernel

$$\Pi_k(z,f) := \int_{\mathbb{R}} f(\pi_k(z)+sk)\ \rho_k(\pi_k(z)+sk)\, ds\ ,\ \ z \in E \tag{2.16}$$

(where ρ_k are as in (A.3)). As usual we denote the conditional expectation of $\nu \in M_1(E)$ w.r.t. a σ–algebra $\Sigma \subset \mathcal{B}(E)$ by $E_\nu[\cdot \,|\, \Sigma]$.

Remark 2.12 Let $k \in K_0$. Note that we have for the σ–algebra $\sigma(\pi_k)$ generated by $\pi_k : E \to E$ that

$$\sigma(K_0 \setminus \{k\}) = \sigma(\pi_k)\ . \tag{2.17}$$

Indeed, obviously $\sigma(K_0 \setminus \{k\}) \subset \sigma(\pi_k)$. But also the dual inclusion holds by [Sch73, Lemma 18, p. 108], since $\pi_k(E)$ as a closed subspace of E is also a Souslin space and $K_0 \setminus \{k\}$ separates the points of $\pi_k(E)$. In particular, (2.17) implies that $\Pi_k(\cdot,f)$ is $\sigma(K_0 \setminus \{k\})$–measurable for all $f \in \mathcal{B}_b(E)$ and $\Pi_k(\cdot,gf) = g\Pi_k(\cdot,f)$ provided g is $\sigma(K_0 \setminus \{k\})$–measurable. Hence, if ν is a probability measure on $(E,\mathcal{B}(E))$, then $E_\nu[f \mid \sigma(K_0 \setminus \{k\})] = \Pi_k(\cdot,f)$ ν–a.e. for all $f \in \mathcal{B}_b(E)$ if and only if $\nu\, \Pi_k = \nu$.

Proposition 2.13 *Let $\nu \in M_1(E)$. Then the following assertions are equivalent:*

(i) $\nu \in \mathcal{M}^a$.

(ii) $E_\nu[f \,|\, \sigma(K_0 \setminus \{k\})] = \Pi_k(\cdot,f)$ ν-a.e. for all $f \in \mathcal{B}_b(E)$, $k \in K_0$.

(iii) $\nu\ \Pi_k = \nu$ for all $k \in K_0$.

Proof. Suppose (i) holds and let $k \in K_0$; $f,g \in \mathcal{B}_b(E)$. Then by Remark 2.6 (ii) and Fubini's theorem

$$\begin{aligned}\int g\circ\pi_k\ f\, d\nu &= \int g\circ\pi_k\ f\ \rho_k\ d\sigma_k^\nu \\ &= \int_E\int_{\mathbb{R}} (g\circ\pi_k)(z+tk)\ f(z+tk)\ \rho_k(z+tk)\, dt\ \nu(dz)\ .\end{aligned}$$

Since $\pi_k(z+tk) = \pi_k(z)$ and $z+tk = \pi_k(z)+(t+{}_{E'}\langle k,z\rangle_E)k$, by translation–invariance of dt the latter is equal to

$$\begin{aligned}&\int_E (g\circ\pi_k)(z)\int_{\mathbb{R}} f(\pi_k(z)+tk)\ \rho_k(\pi_k(z)+tk)\, ds\ \nu(dz) \\ =\ &\int_E g\circ\pi_k\ \Pi_k(\cdot,f)\, d\nu\ .\end{aligned}$$

Hence (ii) follows by the first part of Remark 2.12.

We already know that (ii) $\Leftrightarrow$ (iii) by the last part of Remark 2.12.

Suppose (iii) holds and let $k \in K_0$, $t \in \mathbb{R}$, $f \in \mathcal{B}_b(E)$. Then by (ii) and translation–invariance of dt

$$\int f(z-tk)\,\nu\,(dz) = \int_E \int_{\mathbb{R}} f(\pi_k(z)+sk)\;\rho_k(\pi_k(z)+(s+t)k)\,ds\;\nu\,(dz)\ .$$

By (A.3) the latter is equal to

$$\int \Pi_k(\cdot, f\ a_{tk})\,d\nu = \int f\ a_{tk}\,d\nu\ .$$

Hence $\nu \in \mathcal{M}^a$. $\square$

Though Proposition 2.13 is a useful description of $\mathcal{M}^a$ it is not enough for our purpose. We need Theorem 2.14 below in order to be able to apply the above mentioned general result of Dynkin/Föllmer to obtain the desired representation (as well as $\mathcal{M}^a_{ex} \neq \emptyset$) as an immediate consequence (see Corollary 2.19 below). Theorem 2.14 is of its own interest (cf. Remark 2.18 below) and the first main result of this subsection.

Let $K_0 = \{k_m \mid m \in \mathbb{N}\}$ and $n \in \mathbb{N}$. Let

$$P_{\leq n}(z) := \sum_{i=1}^{n}\ {}_{E'}\langle k_i, z\rangle_E\ k_i,\ \ z\in E\,,\ \text{ and }\ \pi_{>n} := id_E - P_{\leq n}\ .$$

Then $E = P_{\leq n}(E) \oplus \pi_{>n}(E)$ and $\pi_{>n}(E)$ is a closed subspace of E. For $t_1, \ldots, t_n \in \mathbb{R}$ we set $\underline{t}\,\underline{k}_n = t_1k_1 + \cdots + t_nk_n$ and $\underline{k}_n := (k_1, \ldots, k_n)$. Then for $z \in E$ define

$$\tilde{\rho}_{\underline{k}_n}(z) := \rho_{k_n}(z)\left(\int_{\mathbb{R}^{n-1}} \Pi_{i=2}^{n}\ \frac{\rho_{k_i}(z+\sum\limits_{j=i}^{n} s_jk_j)}{\rho_{k_{i-1}}(z+\sum\limits_{j=i}^{n} s_jk_j)}\,ds_2\ldots ds_n\right)^{-1}\ ,$$

$$N_n := \left\{z\in E \mid \int \tilde{\rho}_{\underline{k}_n}(z+s_1k_1+\cdots+s_nk_n)\,ds_1\ldots ds_n \neq 1\right\}\ .$$

Then (as in (2.9), (2.10)) for all $t_1, \ldots, t_n \in \mathbb{R}$

$$N_n + \underline{t}\,\underline{k}_n = N_n\,,\ \text{ and }\ E\setminus N_n + \underline{t}\,\underline{k}_n = E\setminus N_n\ . \tag{2.18}$$

Define

$$\rho_{\underline{k}_n}(z) := \begin{cases} \tilde{\rho}_{\underline{k}_n}(z)\,, & \text{if } z \in E \setminus N_n \\ \rho_{k_n}(z)(2\pi)^{-\frac{n-1}{2}}\, e^{-\frac{1}{2}\sum\limits_{i=1}^{n-1} {}_{E'}\langle k_i, z\rangle_E^2}\,, & \text{if } z \in N_n \end{cases} \tag{2.19}$$

and for $f \in \mathcal{B}_b(E)$, $z \in E$

$$\Pi_{>n}(z, f) := \int f(\pi_{>n}(z) + \underline{s}\,\underline{k}_n)\ \rho_{\underline{k}_n}(\pi_{>n}(z) + \underline{s}\,\underline{k}_n)\, ds_1 \ldots ds_n\ . \tag{2.20}$$

Theorem 2.14 *Let $\nu \in M_1(E)$. Then the following assertions are equivalent:*

(i) $\nu \in \mathcal{M}^a$.

(ii) For all $n \in \mathbb{N}$, $E_\nu[f \mid \mathcal{F}^n] = \Pi_{>n}(\cdot, f)$ ν–a.e. for all $f \in \mathcal{B}_b(E)$.

(iii) $\nu\Pi_{>m} = \nu$ for all $m \in \mathbb{N}$.

For the proof of Theorem 2.14 we need some preparations. Fix $n \in \mathbb{N}$.

Remark 2.15 (i) By definition of $\rho_{\underline{k}_n}$, (A.3), and (2.18), $\Pi_{>n}$ is a probability kernel on $(E, \mathcal{B}(E))$.

(ii) By exactly the same arguments as in Remark 2.12 one sees that for all $n \in \mathbb{N}$

$$\mathcal{F}^n = \sigma(\pi_{>n})\ . \tag{2.21}$$

Hence for every $f \in \mathcal{B}_b(E)$, $\Pi_{>n}(\cdot, f)$ is $\mathcal{F}^n$–measurable (see (2.13)) and

$$\Pi_{>n}(\cdot, gf) = g\ \Pi_{>n}(\cdot, f) \text{ for all } g \in \mathcal{B}_b(E) \text{ which are } \mathcal{F}^n\text{-measurable}\ . \tag{2.22}$$

Proof of Theorem 2.14, (ii) $\Leftrightarrow$ (iii): This is an immediate consequence of Remark 2.15 (ii). $\square$.

Lemma 2.16 *Let $n \in \mathbb{N}$. Then*

$$\Pi_{>n}\ \Pi_{k_n} = \Pi_{>n}\ . \tag{2.23}$$

Proof. We first note that by definition of $\rho_{\underline{k}_n}$, (2.18), and (A.3) we have that for all $z \in E$, $s_1, \dots, s_{n-1}, s \in \mathbb{R}$,

$$\begin{aligned}
&\int \rho_{\underline{k}_n}\left(\pi_{>n}(z) + s_1 k_1 + \cdots + s_n k_n\right) ds_n \cdot \rho_{k_n}\left(\pi_{>n}(z) + s_1 k_1 + \cdots + s_{n-1} k_{n-1} + s k_n\right) \\
= \; & \rho_{\underline{k}_n}(\pi_{>n}(z) + s_1 k_1 + \cdots + s_{n-1} k_{n-1} + s k_n) \;, \qquad (2.24)
\end{aligned}$$

as an elementary calculation shows.

Let now $f \in \mathcal{B}_b(E)$. Then

$$\begin{aligned}
& \Pi_{>n}\left(z, \Pi_{k_n}(\cdot, f)\right) = \; \Pi_{>n}\left(z, \int_{\mathbb{R}} f(\pi_{k_n}(\cdot) + s k_n)\; \rho_{k_n}(\pi_{k_n}(\cdot) + s k_n)\, ds\right) \\
= \; & \int_{\mathbb{R}^n} \int_{\mathbb{R}} f\left(\pi_{>n}(z) + s_1 k_1 + \cdots + s_{n-1} k_{n-1} + s k_n\right) \\
& \qquad \rho_{k_n}(\pi_{>n}(z) + s_1 k_1 + \cdots + s_{n-1} k_{n-1} + s k_n) ds \\
& \qquad\qquad \rho_{\underline{k}_n}(\pi_{>n}(z) + s_1 k_1 + \cdots + s_n k_n)\, ds_1 \dots ds_n \\
= \; & \Pi_{>n}(z, f)
\end{aligned}$$

where the last equality holds because of Fubini's theorem and (2.24). $\square$

Proof of Theorem 2.14, (iii) $\Rightarrow$ (i): Suppose (iii) holds. Then by Lemma 2.16 for all $n \in \mathbb{N}$

$$\nu\; \Pi_{k_n} = \nu\; \Pi_{>n}\; \Pi_{k_n} = \nu\; \Pi_{>n} = \nu \;.$$

Hence (i) follows by Proposition 2.13.

$\square$

For $s_1, \dots, s_n \in \mathbb{R}$, $z \in E$ we define

$$a_{\underline{s}\,\underline{k}_n}(z) := \Pi_{i=1}^n a_{s_i k_i}(z + \sum_{j=i+1}^n s_j k_j) \;. \qquad (2.25)$$

Let now $\nu \in \mathcal{M}^a$ and fix $n \in \mathbb{N}$. We define the multidimensional analogue $\sigma^\nu_{\underline{k}_n}$ of σ^ν_k (cf. Definition 2.3) by

$$\sigma^\nu_{\underline{k}_n}(A) := \int_{\mathbb{R}^n} \nu \circ \tau^{-1}_{\underline{s}\,\underline{k}_n}\; ds_1 \dots ds_n \;,\; A \in \mathcal{B}(E) \qquad (2.26)$$

(see also [He72]). Clearly, $\sigma^\nu_{\underline{k}_n} \circ P^{-1}_{\leq n}$ is just Lebesgue measure on $\mathbb{R}^n$, hence $\sigma_{\underline{k}_n}$ is σ-finite and equal to Lebesgue measure if $E = \mathbb{R}^n$. Furthermore,

$$\sigma^\nu_{\underline{k}_n} \circ \tau^{-1}_{\underline{s}\,\underline{k}_n} = \sigma^\nu_{\underline{k}_n} \;. \qquad (2.27)$$

Lemma 2.17 *Let $n \in \mathbb{N}$, $\nu \in \mathcal{M}^a$. Then:*

(i) $\nu(dz) = \left(\int_{\mathbb{R}^n} a_{\underline{s}\,\underline{k}_n}(z)\, ds_1 \dots ds_n\right)^{-1} \sigma^{\nu}_{\underline{k}_n}(dz).$

(ii) $\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} a_{\underline{s}\,\underline{k}_n}(z + \underline{s'}\,\underline{k}_n)\, ds_1 \dots ds_n\right)^{-1} ds'_1 \dots ds'_n = 1$ *for ν–a.e. $z \in E$.*

(iii) $\left(\int_{\mathbb{R}^n} a_{\underline{s}\,\underline{k}_n}(z)\, ds_1 \dots ds_n\right)^{-1} = \tilde{\rho}_{\underline{k}_n}(z)$ *for all $z \in E$.*

(iv) $\nu(N_n) = 0$. *In particular,* $\nu = \rho_{\underline{k}_n} \cdot \sigma^{\nu}_{\underline{k}_n}$.

Proof. (i): Let $f \in \mathcal{B}_b(E)$, $f \geq 0$. Then by Fubini's theorem

$$
\begin{aligned}
\int f \, d\sigma^{\nu}_{\underline{k}_n} &= \int_{\mathbb{R}^n} \int f(z - \underline{s}\,\underline{k}_n)\; \nu(dz)\, ds_1 \dots ds_n \\
&= \int_{\mathbb{R}^n} \int f(z - s_2 k_2 - \dots - s_n k_n)\, a_{s_1 k_1}(z)\, \nu(dz)\, ds_1 \dots ds_n \\
&\vdots \\
&= \int_{\mathbb{R}^n} \int f(z)\; a_{\underline{s}\,\underline{k}_n}(z)\; \nu(dz)\, ds_1 \dots ds_n \; .
\end{aligned}
$$

Since by (2.25) $(s_1, \dots, s_n, z) \mapsto a_{\underline{s}\,\underline{k}_n}(z)$ is $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(E)$ measurable and strictly positive, assertion (i) follows.

(ii): Let $f \in \mathcal{B}_b(E)$. Then by (i)

$$
\begin{aligned}
\int f \, d\sigma^{\nu}_{\underline{k}_n} &= \int_{\mathbb{R}^n} f(z - \underline{s'}\,\underline{k}_n)\; \nu(dz)\, ds'_1 \dots ds'_n \\
&\underset{(i)}{=} \int_{\mathbb{R}^n} \int f(z - \underline{s'}\,\underline{k}_n)\; \left(\int_{\mathbb{R}^n} a_{\underline{s}\,\underline{k}_n}(z)\, ds_1 \dots ds_n\right)^{-1} \; \sigma^{\nu}_{\underline{k}_n}(dz)\, ds'_1 \dots ds'_n \; .
\end{aligned}
$$

By (2.27) the latter is equal to

$$
\int f(z) \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} a_{\underline{s}\,\underline{k}_n}(z + \underline{s'}\,\underline{k}_n)\, ds_1 \dots ds_n\right)^{-1} ds'_1 \dots ds'_n \; \sigma^{\nu}_{\underline{k}_n}(dz)
$$

and (ii) is proven.

(iii): Let $z \in E$. Then by (A.3)

$$
\begin{aligned}
\tilde{\rho}_{\underline{k}_n}(z) &= \rho_{k_n}(z) \left(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \Pi_{i=2}^n \; \frac{\rho_{k_i}(z + \sum_{j=i}^n s_j k_j)}{\rho_{k_{i-1}}(z + \sum_{j=i}^n s_j k_j)} \; \rho_{k_1}(z + \sum_{j=2}^n s_j k_j + s_1 k_1)\; ds_1 \dots ds_n \right)^{-1} \\
&= \left(\int_{\mathbb{R}^n} \Pi_{i=1}^n \frac{\rho_{k_i}(z + \sum_{j=i}^n s_j k_j)}{\rho_{k_i}(z + \sum_{j=i+1}^n s_j k_j)} \; ds_1 \dots ds_n \right)^{-1}
\end{aligned}
$$

and by (2.12) this implies assertion (iii).

(iv): By (ii) and (iii) it follows that $\nu(N_n) = 0$ and the last part of assertion (iv) is then a consequence of (i) and (iii). □

Proof of Theorem 2.14, (i) $\Rightarrow$ (iii) Let $\nu \in \mathcal{M}^a$, $n \in \mathbb{N}$, and $f \in \mathcal{B}_b(E)$. Then

$$
\begin{aligned}
& \int \Pi_{>n}(z, f)\ \nu(dz) \\
= \ & \int_{\mathbb{R}^n} \int f(\pi_{>n}(z) + \underline{s}\,\underline{k}_n)\ \rho_{\underline{k}_n}(\pi_{>n}(z) + \underline{s}\,\underline{k}_n)\, ds_1 \ldots ds_n\ \nu(dz) \\
= \ & \int \int_{\mathbb{R}^n} (f \cdot \rho_{\underline{k}_n})(\pi_{>n}(z) + ({}_{E'}\langle k_1, z\rangle_E + s_1)k_1 + \cdots + ({}_{E'}\langle k_n, z\rangle_E + s_n)k_n)\, ds_1 \ldots ds_n\ \nu(dz) \\
= \ & \int \int_{\mathbb{R}^n} (f \cdot \rho_{\underline{k}_n})(z + \underline{s}\,\underline{k}_n)\, ds_1 \ldots ds_n\ \nu(dz) \\
= \ & \int f\ \rho_{\underline{k}_n}\ d\sigma^{\nu}_{\underline{k}_n}\ .
\end{aligned}
$$

By Lemma 2.17 (iv) the latter is equal to

$$\int f\, d\nu$$

and (iii) is proven. □

Remark 2.18 By Remark 2.15 the family $(\Pi_{>n})_{n\in\mathbb{N}}$ of probability kernels has all properties of a *specification* in the sense of [P76] w.r.t. to the σ–algebras $\mathcal{F}^n$ (cf. Subsection 4.1 below), except for the so–called consistency condition, i.e.,

$$\Pi_{<n}\ \Pi_{<m} = \Pi_{<n}\ . \tag{2.28}$$

might *not* hold for all $n > m$. Using the terminology of [P76] nevertheless, Theorem 2.14 just says that $\mathcal{M}^a$ is exactly the set of all *Gibbs states* of $(\Pi_{<n})_{n\in\mathbb{N}}$. The above construction can be modified so that (2.28) holds, but the corresponding measures $\Pi_{<n}(z, \cdot)$, $n \in \mathbb{N}$, would no longer have total mass equal to one.

The second main result of this subsection (i.e., the representation theorem) is now an immediate consequence.

Corollary 2.19 *Assume that $(E, \mathcal{B}(E))$ is a standard measurable space (which is e.g. the case if E is a* Lusin space*, i.e., the continuous one-to-one image of a complete separable metric space). There exists a probability kernel $\Pi : E \times \mathcal{B}(E) \to \mathbb{R}$ having the following properties:*

(i) $\Pi(\cdot, f)$ *is $\mathcal{F}^\infty$–measurable.*

(ii) For all $z \in E$, $\Pi(z, \cdot) \in \mathcal{M}^a_{ex}$. In particular, $\mathcal{M}^a_{ex} \neq \emptyset$.

(iii) For all $\nu \in \mathcal{M}^a$

$$E_\nu[f \mid \mathcal{F}^\infty] = \Pi(\cdot, f) \ \nu\text{–a.e. for all } f \in \mathcal{B}_b(E) \ .$$

(iv) If for $z \in E$

$$\Delta(z) := \{y \in E \mid \Pi(z, \cdot) = \Pi(y, \cdot)\} \ ,$$

then $\Pi(z, \Delta(z)) = 1$.

(v) For all $\nu \in M_1(E)$, $\nu\Pi = \nu$ if and only if $\nu \in \mathcal{M}^a$.

(vi) Let $E_a := E/\Delta$ be the quotient space given by the map $\Delta : E \to \mathcal{B}(E)$ defined in (iv) equipped with the corresponding σ–algebra $\mathcal{B}(E_a)$ induced by $\mathcal{B}(E)$. Then for all $\nu \in \mathcal{M}^a$ there exists a unique probability measure m_ν on $(E_a, \mathcal{B}(E_a))$ such that

$$\nu = \int_{E_a} \nu_x \ m_\nu(dx) \, ,$$

where for $x \in E_a$, $\nu_x := \Pi(z, \cdot)$ for $z \in \Delta^{-1}(x)$ (thus $\nu_x \in \mathcal{M}^a_{ex}$ for all $x \in E_a$) and $m_\nu := \nu \circ \Delta^{-1}$.

Proof. By Theorem 2.14 the proof is exactly the same as that of Theorem 2.2 in [P76] (i.e., that of the Dynkin/Föllmer representation). We emphasize that the consistency condition (cf. (2.28)) assumed in [P76], but which may not hold in our case, is not used in the proof once one has reduced the situation to countably many kernels which is a-priori the case here. □

Remark 2.20 (i) Since $\mathcal{M}^a_{ex} \neq \emptyset$, also $\mathcal{M}^{K_0}_{ex,s} \neq \emptyset$ by Theorem 2.10.

(ii) E_a is called the *Dynkin–Föllmer–Martin boundary* of $(\Pi_{>n})_{n \in \mathbb{N}}$.

(iii) In the particular case $E := C(\mathbb{R})$ and for particular a_{tk}, $k \in K_0$, $t \in \mathbb{R}$, Corollary 2.19 has been proved in [RoyYo76] based on Choquet theory. In this paper also a characterization of the elements in $\mathcal{M}^a_{ex}$ in terms of the Markov property is given.

3 Irreducibility of Dirichlet forms, time ergodicity of diffusions, and extremal measures

3.1 A positivity condition

Assume that conditions (A.1) and (A.2) of Subsection 2.2 hold. We additionally introduce the following condition:
(A.4) For all $k \in K_0$ and any compact set $C \subset \mathbb{R}$

$$\int_C a_{sk}(z)^{-1} ds < \infty \qquad \text{for all } z \in E$$

(where K_0, a_{sk} are as in (A.2).)

Remark 3.1 (i) Suppose that conditions (A.1), (A.2), and (A.4) hold. Then for all $\nu \in \mathcal{M}^a$, $k \in K_0$, and any compact $C \subset \mathbb{R}$

$$\int_C \tilde{\rho}_k^{\nu}(z + sk)^{-1} ds < \infty \qquad \text{for } \nu\text{-a.e. } z \in E \tag{3.1}$$

(cf. the proof of Proposition 2.4 (ii)). Furthermore, if (A.3) holds then (A.4) is equivalent to
(A.4)' For all $k \in K_0$ and any compact set $C \subset \mathbb{R}$

$$\int_C \rho_k(z + sk)^{-1} ds < \infty \qquad \text{for all } z \in E \tag{3.2}$$

where ρ_k are as in (A.3).

(ii) Suppose that (A.1) holds and let $K_0 \subset E'$ be an orthonormal basis of H separating the points of E such that there exists $\mu \in \mathcal{M}^{K_0}$. Then by Proposition 2.4 (ii) we can construct versions a_{tk} from $\tilde{a}_{tk}^{\mu}$, $k \in K_0$, $t \in \mathbb{R}$, so that (A.2) – (A.4) hold (cf. Remark 2.6 (iii)).

3.2 Classical Dirichlet forms

In this subsection assume that (A.1), (A.2), and (A.4) hold. Fix $\nu \in \mathcal{M}^a$ and $k \in K_0$. Define

$$\nu_k := \nu \circ \pi_k^{-1} \, . \tag{3.3}$$

Hence by (2.4) (see also the proof of Proposition 2.13, (i) $\Rightarrow$ (ii)) for all $f \in \mathcal{B}_b(E)$

$$\int_E f(z)\nu(dz) = \int_{\pi_k(E)} \int_{\mathbb{R}} f(x+sk)\ \tilde{\rho}_k^\nu(x+sk)\ ds\ \nu_k(dx) \tag{3.4}$$

and, correspondingly,

$$L^2(E;\nu) = \int_{\pi_k(E)}^{\oplus} L^2(\mathbb{R}; \tilde{\rho}_k^\nu(x+sk)ds)\ \nu_k(dx) \tag{3.5}$$

in the sense that each $f \in L^2(E;\nu)$ corresponds to a "field of vectors" $(f_x)_{x\in\pi_k(E)}$ where $f_x := f(x+\cdot k)$, $x \in \pi_k(E)$ (cf. [AR90a] for details and references).

Define $D(\mathcal{E}_{\nu,k})$ to be the set of all $u \in L^2(E;\nu)$ such that for the corresponding element $(u_x)_{x\in\pi_k(E)}$ in $\int_{\pi_k(E)}^{\oplus} L^2(\mathbb{R};\tilde{\rho}_k^\nu(x+sk)ds)\ \nu_k(dx)$ for ν_k-a.e. $x \in \pi_k(E)$ there exists a locally absolutely continuous ds-version $\tilde{u}_x$ of u_x on $\mathbb{R}$ such that $\left(\frac{d\tilde{u}_x}{ds}\right)_{x\in\pi_k(E)} \in \displaystyle\int_{\pi_k(E)}^{\oplus} L^2(\mathbb{R};\tilde{\rho}_k^\nu(x+sk)ds)\nu_k(dx)$. For $u \in D(\mathcal{E}_{\nu,k})$ we define $\frac{\partial^\nu u}{\partial k}$ as the element in $L^2(E;\nu)$ corresponding to $\left(\frac{d\tilde{u}_x}{ds}\right)_{x\in\pi_k(E)}$ according to (3.5). Note that by (3.1)

$$u \mapsto \frac{\partial^\nu u}{\partial k}$$

is a well-defined operator on $L^2(E;\nu)$. Define

$$\mathcal{E}_{\nu,k}(u,v) := \int \frac{\partial^\nu u}{\partial k}\frac{\partial^\nu v}{\partial k} d\nu\ ; \quad u,v \in D(\mathcal{E}_{\nu,k}), \tag{3.6}$$

and

$$\begin{aligned} \mathcal{E}_\nu(u,v) &:= \sum_{k\in K_0} \mathcal{E}_{\nu,k}(u,v)\ ; \\ u,\ v \in D(\mathcal{E}_\nu) &:= \left\{ w \in \bigcap_{k\in K_0} D(\mathcal{E}_{\nu,k}) \Big| \sum_{k\in K_0} \mathcal{E}_{\nu,k}(w,w) < \infty \right\}. \end{aligned} \tag{3.7}$$

Then by [AR90a, Theorem 3.8] $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is a Dirichlet form on $L^2(E;\nu)$ such that $D(\mathcal{E}_\nu)$ contains the *finitely based smooth bounded functions*, i.e., $\widetilde{\mathcal{F}C_b^\infty}^\nu \subset D(\mathcal{E}_\nu)$ where $\widetilde{\mathcal{F}C_b^\infty}^\nu$ are the ν-equivalence classes of

$$\mathcal{F}C_b^\infty := \left\{ F(l_1,\dots,l_m) \Big| m \in \mathbb{N}, F \in C_b^\infty(\mathbb{R}^m),\ l_1,\dots,l_m \in E' \right\}.$$

We would like to emphasize at this point that by Remark 2.1 (ii) and [AR90b, Proposition 2.7] $\nu(U) > 0$ for all non-empty open sets $U \subset E$. Hence every class in $\widetilde{\mathcal{F}C_b^\infty}^\nu$ has *exactly one* representative in $\mathcal{F}C_b^\infty$. Therefore, we shall not distinguish between $\widetilde{\mathcal{F}C_b^\infty}^\nu$ and $\mathcal{F}C_b^\infty$ below. Note that by Remark 3.1 (i) each $k \in K_0$ is admissible for ν in the sense of [AR90a, Definition 3.4] and that [AR90a, condition (3.12)] holds, since $K_0 \subset E'$ is an orthonormal base of H. Hence [AR90a, Theorem 3.8] indeed applies and, furthermore, it implies that

$$\frac{\partial^\nu u}{\partial k}(z) = \frac{d}{ds} u(z+sk)|_{s=0} \quad \text{for } \nu\text{-a.e. } z \in E, \text{ for all } u \in \mathcal{F}C_b^\infty \text{ and all } k \in K_0. \tag{3.8}$$

$(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$, because of its particular shape, is called a *classical Dirichlet form* in [AR90a, Remark 3.9]. For the general definition of a Dirichlet form (not needed here) we refer to [F80], [BH91], [MR92], [FOT94].

3.3 Characterization of irreducibility and a representation theorem

In this subsection suppose that (A.1), (A.2), and (A.4) hold. We recall that $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is called *irreducible* if for any $u \in D(\mathcal{E}_\nu)$ with $\mathcal{E}_\nu(u,u) = 0$ it follows that u is constant ν-a.e. Let $(L_\nu, D(L_\nu))$ be the generator corresponding to $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$, i.e., the unique negative definite self-adjoint operator such that

$$\mathcal{E}_\nu(u,v) = (\sqrt{-L_\nu}u, \sqrt{-L_\nu}v)_{L^2(E;\nu)} \quad \text{for all } u, v \in D(\mathcal{E}_\nu) = D(\sqrt{-L_\nu}). \tag{3.9}$$

Let $T_t^\nu := e^{tL_\nu}$, $t \geq 0$ denote the corresponding strongly continuous contraction semigroup on $L^2(E;\nu)$ (cf. e.g. [MR92, Chap. I, Sections 1,2] for details). We recall the following well-known result:

Proposition 3.2 *Let $\nu \in \mathcal{M}^a$. Then the following assertions are equivalent:*

(i) $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible.

(ii) $(T_t^\nu)_{t\geq 0}$ is irreducible *(i.e., if $u \in L^2(E;\nu)$ such that $T_t^\nu(uf) = uT_t^\nu f$ for all $f \in L^\infty(E;\nu)$ and all $t > 0$, then u is a constant).*

(iii) *If $u \in L^2(E;\nu)$ such that $T_t^\nu u = u$ for all $t > 0$, then u is a constant.*

(iv) $\int \left(T_t^\nu f - \int f d\nu\right)^2 d\nu \underset{t\to\infty}{\longrightarrow} 0$ *for all $f \in L^2(E;\nu)$. (In this case $(T_t^\nu)_{t>0}$ is also sometimes called* ergodic.*)*

(v) *If $u \in D(L_\nu)$ with $L_\nu u = 0$, then u is a constant.*

For a proof of Proposition 3.2 we refer to the Appendix of [AKR95b]. Now we can formulate and prove the main result of this section.

Theorem 3.3 *Let $\nu \in \mathcal{M}^a$. Then the following assertions are equivalent:*

(i) *$(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible.*

(ii) *ν is K_0-ergodic.*

(iii) *$\nu \in \mathcal{M}^a_{ex}$.*

Proof. By Theorem 2.10 (i) we only have to prove the equivalence of assertions (i) and (ii). But this is an immediate consequence of the following two lemmas.

Lemma 3.4 *Let $\nu \in \mathcal{M}^a$, $k \in K_0$ and $f : E \to \mathbb{R}$ be a $\mathcal{B}(E)$-measurable, ν-square-integrable function which is $\{k\}$-invariant w.r.t. ν. Then (if f also denotes the corresponding ν-class) $f \in D(\mathcal{E}_{\nu,k})$ and $\frac{\partial^\nu f}{\partial k} = 0$.*

Proof. By Fubini's theorem and the k-invariance of f we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \int |f(z+tk) - f(z)| \nu(dz) e^{-|t|} dt \\ &= \int_{\pi_k(E)} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x+(s+t)k) - f(x+sk)|\, e^{-|t|}\ dt\ \tilde{\rho}_k^\nu(x+sk)\ ds\, \nu_k(dx) \\ &= \int_{\mathbb{R}} \int_{\pi_k(E)} \int_{\mathbb{R}} |f(x+tk) - f(x+sk)|\, e^{-|t-s|}\ \tilde{\rho}_k^\nu(x+sk)\ ds\ \nu_k(dx) dt\,. \end{aligned}$$

Hence there exists $t \in \mathbb{R}$ such for ν_k-a.e. $x \in \pi_k(E)$

$$f(x+sk) = f(x+tk) \quad ds\text{-a.e. on } \mathbb{R}$$

since $\tilde{\rho}_k^\nu(z) > 0$ for every $z \in E$. Therefore, for ν_k-a.e. $x \in \pi_k(E)$, $s \mapsto f(x+sk)$ has a ds-version $\tilde{f}_x$ which is constant on $\mathbb{R}$, hence $\frac{d\tilde{f}_x}{ds} = 0$ on $\mathbb{R}$. Consequently,

$$\left(\frac{d\tilde{f}_x}{ds}\right)_{x\in\pi_k(E)} = 0$$

and hence $f \in D(\mathcal{E}_{\nu,k})$ with $\frac{\partial^\nu f}{\partial k} = 0$. □

Lemma 3.5 *Let $\nu \in \mathcal{M}^a$ and $u \in D(\mathcal{E}_\nu)$ such that $\mathcal{E}_\nu(u,u) = 0$. Then u is K_0-invariant w.r.t. ν.*

Proof. Let $k \in K_0$ and $t \in \mathbb{R}$. Then by the definition of $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ we have for ν_k-a.e. $x \in \pi_k(E)$

$$u(x+(t+s)k) = u(x+sk) \text{ for } ds\text{-a.e. } s \in \mathbb{R}.$$

Hence

$$\begin{aligned} & \int |u(z+tk) - u(z)|\ \nu(dz) \\ = & \int_{\pi_k(E)} \int_{\mathbb{R}} |u(x+(s+t)k) - u(x+sk)|\ \tilde{\rho}_k^\nu(x+sk)\, ds\ \nu_k(dx) \\ = & \ 0\ . \end{aligned}$$

□

Remark 3.6 (i) We emphasize that a *much* weaker condition than (A.4) on a_{sk} implies the closedness of the quadratic form (3.7) to yield a corresponding Dirichlet form (cf. [AR89b] for details as well as the one-dimensional example below). However, for the validity of (i) $\Leftrightarrow$ (ii) in Theorem 3.3, condition (A.4) cannot be dropped, as the following example shows: Let $E = H = \mathbb{R}$, hence $K_0 = \{1\}$. Let $\alpha \in]0,\infty[$ and let $\rho_\alpha \in C_b^1(\mathbb{R})$ be such that $\rho_\alpha(s) = s^{2\alpha}$ for $s \in]-1,1[$, $\rho_\alpha > 0$ on $\mathbb{R}\setminus\{0\}$ and $\int \rho_\alpha ds = 1$. If for $t \in \mathbb{R}$ we set

$$a_{t\cdot 1}^{(\alpha)}(s) := \frac{\rho_\alpha(s+t)}{\rho_\alpha(s)}\ , \quad s \in \mathbb{R}\ ,$$

it follows that (A.1) – (A.3) are satisfied and by Remark 2.11 (ii) that $\mathcal{M}^{a^{(\alpha)}} = \{\rho_\alpha ds\} = \mathcal{M}_{ex}^{a^{(\alpha)}}$. For any $\alpha \in]0,\infty[$ we can define a corresponding Dirichlet form as follows. Let $D(\mathcal{E}^{(\alpha)})$ be the set of all $u \in L^2(\mathbb{R};\rho_\alpha ds)$ such that u has a locally absolutely continuous ds-version $\tilde{u}$ on $R(\rho_\alpha) := \Big\{t \in \mathbb{R}\Big| \int_{t-\varepsilon}^{t+\varepsilon} \rho_\alpha^{-1} ds < \infty \text{ for some } \varepsilon > 0\Big\}$ and $\frac{d\tilde{u}}{ds} \in L^2(\mathbb{R}; \rho_\alpha ds)$. For $u, v \in D(\mathcal{E}^{(\alpha)})$ define

$$\mathcal{E}^{(\alpha)}(u,v) := \int \frac{d\tilde{u}}{ds}\frac{d\tilde{v}}{ds}\rho_\alpha ds\ .$$

Note that for $\alpha \in]0, \frac{1}{2}[$, $R(\rho_\alpha) = \mathbb{R}$, hence for such α this definition is a special case of (3.7). Again by [AR90a, Theorem 3.8] $(\mathcal{E}^{(\alpha)}, D(\mathcal{E}^{(\alpha)}))$ is a closed Dirichlet form on $L^2(\mathbb{R}; \rho_\alpha ds)$ such that $C_b^\infty(\mathbb{R}) \subset D(\mathcal{E}^{(\alpha)})$. Despite the fact that $\mathcal{M}^{a^{(\alpha)}} = \{\rho_\alpha ds\} = \mathcal{M}_{ex}^{a^{(\alpha)}}$ we have, however, the following

Claim. Let $\alpha \in]0, \infty[$. Then the following assertions are equivalent:
(a) $(\mathcal{E}^{(\alpha)}, D(\mathcal{E}^{(\alpha)}))$ is irreducible.
(b) $\alpha \in]0, \frac{1}{2}[$.
(c) (A.4) is satisfied.

Proof. Clearly, (A.4) is satisfied if and only if $\alpha \in]0, \frac{1}{2}[$. So, if $\alpha \in]0, \frac{1}{2}[$, Theorem 3.3 implies that $(\mathcal{E}^{(\alpha)}, D(\mathcal{E}^{(\alpha)}))$ is irreducible, since $\mathcal{M}^{a^{(\alpha)}} = \{\rho_\alpha ds\} = \mathcal{M}_{ex}^{a^{(\alpha)}}$. If $\alpha \in [\frac{1}{2}, \infty[$, then $R(\rho_\alpha) = \mathbb{R} \setminus \{0\}$, hence for all $\lambda_1, \lambda_2 \in \mathbb{R}$

$$u_{\lambda_1,\lambda_2} = \lambda_1 1\!\!1_{]-\infty,0[} + \lambda_2 1\!\!1_{[0,\infty[} \in D(\mathcal{E}^{(\alpha)})$$

and $\mathcal{E}^{(\alpha)}(u_{\lambda_1,\lambda_2},\ u_{\lambda_1,\lambda_2}) = 0$. So, $(\mathcal{E}^{(\alpha)}, D(\mathcal{E}^{(\alpha)}))$ is not irreducible.

(ii) In [BR95], [AKR95b] a result related to Theorem 3.3 above was obtained. It roughly says the following: Let $\mathcal{G}^b$ denote the (convex) set of all probability measures on $(E, \mathcal{B}(E))$ having the same logarithmic derivative b. Then for $\nu \in \mathcal{G}^b$ the corresponding classical Dirichlet form $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ on $L^2(E;\nu)$ is irreducible if and only if $\nu \in \mathcal{G}_{ex}^b :=$ the set of extreme points in $\mathcal{G}^b$ (cf. [AKR95b, Theorem 1.2] for details). In case both results are applicable by [AR90a, Theorem 4.7] the latter result can be considered as a kind of infinitesimal version of Theorem 3.3 above. However, the types of examples where one or the other result applies can be quite disjoint, since [AKR95b, Theorem 1.2] does not require (A.4) resp. (A.4)' to hold and "bad" zeros of ρ_k, $k \in K_0$, are allowed, but each ρ_k needs to be "differentiable in direction k" for all $k \in K_0$ in a certain sense. E.g. in the example in part (i) above [AKR95b, Theorem 1.2] is applicable if and only if $\alpha \in]\frac{1}{2}, \infty[$. In this case it is easy to see that if

$$\nu_1 := \left(\int_{-\infty}^{0} \rho_\alpha ds\right)^{-1} 1\!\!1_{[-\infty,0[}\,\rho_\alpha ds$$
$$\nu_2 := \left(\int_{0}^{\infty} \rho_\alpha ds\right)^{-1} 1\!\!1_{[0,\infty[}\,\rho_\alpha ds$$

then $\mathcal{G}^b_{ex} = \{\nu_1, \nu_2\}$ and hence $\mathcal{G}^b = \{\lambda\nu_1 + (1-\lambda)\nu_2 \,|\, \lambda \in [0,1]\}$. So, $\rho_\alpha ds$ is not extreme in $\mathcal{G}^b$ and by [AKR95b, Theorem 1.2] we see again that $(\mathcal{E}^{(\alpha)}, D(\mathcal{E}^{(\alpha)}))$ is not irreducible if $\alpha \in]\frac{1}{2}, \infty[$.

(iii) We would like to draw the reader's attention to a subtle point concerning the definition of $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ in (3.7). If we define $D(\mathcal{E}^\circ_\nu)$ as the closure of $\mathcal{F}C^\infty_b$ w.r.t to the norm $(\mathcal{E}_\nu(\cdot,\cdot) + (\cdot,\cdot)_{L^2(E;\nu)})^{1/2}$ and

$$\mathcal{E}^\circ_\nu := \mathcal{E}_\nu|_{D(\mathcal{E}^\circ_\nu)\times D(\mathcal{E}^\circ_\nu)} \ ,$$

then $(\mathcal{E}^\circ_\nu, D(\mathcal{E}^\circ_\nu))$ is again a Dirichlet form on $L^2(E;\nu)$ (cf. [AR90a, Theorem 3.8]). However, this Dirichlet form might in general be smaller, i.e., $D(\mathcal{E}^\circ_\nu) \underset{\neq}{\subset} D(\mathcal{E}_\nu)$. However, in the case $E = H = \mathbb{R}^d$ and if all $\sqrt{\rho_k}$ are locally in the Sobolev space of order 1 in $L^2(\mathbb{R}^d; dx)$ (where $dx :=$ Lebesgue measure on $\mathbb{R}^d$), then both Dirichlet forms coincide due to [RZ94]. This fact is generally described as: "Markov uniqueness" holds for classical Dirichlet forms on $\mathbb{R}^d$ (cf. [RZ92], [ARZ93a – c], [RZ94], [AKR95a]).

(iv) Let $E = H = \mathbb{R}^d$. Then $\mathcal{M}^a = \{\nu\} = \mathcal{M}^a_{ex}$ by Remark 2.11 (ii). Hence if (A.4) holds then $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible on $L^2(\mathbb{R}^d;\nu)$ by Theorem 3.3. This generalizes a corresponding result in [BR95], since by [AR90a, Theorem 4.7, in particular equality (4.13)] (see also [AKuR90, Proposition 2.4]) (A.4) is a weaker assumption than the one imposed in [BR95, Theorem 6.2]. The latter result by [BR95, Theorem 6.15 (ii)] namely also implies the irreducibility of $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ on $L^2(E;\nu)$).

Because of Remark 3.6 (iii) the following representation of any Dirichlet form $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ for $\nu \in \mathcal{M}^a$ on $\mathcal{F}C^\infty_b$ in terms of irreducible ones is really only of interest if $D(\mathcal{E}^\circ_\nu) = D(\mathcal{E}_\nu)$. But it can be stated and proved in the general case.

Theorem 3.7 *Suppose that additionally (A.3) holds. Let $\nu \in \mathcal{M}^a$ and consider the representation*

$$\nu = \int_{E_a} \nu_x \ m_\nu(dx)$$

proved in Corollary 2.19. Then for all $u, v \in \mathcal{F}C_b^\infty$

$$\mathcal{E}_\nu(u,v) = \int_{E_a} \mathcal{E}_{\nu_x}(u,v)\; m_\nu(dx)\ ,$$

and all $(\mathcal{E}_{\nu_x}, D(\mathcal{E}_{\nu_x}))$ *are irreducible Dirichlet forms on* $L^2(E;\nu_x)$.

Proof. Since $\nu_x \in \mathcal{M}^a$ for every $x \in E_a$, all $(\mathcal{E}_{\nu_x}, D(\mathcal{E}_{\nu_x}))$ are indeed well-defined Dirichlet forms on $L^2(E;\nu_x)$. Furthermore, if $u \in \mathcal{F}C_b^\infty$ and $\sigma \in \mathcal{M}^a$, then by (3.8) $\frac{\partial^\sigma u}{\partial k}$ is just the directional derivative of u w.r.t. k and thus independent of σ. Now the assertion follows immediately from Theorem 3.3. $\square$

3.4 Time ergodicity of corresponding diffusions

In this subsection we want to explain the underlying probabilistic counterpart. We suppose that (A.1), (A.2), and (A.4) hold and we assume that E is a separable real Banach space or a conuclear space in the sense of [AR89a, Subsection 3c] (as e.g. the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered Schwartz distributions).

This subsection is an analogue of [AKR95b, Sect. 4]. We emphasize that *all* results in [AKR95b, Sect. 4] not mentioned here (as e.g. the ones related to "Fukushima's quasi-everywhere ergodic theorem" and large deviations) still hold in full generality in our case here, which according to Remark 3.6 (ii) above is quite different from the one analyzed in [AKR95b]. In order not to be too repetitious we only state the two main results below without proofs and refer the reader to [AKR95b] for details and references.

Let $\nu \in \mathcal{M}^a$ and let $(\mathcal{E}_\nu^\circ, D(\mathcal{E}_\nu^\circ))$ be defined as in Remark 3.6 (iii). Let $(L_\nu^\circ, D(L_\nu^\circ))$ be the associated generator (defined analogously to $(L_\nu, D(L_\nu))$ in (3.9)) and $\overset{\circ}{T}{}_t^\nu := e^{tL_\nu^\circ}$, $t > 0$, the corresponding operator semigroup on $L^2(E;\nu)$.

Theorem 3.8 *Let* $\nu \in \mathcal{M}^a$ *with the additional property that* $\int |\,{}_{E'}\langle l, z\rangle_E|\ \nu(dz) < \infty$ *for all* $l \in E'$ *in case* E *is conuclear. Then there exists a diffusion process* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, (\theta_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ *on* E *which is* properly associated *with* $(\mathcal{E}_\nu^\circ, D(\mathcal{E}_\nu^\circ))$, *i.e., for all ($\mu$-versions of)* $f \in L^2(E;\nu)$ *and all* $t \geq 0$ *the function*

$$z \mapsto p_t f(z) := \int_\Omega f(X_t)dP_z,\ \ z \in E\ , \tag{3.10}$$

is an $\mathcal{E}_\nu^\circ$-quasi-continuous version of $\overset{\circ}{T}{}_t^\nu f$. $\mathbb{M}$ is up to ν-equivalence unique (cf. [MR92, Chap. IV, Sect. 6]). In particular, $\mathbb{M}$ is ν-symmetric, *i.e.,* $\int p_t f \; g \, d\nu = \int f \; p_t g \, d\nu$ *for all $f, g \in \mathcal{B}_b(E)$ and all $t > 0$, as well as* conservative, *i.e.,* $p_t 1 = 1$ *$\mathcal{E}_\nu^\circ$-q.e. for all $t \geq 0$. Thus ν is an invariant measure for* $\mathbb{M}$.

As usual we set

$$P_\nu := \int P_z \, \nu(dz)$$

and we recall that P_ν is called *(time) ergodic* if any bounded $\mathcal{F}$-measurable function $G : \Omega \to \mathbb{R}$ which is θ_t-invariant for all $t \geq 0$, is a constant (P_ν-a.e).

Note that trivially the irreducibility of $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ implies that of $(\mathcal{E}_\nu^\circ, D(\mathcal{E}_\nu^\circ))$ but not vice versa.

Theorem 3.9 *Let $\nu \in \mathcal{M}^a$ with the additional property that $\int |{}_{E'}\langle l, z\rangle_E| \; \nu(dz) < \infty$ for all $l \in E'$ in case E is a conuclear space. Consider the following assertions:*
(i) $\nu \in \mathcal{M}_{ex}^a$.
(ii) $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible.
(iii) $(\mathcal{E}_\nu^\circ, D(\mathcal{E}_\nu^\circ))$ is irreducible.
(iv) P_ν is (time) ergodic.
Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). In particular, if $D(\mathcal{E}_\nu) = D(\mathcal{E}_\nu^\circ)$ then (i) – (iv) are equivalent.

Remark 3.10 (i) Let E be a conuclear space and Φ_n with dual spaces Φ_n', $n \in \mathbb{N}$, as in [AR89a, Subsection 3c], in particular $E = \bigcup_{n \in \mathbb{N}} \Phi_n'$. Let $\nu \in \mathcal{M}_a$ and suppose that the following conditions holds:

$$\nu(\Phi_n') = 1 \qquad \text{for some } n \in \mathbb{N}. \tag{3.11}$$

It follows from [AR89a, Proposition 3.7] that (3.11) holds if $\int |{}_{E'}\langle l, z\rangle_E| \, dz < \infty$ for all $l \in E'$. However, by the proof of [AR89a, Theorem 3.9] it follows that both Theorem 3.8 and 3.9 above still hold if the latter condition is replaced by (3.11). The same is true for the corresponding results in [AKR95b] (i.e., for all results in [AKR95b, Sect. 4]).

(ii) $\mathbb{M}$ (as well as $(\overset{\circ}{T}{}_t^\nu)_{t>0}$) is called the *stochastic dynamics* or the *stochastic quantization* of ν.

4 Application to Gibbs states for lattice and continuous models

4.1 Classical lattice systems

Consider the d-dimensional integer lattice $\mathbb{Z}^d$ ($\subset \mathbb{R}^d$). Let $E := \mathbb{R}^{\mathbb{Z}^d}$ be equipped with the product topology. For $\Lambda \subset \mathbb{Z}^d$ and $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$ we identify $\mathbb{R}^{\mathbb{Z}^d}$ with $\mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda^c}$ and write

$$\mathbb{R}^{\mathbb{Z}^d} \ni x = x_\Lambda \times x_{\Lambda^c} \in \mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda^c}$$

for $x = (x_i)_{i \in \mathbb{Z}^d}$, $x_\Lambda = (x_i)_{i \in \Lambda}$, $x_{\Lambda^c} = (x_i)_{i \in \Lambda^c}$. For $\Lambda \subset \mathbb{Z}^d$ let $\sigma(\Lambda)$ denote the σ-algebra generated by all maps $x \mapsto x_i$, $i \in \Lambda$, from $(E, \mathcal{B}(E))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\mathcal{F}$ denote the set of all finite subsets of $\mathbb{Z}^d$ and let $\Phi := (\Phi_\Lambda)_{\Lambda \in \mathcal{F}}$ be a *potential*, i.e., for each $\Lambda \in \mathcal{F}$

$$\Phi_\Lambda := \mathbb{R}^\Lambda \longrightarrow \mathbb{R} \cup \{+\infty\}$$

is $\mathcal{B}(\mathbb{R}^\Lambda)$-measurable. Consider the corresponding *conditional energy* $(E_\Lambda^\Phi)_{\Lambda \in \mathcal{F}}$ defined by

$$E_\Lambda^\Phi(x) := \begin{cases} \sum\limits_{\Lambda' \cap \Lambda \neq \emptyset} \Phi_{\Lambda'}(x_{\Lambda'}) & \text{if } \sum\limits_{\Lambda' \cap \Lambda \neq \emptyset} \left| \Phi_{\Lambda'}(x_{\Lambda'}) \right| < \infty \\ +\infty & \text{otherwise} , \end{cases} \tag{4.1}$$

$x \in E$.

For $\Lambda \in \mathcal{F}$ define a kernel $\pi_\Lambda^\Phi : E \times \mathcal{B}(E) \to [0,1]$ by

$$\pi_\Lambda^\Phi(\xi, A) := \begin{cases} \left[Z_\Lambda^\Phi(\xi) \right]^{-1} \displaystyle\int_{\mathbb{R}^\Lambda} 1\!\!1_A(x_\Lambda \times \xi_{\Lambda^c}) \exp\left[-E_\Lambda^\Phi(x_\Lambda \times \xi_{\Lambda^c}) \right] dx^\Lambda \\ \qquad\qquad \text{if } 0 < Z_\Lambda^\Phi(\xi) < \infty , \\ 0 \qquad\qquad \text{otherwise} , \end{cases} \tag{4.2}$$

$\xi \in E$, $A \in \mathcal{B}(E)$, where $Z_\Lambda^\Phi(\xi) := \int \exp[-E_\Lambda^\Phi(x_\Lambda \times \xi_{\Lambda^c})] dx^\Lambda$, and dx^Λ denotes Lebesgue measure on $\mathbb{R}^\Lambda$. Let $\pi^\Phi := (\pi_\Lambda^\Phi)_{\Lambda \in \mathcal{F}}$. π^Φ is a $(\sigma(\Lambda^c))_{\Lambda \in \mathcal{F}}$ – *specification* in the sense of [P76], i.e., for all Λ, $\Lambda' \in \mathcal{F}$

(S.1) $\pi_\Lambda(x, E) \in \{0, 1\}$ for all $x \in E$.

(S.2) $\pi_\Lambda(\cdot, A)$ is $\sigma(\Lambda^c)$-measurable for all $A \in \mathcal{B}(E)$.

(S.3) $\pi_\Lambda(\cdot, gf) = g\ \pi(\cdot, f)$ for all $g,\ f \in \mathcal{B}_b(E)$, g $\sigma(\Lambda^c)$-measurable.

(S.4) $\pi_{\Lambda'}\pi_\Lambda = \pi_{\Lambda'}$ whenever $\Lambda' \in \mathcal{F}$, $\Lambda \subset \Lambda'$.

Definition 4.1 $\nu \in \mathcal{M}_1(E)$ is called a *Gibbs measure* of π^Φ if for all $\Lambda \in \mathcal{F}$

$$E_\nu[f \mid \sigma(\Lambda^c)] = \pi_\Lambda^\Phi(\cdot,\ f) \quad \nu - a.e.$$

for all $f \in \mathcal{B}_b(E)$ or equivalently (because of (4.3)

$$\nu\pi_\Lambda^\Phi = \nu\ .$$

Let $\mathcal{G}(\Phi)$ denote the set of all Gibbs measures of π^Φ.

We refer to [P76], [G88] for the above set-up. By Theorem (1.33) in [G88] we have that $\mathcal{G}(\Phi)$ is uniquely determined by $(\pi_{\{i\}}^\Phi)_{i\in\mathbb{Z}^d}$, more precisely:

Proposition 4.2 *$\nu \in \mathcal{M}_1(E)$ is a Gibbs measure for π^Φ if and only if $\nu\pi_{\{i\}}^\Phi = \nu$ for all $i \in \mathbb{Z}^d$.*

Remark 4.3 Above we are in the situation of a classical lattice system where the single spin spaces are all equal to $\mathbb{R}^1$. By a simple relabeling the case where the single spin spaces are equal to $\mathbb{R}^d$ can be easily reduced to this case and all results below apply.

Let now $H := l_2(\mathbb{Z}^d)$ and let K_0 be the standard orthonormal basis $\{k_i \mid i \in \mathbb{Z}^d\}$ of H, i.e., $k_i := (\delta_{ij})_{j\in\mathbb{Z}^d}$, $i \in \mathbb{Z}^d$.

Theorem 4.4 *Suppose $\mathcal{G}(\Phi) \neq \emptyset$ and that the following condition holds:*

$$(\Phi\,1) \qquad \sum_{\Lambda'\cap\Lambda\neq\emptyset} \left|\Phi_{\Lambda'}(x_{\Lambda'})\right| < \infty \quad \textit{and}$$

$$\int_{\mathbb{R}^\Lambda} \exp\left[-E_\Lambda^\Phi(x_\Lambda \times \xi_{\Lambda^c})\right] dx^\Lambda < \infty \quad \textit{for all } x,\ \xi \in E \textit{ and all } \Lambda \in \mathcal{F}\ .$$

Define for $i \in \mathbb{Z}^d$, $t \in \mathbb{R}$

$$a_{tk_i}(x) := \exp\left[-\sum_{\Lambda'\cap\{i\}\neq\emptyset} (\Phi_{\Lambda'}(x_{\Lambda'} + tk_i) - \Phi_{\Lambda'}(x_{\Lambda'}))\right]\ , \qquad (4.3)$$

$x \in E$.

Then:

(i) (A.1) — (A.3) hold, $\mathcal{M}^a = \mathcal{G}(\Phi)$, and all results of Section 2 apply.

(ii) Suppose that, in addition, the following condition holds:

(Φ 2) For all $i \in \mathbb{Z}^d$ and all compact sets $C \subset \mathbb{R}$

$$\int_C a_{sk_i}(x)^{-1}\, ds < \infty \qquad \text{for all } x \in E$$

(where a_{sk_i} are as in (4.3)).

Then all results of Section 3 apply.

Proof. We first note that by (Φ1) assumptions (A.1) – (A.3) can be easily checked, and that (Φ2) is just (A.4).

(i): By Propositions 2.13 and 4.2 it follows immediately that $\mathcal{G}(\Phi) = \mathcal{M}^a$.

(ii): Obvious. $\square$

Remark 4.5 (i) Condition (Φ1) always holds for *finite range interactions*, i.e., the case where there exists $N \in \mathbb{N}$ such that $\Phi_\Lambda \equiv 0$ for all $\Lambda \in \mathcal{F}$ such that $\#\Lambda > N$. (Φ2) holds e.g., if $t \mapsto a_{tk_i}(x)$ is lower semi-continuous. Both conditions are obviously fulfilled in the well-known $P(\Phi)_d$-model on the lattice where also it is well-known that $\mathcal{G}(\Phi) \neq \emptyset$ (cf. e.g. [AKR95b, Subsection 5.1] for details). In particular, Theorem 4.4 generalizes all results in [AKR95b, Section 5] on the $P(\Phi)_d$-lattice model since the *temperedness condition* imposed there on the Gibbs states, are dropped here.

(ii) The fact that Gibbs measures can be alternatively described through their "shift" – Radon-Nikodym derivatives is known for a number of specific models (see e.g. [Roy77] for the case of the $P(\Phi)_d$-lattice model). Our point here is to pin down precise conditions in the lattice case to prove this "folklore result" rigorously and to apply our results of Sections 2, 3, which are really new in these situations.

(iii) Clearly, under *temperedness conditions* (cf. e.g. [AKR95b, Section 5]) condition (Φ1) can be relaxed, since then only a smaller class of boundary data ξ has to be considered.

(iv) By [AKR95b, Theorem 5.13] $D(\mathcal{E}_\nu^\circ) = D(\mathcal{E}_\nu)$ for a large class of potentials Φ and Gibbs measures ν.

4.2 Quantum lattice systems

To any $k \in \mathbb{Z}^d$ let there be associated a one–dimensional quantum harmonic oscillator with physical Hilbert space $L^2(\mathbb{R}; dx_k)$ and Hamiltonian

$$H_k := -\frac{1}{2m}\frac{d^2}{dx_k^2} + \frac{a^2}{2}x_k^2 , \qquad m, a^2 > 0 \ .$$

We consider a simple translation invariant system of interacting quantum harmonic oscillators with heuristic Hamiltonian

$$\sum_{k\in\mathbb{Z}^d} H_k + \sum_{\langle k,j\rangle\subset\mathbb{Z}^d} \Phi(x_k, x_j) + \sum_{k\subset\mathbb{Z}^d} V(x_k) \ . \tag{4.4}$$

Here $V(q)$, $q \in \mathbb{R}$, is an anharmonic potential which is given by a function $V \in C(\mathbb{R})$ bounded from below and polynomially bounded, i.e., $V(q) \leq C_V(1 + |q|)^M$, $q \in \mathbb{R}$, for some C_V, $M > 0$. The two-particle interaction $\Phi(x_k, x_j)$ is taken over all nearest–neighbour pairs $\langle k, j\rangle \subset \mathbb{Z}^d$ ($|k - j| = 1$) and, for simplicity, is supposed to be of the form

$$\Phi(x_k, x_j) := \frac{J}{2}(x_k - x_j)^2, \quad J > 0 \ . \tag{4.5}$$

The latter means that we consider harmonic pair interactions with an intensity J. A basic problem of equilibrium quantum statistical physics is the study of temerature (i.e. Gibbs) states corresponding to (4.4). We will follow the Euclidean approach based on a path space representation for Gibbs states (see e.g. the early work of S. Albeverio and R. Høegh-Krohn [AH-K75b] and its further developments in [KlL81], [GloK90]). Under this approach the problem of giving a proper meaning to a Gibbs state G_β of the lattice system (4.4) at inverse temperatur $\beta > 0$ can be reduced to the problem of constructing some Euclidean Gibbs measure ν_β on the space Ω_β of periodic trajectories (loops) on $[0, \beta]$ with values in $\mathbb{R}^{\mathbb{Z}^d} := \times_{k\in\mathbb{Z}^d}\mathbb{R}^1$.

Namely, let S_β be a circle having length β. (It is convenient to interpret S_β as the interval $[0, \beta]$ with the ends identified.) The spaces $C(S_\beta)$ and $L^2(S_\beta)$ consist of functions $\omega(\cdot) : S_\beta \to \mathbb{R}^1$ which are continuous resp. square integrable w.r.t. Lebesgue measure on S_β. If $C(S_\beta)$ is equipped with the sup–norm $\|\cdot\|_{C(S_\beta)}$ and $L^2(S_\beta)$ with its natural norm, for the respective Borel σ–algebras the following relation holds

$$B(C(S_\beta)) = \mathcal{B}(L^2(S_\beta)) \cap C(S_\beta) \ .$$

We define as the configuration space the temperature loop space (TLS)

$$\begin{aligned} \Omega_\beta &:= C(S_\beta)^{\mathbb{Z}^d} \\ &= \{\omega(\cdot) = (\omega(\cdot))_{k\in\mathbb{Z}^d} \mid \omega : S_\beta \to \mathbb{R}^{\mathbb{Z}^d},\ \omega_k(\cdot) \in C(S_\beta)\} \end{aligned} \tag{4.6}$$

equipped with the product topology and corresponding Borel σ–algebra $\mathcal{B}(\Omega_\beta)$. As before let $\mathcal{F}$ denote the set of all $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$. For $\Lambda \subset \mathbb{Z}^d$, the σ-algebra generated by

$$\{\omega \in \Omega_\beta \mid (\omega_k)_{k\in\Lambda} \in B_\Lambda\},\ B_\Lambda \in \mathcal{B}(C(S_\beta)^\Lambda)$$

is denoted by $\sigma(\Lambda)$. Clearly, $\mathcal{B}(\Omega_\beta)$ is the smalles σ–algebra on Ω_β containing $\sigma(\Lambda)$ for all $\Lambda \in \mathcal{F}$.

The Euclidean measure ν_β corresponding to a Gibbs state G_β of the system (4.4) has the heuristic representation

$$d\nu_\beta(\omega(\cdot)) = \frac{1}{Z_\beta} \exp\{-E_\beta(\omega(\cdot))\} \times_{k\in\mathbb{Z}^d, \tau\in S_\beta} d\omega_k(\tau) \tag{4.7}$$

where

$$\begin{aligned} E_\beta(\omega(\cdot)) &= \int_{S_\beta} \Big[\frac{m}{2}\sum_{k\in\mathbb{Z}^d} \dot{\omega}_k^2(\tau) + \frac{a^2}{2}\sum_{k\in\mathbb{Z}^d} \omega_k^2(\tau) \\ &\quad + \frac{J}{2}\sum_{\langle k,j\rangle} (\omega_k(\tau) - \omega_j(\tau))^2 + \sum_{k\in\mathbb{Z}^d} V(\omega_k(\tau))\Big]\, d\tau \end{aligned} \tag{4.8}$$

is the Euclidean action functional of the system and Z_β is a nomalizing factor. In full analogy with classical mechanics, a rigorous meaning can be given to the measure ν_β by the Dobrushin-Lanford-Ruelle formalism as an infinite volume Gibbs distribution with spin space $C(S_\beta)$.

To this end we modify the consideration of Subsection 4.1 as follows. First of all we introduce a reference measure on $E := \Omega_\beta$. Let us consider the Hilbert space $\mathcal{H}_\beta := L^2(S_\beta)$ (with the usual inner product denoted by $\langle\cdot,\cdot\rangle_\beta$ and corresponding norm $\|\cdot\|_\beta$) and the positive self–adjoint operator $A_\beta := -m\Delta_\beta + a^2 1\!\!1$ in $\mathcal{H}_\beta$. Here Δ_β is the Laplace–Beltrami operator on the circle S_β. Let γ_β be a Gaussian measure on $(\mathcal{H}_\beta, \mathcal{B}(\mathcal{H}_\beta))$ with zero mean value and correlation operator $C_\beta = A_\beta^{-1}$, i.e.,

$$\int_{\mathcal{H}_\beta} e^{i\langle\varphi,\omega\rangle_\beta}\, d\gamma_\beta(\omega) = e^{-\frac{1}{2}\langle C_\beta\varphi,\varphi\rangle_\beta}, \quad \varphi \in \mathcal{H}_\beta.$$

We note that $\gamma_\beta(C(S_\beta)) = 1$. For any $\Lambda \subset \mathbb{Z}^d$ we set $d\gamma_\beta^\Lambda(\omega_\Lambda) := \times_{k\in\Lambda} d\gamma_\beta(\omega_k)$. The measures γ_β^Λ on $C(S_\beta)^\Lambda$ will play the role of Lebesgue measure in Subsection 4.1.

The conditional energy $(E_\Lambda^\Phi)_{\Lambda\in\mathcal{F}}$ in this case is defined by

$$\begin{aligned}(E_\Lambda^\Phi)(\omega) \;&:=\; \sum_{k\in\Lambda}\int_0^\beta V(\omega_k(\tau))d\tau + \sum_{\langle k,j\rangle,\, k,j\in\Lambda}\int_0^\beta \Phi(\omega_k(\tau),\omega_j(\tau))d\tau \\ &\quad + \sum_{\langle k,j\rangle,\ k\in\Lambda,\, j\in\Lambda^c}\int_0^\beta \Phi(\omega_k(\tau),\omega_j(\tau))\,d\tau\ ,\end{aligned}$$

$\omega \in E$. The specification $\pi^\Phi = (\pi_\Lambda^\Phi)_{\Lambda\in\mathcal{F}}$ is then defined by the kernels

$$\pi_\Lambda^\Phi(\xi, A) := \frac{1}{Z_\Lambda^\Phi(\xi)}\int_{C(S_\beta)^\Lambda} 1\!\!1_A(\omega_\Lambda\times\xi_{\Lambda^c})e^{-E_\Lambda^\Phi(\omega_\Lambda\times\xi_{\Lambda^c})}\,d\gamma_\beta^\Lambda(\omega_\Lambda)\,, \tag{4.9}$$

$\xi \in E$, $A \in \mathcal{B}(E)$, where $Z_\lambda^\Phi(\xi) := \int_{C(S_\beta)^\Lambda} e^{-E_\Lambda^\Phi(\omega_\Lambda\times\xi_{\Lambda^c})}\,d\gamma_\beta^\Lambda(\omega_\Lambda)$. These kernels satisfy properties (S.1) – (S.4) and Definition 4.1 is applicable in this situation. Let $\mathcal{G}_\beta$ denote the set of all Gibbs measures of π^Φ.

Let now $H := l_2(\mathbb{Z}^d \to \mathcal{H}_\beta)$ $(= l_2(\mathbb{Z}^d)\otimes\mathcal{H}_\beta)$. We construct an orthonormal basis K_0 of H as follows. We define an orthonormal basis $\{e_n \mid n\in\mathbb{Z}\}$ in $\mathcal{H}_\beta = L^2(S_\beta)$ by

$$e_n(\tau) := \begin{cases} \beta^{-\frac{1}{2}}\ , & n = 0\\ (\frac{2}{\beta})^{\frac{1}{2}}\cos(\frac{2\pi}{\beta}n\tau)\ , & n > 0\\ -(\frac{2}{\beta})^{\frac{1}{2}}\sin(\frac{2\pi}{\beta}n\tau)\ , & n < 0\ ,\end{cases}$$

$\tau \in S_\beta$. It is well-known and easy to check that $\{e_n \mid n\in\mathbb{Z}\}$ is an eigenbasis of A_β with corresponding eigenvalues $\lambda_n := (\frac{2\pi}{\beta}n)^2 m + a^2$, $n\in\mathbb{N}$. For any $i = (n,j)\in\mathbb{Z}^{d+1}$ $(n\in\mathbb{Z},\, j\in\mathbb{Z}^d)$ we define

$$k_i := (\delta_{jl})_{l\in\mathbb{Z}^d}\otimes e_n \in H\ ,$$

and set $K_0 := \{k_i \mid i\in\mathbb{Z}^{d+1}\}$. Define for $i=(n,j)\in\mathbb{Z}^{d+1}$, $t\in\mathbb{R}$, $\omega\in E$,

$$a_{tk_i}(\omega) := a_{tk_i}^\Phi(\omega)\ a_{tk_i}^0(\omega)\ , \tag{4.10}$$

where

$$
\begin{aligned}
a^0_{tk_i}(\omega) &:= \frac{d\gamma_\beta^{\mathbb{Z}^d}(\omega + tk_i)}{d\gamma_\beta^{\mathbb{Z}^d}(\omega)} = \frac{d\gamma_\beta(\omega_j + te_n)}{d\gamma_\beta(\omega_j)} \\
&= e^{-t\langle A_\beta e_n, \omega_j\rangle_\beta - \frac{t^2}{2}\langle A_\beta e_n, e_n\rangle_\beta} , \qquad (4.11) \\
a^\Phi_{tk_i}(\omega) &:= \exp\Big\{ - \sum_{j':|j'-j|=1} \int_0^\beta [\Phi(\omega_j(\tau) + te_n(\tau), \omega_{j'}(\tau)) \\
&\quad -\Phi(\omega_j(\tau), \omega_{j'}(\tau))]\, d\tau \Big\} \\
&\quad \cdot \exp\Big\{ - \int_0^\beta [V(\omega_j(\tau) + te_n(\tau)) - V(\omega_j(\tau))]\, d\tau \Big\} . \qquad (4.12)
\end{aligned}
$$

If we introduce (Fourier) variables

$$C(S_\beta) \ni \omega_j \mapsto (\hat{\omega}_j^{(n)} := \langle \omega_j, e_n\rangle_\beta)_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}},$$

then for $\omega \in E$ (4.11) and (4.12) have the form

$$
\begin{aligned}
a^0_{tk_i}(\omega) &= e^{-t\lambda_n \hat{\omega}_j^{(n)} - \frac{t^2}{2}\lambda_n} , \qquad (4.13) \\
a^\Phi_{tk_i}(\omega) &= \exp\left\{ -Jt \sum_{j':|j'-j|=1} (\hat{\omega}_j^{(n)} - \hat{\omega}_{j'}^{(n)}) - t^2 Jd \right\} \\
&\quad \cdot \exp\left\{ - \int_0^\beta [V(\omega_j(\tau) + te_n(\tau)) - V(\omega_j(\tau))]\, d\tau \right\} . \qquad (4.14)
\end{aligned}
$$

Let $\mathcal{M}^a$ be defined as in (A.2) in Subsection 2.2 with E, H, K_0, a_{tk_i} as specified above.

Theorem 4.6 *The set $\mathcal{G}_\beta$ of Euclidean Gibbs measures is non–empty and $\mathcal{G}_\beta = \mathcal{M}^a$. Moreover, (A.1) – (A.4) hold and, therefore, all results of Sections 2 and 3 apply.*

Proof. Let us denote the linear hull of $\{e_n \mid n \in \mathbb{Z}\} \subset C^\infty(S_\beta)$ by $\mathcal{L}_\beta$, i.e., the set of all trigonometric polynomials on S_β. We set $K := [\mathcal{L}_\beta]^{\mathbb{Z}^d}_{fin}$, i.e., $\xi \in K$ if and only if $\xi = (\xi_j)_{j\in\mathbb{Z}^d}$, $\xi_j \in \mathcal{L}_\beta$, $j \in \mathbb{Z}^d$, and for some $R = R(\xi)$, $|j| > R \Rightarrow \xi_j \equiv 0$. Equivalently, we have $K =$ linear hull of K_0. By Remark 2.1 (ii) (i.e., the cocycle property) and (4.10) – (4.12) we conclude that any $\nu \in \mathcal{M}^a$ is K–quasi-invariant with "shift"–Radon–Nikodym derivatives for $\xi \in K$ given by

$$a_\xi(\omega) = a^0_\xi(\omega) a^\Phi_\xi(\omega), \qquad \omega \in E ,$$

where

$$
\begin{aligned}
a_\xi^0(\omega) &:= \frac{d\gamma_\beta^{\mathbb{Z}^d}(\omega+\xi)}{d\gamma_\beta^{\mathbb{Z}^d}(\omega)} , \\
a_\xi^\Phi(\omega) &:= \exp\Big\{ - \sum_{k:|k|\le R(\xi)} \sum_{j:|j-k|=1} \int_0^\beta \Big[\Phi(\omega_k(\tau)+\xi_k(\tau), \omega_j(\tau)+\xi_j(\tau)) \\
&\qquad - \Phi(\omega_k(\tau), \omega_j(\tau))\Big]\, d\tau \Big\} \\
&\qquad \cdot \exp\left\{ - \sum_{k:|k|\le R(\xi)} \int_0^\beta \left[V(\omega_k(\tau)+\xi_k(\tau)) - V(\omega_k(\tau))\right] d\tau \right\} .
\end{aligned}
$$

Under our assumptions by standard superstability arguments (see e.g. [ParY94]) we have that $\mathcal{G}_\beta \neq \emptyset$. For any $\nu \in \mathcal{G}_\beta$ by (4.9) and Definition 4.1 we obtain immediately K–quasi-invariance and that

$$
\frac{d\nu(\omega+\xi)}{d\nu(\omega)} = a_\xi(\omega), \;\; \xi \in K, \;\; \omega \in E ,
$$

i.e., $\nu \in \mathcal{M}^a$.

For the proof of the converse statement we start with any given $\nu \in \mathcal{M}^a$. Fix $j \in \mathbb{Z}^d$, set $\Lambda_j := \{j\} \subset \mathbb{Z}^d$, $\Lambda_j^c := \mathbb{Z}^d \setminus \{j\}$, and let $\nu_{\Lambda_j^c}$ denote the projection of ν onto $C(S_\beta)^{\Lambda_j^c}$. We can desintegrate ν as follows:

$$
\nu(d\omega_j \; d\omega_{\Lambda_j^c}) = \mu_{\omega_{\Lambda_j^c}}(d\omega_j) \; \nu_{\Lambda_j^c}(d\omega_{\Lambda_j^c}) ,
$$

where $\mu_{\omega_{\Lambda_j^c}}(d\omega_j)$ are measures on $(C(S_\beta), \mathcal{B}(C(S_\beta)))$. Let $\Omega_{\beta,j}$ be the set of all $\omega_{\Lambda_j^c} \in C(S_\beta)^{\Lambda_j^c}$ such that we have the following property of quasi–invariance: for any $\xi_j \in \mathcal{L}_\beta$

$$
\frac{d\mu_{\omega_{\Lambda_j^c}}(\omega_j+\xi_j)}{d\mu_{\omega_{\Lambda_j^c}}(\omega_j)} = a_{\xi_j}(\omega_j \times \omega_{\Lambda_j^c}), \quad \mu_{\omega_{\Lambda_j^c}} - a.e. \; \omega_j \in C(S_\beta). \tag{4.15}
$$

Then clearly $\nu_{\Lambda_j^c}(\Omega_{\beta,j}) = 1$ (cf. [Roy75, Proposition 3]). From now on fix $\omega_{\Lambda_j^c} \in \Omega_{\beta,j}$. By [G88, Theorem (1.33)] (cf. also Proposition 4.2) we need to show that (4.15) implies

$$
\mu_{\omega_{\Lambda_j^c}}(d\omega_j) = n_{\Lambda_j}(d\omega_j \mid \omega_{\Lambda_j^c}) , \tag{4.16}
$$

where $n_{\Lambda_j}(d\omega_j \mid \omega_{\Lambda_j^c})$ is the so-called Gibbs measure in the volume Λ_j with boundary condition $\omega_{\Lambda_j^c} \in C(S_\beta)^{\Lambda_j^c}$, i.e.,

$$n_{\Lambda_j}(d\omega_j \mid \omega_{\Lambda_j^c}) := \frac{1}{Z^\Phi_{\Lambda_j}(\omega_{\Lambda_j^c})} e^{-E^\Phi_{\Lambda_j}(\omega_j \times \omega_{\Lambda_j^c})} \gamma_\beta(d\omega_j)$$

(see (4.9)). The explicit form for $E^\Phi_{\Lambda_j}$ gives

$$\begin{aligned} n_{\Lambda_j}(d\omega_j \mid \omega_{\Lambda_j^c}) &= \frac{1}{Z^\Phi_{\Lambda_j}(\omega_{\Lambda_j^c})} \\ &\cdot \exp\{-\frac{J}{2} \sum_{j':|j'-j|=1} \int_0^\beta (\omega_j(\tau) - \omega_{j'}(\tau))^2 d\tau \\ &\cdot \exp\{-\int_0^\beta V(\omega_j(\tau))d\tau\} \; \gamma_\beta(d\omega_j) \; . \end{aligned} \tag{4.17}$$

From (4.17) we have for any $\xi_j \in \mathcal{L}_\beta$

$$\frac{d\, n_{\Lambda_j}(\omega_j + \xi_j \mid \omega_{\Lambda_j^c})}{d\, n_{\Lambda_j}(\omega_j \mid \omega_{\Lambda_j^c})} = a_{\xi_j}(\omega_j \times \omega_{\Lambda_j^c}) \text{ for } \; n_{\Lambda_j}(\cdot \mid \omega_{\Lambda_j^c}) - a.e.\ \omega_j \in C(S_\beta).$$

Now we shall show that the latter equality determines the measure $n_{\Lambda_j}(d\omega_j \mid \omega_{\Lambda_j^c})$ uniquely (see also [Roy75, Proposition 5]). To see this we first of all rewrite the cocycle a_{ξ_j} in the form

$$\begin{aligned} a_{\xi_j}(\omega_j \times \omega_{\Lambda_j^c}) &= \exp\{-2Jd \int_0^\beta \omega_j(\tau)\xi_j(\tau)d\tau + J \int_0^\beta \eta_j(\tau)\xi_j(\tau)d\tau\} \\ &\cdot \exp\{-Jd \int_0^\beta \xi_j^2(\tau)d\tau\} \cdot \exp\{-\int_0^\beta [V(\omega_j(\tau) + \xi_j(\tau)) - V(\omega_j(\tau))]d\tau\} \\ &\cdot \frac{d\gamma_\beta(\omega_j + \xi_j)}{d\gamma_\beta(\omega_j)} =: a_{\xi_j}(\omega_j, \eta_j) \; , \end{aligned} \tag{4.18}$$

where $\eta_j := \sum_{j':|j'-j|=1} \omega_{j'} \in C(S_\beta)$. Now fix η_j and let μ be a probability measure on $(C(S_\beta), \mathcal{B}(C(S_\beta)))$ such that for all $\xi_j \in \mathcal{L}_\beta$

$$\frac{d\mu(\omega_j + \xi_j)}{d\mu(\omega_j)} = a_{\xi_j}(\omega_j,\, \eta_j) \quad \text{for } \mu - a.e.\ \omega_j \in C(S_\beta) \; .$$

We introduce the new measure

$$\sigma(d\omega_j) := \exp\left\{\int_0^\beta V(\omega_j(\tau))d\tau\right\} \; \mu(d\omega_j)$$

which is a σ–finite measure on $\mathcal{B}(C(S_\beta))$; more precisely, σ is finite on any bounded Borel set in $C(S_\beta)$.

By (4.18) we have for all $\xi_j \in \mathcal{L}_\beta$

$$\begin{aligned}\frac{d\sigma(\omega_j+\xi_j)}{d\sigma(\omega_j)} &= \exp\{-2dJ\int_0^\beta \omega_j(\tau)\,\xi_j(\tau)d\tau + J\int_0^\beta \eta_j(\tau)\,\xi_j(\tau)d\tau - \\ &\qquad Jd\int_0^\beta \xi_j^2(\tau)d\tau\}\cdot\frac{d\gamma_\beta(\omega_j+\xi_j)}{d\gamma_\beta(\omega_j)} \quad \text{for } \sigma - a.e.\ \omega_j \in C(S_\beta)\ .\end{aligned} \tag{4.19}$$

For our given $\eta_j \in C(S_\beta)$ we introduce a Gaussian measure on $\mathcal{B}(C(S_\beta))$ by

$$\gamma_\beta^{\eta_j}(d\omega_j) := \frac{1}{Z_{\eta_j}} e^{-Jd\int_0^\beta \omega_j^2(\tau)d\tau + J\int_0^\beta \omega_j(\tau)\eta_j(\tau)d\tau}\,\gamma_\beta(d\omega_j)\ ,$$

where Z_{η_j} is a normalization constant. Then for any $\xi_j \in \mathcal{L}_\beta$

$$\frac{d\gamma_\beta^{\eta_j}(\omega_j+\xi_j)}{d\gamma_\beta^{\eta_j}(\omega_j)} = \frac{d\sigma(\omega_j+\xi_j)}{d\sigma(\omega_j)}\ . \tag{4.20}$$

We claim that σ is a finite measure. To this end consider the image measure σ_1 of σ under the map $\omega_j \mapsto \langle\omega_j, e_1\rangle_\beta$. Then by (4.20) and the product structure of the right hand side of (4.19) σ_1 is quasi–invariant on $\mathbb{R}$ with "shift"–Radon–Nikodym derivatives equal to that of a Gaussian measure $\gamma(s)\,ds$ on $\mathbb{R}$. It is well-known and easy to see that σ_1 is equivalent to ds. (Indeed, since σ is σ-finite, so is σ_1. Hence there exists $\rho_0 \in L^1(\mathbb{R};\nu_1)$, $\rho_0 > 0$ σ_1–a.e., such that $\sigma_2 := \rho_0\cdot\sigma_1$ is a probability measure. Therefore, $\sigma_2 \sim ds$ by Remark 2.6 (iii), and thus $\sigma_1 \sim ds$.) Hence if $\rho := \frac{d\sigma_1}{ds}$, then for all $t \in \mathbb{R}$

$$\frac{\rho(\cdot + t)}{\rho} = \frac{\gamma(\cdot + t)}{\gamma} \qquad ds - a.e.$$

It follows that ρ/γ is shift–invariant ds–a.e., hence a constant ds–a.e. Thus σ_1 is a constant multiple of $\gamma(s)\,ds$, in particular, $\sigma(C(S_\beta)) = \sigma_1(\mathbb{R}) < \infty$.

Now it follows from [Roy75, Proposition 4] (or from [BR95, Theorem 6.1 (ii)]) that

$$\frac{\sigma}{\sigma(C(S_\beta))} = \gamma_\beta^{\eta_j}\ .$$

Consequently, $\mu(d\omega_j)$ is a constant multiple of $n_{\Lambda_j}(d\omega_j \mid \omega_{\Lambda_j^c})$ by (4.17) and, since both are probability measures, they coincide. Thus (4.16) is shown and the equality $\mathcal{G}_\beta = \mathcal{M}^a$ is proved as well as (A.1), (A.2). Also (A.3) holds. The reason is that by Proposition 2.4 all (in-)equalities required in (A.3) hold $\gamma_\beta^{\mathbb{Z}^d}$–a.e. But all functions appearing in (A.3) are continuous in $z \in E = C(S_\beta)^{\mathbb{Z}^d}$ in this case, as is immediately implied by (4.13), (4.14). Since $\gamma_\beta^{\mathbb{Z}^d}(U) > 0$ for any non-empty open $U \subset C(S_\beta)^{\mathbb{Z}^d}$, (A.3) follows. (A.4) is obvious since all exponents in (4.13), (4.14) are finite for *all* $\omega \in C(S_\beta)^{\mathbb{Z}^d}$ and we have continuity in t. This completes the proof. □

Remark 4.7 (i) The special form of Φ in (4.5) was *only* assumed for simplicity and is *not* necessary for Theorem 4.6 to hold. By a slight modification of the above arguments one can show that in the general case $\mathcal{G}_\beta$ at least coincides with all those elements in the corresponding set $\mathcal{M}^a$ which are locally equivalent with $\gamma_\beta^{\mathbb{Z}^d}$. This is sufficient to apply all results from Sections 2 and 3 above. For details we refer to the next section where a much harder, but w.r.t. the "local equivalence" similar case is treated.

(ii) For a large class of measures ν in $\mathcal{G}_\beta$ one knows that $D(\mathcal{E}_\nu^\circ) = D(\mathcal{E}_\nu)$ (cf [AKRT96, Remark 5.1 (i)] and also [ParY96]).

4.3 $P(\Phi)_2$ – Guerra – Rosen – Simon Gibbs states in infinite volume

We first recall the necessary notions. Let $E := \mathcal{D}' := \mathcal{D}'(\mathbb{R}^2)$, i.e., the space of real Schwartz distributions on $\mathbb{R}^2$, $\mathcal{D} := \mathcal{D}(\mathbb{R}^2)$ the associated test function space with the usual topology. $\mathcal{D}'$ equipped with the $\sigma(\mathcal{D}', \mathcal{D})$-topology is a *Lusin space* (i.e., the continuous one-to-one image of a complete separable metric space) and $E' = \mathcal{D}'' = \mathcal{D}$. Let μ_0 be the Gaussian measure on $\mathcal{D}'$ with mean zero and covariance

$$\int_{\mathcal{D}'} \langle \varphi, z\rangle \langle \psi, z\rangle \, \mu_0(dz) = \int\int \varphi(x)(-\Delta + m^2)^{-1}(x-y)\psi(y)\lambda^2(dx)\lambda^2(dy)$$

for $\varphi, \psi \in \mathcal{D} = E'$. Here λ^2 denotes Lebesgue measure on $\mathbb{R}^2$, $x \mapsto (-\Delta + m^2)^{-1}(x)$ is the Green function of $(-\Delta + m^2)$ on $\mathbb{R}^2$, $m > 0$, and $\langle\, ,\, \rangle$: $\mathcal{D} \times \mathcal{D}' \to \mathbb{R}$ is the dualization between $\mathcal{D}'$ and $\mathcal{D}$. Identifying $f \in L^2(\mathbb{R}^2; \lambda^2)$ with the map $\varphi \mapsto \int_{\mathbb{R}^2} \varphi(x) f(x) \lambda^2(dx)$, $\varphi \in \mathcal{D}$, $L^2(\mathbb{R}^2; \lambda^2)$ becomes naturally a subspace of $\mathcal{D}'$ and in this sense

$$\mathcal{D} \subset L^2(\mathbb{R}^2; \lambda^2) \subset \mathcal{D}' \,.$$

i.e., (A.1) holds with $H := L^2(\mathbb{R}^2;\lambda^2)$.

Clearly, for $\varphi_1,\ldots,\varphi_j \in \mathcal{D}$, $\prod_{i=1}^{j}\langle\varphi_i,\cdot\rangle \in L^2(\mathcal{D}';\mu_0)$. Define for $n \in \mathbb{N}$, $P^{(n)} := P^{(\leq n)} - P^{(\leq n-1)}$ with $P^{(\leq n)}$ being the closed linear span of the monomials $\prod_{i=1}^{j}\langle\varphi_i,\cdot\rangle$, $j \leq n$ in $L^2(\mathcal{D}';\mu_0)$. Now if $\varepsilon \in]0,1]$, $h \in L^{1+\varepsilon}(\mathbb{R}^2;\lambda^2)$ and $n \in \mathbb{N}$, define $: z^n : (h)$ to be the unique element in $P^{(n)}$ such that

$$\begin{aligned}
&\int_{\mathcal{D}'} : z^n : (h) \ : \prod_{i=1}^{n}\langle\varphi_i,\cdot\rangle : \ d\mu_0 \\
= \ & n! \int_{\mathbb{R}^2} \prod_{i=1}^{n} \left(\int_{\mathbb{R}^2} (-\Delta + m^2)^{-1}(x-y_i)\,\varphi_i(y_i)\lambda^2(dy_i)\right) h(x)\,\lambda^2(dx)\ ,
\end{aligned}$$

where $:\prod_{i=1}^{n}\langle\ \varphi_i,\cdot\rangle:$ is the orthogonal projection of $\prod_{i=1}^{n}\langle\ \varphi_i,\cdot\rangle$ in $L^2(\mathcal{D}';\mu_0)$ onto $P^{(n)}$ (see [S74, p.12] for an explicit definition of the "Wick product" $:\prod_{i=1}^{n}\langle\varphi_i,\cdot\rangle:$ and [S74, §V.1] for the existence of $: z^n : (h)$). Clearly, for $h_1,\ h_2 \in L^{1+\varepsilon}(\mathbb{R}^2;\lambda^2)$, $\alpha,\ \beta \in \mathbb{R}$

$$: z^n : (\alpha h_2 + \beta h_2) = \alpha\ : z^n : (h_1) + \beta\ : z^n : (h_2) \quad \mu_0 - a.e. \tag{4.21}$$

Let $P : \mathbb{R} \to \mathbb{R}$ be defined by

$$P(s) := \sum_{n=0}^{2N} b_n\, s^n,\ \ s \in \mathbb{R}\ , \tag{4.22}$$

$b_n \in \mathbb{R}$, $N \in \mathbb{N}$ and $b_{2N} > 0$. Define for $U \subset \mathbb{R}^2$, U open, bounded,

$$a_U(z) := \sum_{n=0}^{2N} b_n\ : z^n : (1\!\!1_U)\ , \tag{4.23}$$

where as before $1\!\!1_U$ means indicator function of U. We have that

$$e^{-a_U} \in L^p(\mathcal{D}';\mu_0) \quad \text{for all } p \in [1,\infty[\tag{4.24}$$

(cf. [S74, §5.2] and [GlJ81]).

Remark 4.8 In this paper we only consider the "polynomial case". Similarly, we can handle the "trigonometric" and the "Høegh-Krohn model" (where P in (4.22), (4.23) is replaced by a tigonometric resp. exponential function, cf. [AH-K74], [AH-K79] and also [Ze84] for details).

Define for $U \subset \mathbb{R}^2$, U open,

$$\mathcal{D}(U) := \{\varphi \in \mathcal{D} \mid \operatorname{supp} \varphi \subset U\} ,$$

and let $\sigma(U)$ be the sub-σ-algebra of $\mathcal{B}(\mathcal{D}')$ generated by all $\varphi \in \mathcal{D}(U)$. For $A \subset \mathbb{R}^2$ define

$$\sigma(A) := \bigcap_{\substack{A \subset U \\ U \text{ open}}} \sigma(U) .$$

Definition 4.9 ([GuRoS75, 76]) A probability measure ν on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$ is called a *Guerra–Rosen–Simon (=GRS)–Gibbs state* with coupling constant $\lambda \geq 0$ if for any $U \subset \mathbb{R}^2$, U open and bounded,

(i) $\nu_{|_{\sigma(U)}}$ (i.e., the restriction of ν to $\sigma(U)$) is absolutely continuous with respect to $\mu_{0|_{\sigma(U)}}$.

(ii) $E_\nu[f \mid \sigma(U^c)] = E_\nu[f \mid \sigma(\partial U)]$ ν-a.e. for any $\sigma(U)$-measurable $f : \mathcal{D}' \to \mathbb{R}_+$ (where $U^c := \mathbb{R}^2 \setminus U$ and ∂U means topological boundary of U).

(iii) For every $\sigma(U)$-measurable $f : \mathcal{D}' \to \mathbb{R}_+$

$$E_\nu[f \mid \sigma(\partial U)] = \frac{E_{\mu_0}[f e^{-\lambda a_U} \mid \sigma(\partial U)]}{E_{\mu_0}[e^{-\lambda a_U} \mid \sigma(\partial U)]} \quad \nu - a.e.$$

Let $\mathcal{G}_\lambda$ denote the set of all GRS-Gibbs states with coupling constant $\lambda \geq 0$.

From now on we fix $\lambda \geq 0$.

Remark 4.10 (i) A probability measure μ on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$ is said to have the *local Markov property* if (ii) in Definition 4.7 holds for any $U \subset \mathbb{R}^2$, U open, bounded.

(ii) In [R86] a (local) specification $(\pi_U^\lambda)_{U \in \mathbb{L}}$ was constructed such that the associated Gibbs states are exactly the GRS-Gibbs states above and a representation formula of arbitrary GRS-Gibbs states in terms of extremal Gibbs states was derived. We refer to [R86] for the precise definition of $(\pi_U^\lambda)_{U \in \mathbb{L}}$. We only note here that the system $\mathbb{L}$ of open sets contains all open balls in $\mathbb{R}^2$. Using this specification one can locally define general boundary conditions and interpret the GRS-Gibbs state as the corresponding "infinite volume" limits (see [R86] for details). We emphasize that it is entirely useless to construct some abstract specification so that the associated Gibbs states are exactly the GRS-Gibbs states. The point of [R86] is that the corresponding kernels are given by *explicit* formulae. We shall use this specification to prove Theorem 4.11 below in an essential way.

(iii) By [R86, Theorems 5.4 and 5.6] in Definition 4.7 (i) "absolutely continuous" can be replaced by "equivalent".

(iv) We have that $\mathcal{G}_\lambda \neq \emptyset$ and, in general, $\#G_\lambda > 1$ as shown in the above quoted literature (cf. e.g. [GuRoS75, 76], [FrS77], [R86], [R85]).

For $r \in]0, \infty[$ let U_r denote the open ball in $\mathbb{R}^2$ with radius r. Fix $n, i \in \mathbb{N}$, $i \geq 2$. Let $\mathcal{D}_1(U_{n-\frac{1}{i}})$ be a countable dense subset of $\mathcal{D}(U_{n-\frac{1}{i}})$ (endowed with the topology inherited from $\mathcal{D}$) such that any finite collection of elements in $\mathcal{D}_1(U_{n-\frac{1}{i}})$ is linear independent. Applying Gram–Schmidt orthogonalization we can thus construct $D_{n-\frac{1}{i}} \subset \mathcal{D}(U_{n-\frac{1}{i}})$ such that $D_{n-\frac{1}{i}} \subset D_{n'-\frac{1}{i'}}$, if $n \leq n'$ and $i \leq i'$, and $D_{n-\frac{1}{i}}$ is an orthonormal basis of $L^2(U_{n-\frac{1}{i}}; \lambda^2)$ for all $n, i \in \mathbb{N}$, $i \geq 2$. Set

$$K_0 := \bigcup_{\substack{n,i\in\mathbb{N}\\ i\geq 2}} D_{n-\frac{1}{i}} \; . \tag{4.25}$$

Then $K_0 \subset \mathcal{D}\ (= E')$ and K_0 is an orthonormal basis of $L^2(\mathbb{R}^2; \lambda^2)$ separating the points of $\mathcal{D}'$. For $k \in \mathcal{D}$, $t \in \mathbb{R}$, define for $z \in \mathcal{D}'$

$$a_{tk}(z) := a^0_{tk}(z) \cdot\, a^\lambda_{tk}(z) \tag{4.26}$$

where

$$a^0_{tk}(z) := \exp\left[-t\langle(-\Delta + m^2)\, k, z\rangle - \frac{1}{2}t^2\langle(-\Delta + m^2)\, k, k\rangle\right] \tag{4.27}$$

and

$$a^\lambda_{tk}(z) := \exp\left[-\lambda \sum_{n=0}^{2N} b_n \sum_{i=0}^{n-1} \binom{n}{i} t^{n-i} \;:z^i:\; (k^{n-i})\right] \; . \tag{4.28}$$

The reader should forgive us concerning the abuse of notation concerning (4.23) and (4.26). We emphasize that

$$\text{both } a^0_{tk} \text{ and } a^\lambda_{tk} \text{ is } \sigma(U)\text{–measurable provided } k \in \mathcal{D}(U). \tag{4.29}$$

Let now $\mathcal{M}^a$ be defined as in (A.2) with K_0 and a_{tk} as in (4.26) – (4.28), and let $\mathcal{M}^{a,\mathcal{D}_1}$ be defined correspondingly with $\mathcal{D}_1$ replacing K_0 for any $\mathcal{D}_1 \subset \mathcal{D}$. Define

$$M_1(\mathcal{D}')_{loc,\mu_0} := \{\nu \in M_1(\mathcal{D}') \mid \nu_{|\sigma(U_n)} \sim \mu_{0|\sigma(U_n)} \quad \text{for all } n \in \mathbb{N}\}$$

where "$\sim$" means "equivalent", i.e., the two measures have the same zero sets. For a subset $\mathcal{M} \subset M_1(\mathcal{D}')$ set

$$\mathcal{M}_{loc,\mu_0} := \mathcal{M} \cap M_1(\mathcal{D}')_{loc,\mu_0} \ .$$

Now we can formulate and prove the first main result of this subsection.

Theorem 4.11 $\mathcal{G}_\lambda = \mathcal{M}^{a,\mathcal{D}}_{loc,\mu_0} = \mathcal{M}^a_{loc,\mu_0}$.

Remark 4.12 (i) In [S74, Theorem X.14] a stronger version of "$\mathcal{G}_\lambda = \mathcal{M}^{a,\mathcal{D}}_{loc,\mu_0}$", i.e., $\mathcal{G}_\lambda = \mathcal{M}^{a,\mathcal{S}} \cap \mathcal{M}_0$ (where $\mathcal{S}$ ($\supset \mathcal{D}$) is the space of all smooth functions on $\mathbb{R}^2$ decreasing with all derivatives for $|x| \longrightarrow \infty$ faster than any $x \mapsto (1+|x|^2)^M$, $M < 0$, and $\mathcal{M}_0$ denotes the set of all $\nu \in M_1(\mathcal{D}')$ which are locally absolutely continuous w.r.t. μ_0), is announced as a result of J. Fröhlich in a preprint of 1973. However, in the published version (i.e., [Fr74, 77]) only those measures in $\mathcal{M}^{a,\mathcal{D}}_{loc,\mu_0}$ which are "compactly Π–ergodic" (cf. [Fr77, Definition 2.3]) have been shown to be in $\mathcal{G}_\lambda$ (cf. [Fr77, Corollary 2.13]). Conversely, also only very special measures in $\mathcal{G}_\lambda$ have been shown to belong to $\mathcal{M}^{a,\mathcal{D}}_{loc,\mu_0}$ (cf. [Fr77, Theorems 4.5, 4.8]). In [FrS77, Final remark in Sect. 6] it finally has been shown that all "ultra-regular" (see [FrS77, Sect. 6]) elements in $\mathcal{G}_\lambda$ belong to $\mathcal{M}^{a,\mathcal{D}}_{loc,\mu_0}$. In this paper we present a full proof of Theorem 4.11 based on the results of [R85, 86]. It should also be noted that the formulation of [S74, Theorem X.14] with $\mathcal{S}$ instead of $\mathcal{D}$ is problematic since for $k \in \mathcal{S}$, a^λ_{tk} in (4.28) is no longer locally measurable (since $: z^i : (k^{n-i})$ is $\sigma(\text{supp } k)$ – measurable, while $\text{supp}\, k$ is no longer compact). Hence $\mathcal{M}^{a,\mathcal{S}} \cap \mathcal{M}_0$ will drastically depend on the special μ_0–version chosen for the $L^2(\mathcal{D}';\mu_0)$–class $: z^i : (k^{n-i})$, but $\mathcal{G}_\lambda$ does not. We, therefore, suspect that writing $\mathcal{S}$ for $\mathcal{D}$ is just a misprint in [S74, Theorem X.14].

(ii) Concerning the analogue of Theorem 4.11 in the $P(\Phi)_1$–case, i.e., the case where $\mathbb{R}^2$ above is replaced by $\mathbb{R}^1$ we refer to [CoRe75], [PrYo75], [Roy75], and [RoyYo76].

Proof of Theorem 4.11. For brevity we have to assume the reader to be familiar with the terminology and results in [R86]. We shall use the same notation, hence rename μ_0, μ by P_0, P respectively. a_U from now on will always denote the special version of the right hand side of (4.23) constructed in [R86], which is $]-\infty,\infty[$–valued a.e. w.r.t. P_0 and $\pi_U(z,\cdot)$, $z \in E$ (cf. [R86, Theorem 4.1]). Let π_U, π_U^λ, $\overline{H}_U$, P_U be as ibidem. The proof consists of three claims:

Claim 1. $\mathcal{G}_\lambda \subset \mathcal{M}^{a,\mathcal{D}}_{loc,P_0}$.

Proof. Let $P \in \mathcal{G}_\lambda$. Then $P_{|\sigma(U_n)} \sim P_{0|\sigma(U_n)}$ for all $n \in \mathbb{N}$ by Remark 4.10 (iii). Let $k \in \mathcal{D}$ and $n \in \mathbb{N}$ such that supp $k \subset U_n$. Since P is a Gibbs state for $(\pi^\lambda_{U_n})_{n\in\mathbb{N}}$ (cf. [R86, Theorem 5.4]), it follows for all $f \in \mathcal{B}_b(\mathcal{D}')$

$$\int f(z-k)\ P(dz) = \int \pi^\lambda_{U_n}(z, f(\cdot - k))\ P(dz)\ . \tag{4.30}$$

But by definition of $\pi^\lambda_{U_n}$ (cf. [R86, (4.3) and (2.7)]) for all $z \in E$

$$\begin{aligned}\pi^\lambda_{U_n}(z, f(\cdot - k)) \ &= \ \int f(z' + \overline{H}_{U_n}(z) - k)\, e^{-\lambda a_{U_n}(z' + \overline{H}_{U_n}(z))} P_{U_n}(dz') \\ &\quad \cdot \left(\int e^{-\lambda a_{U_n}(z' + \overline{H}_{U_n}(z))} P_{U_n}(dz') \right)^{-1}\ .\end{aligned}$$

But e.g. by [R86, Proposition 5.10]

$$\frac{(P_{U_n} \circ \tau_k^{-1})(dz)}{P_{U_n}(dz)} = a^0_k(z) \quad \text{for } P_{U_n} - a.e.\ z \in \mathcal{D}'\ ,$$

and $a^0_k(z' + \overline{H}_{U_n}(z)) = a^0_k(z')$ for all z, $z' \in \mathcal{D}'$, since $\overline{H}_{U_n}(z)$ is $(-\Delta + m^2)$–harmonic on U_n. Hence

$$\pi^\lambda_{U_n}(z, f(\cdot - k)) = \pi^\lambda_{U_n}(z, f\ a^0_k\ \ e^{-\lambda[a_{U_n}(\cdot + k) - a_{U_n}]}).$$

Hence by (4.30)

$$\int f(z-k) P(dz) = \int f\ a^0_k\ e^{-\lambda[a_{U_n}(\cdot + k) - a_{U_n}]}\ dP\ .$$

But by e.g. the same arguments that prove [R86, Theorem 3.3] we obtain (since $k \in \mathcal{D}(U_n)$) that

$$e^{-\lambda[a_{U_n}(\cdot + k) - a_{U_n}]} = a^\lambda_k \qquad P - a.e. \tag{4.31}$$

since $P_{|\sigma(U_n)} \sim P_{U_n|\sigma(U_n)}$ (cf. [R86, Theorem 5.6]). Hence $P \in \mathcal{M}^{a,\mathcal{D}}_{loc,P_0}$. $\square$

Claim 2. Let $P \in \mathcal{M}^a_{loc,P_0}$ and $n, n' \in \mathbb{N}$, $n' > n$. Then there exists $\rho^{U_{n'} \setminus U_n} : \mathcal{D}' \to]0, \infty[$, $\sigma(U_{n'} \setminus U_n)$–measurable such that $P = e^{-\lambda a_{U_n}} \rho^{U_{n'} \setminus U_n} P_0$ on $\sigma(U_{n'})$.

Proof. Let $\rho : \mathcal{D}' \to]0, \infty[$ be $\sigma(U_{n'})$–measurable such that

$$P_{|\sigma(U_{n'})} = \rho \, P_{0|\sigma(U_{n'})} \ .$$

By (4.29) and Remark 2.1 (ii) we have that

$$\mathcal{M}^a_{loc,\,\mu_0} = \mathcal{M}^{a,K}_{loc,\,P_0}$$

where K denotes the linear span of K_0. Let $\left\langle D_{n'} \right\rangle$ denote the linear span of $D_{n'}$. Since $P \in \mathcal{M}^a_{loc,P_0}$ and $P_0 \in \mathcal{M}^{a^0}$ (see e.g. [R86, Proposition 5.10]), it follows that for all $k \in \left\langle D_{n'} \right\rangle$

$$a^\lambda_k = \frac{\rho(\cdot + k)}{\rho} \; a^0_k \qquad P_{0|\sigma(U_{n'})} - a.e.$$

Hence as in (4.31) by (4.26) we obtain that

$$\rho \; e^{\lambda a_{U_{n'}}} = \rho(\cdot + k) \; e^{\lambda a_{U_{n'}}(\cdot + k)} \tag{4.32}$$

$P_{0|\sigma(U_{n'})}$–a.e for all $k \in \left\langle D_{n'} \right\rangle$. Fix $N \in \mathbb{N}$ and define

$$\begin{aligned} \rho_N &:= \min(\rho \; e^{\lambda a_{U_{n'}}}, \, N) \ , \\ c_N &:= \left(\int \rho_N dP_0 \right)^{-1} \ , \\ \nu_N &:= c_N^{-1} \rho_N P_0 \ , \end{aligned}$$

and

$$\tilde{\nu}_N(k) := \int_{\mathcal{D}'} e^{i\langle k, z\rangle} \; \nu_N(dz), \qquad k \in \mathcal{D} \ .$$

Then for any $k \in \left\langle D_{n'} \right\rangle$ and $k_1 := (-\Delta + m^2)\, k$

$$\begin{aligned} \frac{d}{dt}\tilde{\nu}_N(tk_1) \quad &= \quad i \int e^{it\langle k, z\rangle} \langle k_1, z\rangle \; \nu_N(dz) \\ &= \quad (-i)\frac{d}{d\varepsilon} \int e^{it\langle k_1, z\rangle} a^0_{\varepsilon k}(z) \; \nu_N(dz)\bigg|_{\varepsilon=0} \\ &\underset{\text{by (4.32)}}{=} \quad (-i)\frac{d}{d\varepsilon} \int e^{it\langle k_1, z - \varepsilon k\rangle} \; \nu_N(dz)\bigg|_{\varepsilon=0} \\ &= \quad t\,\langle k_1, k\rangle \; \tilde{\nu}_N(tk_1) \end{aligned}$$

for all $t \in \mathbb{R}$. Hence, since $\widetilde{\nu}_N(0) = 1$,

$$\begin{aligned} \widetilde{\nu}_N(tk_1) &= e^{-\frac{1}{2}t^2\langle(-\Delta+m^2)k,k\rangle} \\ &= \int e^{it\langle k_1,z\rangle}\ P_0(dz) \end{aligned}$$

for all $t \in \mathbb{R}$. Fix $i \in \mathbb{N}$, $i \geq 2$. Since every $k \in \mathcal{D}(U_{n'-\frac{1}{i}})$ is a limit of a sequence $((-\Delta+m^2)k_j)_{j\in\mathbb{N}}$ in $L^2(\mathbb{R}^2;\lambda^2)$ for certain $k_j \in \left\langle D_{n'}\right\rangle$, $j \in \mathbb{N}$, since ν_N has a bounded density w.r.t. P_0, and since P_0 is Gaussian, it follows that

$$\widetilde{\nu}_N(k) = \int e^{i\langle k,z\rangle}\ P_0(dz)$$

for all $k \in \mathcal{D}(U_{n'-\frac{1}{i}})$. Consequently,

$$\nu_{N|\sigma(U_{n'-\frac{1}{i}})} = P_{0|\sigma(U_{n'-\frac{1}{i}})}\ . \tag{4.33}$$

Let $k \in \mathcal{D}(U_{n'})$. Then by (a slight modification of the arguments leading to) [R86, Proposition 5.3] there exists a $\sigma(\operatorname{supp} k \cap (U_{n'} \setminus U_n))$-measurable random variable $X_1 \in \bigcap_{p\geq 1} L^p(\mathcal{D}'; P_0)$ and a $\sigma(U_n)$-measurable $X_2 \in \bigcap_{p\geq 1} L^p(\mathcal{D}'; P_0)$ such that

$$\langle k, \cdot\rangle = X_1 + X_2 \qquad P_0 - a.e.$$

Hence (since $\sigma(U_{n'} \setminus U_n) \subset \sigma(U_n^c)$)

$$\begin{aligned} & \int e^{i\langle k,z\rangle}\ E_{P_0}[\rho_N \mid \sigma(U_{n'} \setminus U_n)](z)\ P_0(dz) \\ =\ & \int e^{iX_1}\ E_{P_0}[e^{iX_2} \mid \sigma(U_n^c)]\ E_{P_0}[\rho_N \mid \sigma(U_{n'} \setminus U_n)]\ dP_0 \\ =\ & \int e^{iX_1}\ E_{P_0}[e^{iX_2} \mid \sigma(\partial U_n)]\ E_{P_0}[\rho_N \mid \sigma(U_{n'} \setminus U_n)]\ dP_0 \\ =\ & \int e^{iX_1}\ E_{P_0}[e^{iX_2} \mid \sigma(\partial U_n)]\ \rho_N\ dP_0 \\ =\ & c_N \int e^{iX_1}\ E_{P_0}[e^{iX_2} \mid \sigma(\partial U_n)]\ dP_0 \\ =\ & c_N \int E_{P_0}[e^{i(X_1+X_2)} \mid \sigma(U_n^c)]\ dP_0 \\ =\ & c_N \int e^{i\langle k,z\rangle} P_0(dz) \\ =\ & \int e^{i\langle k,z\rangle}\rho_N(z)\ P_0(dz)\ , \end{aligned} \tag{4.34}$$

where we used the (local) Markov property of P_0 (see e.g. [R85, Theorem 8.10]) in the second as well as in the fifth step, and (4.33) in the last as well as the fourth step. (Concerning the latter the reader should note that supp $k \subset U_{n'-\frac{1}{i}}$ for some large enough $i \in \mathbb{N}$.) (4.34) clearly implies that ρ_N is P_0–a.s. equal to a $\sigma(U_{n'} \setminus U_n)$–measurable function, hence so is $\rho \cdot e^{\lambda a_{U_{n'}}}$. But by [R86, Theorem 4.1] there exists a $\sigma(U_{n'} \setminus U_n)$–measurable function, denoted by $a_{U_n, U_{n'} \setminus U_n}$ in [R86], such that

$$a_{U_{n'}} = a_{U_n} + a_{U_n, U_{n'} \setminus U_n} \qquad P_0 - a.e.$$

Hence

$$\rho = e^{-\lambda a_{U_n}} \rho^{U_{n'} \setminus U_n} \qquad P_0 - a.e.$$

with $\rho^{U_{n'} \setminus U_n} := \rho e^{\lambda a_{U_{n'}}} \, e^{-\lambda a_{U_n, U_{n'} \setminus U_n}}$ $\sigma(U_{n'} \setminus U_n)$–measurable and the claim is proven. □

Claim 3. $\mathcal{M}^a_{loc,P_0} \subset \mathcal{G}_\lambda$.

Proof. Let $P \in \mathcal{M}^a_{loc,P_0}$. Let $n \in \mathbb{N}$ and $k \in \mathcal{D}$. By [R86, Lemma 5.15] there exists $n' \in \mathbb{N}$ such that $z \mapsto \pi^\lambda_U(z, e^{i\langle k, \cdot \rangle})$ is $\sigma(U_{n'})$ – measurable. Hence by Claim 2

$$\begin{aligned}
& \int \pi^\lambda_{U_n}(z, e^{\langle k, \cdot \rangle}) \; P(dz) \\
= & \int \pi^\lambda_{U_n}(z, e^{\langle k, \cdot \rangle}) \; e^{-\lambda a_{U_n}(z)} \rho^{U_{n'} \setminus U_n}(z) P_0(dz) \\
= & \int \frac{\pi_{U_n}(z, e^{\langle k, \cdot \rangle} \; e^{-\lambda a_{U_n}})}{\pi_{U_n}(e^{-\lambda a_{U_n}})} \; \pi_{U_n}(e^{-\lambda a_{U_n}}) \; \rho^{U_{n'} \setminus U_n}(z) \, P_0(dz) \\
= & \int e^{i\langle k, \cdot \rangle} \; e^{-\lambda a_{U_n}} \; \rho^{U_{n'} \setminus U_n} \, dP_0 \\
= & \int e^{\langle k, \cdot \rangle} \; dP \; ,
\end{aligned} \tag{4.35}$$

where we used the definition of $\pi^\lambda_{U_n}$ (cf. [R86, (4.3)] in the second step and the fact that P_0 is a Gibbs state for π_{U_n}, $U \subset \mathbb{R}^2$, U open (cf. [R85, Theorem 8.5]) in the second and third step. By the consistency condition satisfied by $(\pi^\lambda_U)_{U \in \mathbb{L}}$ (cf. [R86, Theorem 4.3 and (1.4)]) and since $U_n \nearrow \mathbb{R}^2$, it follows that (4.35) holds for any $U \in \mathbb{L}$ replacing U_n. Hence $P \in \mathcal{G}_\lambda$ (cf. Remark 4.10 (ii)). □

Since clearly $\mathcal{M}^{a,\mathcal{D}}_{loc,P_0} \subset \mathcal{M}^a_{loc,P_0}$, the assertion of the theorem now follows from Claim 1 and Claim 3. $\square$

Lemma 4.13 $\mathcal{M}^a_{ex} \cap M_1(\mathcal{D}')_{loc,\mu_0} = \left(\mathcal{M}^a_{loc,\mu_0}\right)_{ex} = (\mathcal{G}_\lambda)_{ex}$ *(where for a convex set $\mathcal{M} \subset M_1(\mathcal{D}')$ the corresponding set of extreme points is denoted by $\mathcal{M}_{ex}$). In other words: $\mathcal{M}^a_{loc,\mu_0}$ is a face in $\mathcal{M}^a$.*

Proof. Obviously, $\mathcal{M}^a_{ex} \cap M_1(\mathcal{D}')_{loc,\mu_0} \subset (\mathcal{M}^a_{loc,\mu_0})_{ex} = (\mathcal{G}_\lambda)_{ex}$ (where the last equality follows from Theorem 4.11). Now let $\nu \in (\mathcal{M}^a_{loc,\mu_o})_{ex}$. Since $\nu \in \mathcal{M}^a \cap M_1(\mathcal{D},)_{loc,\mu_0}$, we only have to show that ν is extreme in $\mathcal{M}^a$. So, let $\nu_1, \nu_2 \in \mathcal{M}^a$ and $\alpha \in]0,1[$ such that

$$\nu = \alpha\nu_1 + (1-\alpha)\nu_2 \ .$$

Then both ν_1 and ν_2 are absolutely continuous w.r.t. ν, hence $\widetilde{\nu}_1 := \frac{1}{2}(\nu_1+\nu)$, $\widetilde{\nu}_2 := \frac{1}{2}(\nu_2+\nu)$ are both equivalent to ν and $\widetilde{\nu}_1$, $\widetilde{\nu}_2 \in \mathcal{M}^a$. Consequently, $\widetilde{\nu}_1, \widetilde{\nu}_2 \in \mathcal{M}^a_{loc,\mu_0}$. But

$$\nu = \alpha\widetilde{\nu}_1 + (1-\alpha)\widetilde{\nu}_2 \ .$$

Since $\nu \in (\mathcal{M}^a_{loc,\mu_0})_{ex}$ it follows that $\nu = \widetilde{\nu}_1 = \widetilde{\nu}_2$, hence $\nu = \nu_1 = \nu_2$. Consequently,

$$\nu \in \mathcal{M}^a_{ex} \ .$$

$\square$

Now we can formulate and prove one of the main results of this paper. To a large extent the theory developed in Sections 2, 3 was designed to apply to this situation.

Theorem 4.14 *(i) The following assertions are equivalent for a GRS–Gibbs state ν (i.e., $\nu \in \mathcal{G}_\lambda$):*

(a) $\nu \in (\mathcal{G}_\lambda)_{ex}$.

(b) ν is K_0–ergodic.

(c) $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is irreducible.

(ii) Let $\nu \in (\mathcal{G}_\lambda)_{ex}$ such that $\nu(\mathcal{S}'(\mathbb{R}^2)) = 1$. Then there exists a diffusion process $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in\mathcal{D}'})$ on $\mathcal{D}'$ which is properly associated with $(\mathcal{E}^\circ_\nu, D(\mathcal{E}^\circ_\nu))$. Furthermore, $(\mathcal{E}^\circ_\nu, D(\mathcal{E}^\circ_\nu))$ is irreducible, $(\overset{\circ}{T}{}^\nu_t)_{t\geq 0}$ is ergodic, and P_ν is time–ergodic.

Proof. Clearly, assumptions (A.1) and (A.2) are satisfied. (A.3), (A.4) might not hold for arbitrary μ_0–versions of a_{tk} in (4.26) (more precisely, of a_{tk}^{λ} in (4.28)). But we can apply Proposition 2.4 (i), (ii) to construct appropriate μ_0–versions that satisfy (A.3), (A.4). Because of (4.29) the set $\mathcal{M}^{a}_{loc,\mu_0}$ remains unchanged when going over from the initial to these new versions.
(i): "(a)$\Leftrightarrow$(b)" now follows from Theorems 2.10 (ii), 4.11, and Lemma 4.13. "(b)$\Leftrightarrow$(c)" follows from Theorems 3.3 and 4.11.
(ii): Since $\nu \in (\mathcal{G}_\lambda)_{ex}$, we know by (i) that ν is K_0-ergodic. Let Φ_n with dual spaces Φ_n', $n \in \mathbb{N}$, be as in [AR89a, Subsection 3d)], in particular $L^2(\mathbb{R}^2;\lambda^2) \subset \Phi_n'$ for all $n \in \mathbb{N}$ and

$$\mathcal{S}'(\mathbb{R}^2) = \bigcup_{n\in\mathbb{N}} \Phi_n' \ .$$

Since $K_0 \subset \Phi_n'$, $1\!\!1_{\Phi_n'}$ is K_0–invariant ν-a.e. for all $n \in \mathbb{N}$. By assumption $\nu(\bigcup \Phi_n') = 1$, hence $\nu(\Phi_n') = 1$ for some $n \in \mathbb{N}$ since ν is K_0–ergodic. Now the assertions follow by Theorems 3.8, 3.9 and Remark 3.10 (i). $\square$

Remark 4.15 (i) Using Corollary 2.19 we could also prove that each $\nu \in \mathcal{G}_\lambda$ can be represented in terms of extreme elements. But this is well-known by [R86, Theorem 7.4]. However, $(\Pi_{>n})_{n\in\mathbb{N}}$ (defined in (2.20)) by Theorem 2.14 is a *new* specification which (as well as $(\Pi_k)_{k\in K_0}$ (defined in (2.16)) by Proposition 2.13) describes $\mathcal{G}_\lambda$ completely, in terms of conditional probabilities on $\mathbb{R}^n$, $n \in \mathbb{N}$ (resp. merely $\mathbb{R}$).

(ii) The assumption $\nu(\mathcal{S}'(\mathbb{R}^2)) = 1$ in Theorem 4.14 (ii) was only made to apply Theorems 3.8, 3.9 and Remark 3.10 in a clean way. The difficulty is that so far the existence of the diffusion process $\mathbb{M}$ (i.e., the *stochastic quantization* of ν) was only proved rigorously under this assumption. We expect, however, that this can be done also in the general case and then the above assumption on the support of ν can be dropped. If $\mathcal{G}_{\lambda,\mathcal{S}'}$ denotes the subset of $\mathcal{G}_\lambda$ consisting of all ν with $\nu(\mathcal{S}'(\mathbb{R}^2)) = 1$, then it is well-known that $\mathcal{G}_{\lambda,\mathcal{S}'} \neq \emptyset$ (cf. [GlJ81]). The proof of Lemma 4.13 shows that $\mathcal{G}_{\lambda,\mathcal{S}'}$ is a face of $\mathcal{G}_\lambda$ (i.e., $(\mathcal{G}_{\lambda,\mathcal{S}'})_{ex} \subset (\mathcal{G}_\lambda)_{ex}$). [R86, Theorem 7.4] or Corollary 2.19 then immediately imply that $(\mathcal{G}_{\lambda,\mathcal{S}'})_{ex} \neq \emptyset$, and Theorem 4.14 (ii) applies to each $\nu \in (\mathcal{G}_{\lambda,\mathcal{S}'})_{ex}$ i.e., the corresponding stochastic quantizations are time-ergodic diffusions.

(iii) If $\mathcal{G}_{\lambda,u.r.} \subset \mathcal{G}_{\lambda,\mathcal{S}'}$ denotes the set of all *ultra-regular* GRS – Gibbs states (cf [FrS77]), then all assertions about $\mathcal{G}_{\lambda,\mathcal{S}'}$ in (ii) also hold for $\mathcal{G}_{\lambda,u.r.}$.

In particular, if $\mathcal{G}_{\lambda,u.r.} = \{\nu\}$ (which holds for small enough λ according to a result in [AH-KZe89a,b]), then $\nu \in (\mathcal{G}_{\lambda,\mathcal{S}'})_{ex}$ and Theorem 4.14 (ii) applies to this ν.

(iv) If ν is as in Theorem 4.14 (ii), the results in [Mü92, 94] about large deviations of the occupation density of $\mathbb{M}$ from ergodic behaviour apply.

(v) The question about Markov uniqueness for $\nu \in \mathcal{G}_\lambda$, i.e., whether $D(\mathcal{E}_\nu^\circ) = D(\mathcal{E}_\nu)$ (cf. Remark 3.6 (iii)), is entirely open.

(vi) Also the stochastic dynamics for $P(\Phi)_2$ – fields $\nu_{f.vol.}$ in finite volume was constructed (cf. [J-LMi85], [Mi86], [BoCMi88], [AR89a,b], [AR91], [RZ92], [GaGo95]). As also mentioned in most of these references in this case, if the regularization parameter α (cf [RZ92, Sect. 7]) is *strictly* positive, the ergodicity of the corresponding semigroup resp. the time-ergodicity of the process resp. the irreducibility of the Dirichlet form easily follows from known results. Let us sketch one line of arguments that proves this: By [RZ92, Theorem 7.5] $D(\mathcal{E}_{f.vol.}^\circ) = D(\mathcal{E}_{f.vol.})$. So, all extremality / irreducibility / ergodicity properties of the involved objects are equivalent (cf. Theorem 3.9 above). By [ARZ93a, Theorem 1.3] (cf. also the remarks subsequent to Theorem 1.5 ibidem) and [F82, Corollary to Theorem 2] it follows that the Dirichlet form $(\mathcal{E}_{f.vol.}, D(\mathcal{E}_{f.vol.}))$ is irreducible and everything is proved. We note that though in [F82] the state space is assumed to be locally compact and the Dirichlet form to be regular, the results in [F82] nevertheless apply in this case by the general "local compactification / regularization method" in [MR92, Chap. VI] (cf. [Mü92] for details concerning the implementation of the latter method for the results in [F82]).

(vii) Concerning the analogous question in case of "time-zero fields" (see e.g. [AR91, Subsection 7.II.b)]) about the ergodicity of the associated stochastic quantization, we finally note the following: if $\nu \in \mathcal{G}_{\lambda,\mathcal{S}'}$ is replaced by a time-zero quantum field ν_0 in $\mathcal{S}'(\mathbb{R}^1)$, the corresponding Dirichlet form $(\mathcal{E}_\nu^\circ, D(\mathcal{E}_\nu^\circ))$ on $L^2(\mathcal{S}'(\mathbb{R}); \nu_0)$ exists by [AR90a, Subsection 5] resp. [AH-K77] and the corresponding stochastic quantization, i.e., a diffusion $\mathbb{M}_0$ as above, was constructed in [AR91, Subseciton 7.II.b)]. If ν_0 comes from a space-time field ν on $\mathcal{S}'(\mathbb{R}^2)$ satisfying the *global Markov property* (cf. [AH-KZe89b]), then by the famous "mass–gap result" of J. Glimm and A. Jaffe (cf. [GlJ81]) a certain Dirichlet form on $L^2(\mathcal{S}'(\mathbb{R}); \nu_0)$ which extends $(\mathcal{E}_\nu^\circ, D(\mathcal{E}_\nu^\circ))$ is irreducible, hence so is $(\mathcal{E}_\nu^\circ, D(\mathcal{E}_\nu^\circ))$. In particular, the corresponding semigroup $(\overset{\circ}{T}{}_t^{\nu_0})_{t \geq 0}$ on $L^2(\mathcal{S}'(\mathbb{R}); \nu_0)$ is ergodic, and $\mathbb{M}_0$ is time-ergodic.

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